Brouwer's fan theorem and convexity

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March 18, 2017

Key words Constructive mathematics, fan theorem, convex functions
MSC (2010) 03F60, 26E40, 52A41

Abstract

In the framework of Bishop's constructive mathematics we introduce co-convexity as a property of subsets B of $\{0,1\}^*$, the set of finite binary sequences, and prove that co-convex bars are uniform. Moreover, we establish a canonical correspondence between detachable subsets B of $\{0,1\}^*$ and uniformly continuous functions f defined on the unit interval such that B is a bar if and only if the corresponding function f is positive-valued, B is a uniform bar if and only if f has positive infimum, and B is co-convex if and only if f satisfies a weak convexity condition.

1 Introduction

It is well-known that Brouwer's fan theorem for detachable bars implies that every uniformly continuous positive-valued function defined on the unit interval has positive infimum, see [9]. In [3, Theorem 1] we have shown that if the function is convex, the fan theorem is no longer required:

Theorem 1. *Suppose that $f : [0,1] \to \,]0,\infty[$ is uniformly continuous and convex. Then f has positive infimum.*

Thus the question arises whether there is a constructively valid 'convex' version of the fan theorem. To this end, we will define 'co-convexity' as a property of subsets B of $\{0,1\}^*$, and show in Theorem 2 that there indeed is such a result.

How is this related to convex functions as in Theorem 1? In their seminal paper [9], Julian and Richman showed that for every detachable subset B of $\{0,1\}^*$ there exists a uniformly continuous function $f : [0,1] \to [0,\infty[$ such that

(i) B is a bar $\Leftrightarrow$ f is positive-valued

(ii) B is a uniform bar $\Leftrightarrow$ f has positive infimum.

Conversely, for every uniformly continuous function $f : [0,1] \to [0,\infty[$ there exists a detachable subset B of $\{0,1\}^*$ such that (i) and (ii) hold. Our aim is to include the following correspondence

(iii) B is co-convex $\Leftrightarrow$ f is weakly convex

into that list, where weak convexity of functions generalises convexity. The way we achieve our aim shows some similarities with the proofs presented in [2] and [9], but in the crucial parts we need to proceed differently in order to include (iii), in particular when deriving the function f with properties (i)–(iii) for some given detachable set B. Interestingly, in the latter case this alternative way also yields a very elementary proof of the corresponding result in [9], which may be of interest of its own. Another consequence of the derived correspondence is a more general version of Theorem 1, see Corollary 1.

2 Co-convex bars are uniform

Let $\{0,1\}^*$ be the set of all finite binary sequences u, v, w and $\{0,1\}^{\mathbb{N}}$ the set of all infinite binary sequences α, β, γ. The *length* $|u|$, the *concatenation* $u*v$, and the *restriction* $\overline{\alpha}k$ are defined as usual, see for instance [2]. If $|u| = n$, we denote the components of u by $u_0, \ldots, u_{n-1}$. Note that $\overline{\alpha}0 = \emptyset$, where $\emptyset$ is the empty sequence. A subset B of $\{0,1\}^*$ is *closed under extension* if $u * v \in B$ for all $u \in B$ and for all v. A sequence α *hits* B if there exists an n such that $\overline{\alpha}n \in B$. B is a *bar* if every α hits B. B is a *uniform bar* if there exists N such that for every α there exists an $n \leq N$ such that $\overline{\alpha}n \in B$. Often one requires B to be *detachable*, that is for every u the statement $u \in B$ is decidable. Brouwer's *fan theorem* for detachable bars is the following statement, see [6].

FAN Every detachable bar is a uniform bar.

Define the *upper closure* B' of B by

$$B' = \{u \mid \exists k \leq |u| \, (\overline{u}k \in B)\} .$$

Note that B is a (detachable) bar if and only if B' is a (detachable) bar and B is a uniform bar if and only B' is a uniform bar. Therefore, we may assume that bars are closed under extension. Set

$$u < v \overset{def}{\Leftrightarrow} |u| = |v| \wedge \exists k < |u| \, (\overline{u}k = \overline{v}k \wedge u_k = 0 \wedge v_k = 1)$$

and

$$u \leq v \overset{def}{\Leftrightarrow} u = v \vee u < v.$$

Definition. *A subset B of $\{0,1\}^*$ is* co-convex *if for every α which hits B there exists an n such that either*

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \textit{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B\,.$$

