Problem Sheet 13

Hand-in deadline: 26.12.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let $\mathcal{A}$ be a C*-algebra with a unit, denoted 1, and ω be a state, i.e. a positive linear functional over $\mathcal{A}$ such that $\omega(1) = 1$.

(i) Prove the following identities:

$$\omega(A^*) = \overline{\omega(A)}, \qquad |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),$$
$$|\omega(A^*BA)| \leq \omega(A^*A)\|B\|$$

(Hint: consider the quadratic form $\lambda \mapsto \omega((A+\lambda B)^*(A+\lambda B)))$

(ii) Let $\mathcal{N} := \{A \in \mathcal{A} : \omega(A^*A) = 0\}$. Prove that $A \in \mathcal{A}, N \in \mathcal{N}$ implies $AN \in \mathcal{N}$, i.e. $\mathcal{N}$ is a left ideal

(iii) Let $h := \mathcal{A}/\mathcal{N}$, and denote ψ_A the equivalence class of $A \in \mathcal{A}$, namely $\psi_A := \{\tilde{A} \in \mathcal{A} : \exists N \in \mathcal{N} : \tilde{A} = A + N\}$. Prove that the bilinear form over h

$$(\psi_A, \psi_B) \longmapsto \langle \psi_A, \psi_B \rangle := \omega(A^*B)$$

is well-defined (i.e. the r.h.s. is independent of the chosen representative of the classes ψ_A, ψ_B) and defines a scalar product

(iv) Hence h equipped with $\langle \cdot, \cdot \rangle$ is a pre-Hilbert space. Let $\mathcal{H}$ denote its completion. Prove that the linear map $\pi : \mathcal{A} \to \mathcal{L}(h)$ defined by

$$\pi(A)\psi_B := \psi_{AB}$$

is bounded, and that it is a *-homomorphism, namely

$$\pi(A^*) = \pi(A)^*, \qquad \pi(AB) = \pi(A)\pi(B).$$

(v) Finally, let $\mathcal{H} \ni \Omega := \psi_1$, where $1 \in \mathcal{A}$. Prove that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle.$$

The triple $(\mathcal{H}, \pi, \Omega)$ is the *GNS representation* of $\mathcal{A}$ associated with ω.

Ex. 2: Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = d < \infty$. Let

$$\Omega = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathcal{H} \otimes \mathcal{H}$$

be a maximally entangled state (here $\{|i\rangle\}_{i=1}^{d}$ is a basis of $\mathcal{H}$). Show that every $\psi \in \mathcal{H} \otimes \mathcal{H}$ can be written as

$$\psi = (\mathbf{1}_{\mathcal{H}} \otimes R)\Omega$$

for some $R : \mathcal{H} \to \mathcal{H}$.

Ex. 3: Let $\mathcal{H} = \mathbb{C}^2$. All hermitian $\rho \in \mathcal{B}(\mathcal{H})$ are of the form

$$\rho = \frac{1}{2}\left(c\mathbf{1}_{2\times 2} + \vec{a} \cdot \vec{\sigma}\right),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $c \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$. For which $c, \vec{a}$ is ρ a density matrix? For which is it a pure state?

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