Brouwer's fan theorem and convexity

Josef Berger and Gregor Svindland

LMU Munich

JAIST Logic Seminar Series

16 March 2017

Part 1) Brouwer's fan theorem

- $\{0,1\}^*$ the set of finite binary sequences u, v, w
- $|u|$ the length of u, i.e.

$$\text{for } u = (u_0, \ldots, u_{n-1}) \text{ we have } |u| = n$$

- $u * v$ the concatenation of u and v, i.g.

$$(0,1) * (0,0,1) = (0,1,0,0,1)$$

- α, β, γ infinite binary sequences
- $\overline{\alpha}n$ the restriction of α to the first n elements, i.e.

$$\overline{\alpha}n = (\alpha_0, \ldots, \alpha_{n-1})$$

$\mathrm{B} \subseteq \{0,1\}^*$ is

- *detachable* if $\forall u\,(u \in \mathrm{B} \vee u \notin \mathrm{B})$
- a *bar* if $\forall \alpha\, \exists n\, (\overline{\alpha}n \in \mathrm{B})$ ("every α *hits B*")
- a *uniform bar* if $\exists N\, \forall \alpha\, \exists n \leq N\, (\overline{\alpha}n \in \mathrm{B})$

FAN every detachable bar is a uniform bar

Proposition 1

(Julian, Richman 1984) The following are equivalent:

- FAN
- $f : [0,1] \to \mathbb{R}^+ \ u/c \ \Rightarrow \ \inf f > 0$

Proposition 2

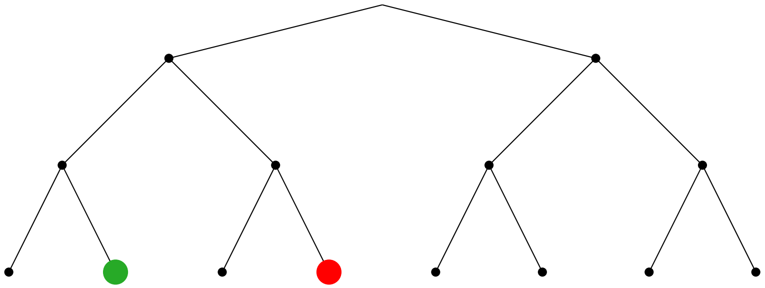
(B., Svindland 2016)

$f : [0,1] \to \mathbb{R}^+ \ u/c + \textbf{convex} \ \Rightarrow \ \inf f > 0$

Question:

Is there a corresponding extra condition on bars such that FAN becomes constructively valid?

Part 2) Co-convex sets



$$(0, 0, 1) < (0, 1, 1)$$

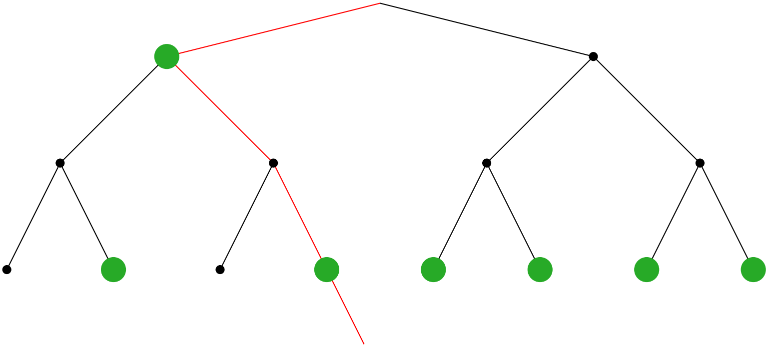
$$u < v \overset{def}{\Leftrightarrow} |u| = |v| \wedge \exists i < |u| \, (\overline{u}i = \overline{v}i \wedge u_i = 0 \wedge v_i = 1)$$

$$u \leq v \overset{def}{\Leftrightarrow} u < v \ \vee \ u = v$$

Definition

A subset B of $\{0,1\}^*$ is co-convex *if for every α which hits B there exists an n such that either*

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \textit{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B\,.$$



Proposition 3

Every co-convex bar is a uniform bar.

Question:

How is this related to positive convex functions having positive infimum?

Part 3) From bars to functions

- A subset S of a metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ there exist $s_1, \ldots, s_n \in S$ such that

$$\forall s \in S \, \exists i \in \{1, \ldots, n\} \, (d(s, s_i) < \varepsilon)$$

  and *compact* if it is totally bounded and *complete* (i.e. every Cauchy sequence in S has a limit in S).
- The mapping

$$(\alpha, \beta) \mapsto \inf \left\{ 2^{-k} \mid \overline{\alpha}k = \overline{\beta}k \right\}$$

  defines a compact metric on $\{0, 1\}^{\mathbb{N}}$.

