Convexity and constructive minima

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X compact metric space and $f : X \to \mathbb{R}$ uniformly continuous

the **infimum** of f is given by

$$\inf f = \inf \{f(x) \mid x \in X\}$$

x is a **minimum point** of f if

$$f(x) = \inf f$$

f has **at most one minimum point** if

$$d(x, y) > 0 \;\;\Rightarrow\;\; \inf f < f(x) \;\vee\; \inf f < f(y)$$

In Bishop's constructive mathematics, the following statements are equivalent:

(FAN) Brouwer's fan theorem for decidable bars

(POS) each uniformly continuous map

$$f : X \to \mathbb{R}^+$$

on a compact metric space has positive infimum

(MIN!) each uniformly continuous map

$$f : X \to \mathbb{R}$$

on a compact metric space with at most one minimum point has a minimum point recently!

In a convex setting, the statements (FAN) and (POS) are constructively provable.

Constructive version of (FAN)

Proposition 1

Every co-convex bar is a uniform bar.

Josef Berger and Gregor Svindland, *Brouwer's fan theorem and convexity.* Submitted

Constructive version of (POS)

A function $f : C \to \mathbb{R}$, where C is a convex subset of $\mathbb{R}^m$, is called *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leq \max(f(x), f(y))$$

for all $\lambda \in [0,1]$ and $x, y \in C$.

Proposition 2

If $C \subseteq \mathbb{R}^m$ is compact and convex and

$$f : C \to \mathbb{R}^+$$

is quasi-convex and uniformly continuous, then $\inf f > 0$.

Josef Berger and Gregor Svindland, *Convexity and constructive infima.* Arch. Math. Logic 55 (2016) 873–881

new!

In a convex setting, the statement (MIN!) is constructively provable.

Constructive version of (MIN!)

Proposition 3

Fix a convex and compact subset C of $\mathbb{R}^m$ and suppose that $f : C \to \mathbb{R}$ has at most one minimum point and is quasi-convex and uniformly continuous. Then f has a minimum point.

Proof of Proposition 3

Suppose that $f : C \to \mathbb{R}$ has at most one minimum point. We can assume that $\inf f = 0$.

Lemma 1

Suppose that A and B are compact convex subsets of C such that $d(a, b) > 0$ for all $a \in A$ and $b \in B$. Then

$$\inf f \restriction A > 0 \quad \vee \quad \inf f \restriction B > 0\,.$$

Proof.

Apply Proposition 2 to the function

$$A \times B \ni (a, b) \mapsto \max\left(f(a), f(b)\right)\,.$$

$\square$

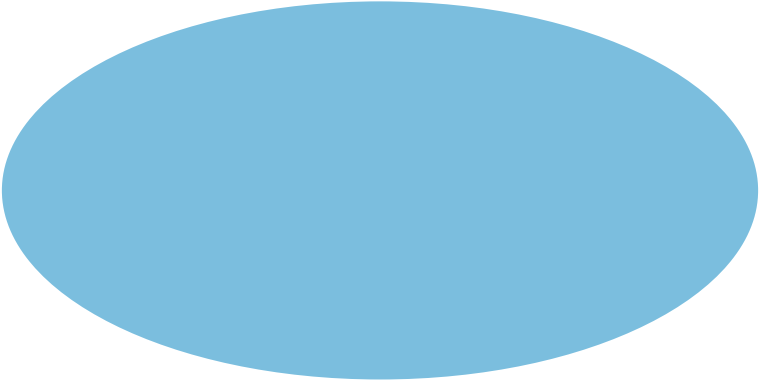
Proof of Proposition 3

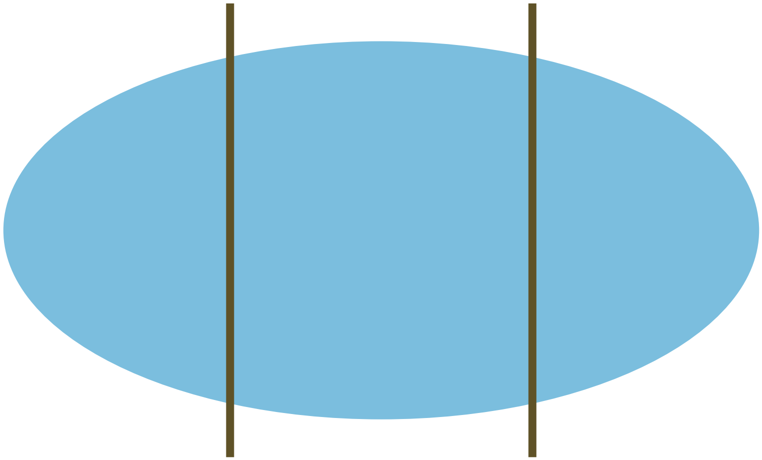
The **diameter** of a compact set X is defined by