Note that, for detachable B, co-convexity follows from the convexity of the complement of B, where $C \subseteq \{0,1\}^*$ is *convex* if for all u, v, w we have

$$u \leq v \leq w \wedge u, w \in C \Rightarrow v \in C.$$

Theorem 2. *Every co-convex bar is a uniform bar.*

Proof. Fix a co-convex bar B. Since the upper closure of B is also co-convex, we can assume that B is closed under extension. Define

$$C = \{u \mid \exists n \, \forall w \in \{0,1\}^n \, (u * w \in B)\}\,.$$

Note that $B \subseteq C$ and that C is closed under extension as well. Moreover, B is a uniform bar if and only if there exists an n such that $\{0,1\}^n \subseteq C$.

First, we show that

$$\forall u \, \exists i \in \{0,1\} \, (u * i \in C)\,. \tag{1}$$

Fix u. For

$$\beta = u * 1 * 0 * 0 * 0 * \dots$$

there exist an l such that either

$$\left\{v \mid v \leq \overline{\beta}l\right\} \subseteq B,$$

or

$$\left\{v \mid \overline{\beta}l \leq v\right\} \subseteq B.$$

Since B is closed under extension, we can assume that $l > |u| + 1$. Fix m with $l = |u| + 1 + m$. In the first case, we can conclude that

$$u * 0 * w \in B$$

for every w of length m, which implies that $u * 0 \in C$. In the second case, we obtain

$$u * 1 * w \in B$$

for every w of length m, which implies that $u * 1 \in C$. This concludes the proof of (1).

By countable choice, there exists a function $F : \{0,1\}^* \to \{0,1\}$ such that

$$\forall u \, (u * F(u) \in C)\,.$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha}n).$$

Next, we show by induction on n that

$$\forall n \, \forall u \in \{0,1\}^n \, (u \neq \overline{\alpha}n \Rightarrow u \in C)\,. \tag{2}$$

If $n = 0$, the statement clearly holds, since in this case the statement $u \neq \overline{\alpha}n$ is false. Now fix some n such that (2) holds. Moreover, fix $w \in \{0,1\}^{n+1}$ such that $w \neq \overline{\alpha}(n+1)$.

case 1. $\overline{w}n \neq \overline{\alpha}n$. Then $\overline{w}n \in C$ and therefore $w \in C$.

case 2. $w = \overline{\alpha}n * (1 - \alpha_n) = \overline{\alpha}n * F(\overline{\alpha}n)$. This implies $w \in C$. So we have established (2).

There exists an n such that $\overline{\alpha}n \in B$. Applying (2) to this n, we can conclude that every u of length n is an element of C, thus B is a uniform bar. □

Remark 1. *Note that we do not need to require that the co-convex bar in Theorem 2 is detachable.*

3 From detachable sets to functions

A subset S of a metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ there exist $s_1, \dots, s_n \in S$ such that

$$\forall s \in S \, \exists i \in \{1, \dots, n\} \, (d(s, s_i) < \varepsilon)$$

and *compact* if it is totally bounded and *complete* (i.e. every Cauchy sequence in S has a limit in S). Proofs of the following basic statements can be found in [7, Section 2.2].

Lemma 1. *(i) If S is totally bounded, then for all $x \in X$ the distance*

$$d(x, S) = \inf \{d(x, s) \mid s \in S\}$$

exists and the function $x \mapsto d(x, S)$ is uniformly continuous.

(ii) Uniformly continuous images of totally bounded sets are totally bounded.

(iii) If S is totally bounded and $f : S \to \mathbb{R}$ is uniformly continuous, then

$$\inf f = \inf \{f(s) \mid s \in S\}$$

exists.

We will use the metrics

$$d_1(s,t) = |s-t|, \quad d_2((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|$$

on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The mapping

$$(\alpha,\beta) \mapsto \inf \left\{2^{-k} \mid \overline{\alpha}k = \overline{\beta}k\right\}$$

defines a compact metric on $\{0,1\}^{\mathbb{N}}$. See [6, Chapter 5] for an introduction to basic properties of this metric space. Define a uniformly continuous function $\kappa : \{0,1\}^{\mathbb{N}} \to [0,1]$ by

$$\kappa(\alpha) = 2 \cdot \sum_{k=0}^{\infty} \alpha_k \cdot 3^{-(k+1)}.$$

The following lemma immediately follows from the definition of κ.

