Functional Representations of Partial Orders

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Preliminary general definitions

Let X be an inhabited set, and $\succ$ a binary relation on X that is *asymmetric:*

$$\forall_{x,y\in X}(x \succ y \Rightarrow \neg(y \succ x)).$$

Def. Two associated binary relations on X:

preference-indifference: $x \succcurlyeq y \equiv \forall_{z\in X}(y \succ z \Rightarrow x \succ z)$;

indifference: $x \sim y \equiv (x \succcurlyeq y \wedge y \succcurlyeq x))$.

Note that

$$x \succcurlyeq y \Rightarrow (y \succ x \Rightarrow x \succ x) \Rightarrow \neg(y \succ x).$$

Def. The *upper contour set* and the *strict upper contour set* of x in X, relative to $\succ$, are, respectively

$$
\begin{aligned}
[x, \rightarrow) &\equiv \{y \in X : y \succcurlyeq x\}\,, \\
(x, \rightarrow) &\equiv \{y \in X : y \succ x\}\,.
\end{aligned}
$$

Likewise, we define the *lower contour set* and the *strict lower contour set* of x by

$$
\begin{aligned}
(\leftarrow, x] &\equiv \{y \in X : x \succcurlyeq y\}\,, \\
(\leftarrow, x) &\equiv \{y \in X : y \succ x\}\,.
\end{aligned}
$$

Def. A *partial utility function* for the asymmetric relation $\succ$ on X is a mapping $u : X \to \mathbf{R}$ such that

$$\forall_{x,y \in X}(x \succ y \Rightarrow u(x) > u(y)).$$

If the implication can be replaced by '$\Leftrightarrow$', then u becomes a *utility function* for $\succ$. A necessary condition for this replacement is that of *cotransitivity* (or *negative transitivity*):

$$\forall_{x,y \in X}(x \succ y \Rightarrow \forall_{z \in X}(x \succ z \vee z \succ y)).$$

An old problem with many classical solutions:

> *Find necessary and sufficient conditions for the existence—and, when X is a topological space, continuity—of (partial) utility functions representing asymmetric orders.*

According to Fred Roberts,

> *If possible, the proof of a representation theorem should be constructive; it should not only show us that a representation is possible, but it should show us how actually to construct it.*

F.S. Roberts: *Measurement Theory*, Addison-Wesley, 1979.

Preference systems

Def. Let X be an inhabited set, δ a positive number or ∞, and

$$\mathcal{F} \equiv (\succ_\varepsilon)_{0<\varepsilon<\delta}$$

a family of binary relations on X indexed by the interval $I \equiv (0,\delta)$. We set

$$x \succ y \Leftrightarrow \exists_\varepsilon (x \ \succ_\varepsilon y)$$

and

$$x \succcurlyeq y \Leftrightarrow \forall_z (y \succ z \Rightarrow x \succ z).$$

We call $\mathcal{F}$ a *preference system* on X if the following hold:

- the relation $\succ$ is asymmetric;

- if $\varepsilon, \varepsilon' \in I$, $\varepsilon' \leq \varepsilon$, and $x \succ_\varepsilon y$, then $x \succ_{\varepsilon'} y$;

- if $x \succcurlyeq x' \succ_\varepsilon y' \succcurlyeq y$, then $x \succ_\varepsilon y$;

- if $\varepsilon, \varepsilon'$, and $\varepsilon + \varepsilon'$ belong to I and $x \succ_\varepsilon y$, then
$$\forall_{z \in X}(x \succ_\varepsilon z \vee z \succ_{\varepsilon'} y).$$

Note that the last property implies the cotransitivity of $\succ$—and hence the transitivity of both $\succ$ and $\succcurlyeq$—and that

$$\forall_{x,y \in X}(x \succcurlyeq y \Leftrightarrow \neg(y \succ x)).$$

Denote by B_n the open ball with centre 0 and radius r in $\mathbf{R}^N$.

