Mathematisches Institut der Universität München

Prof. Dr. S. Bachmann
Dr. R. Helling

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Problem Sheet 4

Hand-in deadline: 10.11.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

For this sheet, you may recall the following three facts:

(F1) For $\lambda_0 \in \mathbb{R}$, the *Dirac measure at* λ_0, denoted ν_{λ_0}, is defined by

$$\nu_{\lambda_0}(M) = \begin{cases} 1 & \text{if } \lambda_0 \in M \\ 0 & \text{otherwise} \end{cases}$$

for any Borel set M. This implies $\int_{\mathbb{R}} f(\lambda) d\nu_{\lambda_0}(\lambda) = f(\lambda_0)$ for any continuous function f.

(F2) A finite measure is uniquely determined by its moments: If $\int \lambda^n d\mu(\lambda) = \int \lambda^n d\tilde{\mu}(\lambda)$ for all $n \in \mathbb{N}$ and they are all finite, then[1] $\mu = \tilde{\mu}$.

(F3) The projection-valued measure P^A associated to $A = A^*$ is unique.

Ex. 1: Consider the *parity operator* $\Pi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $\psi(x) \mapsto \psi(-x)$.

(i) Show that Π is self-adjoint.

(ii) Compute its projection-valued measure P^Π.

(iii) Show that $\lambda \in \operatorname{supp}(P^\Pi)$ if and only if $\Pi - \lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have a bounded inverse.

Ex. 2: Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Let λ_0 be an eigenvalue and $\psi \in \mathcal{D}(A)$ a corresponding eigenvector, namely $A\psi = \lambda_0 \psi$.

(i) Compute μ_ψ.

(ii) Show that

$$\operatorname{Ran} P^A(\{\lambda_0\}) = \operatorname{Ker}(A - \lambda_0).$$

Hint: Verify first that $\operatorname{Ran} P^A(\{\lambda_0\}) \subseteq \operatorname{Ker}(A - \lambda_0)$. For the converse, use (i).

[1] In fact, one needs the following additional condition. Denote $M_k = \int \lambda^k d\mu(\lambda)$; the series $\sum_k M_k r^k / k!$ must have a positive radius of convergence.

Ex. 3: Let $\mathcal{H} = L^2([0,1])$, $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and consider, for any $t \in \mathbb{R}$: $(U_\alpha(t))f(x) = \alpha^{[x-t]} f((x-t))$, where $z = [z] + (z)$ is the decomposition in the integer part and the rest of z.

(i) Show that $t \mapsto U(t)$ is a strongly continuous one-parameter unitary group.

(ii) What is the generator of U_α?

Ex. 4: Let $\mathcal{H} = L^2([0,1])$, and let A be the self-adjoint operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathcal{D}(A) = \{f \in H^2([0,1]) : f(0) = f(1) = 0\}$.

(i) Prove that for every $f \in D(A)$, we have

$$\|f\|_\infty^2 \leq \frac{\epsilon}{2}\|f''\|^2 + \frac{1}{2\epsilon}\|f\|^2$$

for every $\epsilon > 0$.

(ii) Let $q \in L^2([0,1])$. Prove that the multiplication operator M_q is relatively bounded with respect to A with A-bound zero.

Jean-Claude Cuenin