Lemma 2. *For all α, β and n, we have*

- $\overline{\alpha}n = \overline{\beta}n \;\;\Rightarrow\;\; |\kappa(\alpha) - \kappa(\beta)| \leq 3^{-n}$
- $\overline{\alpha}n \neq \overline{\beta}n \;\;\Rightarrow\;\; |\kappa(\alpha) - \kappa(\beta)| \geq 3^{-n}$
- $\overline{\alpha}n < \overline{\beta}n \;\;\Rightarrow\;\; \kappa(\alpha) < \kappa(\beta)$.

For the rest of this section, we fix a detachable subset B of $\{0,1\}^*$. We assume that $\varnothing \notin B$ and that B is closed under extension. Define

$$\eta_B : \{0,1\}^{\mathbb{N}} \to [0,1]\,,\ \alpha \mapsto \inf \left\{3^{-k} \mid \overline{\alpha}k \notin B\right\}. \tag{3}$$

Lemma 3. *The function η_B is well-defined, i.e. the infimum in (3) exists, and uniformly continuous. If $\eta_B(\alpha) > 0$, there exists k such that*

(1) $\overline{\alpha}k \notin B$

(2) $\overline{\alpha}(k+1) \in B$

(3) $\eta_B(\alpha) = 3^{-k}$.

Moreover,

$$\overline{\alpha}n \in B \;\Leftrightarrow\; \eta_B(\alpha) \geq 3^{-n+1} \;\Leftrightarrow\; \eta_B(\alpha) > 3^{-n}$$

for all α and n.

Proof. Set $S = \left\{3^{-k} \mid \overline{\alpha}k \notin B\right\}$. Note that $1 \in S$ and that 0 is a lower bound of S. By [7, Corollary 2.1.19], it suffices to show that for rationals $p < q$ either p is a lower bound of S or there exists $s \in S$ with $s < q$. If $p \leq 0$, p is a lower bound of S. Now assume that $0 < p$. Then there exists k with $3^{-k} < p$. If $\overline{\alpha}k \notin B$, there exist $s \in S$ (choose $s = 3^{-k}$) with $s < q$. If $\overline{\alpha}k \in B$, we can compute the minimum s_0 of S. If $p < s_0$, p is a lower bound of S; if $s_0 < q$, there exists $s \in S$ (choose $s = s_0$) with $s < q$.

If $\inf S > 0$, there exists l such that $3^{-l} < \inf S$. Therefore, $\overline{\alpha}l \in B$. Let k be the largest number such that $\overline{\alpha}k \notin B$.

Assume that $\overline{\alpha}n \in B$. Let l be the largest natural number with $\overline{\alpha}l \notin B$. Then $l \leq n-1$ and thus $\inf S = 3^{-l} \geq 3^{-n+1}$.

Assume that $\inf S > 3^{-n}$. Then there exists k with (1), (2), and (3). We obtain $k < n$ and therefore $\overline{\alpha}n \in B$. □

Set

$$C = \left\{\kappa(\alpha) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}$$

and

$$K = \left\{(\kappa(\alpha), \eta_B(\alpha)) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}.$$

Lemma 4. *The sets C and K are compact.*

Proof. Both sets are uniformly continuous images of the compact set $\{0,1\}^{\mathbb{N}}$ and therefore totally bounded, by Lemma 1. Suppose that $\kappa(\alpha^n)$ converges to t and $\eta_B(\alpha^n)$ converges to s. By Lemma 2, the sequence (α^n) is Cauchy, therefore it converges to a limit α. Then $\kappa(\alpha^n)$ converges to $\kappa(\alpha)$ and $\eta_B(\alpha^n)$ converges to $\eta_B(\alpha)$, therefore $t = \kappa(\alpha)$ and $s = \eta_B(\alpha)$. Thus we have shown that both C and K are complete. □

We now have all ingredients needed to give a simple short proof the following result from [9]:

Proposition 1. *There exists a uniformly continuous function $f_B : [0,1] \to \mathbb{R}$ such that*

(i) B is a bar $\Leftrightarrow$ f_B is positive-valued

(ii) B is a uniform bar $\Leftrightarrow$ $\inf f_B > 0$.

