The Hahn-Banach theorem and infima of convex functions

Hajime Ishihara

School of Information Science
Japan Advanced Institute of Science and Technology
(JAIST)
Nomi, Ishikawa 923-1292, Japan

first CORE meeting, LMU Munich, 7 May, 2016

Normed spaces

Definition

A normed space is a linear space E equipped with a norm $\|\cdot\| : E \to \mathbf{R}$ such that

- $\|x\| = 0 \leftrightarrow x = 0$,
- $\|ax\| = |a|\|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$,

for each $x, y \in E$ and $a \in \mathbf{R}$.

Note that a normed space E is a metric space with the metric

$$d(x, y) = \|x - y\|.$$

Definition

A Banach space is a normed space which is complete with respect to the metric.

Examples

For $1 \leq p < \infty$, let

$$l^p = \{(x_n) \in \mathbf{R}^{\mathbf{N}} \mid \sum_{n=0}^{\infty} |x_n|^p < \infty\}$$

and define a norm by

$$\|(x_n)\| = (\sum_{n=0}^{\infty} |x_n|^p)^{1/p}.$$

Then l^p is a (separable) Banach space.

Examples

Classically the normed space

$$l^\infty = \{(x_n) \in \mathbf{R}^\mathbf{N} \mid (x_n) \text{ is bounded}\}$$

with the norm

$$\|(x_n)\| = \sup_n |x_n|$$

is an inseparable Banach space.

However, constructively, it is not a normed space.

Linear mappings

Definition

A mapping T between linear spaces E and F is linear if

- $T(ax) = aTx$,
- $T(x+y) = Tx + Ty$

for each $x, y \in E$ and $a \in \mathbf{R}$.

Definition

A linear functional f on a linear space E is a linear mapping from E into $\mathbf{R}$.

Bounded linear mappings

Definition

A linear mapping T between normed spaces E and F is bounded if there exists $c \geq 0$ such that

$$\|Tx\| \leq c\|x\|$$

for each $x \in E$.

Proposition

Let T be a linear mapping between normed spaces E and F. Then the following are equivalent.

- *T is continuous,*
- *T is uniformly continuous,*
- *T is bounded.*

Normable linear mappings

Classicaly, for a bounded linear mapping T between normed spaces, its norm

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$$

always exists, and hence the set E' of all bounded linear fuctionals on a normed space E forms a Banach space.

However it is not always the case constructively.

Definition

A linear mapping T between normed spaces is normable if

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$$

exists.

Remark

Let E^* be the set of all normable linear fuctionals on a normed space E.

Open Problem

Under what condition does E^* become a linear space?

Note that $(l^p)^*$ is a linear space for $1 < p < \infty$, and H^* is a linear space for a Hilbert space H.

Convex and sublinear functions

Definition

A function p form a linear space E into $\mathbf{R}$ is convex if

$$p(\lambda x + (1-\lambda)y) \leq \lambda p(x) + (1-\lambda)p(y)$$

for each $x, y \in E$ and $\lambda \in [0,1]$.

Definition

A function p form a linear space E into $\mathbf{R}$ is sublinear if

- $p(ax) = ap(x)$,
- $p(x+y) \leq p(x) + p(y)$

for each $x, y \in E$ and $a \in \mathbf{R}$ with $a \geq 0$.

Note that sublinear functions are convex.

Classical Hahn-Banach theorem

Theorem (Hahn-Banach Theorem, Rudin 1991)

Let p be a sublinear function on a linear space E, let M be a subspace of E, and let f be a linear functional on M such that $f(x) \leq p(x)$ for each $x \in M$. Then there exists a linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $g(y) \leq p(y)$ for each $y \in E$.

Corollaries

Corollary

Let M be a subspace of a normed space E, and let f be a bounded linear functional on M. Then there exists a bounded linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.

Corollary

Let x be a nonzero element of a normed space E. Then there exists a bounded linear functional f on E such that $f(x) = \|x\|$ and $\|f\| = 1$.

Constructive Hahn-Banach theorem

Theorem (Bishop 1967)

Let M be a subspace of a separable normed space E, and let f be a nonzero normable linear functional on M. Then for each $\epsilon > 0$ there exists a normable linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| \leq \|f\| + \epsilon$.

Corollary

Let x be a nonzero element of a separable normed space E. Then for each $\epsilon > 0$ there exists a normable linear functional f on E such that $f(x) = \|x\|$ and $\|f\| \leq 1 + \epsilon$.