Theorem (dsb, JME **9**, 1982). *Let X be a locally compact, convex subset of $\mathbf{R}^N$, δ a positive number of ∞, and $\mathcal{F} \equiv (\succ_\varepsilon)_{0<\varepsilon<\delta}$ a preference system on X. Suppose that there exists a dense subset A of X such that for each $n \in \mathbf{N}^+$ and each $x \in A \cap B_n$, the upper and lower contour sets of x are totally bounded. Suppose also that certain natural$^{(*)}$ continuity and local nonsatiation conditions hold for the family $\mathcal{F}$. Then there exists a continuous (utility) function $u : X \to \mathbf{R}$ such that*

$$\forall_{x,y\in X}(x \succ y \Leftrightarrow u(x) > u(y).$$

(*) You will surely forgive me for not burdening you with the full details of what 'natural' means here.

A reasonable question:

Under the conditions of this theorem, can we find, for each $n \geq 1$*, a mapping* $\alpha_n : (0, \delta) \to \mathbf{R}$ *such that*

$$\forall_{x,y \in B_n} \forall_{\varepsilon \in (0,\delta)} (x \succ_\varepsilon y \Leftrightarrow u(x) > u(y) + \alpha_n(\varepsilon))?$$

No. Take $N = 1$, $X = [0, \infty)$, $\delta = \pi/16$, and for each $\varepsilon \in (0, \delta)$, $x \succ_\varepsilon y$ if and only if

either $x > y + \varepsilon$

or else $x, y \in [0, \pi/2]$ and $\sin x > \sin y + \varepsilon^2$.

Preference relations—a better way

Recall these two properties applicable to a binary relation $\succ$ on X:

$x \succ y \Rightarrow \neg(y \succ x)$ (asymmetry)

$x \succ y \Rightarrow \forall_{z \in X}(x \succ z \vee z \succ y)$ (cotransitivity)

Def. An asymmetric, cotransitive binary relation $\succ$ on X is called a *strict weak order* or *preference relation* on X.

Now specialise to the case where (X, ρ) is a metric space.

Def. The preference relation $\succ$ is *continuous* if for each x in X both the strict upper contour set $(x, \rightarrow)$ and the strict lower contour set $(\leftarrow, x)$ are open.

If $\succ$ is represented by a continuous utility function, then it is a continuous strict weak order.

Fix a strict weak order $\succ$ on (X, ρ).

Def. Let K be a compact subset of X. We say that $\succ$ is *uniformly continuous on* K if for all $a, b \in X$ with $a \succ b$, there exists $r > 0$ such that

$$\forall_{x,y \in K} (\rho(x, y) < r \Rightarrow a \succ x \vee y \succ b).$$

For example, $>$ is uniformly continuous on each compact subset of $\mathbf{R}$.

Uniformly continuity on each compact subset of X is an extension of cotransitivity.

If $\succ$ is continuous, then classically (but **not** recursively) it is uniformly continuous on each compact subset of X.

If $\succ$ is represented by a continuous$^{(*)}$ utility function, then $\succ$ is uniformly continuous on each compact subset of X.

(*) i.e. uniformly continuous on compact sets

Def. We say that $\succ$ is *locally nonsatiated at* the point $x \in X$ if

$$\forall_{\varepsilon>0} \exists_{y \in X} (\rho(x,y) < \varepsilon \wedge y \succ x);$$

and that it is *locally nonsatiated* (on X) if it is locally nonsatiated at each point x of X.

Def. We say that $\succ$ is *uniformly local nonsatiated near the compact set* $K \subset X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall_{a,b\in K}(\rho(a,b) < \delta \Rightarrow \forall_{x\in K}(x \succ a \Rightarrow \exists_{y\succ b}(\rho(x,y) < \varepsilon)))$$

Clearly, this property implies that $\succ$ is locally nonsatiated at each point of the compact set K.

Brouwer's fan theorem for detachable bars ensures that uniform local nonsatiation near each compact $K \subset X$ is intuitionistically equivalent to local nonsatiation.

The constructive Arrow-Hahn-Mirrlees theorem

Theorem (dsb, Indag. Math., 1989). *Let X be an inhabited, locally compact, convex subset of $\mathbf{R}^N$, and let $\succ$ be a preference relation on X that is*

uniformly continuous on each compact $K \subset X$ and

uniformly locally nonsatiated near each compact $K \subset X$.