The proof of Proposition 1 uses Bishop's lemma:

Lemma 5. *(see [5, Ch. 4, Lemma 3.8]) Let A be a compact subset of a metric space (X, d), and x a point of X. Then there exists a point a in A such that $d(x, a) > 0$ entails $d(x, A) > 0$.*

Proof of Proposition 1. Define

$$f_B : [0, 1] \to [0, \infty[\,,\ t \mapsto d_2((t, 0), K). \tag{4}$$

Assume that B is a bar. Fix $t \in [0, 1]$. In view of Bishop's lemma and the compactness of K, it is sufficient to show that

$$d_2((t, 0), (\kappa(\alpha), \eta_B(\alpha))) > 0$$

for each α. This follows from $\eta_B(\alpha) > 0$.
Now assume that f_B is positive-valued. Fix α. Since

$$d_2((\kappa(\alpha), 0), K) = f_B(\kappa(\alpha)) > 0,$$

we can conclude that

$$d_2((\kappa(\alpha), 0), (\kappa(\alpha), \eta_B(\alpha))) > 0.$$

Thus $\eta_B(\alpha)$ is positive which implies that α hits B by Lemma 3.
The second equivalence follows from Lemma 3 and the fact that $\inf f_B = \inf \eta_B$. □

In order to include convexity in the list of Proposition 1, we need to define weakly convex functions:

Definition. *Let S be a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is* weakly convex *if for all $t \in S$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either*

$$\forall s \in S\,(s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon)$$

or

$$\forall s \in S\,(t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon)\,.$$

Remark 2. *(i) Note that in particular uniformly continuous (quasi-)convex functions $f : [0, 1] \to \mathbb{R}$ are weakly convex. To this end, we recall that f is* convex *if we have*

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda)f(t)$$

and quasiconvex *if we have*

$$f(\lambda s + (1-\lambda)t) \leq \max\{f(s), f(t)\}$$

for all $s, t \in [0,1]$ and all $\lambda \in [0,1]$. Clearly, convexity implies quasiconvexity. Now assume that f is quasiconvex. Fix $t \in [0,1]$ and assume that $f(t) > 0$. Set $\varepsilon = f(t)/2$. The assumption that both

$$\inf\{f(s) \mid s \in [0,t]\} < f(t) \quad \textit{and} \quad \inf\{f(s) \mid s \in [t,1]\} < f(t)$$

is absurd, because in that case by uniform continuity there exists $s < t < s'$ such that $f(s) < f(t)$ and $f(s') < f(t)$. Compute $\lambda \in (0,1)$ such that $t = \lambda s + (1-\lambda)s'$, and note that quasiconvexity of f implies $f(t) \leq \max\{f(s), f(s')\} < f(t)$ which is absurd. Hence, it follows that either $\inf\{f(s) \mid s \in [0,t]\} > \varepsilon$ or $\inf\{f(s) \mid s \in [t,1]\} > \varepsilon$.

(ii) Positive functions and monotone functions are weakly convex. Moreover, pointwise continuous functions on $[0,1]$ which are decreasing on $[0,s]$ and increasing on $[s,1]$ for some s are weakly convex. See [8] for a detailed discussion of various notions of convexity.

(iii) If f is weakly convex, then the set $\{t \mid f(t) \leq 0\}$ is convex. With classical logic, the reverse implication holds as well, if f is continuous. This illustrates that weak convexity is indeed a convexity property.

(iv) Fix a dense subset D of $[0,1]$. A uniformly continuous function $f : [0,1] \to \mathbb{R}$ is weakly convex if and only its restriction to D is weakly convex.

Set

$$-C = \{t \in [0,1] \mid d_1(t,C) > 0\}.$$

Even though the proof of Proposition 1 already shows the main idea, when adding the statement

(iii) B is co-convex $\Leftrightarrow$ f_B is weakly convex

to Proposition 1, we cannot argue with f_B as defined in (4), because the property of weak convexity does not make much sense in that case, since f_B is positive on $-C$. Therefore, we introduce a new function g_B by

$$g_B : [0,1] \to \mathbb{R},\ t \mapsto f_B(t) - d_1(t,C).$$

Theorem 3. *(i) B is a bar $\Leftrightarrow$ g_B is positive-valued* *(ii)* B *is a uniform bar* $\;\Leftrightarrow\;$ $\inf g_B > 0$

(iii) B *is co-convex* $\;\Leftrightarrow\;$ g_B *is weakly convex*

For the proof of Theorem 3 we need a few auxiliary results. It is readily verified that:

Lemma 6. *For all α, n, and t we have*

- $g_B(\kappa(\alpha)) = f_B(\kappa(\alpha)) \leq \eta_B(\alpha)$
- $g_B(\kappa(\alpha)) > 3^{-n} \;\Rightarrow\; \overline{\alpha}n \in B \;\Rightarrow\; g_B(\kappa(\alpha)) \geq 3^{-n}$
- $d_1(t, C) \leq f_B(t)$.

