A constructive version of the weak König lemma

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- Constructive mathematics: when proving a statement A, it is not sufficient to derive a contradiction from the assumption that its negation $\neg A$ is false.
- Many statements which are normally lemmas (provable) turn into axioms (not provable).
- For example: $A \vee \neg A$, $\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$, $\neg\neg A \Rightarrow A$
- More examples: the weak König lemma, the fan theorem
- We compare such axioms and investigate which axioms are necessary and sufficient to prove certain theorems. Constructive reverse mathematics (Ishihara).
- This has applications, for example in economics, because a mathematical theorem is easier to apply if we can prove it constructively.

Josef Berger and Gregor Svindland, *A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle*. Annals of Pure and Applied Logic 167 (2016) 1161–1170

Notation

- u, v, w finite binary sequences

$$u = (u_0, u_1, u_2, u_3, u_4) = (0, 1, 0, 0, 1)$$

- $|u|$ length of u

$$|(0, 1, 0)| = 3$$

- $u * w$ concatenation

$$(0, 1, 0) * (0, 1) = (0, 1, 0, 0, 1)$$

- α, β infinite binary sequences

- $\overline{\alpha}n = (\alpha_0, \ldots, \alpha_n)$

Brouwer's fan theorem

- $\{0,1\}^*$ set of all finite binary sequences
- $B \subseteq \{0,1\}^*$ is a bar if

$$\forall \alpha \, \exists n \, (\overline{\alpha} n \in B)$$

- $B \subseteq \{0,1\}^*$ is a uniform bar if

$$\exists N \, \forall \alpha \, \exists n \leq N \, (\overline{\alpha} n \in B)$$

FAN Every bar is a uniform bar.

Lecture on the fan theorem:

www.mathematik.uni-muenchen.de/~jberger/fan.php

The weak König lemma

$T \subseteq \{0,1\}^*$ is an infinite tree if

- $\forall u \in T \, \forall n \leq |u| \, (\overline{u}n \in T)$
- $\forall n \, \exists u \in \{0,1\}^n \cap T$

α is a path of T if every restriction $\overline{\alpha}n$ of α belongs to T

WKL Every infinite tree has a path.

Proposition (Ishihara 2006)

$\mathrm{WKL} \Rightarrow \mathrm{FAN}$

In constructive reverse mathematics, many theorems are equivalent to WKL or to FAN.

A restricted version of WKL

T has at most one path if for all α, β

$$\exists n\,(\alpha_n \neq \beta_n) \;\Rightarrow\; \exists m\,\left(\overline{\alpha}m \notin T \;\vee\; \overline{\beta}m \notin T\right)$$

$\mathrm{WKL!}$ Every infinite tree with at most one path has a path.

Proposition (B., Ishihara 2006)

$\mathrm{WKL!} \Leftrightarrow \mathrm{FAN}$

Josef Berger and Hajime Ishihara, *Brouwer's fan theorem and unique existence in constructive analysis*. Math. Log. Quart. 51, No. 4 (2005) 360–364

Helmut Schwichtenberg, *A direct proof of the equivalence between Brouwer's fan theorem and König's lemma with a uniqueness hypothesis*. J. of Universal Computer Science 11 (2005)

Joan Rand Moschovakis, *Another Unique Weak König's Lemma*. Logic, Construction, Computation, U. Berger, H. Diener, P. Schuster, M. Seisenberger (Eds.), Ontos mathematical logic, Transaction Pub (2012)

FAN $\Rightarrow$ WKL! revisited

Fix an infinite tree T with at most one path. We show that there exists n such that

$$\forall u \in \{0,1\}^n \, (0 * u \notin T) \quad \vee \quad \forall u \in \{0,1\}^n \, (1 * u \notin T) \, . \qquad \spadesuit$$

This gives a clear instruction for constructing a path: starting with the empty sequence, one of the two options to continue is definitely wrong. Iterating this procedure yields a path of T.

Define $B \subseteq \{0,1\}^*$ by

$$u \in B \ \ :\Leftrightarrow \ \ 0 * u \notin T \ \vee \ \forall v \in \{0,1\}^{|u|} \, (1 * v \notin T) \, .$$

Note that B is closed under extension. We can show that B is a bar. By FAN, the set B is a uniform bar. Thus there exists n such that $\{0,1\}^n \subseteq B$. Now consider two cases. If

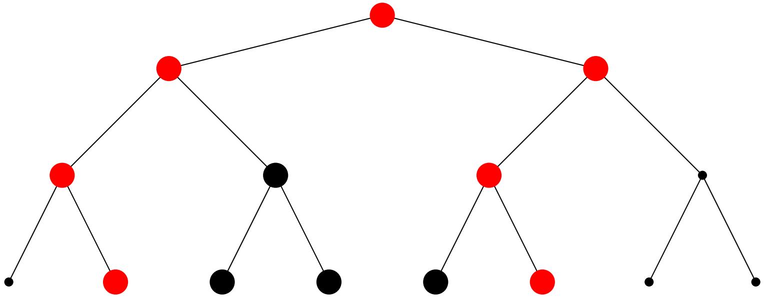
$$\forall u \, (|u| = n \ \Rightarrow \ 0 * u \notin T)$$

then ♠ holds anyway. Now assume that there exists u of length n such that $0 * u \notin T$. By the definition of B, and since $u \in B$, we obtain

$$\forall v \in \{0,1\}^n \, (1 * u \notin T) \, ,$$

which also implies ♠.

Convex trees



A constructive version of WKL

Proposition (B.,S. 2017)

Every infinite convex tree with at most one path has a path.

Proof.

The bar B in the proof of $\mathrm{FAN} \Rightarrow \mathrm{WKL!}$ has the following property: its complement in $\{0,1\}^*$, i.e. the set $\{0,1\}^* \setminus B$, is convex. The uniformity of such bars is provable constructively. $\square$

Corollary

RCA_0 proves that every infinite convex tree has a path.

A constructive version of FAN

Proposition (B.,S. 2017)

Every bar with convex complement is a uniform bar.

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