Suppose also that $[x, \rightarrow)$ is locally compact for each x in a dense subset A of X. Then

(i) *for each* x *in* X, $[x, \rightarrow)$ *is locally compact;*

(ii) *for each* $x \in X$ *and each compact* $K \subset X$,

$$u(K, x) \equiv \sup \{\rho(\xi, [x, \rightarrow) : \xi \in K\}$$

exists.

Moreover, if $X = \bigcup_{n \geqslant 1} K_n$, *where each* K_n *is compact, then*

$$u_{\succ}(x) \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{u(K_n, x)}{1 + u(K_n, x)}$$

defines a continuous utility function $u : X \rightarrow [0, 1]$ *representing* $\succ$.

Def. A preference relation $\succ$ on a metric space X is *admissible* if it is uniformly locally nonsatiated near each compact set and is represented by a continuous utility function.

We also gave conditions that ensure the uniform continuity of the mappings

$$\succ \rightsquigarrow u_{\succ} \text{ on } \mathbf{\Sigma}$$

and

$$(\succ, x) \rightsquigarrow u_{\succ}(x) \text{ on } \mathbf{\Sigma} \times K,$$

where $\mathbf{\Sigma}$ is a certain set of admissible preference relations and $K \subset X$ is compact.

Intransitivity of preferences

In the economics of real life, preference-indifference need not be transitive: consider $\succ$ defined on $\mathbf{R}$ by

$$\forall_{x,y}(x \succ y \Leftrightarrow x > y + 1).$$

Note that this can be represented by the interplay of the two functions $u : x \rightsquigarrow x$ and $v : x \rightsquigarrow x + 1$, in the sense that

$$x \succ y \Leftrightarrow u(x) > v(y).$$

Another representation question:

Given an asymmetric binary relation on $X \subset \mathbf{R}^N$, under what conditions can we find mappings $u, v : X \to \mathbf{R}$ such that

$$\forall_{x,y \in X} (x \succ y \Leftrightarrow u(x) > v(y))?$$

Def. If $\succ$ is an asymmetric binary relation on an inhabited set X, then $\not\prec$ is *pseudotransitive* if

$$\forall_x \forall_{x'} \forall_y \forall_{y'} (x \succ x' \not\prec y' \succ y \Rightarrow x \succ y).$$

This condition is *classically* equivalent to the Scott-Fishburn *interval order property:*

$$\forall_x \forall_{x'} \forall_y \forall_{y'} ((x \succ y \vee x' \succ y') \Rightarrow (x \succ y' \vee x' \succ y)).$$

Proposition. *If $\succ$ is an asymmetric binary relation on X such that $\not\prec$ is pseudotransitive, then $\succ$ is transitive and*

$$\forall_x \forall_y \forall_z ((x \succ y \not\prec x \vee x \not\prec y \succ z) \Rightarrow x \succ z)).$$

Def. Let $\succ$ be an asymmetric binary relation on a metric space (X, ρ). We say that $\succcurlyeq$ is *strongly pseudotransitive* if for all sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ converging to x, y respectively in X,

$$\forall_n (x_n \succ y_n \succcurlyeq a \succ b) \Rightarrow x \succ b.$$

Strongly pseudotransitive implies transitive.

Proposition. *If $\succ$ is an asymmetric, locally nonsatiated binary relation on a metric space, and $\succcurlyeq$ is strongly pseudotransitive, then $\succcurlyeq$ is transitive.*

Theorem (classical; dsb, JME **11**, 1983). *Let* X *be an inhabited, closed, convex subset of* $\mathbf{R}^N$*, and let* $\succ$ *be an open, asymmetric binary relation on* X *such that* $\succcurlyeq$ *is strongly pseudotransitive. Then there exist mappings* $u, v : X \to \mathbf{R}$ *such that*

$$\forall_{x,y \in X}(x \succ y \Leftrightarrow u(x) > v(y).$$

Under reasonable conditions on $\succ$, the mappings u and v are uniformly continuous on compact subsets of X.

An invitation: Find the constructive content of the foregoing theorem.