Lemma 7. *The set $-C$ is dense in $[0,1]$. For every $t \in -C$ there exist unique elements a, a' of C such that*

(a) $t \in \,]a, a'[\, \subseteq -C$.

(b) $d_1(t, C) = \min\left(d_1(t, a), d_1(t, a')\right)$

Moreover, setting $\gamma = \kappa^{-1}(a)$ and $\gamma' = \kappa^{-1}(a')$, we obtain

(c) $\forall n \left(\overline{\gamma}n \in B \;\wedge\; \overline{\gamma'}n \in B \;\Rightarrow\; g_B(t) \geq 3^{-n}\right)$

(d) if $d_1(t, a) < d_1(t, a')$*, then*

$$\gamma \text{ hits } B \;\Leftrightarrow\; g_B(t) > 0 \;\Leftrightarrow\; \inf\left\{g_B(s) \mid a \leq s \leq t\right\} > 0$$

(e) if $d_1(t, a') < d_1(t, a)$*, then*

$$\gamma' \text{ hits } B \;\Leftrightarrow\; g_B(t) > 0 \;\Leftrightarrow\; \inf\left\{g_B(s) \mid t \leq s \leq a'\right\} > 0.$$

Proof. Fix $t \in [0,1]$ and $\delta > 0$. If $d_1(t, C) > 0$, then $t \in -C$. Now assume that there exists an α such that $d_1(t, \kappa(\alpha)) < \delta/2$. There exists an u such that $d_1(\kappa(\alpha), t_u) < \delta/2$, where

$$t_u = \tfrac{1}{2} \cdot \kappa(u * 0 * 1 * 1 * 1 * \ldots) + \tfrac{1}{2} \cdot \kappa(u * 1 * 0 * 0 * 0 * \ldots).$$

Note that $t_u \in -C$ and that $d_1(t, t_u) < \delta$. So $-C$ is dense in $[0,1]$.

Fix $t \in -C$. Since for any α it is decidable whether $\kappa(\alpha) > t$ or $\kappa(\alpha) < t$, the sets $C_{<t} = \{s \in C \mid s < t\}$ and $C_{>t} = \{s \in C \mid s > t\}$ are compact. Let a be the maximum of $C_{<t}$ and let a' be the minimum of $C_{>t}$. Clearly, a and a' fulfil (a) and (b).

(c): Fix n and assume that both $\overline{\gamma}n \in B$ and $\overline{\gamma'}n \in B$. For any α with $\kappa(\alpha) < t$ we have

$$d_2((t,0),(\kappa(\alpha),\eta_B(\alpha))) - d_1(t,C) \geq \kappa(\gamma) - \kappa(\alpha) + \eta_B(\alpha)$$

and similarly for any β with $\kappa(\beta) > t$ we have

$$d_2((t,0),(\kappa(\beta),\eta_B(\beta))) - d_1(t,C) \geq \kappa(\beta) - \kappa(\gamma') + \eta_B(\beta).$$

If $\overline{\alpha}n = \overline{\gamma}n$, then $\overline{\alpha}n \in B$ and we can conclude thar $\eta_B(\alpha) \geq 3^{-n+1}$, by Lemma 3. If $\overline{\alpha}n \neq \overline{\gamma}n$, then $\kappa(\gamma) - \kappa(\alpha) \geq 3^{-n}$, by Lemma 2. The analogous considerations for β conclude the proof that $g_B(t) \geq 3^{-n}$.

(d): Set $\iota = d_1(t,a') - d_1(t,a) > 0$. Suppose that there is n such that $\overline{\gamma}n \in B$. Set $\varepsilon = \min(\iota, 3^{-n})$. Fix s with $a \leq s \leq t$. We show that $g_B(s) \geq \varepsilon$. To this end, note that $d_1(s,C) = d_1(s,a)$. Hence, for all β such that $\kappa(\beta) \geq a'$ we have

$$d_2((s,0),(\kappa(\beta),\eta_B(\beta))) - d_1(s,C) \geq \iota.$$

If α satisfies $\kappa(\alpha) \leq a$, we have

$$d_2((s,0),(\kappa(\alpha),\eta_B(\alpha))) - d_1(s,C) = s - \kappa(\alpha) + \eta_B(\alpha) - d_1(s,C) =$$

$$\kappa(\gamma) - \kappa(\alpha) + \eta_B(\alpha) \geq 3^{-n}.$$

Thus, $g_B(s) \geq \varepsilon$.