Hilbert spaces

Definition

An inner product space is a linear space E equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbf{R}$ such that

- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \leftrightarrow x = 0$,
- $\langle x, y \rangle = \langle y, x \rangle$,
- $\langle ax, y \rangle = a\langle x, y \rangle$,
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

for each $x, y, z \in E$ and $a \in \mathbf{R}$.

Note that an inner product space E is a normed space with the norm

$$\|x\| = \langle x, x \rangle^{1/2}.$$

Definition

A Hilbert space is an inner product space which is a Banach space.

Example

Let

$$l^2 = \{(x_n) \in \mathbf{R}^{\mathbf{N}} \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$$

and define an inner product by

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} x_n y_n.$$

Then l^2 is a Hilbert space.

Riesz theorem

Proposition (Bishop 1967)

Let f be a bounded linear functional on a Hilbert space H. Then f is normable if and only if there exists $x_0 \in H$ such that

$$f(x) = \langle x, x_0 \rangle$$

for each $x \in H$.

Corollary

Let x be a nonzero element of a Hilbert space H. Then there exists a normable linear functional f on H such that $f(x) = \|x\|$ and $\|f\| = 1$.

The Hahn-Banach theorem in Hilbert spaces

Theorem

Let M be a subspace of a Hilbert space H, and let f be a normable linear functional on M. Then there exists a normable linear functional g on H such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.

Proof.

Let $\overline{M}$ be the closure of M. Then there exists a normable extension $\overline{f}$ of f on $\overline{M}$. Since $\overline{M}$ is a Hilbert space, there exists $x_0 \in \overline{M}$ such that

$$\overline{f}(x) = \langle x, x_0 \rangle$$

for each $x \in \overline{M}$. Let $g(x) = \langle x, x_0 \rangle$ for each $x \in H$. Then it is straightforwad to show that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$. □

Differentiations

Let $f : E \to \mathbf{R}$ be a real-valued function on a normed space E.

Definition

f is Gâteaux differentiable at $x \in E$ with the derivative $g : E \to \mathbf{R}$ if for each $y \in E$ with $\|y\| = 1$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall t \in \mathbf{R}(|t| < \delta \to |f(x + ty) - f(x) - tg(y)| < \epsilon|t|).$$

Note that if f is convex, then g is linear.

Definition

f is Fréchet differentiable at $x \in E$ with the derivative $g : E \to \mathbf{R}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall y \in E \forall t \in \mathbf{R}(\|y\| = 1 \wedge |t| < \delta \to |f(x+ty) - f(x) - tg(y)| < \epsilon|t|).$$

Gâteaux differentiable norm

Proposition (I 1989)

Let x be a nonzero element of a normed linear space E whose norm is Gâteaux differentiable at x. Then there exists a unique normable linear functional f on E such that $f(x) = \|x\|$ and $\|f\| = 1$.

Proof.

Take the derivative f of the norm at x. □

Remark

The norm of l^p for $1 < p < \infty$ and the norm of a Hilbert space are Gâteaux (even Fréchet) differentiable at each $x \in E$ with $x \neq 0$.

Uniformly convex spaces

Definition

A normed space E is uniformly convex if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| \geq \epsilon \to \|(x + y)/2\| \leq 1 - \delta$$

for each $x, y \in E$ with $\|x\| = \|y\| = 1$.

Proposition (Bishop 1967)

Let f be a nonzero normable linear functional on a uniformly convex Banach space E. Then there exists $x \in E$ such that $f(x) = \|f\|$ and $\|x\| = 1$.

Remark

l^p for $1 < p < \infty$ and a Hilbert space are uniformly convex.

The Hahn-Banach theorem in Banach spaces

The norm of a normed space E is Gâteaux (Fréchet) differentiable if it is Gâteaux (Fréchet) differentiable at each $x \in E$ with $\|x\| = 1$.

Theorem (I 1989)

Let M be a subspace of a uniformly convex Banach space E with a Gâteaux differentiable norm, and let f be a normable linear functional on M. Then there exists a unique normable linear functional g on H such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.

Proof.