- If S is totally bounded, then for all $x \in X$ the distance

$$d(x, S) = \inf \{d(x, s) \mid s \in S\}$$

  exists and the function $x \mapsto d(x, S)$ is uniformly continuous.
- Uniformly continuous images of totally bounded sets are totally bounded.
- If S is totally bounded and $f : S \to \mathbb{R}$ is uniformly continuous, then

$$\inf f = \inf \{f(s) \mid s \in S\}$$

  exists.

Proposition 4

(Julian, Richman 1984)

For every detachable subset B of $\{0,1\}^*$ *there exists a uniformly continuous function* $f_B : [0,1] \to \mathbb{R}$ *such that*

- *B is a bar* $\Leftrightarrow$ f_B *is positive-valued*
- *B is a uniform bar* $\Leftrightarrow$ $\inf f_B > 0$.

Easy proof of Proposition 4, based on compactness

We can assume that B is closed under extension and does not contain the empty sequence.

$$\eta_B(\alpha) := \inf \left\{ 3^{-k} \mid \overline{\alpha}k \notin B \right\}$$

$$\overline{\alpha}n \in B \iff \eta_B(\alpha) > 3^{-n}$$

$$\kappa(\alpha) := \sum_{k=0}^{\infty} \alpha_k \cdot 3^{-(k+1)}$$

The set

$$K = \left\{ (\kappa(\alpha), \eta_B(\alpha)) \mid \alpha \in \{0,1\}^{\mathbb{N}} \right\} \subseteq \mathbb{R}^2$$

is compact.

$$f_B : [0, 1] \to \mathbb{R},\ t \mapsto d((t, 0), K)$$

Then the following are equivalent:

- B is a uniform bar
- $\inf \eta_B > 0$
- $\inf f_B > 0$

Assume that B is a bar. Fix $t \in [0,1]$. In view of Bishop's lemma and the compactness of K, it is sufficient to show that

$$d((t,0),(\kappa(\alpha),\eta_B(\alpha))) > 0$$

for each α. This follows from $\eta_B(\alpha) > 0$.

Now assume that f_B is positive-valued. Fix α. Since

$$d((\kappa(\alpha),0),K) = f_B(\kappa(\alpha)) > 0,$$

we can conclude that

$$\eta_B(\alpha) = d((\kappa(\alpha),0),(\kappa(\alpha),\eta_B(\alpha))) > 0,$$

which implies that α hits B.

∎

Part 4) Weakly convex functions

Our aim is to include

- B is co-convex $\Leftrightarrow$ f_B is weakly convex

into Proposition 4.

Definition

For $S \subseteq \mathbb{R}$, a function $f : S \to \mathbb{R}$ is weakly convex *if for all $t \in S$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either*

$$\forall s \in S \, (s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon)$$

or

$$\forall s \in S \, (t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon) \, .$$

Define g_B by

$$g_B : [0,1] \to \mathbb{R},\ t \mapsto f_B(t) - d(t, C),$$

where

$$C = \left\{ \kappa(\alpha) \mid \alpha \in \{0,1\}^{\mathbb{N}} \right\}.$$

- B is a bar $\Leftrightarrow$ g_B is positive-valued
- B is a uniform bar $\Leftrightarrow$ $\inf g_B > 0$.

- $g_B(\kappa(\alpha)) = f_B(\kappa(\alpha)) \leq \eta_B(\alpha)$
- $g_B(\kappa(\alpha)) > 3^{-n} \ \Rightarrow\ \overline{\alpha}n \in B \ \Rightarrow\ g_B(\kappa(\alpha)) \geq 3^{-n}$

Assume that g_B is weakly convex. Fix α which hits B. Then $g_B(\kappa(\alpha)) > 0$. There exists an n such that

$$\forall s \left(s \leq \kappa(\alpha) \;\; \Rightarrow \;\; g_B(s) > 3^{-n}\right)$$

or

$$\forall s \left(\kappa(\alpha) \leq s \;\; \Rightarrow \;\; g_B(s) > 3^{-n}\right).$$

Assume the first case. Fix v with $v \leq \overline{\alpha}n$. Then $\kappa(v) \leq \kappa(\alpha)$. If $v \notin B$, then

$$g_B(\kappa(v)) \leq 3^{-n}.$$

This contradiction shows that

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B.$$

Thus B is co-convex.