It remains to show that $g_B(t) > 0$ implies that γ hits B. If $g_B(t) > 0$, then

$$f_B(t) > d_1(t,C) = d_1(t,a)$$

and

$$f_B(t) \leq d_2((t,0),(a,\eta_B(\gamma))) = d_1(t,\kappa(\gamma)) + a,$$

so $\eta_B(\gamma) > 0$. Apply Lemma 3.

(e): This is proved analogously to (d). □

The next lemma is very easy to prove, we just formulate it to be able to refer to it.

Lemma 8. *For real numbers $x < y < z$ and $\delta > 0$ there exists a real number y' such that*

(i) $x < y' < z$

(ii) $d_1(y,y') < \delta$

(iii) $d_1(x, y') < d_1(y', z)$ *or* $d_1(x, y') > d_1(y', z)$.

For a function F defined on $\{0,1\}^{\mathbb{N}}$, set

$$F(u) = F(u * 0 * 0 * 0 * \ldots). \tag{5}$$

Proof of Theorem 3. *(i)* "$\Rightarrow$". Suppose that B is a bar and fix t. By Proposition 1 we obtain $f_B(t) > 0$. If $d_1(t, C) < f_B(t)$, then $g_B(t) > 0$, by the definition of g_B. If $0 < d_1(t, C)$, we can apply Lemma 7 to conclude that $g_B(t) > 0$.

(i) "$\Leftarrow$". If g_B is positive-valued, then f_B is positive-valued as well and Proposition 1 implies that B is a bar.

(ii) "$\Rightarrow$". Suppose that B is a uniform bar. Then, by Proposition 1, $\varepsilon := \inf f_B > 0$. There exists $\delta > 0$ such that

$$|s - t| < \delta \;\;\Rightarrow\;\; |g_B(s) - g_B(t)| < \varepsilon/2$$

for all s and t and there exists an n such that $\{0,1\}^n \subseteq B$. Then for all t we can show that

$$g_B(t) \geq \min\left(\varepsilon/2, 3^{-n}\right),$$

using case distinction $d_1(t, C) < \delta$ or $d_1(t, C) > 0$ and Lemma 7.

(ii) "$\Leftarrow$". If $\inf g_B > 0$, then $\inf f_B > 0$, and Proposition 1 implies that B is a uniform bar.

(iii) "$\Rightarrow$". Assume that B is co-convex. In view of Remark 2 and Lemma 7, it is sufficient to show that the restriction of g_B to $-C$ is weakly convex. Fix $t \in -C$ and assume that $g_B(t) > 0$. Choose γ and γ' according to Lemma 7. In view of Lemma 8 and the uniform continuity of g_B, we may assume without loss of generality that either

$$d_1(\kappa(\gamma), t) < d_1(t, \kappa(\gamma')) \quad \text{or} \quad d_1(\kappa(\gamma), t) > d_1(t, \kappa(\gamma')).$$

Consider the first case. The second case can be treated analogously. By Lemma 7 we obtain

$$\iota = \inf\{g_B(s) \mid \kappa(\gamma) \leq s \leq t\} > 0.$$

In particular, $g_B(\kappa(\gamma)) > 0$, so γ hits B. There exists an n such that either

$$\{v \mid v \leq \overline{\gamma} n\} \subseteq B \tag{6}$$

or

$$\{v \mid \overline{\gamma}n \leq v\} \subseteq B\,. \tag{7}$$

Set $\varepsilon = \min\left(\iota, 3^{-n}\right)$. In case (6), we show that

$$\forall s \in -C\,(s \leq t \;\;\Rightarrow\;\; g_B(s) \geq \varepsilon)\,,$$

as follows. Assume that there exists $s \in -C$ with $s \leq t$ such that $g_B(s) < \varepsilon$. Then, by the definition of ι, we obtain that $s < \kappa(\gamma)$. Applying Lemma 7 again, we can choose α and α' such that

