Problem Sheet 14

Hand-in deadline: 04.02.2016 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: In the following, $\mathcal{H}$ is a finite dimensional Hilbert space. A uniformly continuous *quantum Markov semigroup* (QMS) of $\mathcal{A} := \mathcal{B}(\mathcal{H})$ is a norm continuous one-parameter semigroup $\mathbb{R}_+ \ni t \mapsto \mathcal{T}_t$ of completely positive (CP) linear maps such that $\mathcal{T}_t(1) = 1$ and $\mathcal{T}_0 = \mathrm{id}$. The purpose of this exercise is to prove the following theorem of Gorini-Kossakowski-Sudarshan and Lindblad.

Let $\mathcal{L}$ be the generator of the QMS, $\mathcal{T}_t = \mathrm{e}^{t\mathcal{L}}$.

Note the following which can be seen by identifying $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ with $\mathcal{M}_n(\mathcal{A})$: Φ is a CP map iff any $n \in \mathbb{N}$, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\sum_{i,j} \langle \psi_i, \Phi(X_i^* X_j)\psi_j \rangle \geq 0.$$

Theorem. *If $\mathcal{T}_t$ is a QMS, then there exists a family $L_k \in \mathcal{A}$ and a hermitian $H = H^* \in \mathcal{A}$ such that*

$$\mathcal{L}(X) = \mathrm{i}[H, X] - \frac{1}{2}\sum_k (L_k^* L_k X - 2L_k^* X L_k + X L_k^* L_k)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Reciprocally, any map $\mathcal{L}$ of this form generates a QMS.

1. For any unital CP map Φ, prove that for any $n \in \mathbb{N}$, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\sum_{i,j} \langle \psi_i, (\Phi(X_i^* X_j) - \Phi(X_i)^* \Phi(X_j))\, \psi_j \rangle \geq 0.$$

   (Hint: use Stinespring's dilation)

2. Prove that the generator $\mathcal{L}$ of a QMS is conditionally completely positive (CCP), namely for any $n \in \mathbb{N}$, any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$ such that $\sum_i X_i \psi_i = 0$,

$$\sum_{i,j} \langle \psi_i, \mathcal{L}(X_i^* X_j)\psi_j \rangle \geq 0.$$

   Moreover, show that

$$\mathcal{L}(1) = 0.$$

3. Prove that for any CCP map such that $\mathcal{L}(X^*) = \mathcal{L}(X)^*$, there is an operator $G \in \mathcal{A}$ and a CP map Φ such that
$$\mathcal{L}(X) = G^*X + \Phi(X) + XG$$
(Hint: Define G^* as follows: Let $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$ and
$$G^*\psi := \mathcal{L}\left(|\psi\rangle\langle\psi_0|\right)\psi_0 - \frac{1}{2}\left\langle\psi_0, \mathcal{L}\left(|\psi_0\rangle\langle\psi_0|\right)\psi_0\right\rangle\psi;$$
Moreover, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n$, let $\psi_{n+1} := \psi_0$ and choose a suitable X_{n+1}; Check that Φ defined above is CP)

4. With this, prove the necessity part of the theorem

5. Prove the sufficiency part of the theorem (Hint: $\mathcal{L} = \sum_{j\geq 0}\mathcal{L}_j$ where
$$\begin{aligned}\mathcal{L}_0^{(1)}(X) &:= \mathrm{i}[H, X]\\ \mathcal{L}_0^{(2)}(X) &:= -\frac{1}{2}\sum_k \left(L_k^*L_kX + XL_k^*L_k\right)\\ \mathcal{L}_j(X) &:= +L_j^*XL_j, \qquad j \geq 1.\end{aligned}$$
Use Trotter's product formula)

Ex. 2: Let $\mathcal{H}$ be a finite dimensional Hilbert space. A collection of positive operators $E_i \in \mathcal{B}(\mathcal{H})$ is called a positive operator valued measure (POVM) if
$$\sum_i E_i = id_H$$
Show that there is a Hilbert space $\mathcal{K}$ and a map $V : \mathcal{K} \to \mathcal{H}$ such hat V^*E_iV are orthogonal projections.

Application: The states $f_1 = |1\rangle$ and $f_2 = (|0\rangle + |1\rangle)/\sqrt{2}$ cannot be distinguished by a measurement since their scalar product does not vanish. Devise a measurement scheme (by giving a hermitean operator on a Hilbert space, possibly using an auxiliary qubit, that has three different eigenvalues) that has three outcomes: a, b, c, where a means the state f_1 was identified with certainty, b means f_2 was identified with certainty and c means no information was obtained. Do this such that the probability of obtaining c is minimal if the a priori probability of having f_1 or f_2 is equal. (*Hint:* if P_1 is the projection to the 1-dimensional subspace in $\mathcal{H}$ that is orthogonal to f_2, measuring a non-zero value for P_1 lets you know that the state was f_1 for certain.)