Mathematisches Institut der Universität München

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Problem Sheet 10

Hand-in deadline: 22.12.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: (i) Let $\phi \in C^\infty(\mathbb{R}^d)$ be a real-valued phase function, let $a \in C_c^\infty(\mathbb{R}^d)$, and define

$$I(\lambda) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\pi\lambda\phi(x)} a(x)\,\mathrm{d}x, \quad \lambda > 0.$$

Let $p \in \mathbb{R}^d$, and assume $\nabla\phi(p) \neq 0$. Prove: If a is supported in a small neighborhood of p, then for any $k \in \mathbb{N}$ and $N > 0$,

$$\left|\frac{\mathrm{d}^k I(\lambda)}{\mathrm{d}\lambda^k}\right| \leq C_{k,N}\lambda^{-N}$$

(ii) Assume that $\nabla\phi(p) = 0$ and that $H_\phi(p)$ (the Hessian of ϕ) is non-degenerate. Prove: If a is supported in a small neighborhood of p, then for any $k \in \mathbb{N}$,

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\mathrm{e}^{\mathrm{i}\pi\lambda\phi(p)} I(\lambda))\right| \leq C_k \lambda^{-\frac{d}{2}-k}.$$

Hint: Use the stationary phase formula:

$$I(\lambda) = \mathrm{e}^{-\mathrm{i}\pi\lambda\phi(p)}\mathrm{e}^{-\mathrm{i}\pi\frac{\sigma}{4}} 2^{\frac{d}{2}} (\det H_\phi(p))^{-\frac{1}{2}} \lambda^{-\frac{d}{2}} \left(a(p) + \sum_{j=1}^{N} \lambda^{-j}\mathcal{D}_j a(p) + \mathcal{O}(\lambda^{-(N+1)})\right).$$

Here, σ is the signature of $H_\phi(p)$, and $\mathcal{D}_j$ are certain differential operators of order $\leq 2j$ with smooth coefficients.

(iii) Let S^{d-1} be the unit sphere in $\mathbb{R}^d$, and let σ be the induced Lebesgue measure on S^{d-1}. Define the Fourier transform of this measure by

$$\widehat{\sigma}(k) := \int_{S^{d-1}} \mathrm{e}^{-2\pi\mathrm{i}k\cdot x}\,\mathrm{d}\sigma(x), \quad k \in \mathbb{R}^d.$$

Prove that $|\widehat{\sigma}(k)| \leq C(1+|k|)^{-\frac{d-1}{2}}$.

(iv) (optional!) Let $\chi_{B(0,1)}$ be the characteristic function of the unit ball $B(0,1)\subset\mathbb{R}^d$. Prove that $|\widehat{\chi_{B(0,1)}}|\leq C(1+|k|)^{-\frac{d+1}{2}}$.

Remark: (iv) can be used to prove the following result on the distribution of lattice points in $\mathbb{R}^d$: $|B(0,\lambda)\cap\mathbb{Z}^d|=|B(0,1)|\lambda^d+\mathcal{O}(\lambda^{d-2+2/(d+1)})$.

Ex. 2: Let $\phi\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ be real-valued, $a\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, and consider the operator $T_\lambda : L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ given by

$$(T_\lambda f)(x):=\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}\pi\lambda\phi(x,y)}a(x,y)f(y)\,\mathrm{d}y.$$

Moreover, assume that the 'mixed Hessian' $(\partial^2\phi/\partial_{x_i}\partial_{y_j})$ is non-degenerate on the support of a.

(i) Show that $T_\lambda^*T_\lambda$ is an integral operator, and compute its kernel $K_\lambda(x,y)$. Prove that $|K_\lambda(x,y)|\leq C_N(1+\lambda|x-y|)^{-N}$ for any $N>0$.

(ii) Prove that $\|T_\lambda\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}\leq C\lambda^{-\frac{d}{2}}$.
Hint: Apply Schur's test (Sheet 6, Ex. 2 (i)) to $T_\lambda^*T_\lambda$.

(iii) By a scaling argument, prove that (ii) implies the boundedness of the Fourier transform on $L^2(\mathbb{R}^d)$. (Hence, (ii) may be viewed as a variable coefficient version of Plancherel's theorem.)

(iv) In the same vein, prove that (ii) implies the Hausdorff-Young inequality: For $1\leq p\leq 2$, there exists $C_p>0$ such that $\|\widehat{f}\|_{p'}\leq C_p\|f\|_p$.
(*Hint:* Interpolate (Riesz-Thorin!) (ii) with the trivial estimate for $p=1$).

Ex. 3: In the following, the Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ are assumed to be finite-dimensional.

(i) Let ρ_A a density matrix in $\mathcal{H}_A$. Prove that there exists another Hilbert space $\mathcal{H}_B$ (what is its minimal dimension?) and a unit vector $\psi\in\mathcal{H}_A\otimes\mathcal{H}_B$ such that

$$\rho=\operatorname{tr}_B|\psi\rangle\langle\psi|.$$

(ii) Prove that for any matrix a in $\mathcal{H}_A$ and for any density matrix ρ in $\mathcal{H}_A\otimes\mathcal{H}_B$, we have

$$\operatorname{tr}(\rho(a\otimes\mathrm{id}))=\operatorname{tr}_A(a(\operatorname{tr}_B\rho)).$$

(iii) Let ρ be a density matrix in $\mathcal{H}_A\otimes\mathcal{H}_B$. Assume that the reduced density $\operatorname{tr}_B\rho$ is pure, i.e. $\operatorname{tr}_B\rho=|\psi\rangle\langle\psi|$ for some unit vector $\psi\in\mathcal{H}_A$. Prove that there exists a density matrix σ in $\mathcal{H}_B$ such that

$$\rho=|\psi\rangle\langle\psi|\otimes\sigma.$$

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