$$s \in \left]\kappa(\alpha), \kappa(\alpha')\right[\subseteq -C.$$

Then $\overline{\alpha}n \leq \overline{\alpha'}n \leq \overline{\gamma}n$, therefore both $\overline{\alpha}n$ and $\overline{\alpha'}n$ are in B. This implies $g_B(s) \geq 3^{-n}$, which is a contradiction. In case (7), a similar argument yields

$$\forall s \in -C\,(t \leq s \;\;\Rightarrow\;\; g_B(s) \geq \varepsilon)\,.$$

<u>*(iii)* "$\Leftarrow$"</u>. Assume that g_B is weakly convex. Fix α and suppose that α hits B. Then Lemma 6 implies that $g_B(\kappa(\alpha)) > 0$. There exists an n with $\overline{\alpha}n \in B$ such that

$$\forall s\left(s \leq \kappa(\alpha) \;\;\Rightarrow\;\; g_B(s) > 3^{-n}\right)$$

or

$$\forall s\left(\kappa(\alpha) \leq s \;\;\Rightarrow\;\; g_B(s) > 3^{-n}\right).$$

Assume the first case. Fix v with $v \leq \overline{\alpha}n$. Then $\kappa(v) \leq \kappa(\alpha)$. If $v \notin B$, then Lemma 3 yields

$$g_B(\kappa(v)) = f_B(\kappa(v)) \leq \eta_B(v) \leq 3^{-n}.$$

This contradiction shows that

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B.$$

Now, consider the second case. Fix v with $\overline{\alpha}n < v$. Then $\kappa(\alpha) \leq \kappa(v)$. If $v \notin B$, then $g_B(\kappa(v)) \leq 3^{-n}$. This contradiction shows that

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

□

4 From functions to detachable sets

When constructing a set B from a function f, it is more handy to work with an altered κ. Set

$$\kappa' : \{0,1\}^{\mathbb{N}} \to [0,1]\,,\ \alpha \mapsto \sum_{k=0}^{\infty} \alpha_k \cdot 2^{-(k+1)}.$$

One cannot prove that κ' is surjective, but we can use [1, Lemma 1] to overcome this, partially.

Lemma 9. *Let S be a subset of $[0,1]$ such that*

$$\forall \alpha\, \exists \varepsilon > 0\, \forall x \in [0,1]\, (|x - \kappa'(\alpha)| < \varepsilon \Rightarrow x \in S)\ .$$

Then $S = [0,1]$.

The next lemma is a typical application of Lemma 9.

Lemma 10. *Fix a uniformly continuous function $f : [0,1] \to \mathbb{R}$ and define*

$$F : \{0,1\}^{\mathbb{N}} \to \mathbb{R},\ \alpha \mapsto f(\kappa'(\alpha)).$$

Then

(i) f is positive-valued $\Leftrightarrow$ F is positive-valued

(ii) $\inf f > 0 \ \Leftrightarrow\ \inf F > 0$.

Proof. In (i), the direction "$\Rightarrow$" is clear. For "$\Leftarrow$", apply Lemma 9 to the set

$$S = \{t \in [0,1] \mid f(t) > 0\}\,.$$

The case (ii) follows from the density of the image of κ' in $[0,1]$ and the uniform continuity of f. □

In the following proposition, we use a similar construction as in [2].

Theorem 4. *For every uniformly continuous function*

$$f : [0,1] \to \mathbb{R}$$

there exists a detachable subset B of $\{0,1\}^*$ which is closed under extension such that

(i) B is a bar $\Leftrightarrow$ f is positive-valued

(ii) B is a uniform bar $\Leftrightarrow$ $\inf f > 0$

(iii) B is co-convex $\Leftrightarrow$ *f is weakly convex.*

Proof. Since the function

$$F : \{0,1\}^{\mathbb{N}} \to [0,\infty[\,,\ \alpha \mapsto f(\kappa'(\alpha))$$

is uniformly continuous, there exists a strictly increasing function $M : \mathbb{N} \to \mathbb{N}$ such that

$$|F(\alpha) - F(\overline{\alpha}(M(n)))| < 2^{-n}$$

for all α and n, recalling the convention given in (5). Since M is strictly increasing, for every k the statement

$$\exists n\,(k = M(n))$$

is decidable. Therefore, for every u we can choose $\lambda_u \in \{0,1\}$ such that

$$\begin{array}{lll} \lambda_u = 0 & \Rightarrow & \forall n\,(|u| \neq M(n)) \ \vee \ \exists n\,\big(|u| = M(n) \ \wedge \ F(u) < 2^{-n+2}\big) \\ \lambda_u = 1 & \Rightarrow & \exists n\,\big(|u| = M(n) \ \wedge \ F(u) > 2^{-n+1}\big)\,. \end{array}$$