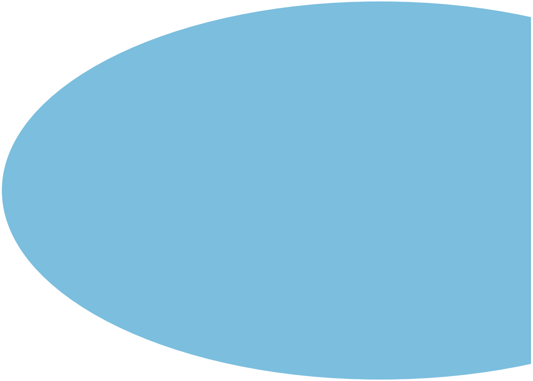
$$\operatorname{diam} X = \sup\left\{d(x,y) \mid x,y \in X\right\}.$$

We construct a sequence (C_n) of subsets of C with vanishing diameter such that $\inf f \restriction C_n = 0$.

Then, we fix $x_n \in C_n$ with $f(x_n) < 1/n$. The sequence (x_n) is Cauchy and its limit is a minimum point of f.







Finite dimensional spaces

A normed space V is *finite-dimensional* if there exist $b_1, \ldots, b_m \in V$ such that the linear mapping

$$\kappa : \mathbb{R}^m \to V, \lambda \mapsto \sum_{i=1}^{m} \lambda_i b_i$$

is bijective. (Injective in the sense that $\|\lambda\| > 0$ implies $\|\kappa(\lambda)\| > 0$.) In this case, both κ and its inverse κ^{-1} are uniformly continuous.

Finite dimensional spaces

Proposition 4

Fix a convex and compact subset C of a finite-dimensional normed space and suppose that $f : C \to \mathbb{R}$ has at most one minimum point and is quasi-convex and uniformly continuous. Then f has a minimum point.

Quasiproximinal sets

Let Y be a subset of a normed linear space X. Fix $a \in X$. Let f_a^Y be the function
$$f_a^Y : Y \ni y \mapsto d(y, a).$$

The set Y is **quasiproximinal** if for every $a \in X$ the implication

$$f_a^Y \text{ has at most one minimum point} \quad \Rightarrow \quad f_a^Y \text{ has a minimum point}$$

is valid.

A corollary: the constructive fundamental theorem of approximation theory

Corollary 1

Every finite-dimensional subspace V of a real normed space X is quasiproximinal.

Douglas S. Bridges, *A Constructive Proximinality Property of Finite-dimensional Linear Subspaces.* Rocky Mountain Journal of Mathematics 11, Number 4 (1981) 491–497

Proof of Corollary 1

Fix $a \in X$ and suppose that $f = f_a^V$ has at most one minimum point. Fix $b \in V$. Set

$$V_0 = \{v \in V \mid d(v, b) \leq 3 \cdot d(a, b)\} .$$

Then V_0 is convex and compact. The function $f \restriction V_0$ is uniformly continuous, quasi-convex and has at most one minimum point. Proposition 4 implies that $f \restriction V_0$ has a minimum point v_0, which is also a minimum point of f.