We may assume without loss of generality that $\|f\| = 1$. Let $\overline{M}$ be the closure of M. Then there exists a normable extension $\overline{f}$ of f on $\overline{M}$. Since $\overline{M}$ is a uniformly convex Banach, there exists $x \in \overline{M}$ such that $\overline{f}(x) = \|x\| = 1$. Take the derivative g of the norm at x. □

The separation theorem in Banach spaces

Theorem (I 1989)

Let C and D be subsets of a uniformly convex Banach space E with a Gâteaux differentiable norm, whose algebraic difference

$$D - C = \{y - x \mid y \in D, x \in C\}$$

is located and convex, and whose mutual distance

$$d = \inf\{\|y - x\| \mid y \in D, x \in C\}$$

is positive. Then there exists a normable linear functional f on E such that $\|f\| = 1$ and

$$f(y) \geq f(x) + d$$

for each $y \in D$ and $x \in C$.

Remark

Open Problem

With a differentability condition on p, does the following full Hahn-Banach theorem hold?

Let p be a sublinear function on a linear space E, let M be a subspace of E, and let f be a linear functional on M such that $f(x) \leq p(x)$ for each $x \in M$. Then there exists a linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $g(y) \leq p(y)$ for each $y \in E$.

The above full Hahn-Banach theorem has various applications in game theory (for example, von Neumann minimax theorem).

Locating subsets

Definition

A subset S of a normed space E is located if

$$\inf_{y \in S} \|x - y\|$$

exists for each $x \in E$.

Remark

Locating a subset S of a normed space E amounts to computing an infimum of the convex function $f : S \to \mathbf{R}$ defined by

$$f : y \mapsto \|x - y\|.$$

Weakly totally bounded sets

Definition
A subset S of a normed space E is weakly totally bounded if

$$\{f(x) \mid x \in S\}$$

is a totally bounded subset of $\mathbf{R}$ for each normable linear functional f on E.

Lemma
A convex subset C of a normed space E is weakly totally bounded if and only if $\sup\{f(x) \mid x \in C\}$ *exists for each normable linear functional f on E.*

Locating subsets in Hirbert spaces

Proposition (I 2001)

Let C be a bounded, convex subset of an inner product space H. Then C is located if and only if C is weakly totally bounded.

Proposition (I 2001)

Let C be a convex subset of a Hilbert space H. If C is weakly totally bounded, then C is located.

Uniformly differentiable convex functions

Definition

A real-valued convex function f on a convex subset C of a normed space E is uniformly differentiable if there exists $g : x \mapsto g_x$ from C into E^* such that for each $b > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x\| \leq b \wedge \|y\| = 1 \wedge |t| < \delta \rightarrow |f(x + ty) - f(x) - tg_x(y)| < \epsilon|t|$$

for each $x, y \in E$ and $t \in \mathbf{R}$.

Remark

Every uniformly differentiable convex function is Fréchet differentiable.

Uniformly smooth normed spaces

Definition

A normed space E is uniformly smooth if its norm is uniformly defferentiable on $x \in E$ with $\|x\| = 1$.

Remark

l^p for $1 < p < \infty$ and a Hilbert space H are uniformly smooth Banach spaces.

Locating subsets in Banach spaces

Proposition (I-Vîță 2003)

Let C be a bounded, convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. If C is located, then C is weakly totally bounded.

Proposition (I-Vîță 2003)

Let C be a bounded, convex subset of a uniformly convex and uniformly smooth Banach space E. If C is weakly totally bounded, then C is located.

Corollary

Let C be a bounded, convex subset of a uniformly convex and uniformly smooth Banach space E. Then C is located if and only if C is weakly totally bounded.

Infima of convex functions

Lemma (Bridges-I-Vîță 2004)

Let C be a bounded, weakly totally bounded, convex subset of a normed space E, and let $f : C \to \mathbf{R}$ be a uniformly differentiable convex function. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in C$, then either

- $f(x) \leq f(y) + \epsilon$ *for all* $y \in C$,
- *or there exists* $z \in C$ *such that* $f(z) < f(x) - \epsilon\delta$.

Theorem (Bridges-I-Vîță 2004)

Let C be a bounded, weakly totally bounded, convex subset of a normed space E, and let $f : C \to \mathbf{R}$ be a uniformly differentiable convex function that is bounded from below. Then $\inf f$ exists.

An application

Corollary

Let C be a bounded, weakly totally bounded, convex subset of a uniformly smooth normed space E, and let $x_1, \ldots x_n \in E$. Then

$$\inf_{y \in C} \sum_{i=1}^{n} \|x_i - y\|^2$$

exists.

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