The set

$$B = \{u \in \{0,1\}^* \mid \exists l \leq |u|\,(\lambda_{\overline{u}l} = 1)\}$$

is detachable and closed under extension. Note that

$$F(\alpha) \geq 2^{-n+3} \ \Rightarrow \ \overline{\alpha}(M(n)) \in B \ \Rightarrow \ F(\alpha) \geq 2^{-n} \tag{8}$$

for all α and n. In view of Lemma 10, (8) yields (i) and (ii).

Assume that B is co-convex. Fix $t \in [0,1]$ and assume that $f(t) > 0$. By part (ii) of Remark 2, we may assume that t is a rational number, which implies that there exists α such that $\kappa'(\alpha) = t$. Now $F(\alpha) > 0$ implies that α hits B. Therefore, there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B$$

or

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

In the first case, we show that

$$\inf\{f(s) \mid s \in [0,t]\} \geq \min\big(2^{-n}, F(\alpha)\big)\,. \tag{9}$$

Assume that there exists $s \leq t$ such that $f(s) < 2^{-n}$ and $f(s) < F(\alpha)$. The latter implies that $s < t$. Choose a β with the property that $\kappa'(\beta)$ is close enough to s such that

$$\kappa'(\beta) < \kappa'(\alpha) \tag{10}$$

and

$$F(\beta) = f(\kappa'(\beta)) < 2^{-n}. \tag{11}$$

Now (8) and (11) imply that $\overline{\beta}n \notin B$. On the other hand, (10) implies that $\overline{\beta}n \leq \overline{\alpha}n$ and therefore $\overline{\beta}n \in B$. This is a contradiction, so we have shown (9).

In the case

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B$$

we can similarly show that

$$\inf \{f(s) \mid s \in [t, 1]\} \geq \min\left(2^{-n}, F(\alpha)\right).$$

Now assume that f is weakly convex. Fix an α which hits B. Then there exists an n with $\overline{\alpha}(M(n)) \in B$ and (8) implies that $f(\kappa'(\alpha)) > 0$. We choose n large enough such that either

$$\inf \{f(t) \mid t \in [0, \kappa'(\alpha)]\} \geq 2^{-n+3}$$

or

$$\inf \{f(t) \mid t \in [\kappa'(\alpha), 1]\} \geq 2^{-n+3}.$$

Applying (8) again, we obtain

$$\{v \mid v \leq \overline{\alpha}(M(n))\} \subseteq B$$

in the first case and

$$\{v \mid \overline{\alpha}(M(n)) \leq v\} \subseteq B.$$

in the second. Therefore, B is co-convex.

□

The following corollary follows immediately.

Corollary 1. *Every uniformly continuous weakly convex function* $f : [0, 1] \to\ ]0, \infty[$ *has positive infimum.*

In [3] we in fact proved a stronger result than Theorem 1, namely that any positive-valued uniformly continuous quasi-convex function f defined on a convex compact subset C of $\mathbb{R}^n$ has positive infimum. One verifies that such functions are in particular *weakly convex* in the following sense: for every hyperplane H such that both halfspaces H^1 and H^2 intersect C, the implication

$$\inf \{f(x) \mid x \in C \cap H\} > 0 \;\; \Rightarrow \;\; \exists i \in \{0,1\} \inf \left\{f(x) \mid x \in C \cap H^i\right\} > 0$$

holds. An inspection of the proof given in [3], which is an inductive argument over the dimension, shows that Corollary 1 as a base clause and then applying the same techniques as presented in [3] in fact yields the following result:

Fix a convex and compact subset C of $\mathbb{R}^n$ and suppose that $f : C \to \,]0,\infty[$ is uniformly continuous and weakly convex. Then f has positive infimum.

Many functions are weakly convex, so in many situations where we normally need the fan theorem we actually can do without—mathematics in convex environments has some innate constructive nature. For example, the proof in [4] of the equivalence of the fundamental theorem of asset pricing and Markov's principle is based on the fact that the Euclidean norm is a convex function.

Acknowledgments. *We thank the Excellence Initiative of the LMU Munich, the Japan Advanced Institute of Science and Technology, and the European Commission Research Executive Agency for supporting the research.*

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