Proof of Corollary 1

Fix $a \in X$ and suppose that $f = f_a^V$ has at most one minimum point. Fix $b \in V$. Set

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Then V_0 is convex and compact. The function $f \upharpoonright V_0$ is uniformly continuous, quasi-convex and has at most one minimum point. Proposition 4 implies that $f \upharpoonright V_0$ has a minimum point v_0, which is also a minimum point of f. For any $v \in V$, either $v \in V_0$ or

$$2 \cdot d(a, b) \leq d(v, b) \leq d(v, a) + d(a, b),$$

which implies

$$f(v_0) \leq f(b) \leq f(v).$$

(FAN) $\Leftrightarrow$ (POS)

Josef Berger and Hajime Ishihara, *Brouwer's fan theorem and unique existence in constructive analysis*. Math. Log. Quart. 51, No. 4 (2005) 360–364

William H. Julian and Fred Richman, *A uniformly continuous function on* $[0,1]$ *that is everywhere different from its infimum*. Pacific Journal of Mathematics 111, No 2 (1984) 333–340

(FAN) $\Leftrightarrow$ (MIN!)

- Josef Berger and Hajime Ishihara, *Brouwer's fan theorem and unique existence in constructive analysis*. Math. Log. Quart. 51, No. 4 (2005) 360–364
- Josef Berger, Douglas Bridges, and Peter Schuster, *The fan theorem and unique existence of maxima.* The Journal of Symbolic Logic, Volume 71, Number 2 (2006) 713–720
- Peter Schuster, 'Unique solutions.' Math. Log. Quart. 52 (2006), 534–539; Corrigendum: Math. Log. Quart. 53 (2007), 214

Brouwer's fan theorem

- $\{0,1\}^*$ the set of finite binary sequences u, v, w
- $|u|$ the length of u, i.e.

$$\text{for } u = (u_0, \ldots, u_{n-1}) \text{ we have } |u| = n$$

- $u * v$ the concatenation of u and v, i.g.

$$(0,1) * (0,0,1) = (0,1,0,0,1)$$

- α, β, γ infinite binary sequences
- $\overline{\alpha}n$ the restriction of α to the first n elements, i.e.

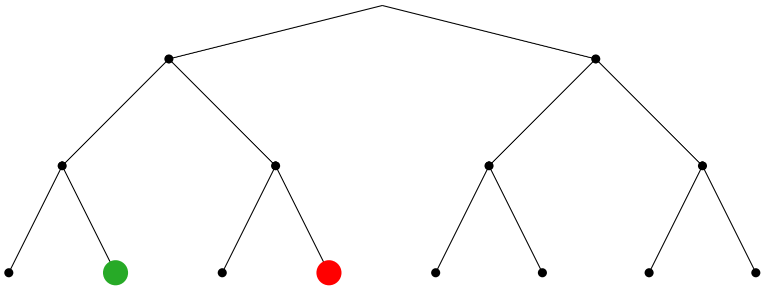
$$\overline{\alpha}n = (\alpha_0, \ldots, \alpha_{n-1})$$

$B \subseteq \{0,1\}^*$ is

- *detachable* if $\forall u\,(u \in B \vee u \notin B)$
- a *bar* if $\forall\alpha\,\exists n\,(\overline{\alpha}n \in B)$ ("every α *hits* B")
- a *uniform bar* if $\exists N\,\forall\alpha\,\exists n \leq N\,(\overline{\alpha}n \in B)$

FAN every detachable bar is a uniform bar

Constructive version of (FAN)



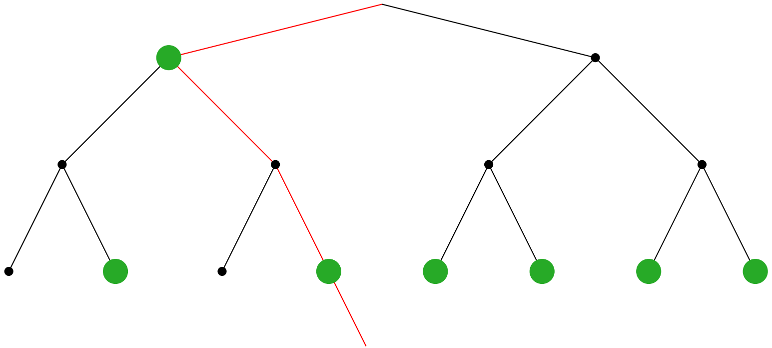
$$(0,0,1) < (0,1,1)$$

$u \leq v \;:\Leftrightarrow\; u < v \;\vee\; u = v$

Constructive version of (FAN)

A subset B of $\{0,1\}^*$ is *co-convex* if for every α which hits B there exists an n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \text{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B\,.$$



Constructive version of (FAN)

Proposition 1

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