$$-C = \{t \in [0,1] \mid d(t, C) > 0\}.$$

- $-C$ is dense in $[0,1]$
- for every $t \in -C$ there exists a unique pair γ, γ' such that

$$t \in (\kappa(\gamma), \kappa(\gamma')) \subseteq -C$$

- if $\overline{\gamma} n \in B$ and $\overline{\gamma'} n \in B$, then $g_B(t) \geq 3^{-n}$
- if $d(\kappa(\gamma), t) < d(t, \kappa(\gamma'))$, then

$$\gamma \text{ hits } B \quad \Leftrightarrow \quad g_B(\kappa(\gamma)) > 0 \quad \Leftrightarrow$$

$$g_B(t) > 0 \quad \Leftrightarrow \quad \inf \{g_B(s) \mid \kappa(\gamma) \leq s \leq t\} > 0$$

Assume that B is co-convex. We show that the restriction of g_B to $-C$ is weakly convex. Fix $t \in -C$ such that $g_B(t) > 0$. There exist γ and γ' such that

$$t \in (\kappa(\gamma), \kappa(\gamma')) \subseteq -C.$$

We can assume that $d(\kappa(\gamma), t) < d(t, \kappa(\gamma'))$, therefore

$$\inf \{g_B(s) \mid \kappa(\gamma) \leq s \leq t\} > 0$$

and γ hits B. There exists an n such that either

$$\underbrace{\{v \mid v \leq \overline{\alpha}n\} \subseteq B}_{\Rightarrow \inf\{g_B(s) \mid s \in [0,t]\} > 0} \quad \text{or} \quad \underbrace{\{v \mid \overline{\alpha}n \leq v\} \subseteq B}_{\Rightarrow \inf\{g_B(s) \mid s \in [t,1]\} > 0}.$$

■

Part 5) From functions to bars

A construction from (B., Ishihara 2005) works. Since the function

$$F : \{0,1\}^{\mathbb{N}} \to [0,\infty[\,,\ \alpha \mapsto f\left(\sum_{k=0}^{\infty} \alpha_k \cdot 2^{-(k+1)})\right)$$

is uniformly continuous, there exists a strictly increasing function $M : \mathbb{N} \to \mathbb{N}$ such that

$$|F(\alpha) - F(\overline{\alpha}(M(n)))| < 2^{-n}$$

for all α and n. Since M is strictly increasing, for every k the statement

$$\exists n\,(k = M(n))$$

is decidable.

For every u we can choose $\lambda_u \in \{0,1\}$ such that

$$\lambda_u = 0 \Rightarrow \forall n \, (|u| \neq M(n)) \vee \exists n \left(|u| = M(n) \wedge F(u) < 2^{-n+2}\right)$$
$$\lambda_u = 1 \Rightarrow \exists n \left(|u| = M(n) \wedge F(u) > 2^{-n+1}\right).$$

The set

$$B = \{u \in \{0,1\}^* \mid \exists l \leq |u| \, (\lambda_{\overline{u}l} = 1)\}$$

is detachable and closed under extension. Note that

$$F(\alpha) \geq 2^{-n+3} \;\; \Rightarrow \;\; \overline{\alpha}(M(n)) \in B \;\; \Rightarrow \;\; F(\alpha) \geq 2^{-n}$$

for all α and n.

$\blacksquare$

Remarks

- weak convexity of f is a weak property
- many functions are weakly convex
- the use of Brouwer's fan theorem as an axiom is often unnecessary
- applications in constructive reverse mathematics, e.g. when calibrating theorems in finance

- Josef Berger and Hajime Ishihara, *Brouwer's fan theorem and unique existence in constructive analysis*, Math. Log. Quart. 51, No. 4 (2005), 360–364
- Josef Berger and Gregor Svindland, *Convexity and constructive infima*, Archive for Mathematical Logic (2016)
- Josef Berger and Gregor Svindland, *A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle*, Annals of Pure and Applied Logic (2016)
- William Julian and Fred Richman, 'A uniformly continuous function on $[0, 1]$ that is everywhere different from its infimum.' *Pacific J. Math.* 111 (1984), 333–340

Bishop's lemma

Let A be a complete, located subset of a metric space (X, d), and x a point of X. Then there exists a point a in A such that $d(x, a) > 0$ entails $d(x, A) > 0$.