MATHEMATISCHES INSTITUT
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Problem Sheet 11

Hand-in deadline: 12.01.2016 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: A classical charged particle of mass m in a magnetic field has a Lagrangian $L = m/2|\dot{x}|^2 + q\dot{x} \cdot A(x) - q\phi(x)$. Give an argument that the path integral obeys Schrödinger's equation (now including the vector potential). To do this, consider time evolution for a small time interval $\Delta t = \epsilon$ and expand to $\mathcal{O}(\epsilon)$. A priori it is not clear if the vector potential has to be evaluated at the initial point x' or the final point x in the path integral. Therefore evaluate it as $A(x+\eta(x'-x))$ for some $\eta \in [0,1]$. You will find that η does not drop out of the final result. Relate this to a familiar ambiguity of the canonical quantisation procedure! η can be fixed by imposing self-adjointness on the Hamilton operator.

Ex. 2: Consider the vector potential given by $A(x, y, z) = \left(\frac{by}{x^2+y^2}, -\frac{bx}{x^2+y^2}, 0\right)$ for $(x, y, z) \in \mathbb{R}^3 \setminus \{x = y = 0\}$, where $b > 0$. Sketch the vector potential in the xy-plane and compute the magnetic field strength $B = \mathrm{curl}A$. Find a gauge transformation which locally (in an open region) maps this vector potential to $A'(x) = 0$. Can this be done globally? Now imagine a double slit type experiment in which the z-axis is shielded by an infinitely strong potential wall and the electrons can go either clockwise or anti-clockwise around the z-axis. For $b = 0$, the amplitudes for these possibilities are ψ_L and ψ_R, respectively. Use the path integral to argue how these amplitudes and thus the observable $|\psi_L + \psi_R|^2$ change for $b \neq 0$. This effect (which has been confirmed in experiment) demonstrates the quantum mechanical observability of the vector potential (and not just the magnetic field strength).

Ex. 3: One could imagine that similar to electric charges there are as well magnetic charges. For concreteness, assume there is a magnetic charge of strength g located at the origin. Then one has

$$\nabla \cdot B(x) = g\delta^{(3)}(x).$$

Thus away from the origin, B can still be written as $B = \nabla \times A$. However, A is no longer defined everywhere at once. As it turns out, one can define a A_N everywhere except on the negative z-axis and another A_S everywhere except on the positive z-axis (the concrete form is not needed here). On the overlap of the domains, they are locally related by a gauge transformation $A_N = A_S + \nabla\Lambda$. Use the path integral for a particle with charge q to compare (for example via a double slit type experiment) the phase of a particle going around the z axis to one which stays at rest. The actual value does not matter but work out a condition on g and q that for the phase difference it does not matter if the path integral is evaluated using A_N or A_S (Hint: Use Stoke's theorem). This condition shows that if there is a single monopole in the universe, all electric charges have to be integer multiples of an elementary charge.

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