Mathematisches Institut der Universität München

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Problem Sheet 1

Hand-in deadline: 20.10.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let $\mathcal{H}$ be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

(i) Let $\widetilde{\mathcal{H}}$ be a dense subspace of $\mathcal{H}$, and let P and Q be two operators on $\mathcal{H}$ mapping $\widetilde{\mathcal{H}}$ into itself and such that $\langle \phi, P\psi \rangle = \langle P\phi, \psi \rangle$ and $\langle \phi, Q\psi \rangle = \langle Q\phi, \psi \rangle$ for all $\phi, \psi \in \widetilde{\mathcal{H}}$. Fix $\psi \in \widetilde{\mathcal{H}}$, $\|\psi\|_2 = 1$, and define

$$\langle P \rangle = \langle \psi, P\psi \rangle, \quad \Delta P := \sqrt{\langle (P - \langle P \rangle)^2 \rangle}, \tag{1}$$

and similarly for Q. Prove that

$$(\Delta Q)(\Delta P) \geq \frac{1}{2} |\langle [Q, P] \rangle|$$

where $[Q, P] := QP - PQ$ is the commutator.

(ii) Prove that equality holds in (1) if and only if $(Q - \langle Q \rangle)\psi = \mathrm{i}\lambda(P - \langle P \rangle)\psi$ for some $\lambda \in \mathbb{R}$.

(iii) Assume that $[Q, P] = \mathrm{i}\mathbb{1}$ (as an identity on $\widetilde{\mathcal{H}}$). Prove that P, Q cannot both be bounded operators on $\widetilde{\mathcal{H}}$. (*Hint:* Compute $[Q^n, P]$ for $n \in \mathbb{N}$.)

Now let $\mathcal{H} = L^2(\mathbb{R})$ and $\widetilde{\mathcal{H}} = \mathcal{S}(\mathbb{R})$ (the class of Schwartz functions[1]). Moreover, let Q be the multiplication operator with the independent variable x, and $P = -\mathrm{i}\frac{\partial}{\mathrm{d}x}$.

(iv) Deduce from (1) that

$$(\Delta Q)(\Delta P) \geq \frac{1}{2} \quad \text{(Heisenberg's uncertainty relation).}$$

(v) Prove that equality in Heisenberg's uncertainty relation holds if and only if ψ is a function that, apart from translation and modulation by a phase, has the form $\psi(x) = c\mathrm{e}^{-ax^2}$ for some $c \in \mathbb{R}$ and $a > 0$.

[1] Recall that $\mathcal{S}(\mathbb{R}) = \{\psi \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} \psi(x) \right| < \infty \text{ for all } n, m \in \mathbb{N}\}$.

(vi) Prove that there is no upper bound for

$$\langle \psi, (Q - \langle \psi, Q, \psi \rangle)^2 \psi \rangle \langle \psi, (P - \langle \psi, P, \psi \rangle)^2 \psi \rangle$$

uniformly in $\psi \in L^2(\mathbb{R})$, $\|\psi\|_2 = 1$.

Ex. 2: Set $[A, B] := AB - BA$.

(i) Verify that $\frac{2}{|x|} = \sum_{j=1}^{3} [\partial_j, \frac{x_j}{|x|}]$, and use it to prove the inequality

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x|} \, \mathrm{d}x \leq \|\nabla \varphi\|_2 \|\varphi\|_2, \quad (\varphi \in C_0^\infty(\mathbb{R}^3)).$$

(ii) Extend the inequality in (i) to all $\varphi \in H^1(\mathbb{R}^3)$.[2] You may use without proof that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$.

(iii) Prove that for all $Z \geq 0$,

$$\int_{\mathbb{R}^3} |\nabla \varphi(x)|^2 \, \mathrm{d}x - Z \int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x|} \, \mathrm{d}x \geq -\frac{1}{4} Z^2 \|\varphi\|_2^2, \quad (\varphi \in H^1(\mathbb{R}^3)),$$

and find $\varphi \in H^1(\mathbb{R}^3)$ which saturates the inequality.

Ex. 3: Let $\mathcal{H}$ be a Hilbert space and $M \subset \mathcal{H}$ a subset. Prove the following statements.

(i) $M^\perp$ is a closed subspace of $\mathcal{H}$,

(ii) $M^{\perp\perp} = \overline{\mathrm{span}(M)}$,

(iii) If M is a subspace, then $M^{\perp\perp} = \overline{M}$.

Ex. 4: Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$ and let $\phi \in \mathcal{H}$. Prove that the following are equivalent:

(i) ϕ_n converges to ϕ in the norm of $\mathcal{H}$.

(ii) ϕ_n converges weakly to ϕ and $\lim_{n \to \infty} \|\phi_n\| = \|\phi\|$.

Show that, in general, weak convergence does not imply norm convergence.

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[2] Recall that $H^1(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \partial_j f \in L^2(\mathbb{R}^3) \text{ for } j = 1, 2, 3\}$ where $\partial_j f$ is the weak derivative. The norm in $H^1(\mathbb{R}^3)$ is given by $\|f\|_{H^1(\mathbb{R}^3)}^2 = \|f\|_2^2 + \sum_{j=1}^{3} \|\partial_j f\|_2^2$.