Problem Sheet 3

Hand-in deadline: 03.11.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let $\{\chi_j\}_{j=1^m}$ be a family of bounded functions in $C^\infty(\mathbb{R}^3)$ such that $\sum_{j=1}^m \chi_j(x)^2 = 1$ for all $x \in \mathbb{R}^3$. Prove the following identity (known as the IMS localization formula [1]):

$$-\Delta\psi = \sum_{j=1}^m \chi_j(-\Delta)(\chi_j\psi) - \sum_{j=1}^m |\nabla\chi_j|^2\psi, \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^3). \tag{1}$$

Ex. 2: Consider the two Hamiltonians

$$H = -\Delta - \frac{1}{|x|}, \quad H^{(x_0)} = -\Delta - \frac{1}{|x|} - \frac{1}{|x-x_0|}$$

acting on wave functions of the variable $x \in \mathbb{R}^3$. Here, $x_0 \in \mathbb{R}^3 \setminus \{0\}$ is a parameter. Define the respective ground state energies as

$$\mathcal{E}_{\mathrm{GS}} := \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|=1}} \langle\psi, H\psi\rangle, \quad \mathcal{E}_{\mathrm{GS}}^{(x_0)} := \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|=1}} \langle\psi, H^{(x_0)}\psi\rangle$$

where $\mathcal{M} := \{\psi \in L^2(\mathbb{R}^3) : \nabla\psi, |\cdot|^{-1}\psi \in L^2(\mathbb{R}^3)\}$. Recall that $\mathcal{E}_{\mathrm{GS}} = -\frac{1}{4}$ (the ground state being $\psi(x) = C\mathrm{e}^{-\frac{|x|}{2}}$).

(i) Prove that

$$\mathcal{E}_{\mathrm{GS}}^{(x_0)} \leq \mathcal{E}_{\mathrm{GS}} - \frac{1}{2}\mathrm{e}^{-|x_0|} \quad \text{for all } x_0 \in \mathbb{R}^3.$$

(*Hint:* Choose a suitable trial function.)

(ii) Prove that there exist constants $c, R > 0$ such that

$$\mathcal{E}_{\mathrm{GS}}^{(x_0)} \geq \mathcal{E}_{\mathrm{GS}} - \frac{c}{|x_0|}, \quad |x_0| \geq R.$$

(*Hint:* Use the IMS formula (1).)

[1] IMS stands for Ismagilov, Morgan, Simon. Incidentally, its importance in the context of atomic Schrödinger operators was discovered by I.M. Sigal.

Ex. 3: (i) Let $\mathcal{H} := L^2([-1,1])$, and let $\mathcal{H} \ni \psi = |\cdot|$ (i.e. $\psi(x) = |x|$). Compute the weak derivative $\varphi = \psi'$ and show that it can be represented as

$$\varphi[v] = \int_{-1}^{1} \varphi(x)v(x)\,\mathrm{d}x, \quad \text{for all } v \in C_0^\infty([-1,1])$$

with $\varphi \in L^2([-1,1])$.

(ii) Compute the weak derivative $\xi = \varphi'$ and show that there is no function in $L^2([-1,1])$ representing ξ. (*Hint:* Test ξ against functions $v \in C_0^\infty([-1,1])$ with $\operatorname{supp}(v) \cap \{0\} = \emptyset$.)

Ex. 4: Let $\mathcal{H} := L^2([0,1])$, and let P_0, $\widetilde{P}$, P_α ($|\alpha| = 1$) be defined as in the lecture. Recall that $P_0^* = \widetilde{P}$.

(i) Let $\mathcal{D}_\pm := \ker(P_0^* \mp \mathrm{i})$. Show that $\dim \mathcal{D}_\pm = 1$ and exhibit two vectors $\psi_\pm \in \mathcal{D}_\pm$ with $\|\psi_+\| = \|\psi_-\|$.

(ii) Show that [2]

$$\mathcal{D}(P_0) \dotplus \mathcal{D}_+ \dotplus \mathcal{D}_- = \mathcal{D}(P_0^*).$$

(iii) Let $U_e : \mathcal{D}_+ \to \mathcal{D}_-$ be a unitary map, and let $\mathcal{D}_e := (I + U_e)\mathcal{D}_+$. We define $P_e : \mathcal{D}(P_e) \to \mathcal{H}$ by

$$\mathcal{D}(P_e) := \mathcal{D}(P_0) \dotplus \mathcal{D}_e,$$
$$\varphi + \varphi_+ + U_e\varphi_+ \mapsto P_0\varphi + \mathrm{i}\varphi_+ - \mathrm{i}U_e\varphi_+.$$

Show that

$$P_0 \subsetneq P_e \subseteq P_e^* \subsetneq P_0^*.$$

(v) Conclude that $P_e = P_e^*$ (*Hint:* Count dimensions.)

(vi) Show that for any choice of U_e in (iii) there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that

$$\psi \in \mathcal{D}_e \iff \psi \in \mathcal{D}(P_\alpha).$$

Jean-Claude Cuenin

[2] Here, for two subspaces M, N of $\mathcal{H}$, the subspace $M \dotplus N$ denotes the algebraic direct sum, i.e. $M \dotplus N = \{\psi + \chi \in \mathcal{H} : \psi \in M, \chi \in N\}$ and $M \cap N = \{0\}$.