Problem Sheet 2

Hand-in deadline: 27.10.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let $d \geq 3$, and assume that $V = V_1 + V_2$ where $V_1 \in L^{d/2}(\mathbb{R}^d)$ and $V_2 \in L^\infty(\mathbb{R}^d)$. Consider the energy

$$E(\psi) = \int_{\mathbb{R}^d} \left(|\nabla\psi(x)|^2 + V(x)|\psi(x)|^2 \right) \mathrm{d}x, \quad \psi \in H^1(\mathbb{R}^d).$$

(i) Prove that $E(\psi)$ is finite. You may use (without proof) the Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla\psi\|_2^2 \geq S_d \|\psi\|^2_{\frac{2d}{d-2}}, \quad \psi \in H^1(\mathbb{R}^d), \tag{1}$$

where $d \geq 3$ and $S_d > 0$.

(ii) Prove that there exists a constant $\lambda > 0$ such that for all $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\|_2 = 1$, we have

$$E(\psi) \geq \frac{1}{2} T(\psi) - \|V_2\|_\infty - \lambda. \tag{2}$$

Deduce a lower bound for E_0.

Hint: Prove (2) first for the case $\|(V_1)_-\|_{d/2} \leq \epsilon$ with $\epsilon > 0$ sufficiently small. Then prove that for any given $\epsilon > 0$ there exists $\lambda > 0$ such that $\|((V_1)_- + \lambda)_-\|_{d/2} \leq \epsilon$. You may find the following version of Chebyshev's inequality useful,

$$|\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}| \leq \lambda^{-d/2} \|f\|^{d/2}_{d/2}, \quad f \in L^{d/2}(\mathbb{R}^d).$$

(iii) Prove that there exists $C > 0$ such that for all $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\|_2 = 1$, we have

$$T(\psi) \leq 2E(\psi) + C.$$

(iv) Assume that $d = 3$ and $V(x) = -|x|^{-1}$. Use (i) to find a numerical lower bound for E_0. The sharp constant in (1) is $S_3 = \frac{3}{4}(4\pi^2)^{2/3}$.

Ex. 2: (i) Let $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bounded sesquilinear form such that

$$|b(\psi, \varphi)| \leq C\|\psi\|\|\varphi\|.$$

Prove that there exists a unique bounded linear operator B such that

$$b(\psi, \varphi) = \langle \psi, B\varphi \rangle \quad \text{and} \quad \|B\| \leq C.$$

(ii) Let $\mathcal{H} = L^2([0, 1])$ and

$$b(\psi, \varphi) = \int_0^1 \left(\int_0^x \overline{\psi(t)} \, \mathrm{d}t \right) \left(\int_0^x \varphi(t) \, \mathrm{d}t \right) \mathrm{d}x.$$

Prove that b is bounded and determine the corresponding operator B.

Ex. 3: Let $T : \mathcal{H} \supset \mathcal{D} \to \mathcal{H}$ be a bounded linear operator that is densely defined (i.e. $\overline{\mathcal{D}} = \mathcal{H}$). Prove that T extends uniquely to a bounded linear operator $\widetilde{T} : \mathcal{H} \to \mathcal{H}$ and that $\|\widetilde{T}\| = \|T\|$.

Ex. 4: Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Define the multiplication operator

$$M_f : \mathcal{D}(M_f) \to L^2(\mathbb{R}^d), \quad (M_f\psi)(x) := f(x)\psi(x),$$

where

$$\mathcal{D}(M_f) := \{\psi \in L^2(\mathbb{R}^d) : f\psi \in L^2(\mathbb{R}^d)\}.$$

Prove the following:

(i) $M_f^* = M_{\overline{f}}$.

(ii) M_f is bounded if and only if $f \in L^\infty(\mathbb{R}^d)$.

Jean-Claude Cuenin