Mathematisches Institut der Universität München

Prof. Dr. S. Bachmann
Dr. R. Helling

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Problem Sheet 12

Hand-in deadline: 19.01.2016 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Consider the bipartite system $\mathcal{H}_S \otimes \mathcal{H}_E$, with $\dim\mathcal{H}_S < \infty$. Let $A = A^*$, resp. $B = B^*$, be self-adjoint operators on $\mathcal{H}_S$, resp. $\mathcal{H}_E$. Their spectral decomposition shall be denoted

$$A = \sum_i a_i P_{\phi_i}, \qquad B = \int \lambda dP(\lambda),$$

where $a_j \neq a_k$ whenever $j \neq k$, and P_{ϕ_i} is the orthogonal projector on the eigenvector $\phi_i \in \mathcal{H}_S$. The Hamiltonian of the system is given by $H = A \otimes B$, and its initial state is $\rho \otimes |\psi\rangle\langle\psi|$, for an arbitrary normalized $\psi \in \mathcal{H}_E$. Let

$$\rho(t) := \mathrm{Tr}_{\mathcal{H}_E}\left(\mathrm{e}^{-\mathrm{i}Ht}(\rho \otimes P_\psi)\mathrm{e}^{\mathrm{i}Ht}\right)$$

and let $d\nu_\psi(\lambda) := d\langle\psi, P(\lambda)\psi\rangle$ be the spectral measure associated with the initial state of the environment ψ. (i) Compute $\rho_{jk}(t) := \langle\phi_j, \rho(t)\phi_k\rangle$, and show that $\rho_{jj}(t) = \rho_{jj}(0)$ for all j.

(ii) Prove the following: If ν_ψ is an absolutely continuous measure (i.e. $d\nu_\psi(\lambda) = f(\lambda)d\lambda$ for a measurable function $f \in L^1(\mathbb{R})$), then $\rho_{jk}(t) \to 0$ as $t \to \infty$ for all $j \neq k$.
(*Hint:* Riemann-Lebesgue's lemma)

(iii) Assume now that

$$\nu_\psi = \alpha\nu_{\lambda_0} + (1-\alpha)\nu_{\mathrm{AC}}$$

where $0 < \alpha \leq 1$, ν_{λ_0} is a Dirac measure at λ_0 (see Problem sheet 4) and ν_{AC} is an absolutely continuous measure. Show that $|\rho_{jk}(t)| \to \alpha$ as $t \to \infty$ for all $j \neq k$.

(iv) Application: A spin coupled to a particle on the line.
Let $\mathcal{H}_S = \mathbb{C}^2, \mathcal{H}_E = L^2(\mathbb{R})$ and $A = (g/2)\sigma^z, B = x$ where $g > 0$, i.e.

$$H = \frac{g}{2}\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

and let the initial state of the particle be a Lorentzian

$$\psi(x) = \sqrt{\frac{\gamma}{\pi}}\frac{1}{x + \mathrm{i}\gamma}, \qquad \gamma > 0.$$

Prove that

$$\rho(t) = \begin{pmatrix} \rho_{00} & \rho_{01}\mathrm{e}^{-g\gamma t} \\ \overline{\rho_{01}}\mathrm{e}^{-g\gamma t} & 1-\rho_{00} \end{pmatrix}.$$

Ex. 2: Consider a C^* algebra $\mathcal{A}$ with unit element I, and define the spectrum of an element $a \in \mathcal{A}$ by

$$\sigma(a) := \{\lambda \in \mathbb{C} : (\lambda I - a) \text{ has no inverse in } \mathcal{A}\}.$$

The spectral radius of a is defined as

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

(i) Prove that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \|a\|,$$

In particular, the above limit always exists and $\sigma(A)$ is non-empty.
(*Hint:* To prove $r(A) \leq \liminf_n \|a^n\|^{1/n}$, study the series $\lambda^{-1}\sum_n (a/\lambda)^n$. To prove $r(a) \geq \limsup_n \|a^n\|^{1/n}$, determine the radius of convergence of this series.)

(ii) Assume that a is normal, i.e. $aa^* = a^*a$. Prove that $r(a) = \|a\|$.

Ex. 3: Classify all C^*-algebras $\mathcal{A}$ such that $\mathcal{A} \subset \mathrm{Mat}(2\times 2, \mathbb{C})$.

Remark: With a bit of more work one can classify all matrix C^*-algebras in the same spirit and find them to be unitarily equivalent to block-diagonal matrices where each block is a full matrix algebra. In addition one can show that in fact all finite-dimensional C^*-algebras are unitarily equivalent to matrix algebras.

Jean-Claude Cuenin