The Computational Content of the Hahn-Banach Theorem: Some Known Results – Some Open Questions

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The Hahn-Banach Theorem

Theorem

Let X be a normed space and $Y \subseteq X$ a linear subspace. Every linear bounded functional $f : Y \to \mathbb{R}$ admits a linear bounded extension $g : X \to \mathbb{R}$ such that $||f|| = ||g||$.

Here $||f|| := \sup_{||x|| \leq 1} |f(x)|$ denotes the usual operator norm.

We are interested in questions such as:

- How difficult is it to find some suitable g, given f, i.e., how difficult is it to compute the map $f \mapsto g$?
- If we fix a computable f, how difficult can a suitable g be to compute?
- How does all this depend on properties of the space X and the subspace Y?

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The following are equivalent:

- *The Hahn-Banach Theorem (for separable spaces).*
- *Weak Kőnig's Lemma.*

This "equivalence" was proved in the following settings:

- Over RCA_0 in reverse mathematics (Brown, Simpson 1986).
- In Bishop's style constructive analysis (Ishihara 1990).
- In the Weihrauch lattice (Gherardi, Marcone 2009).

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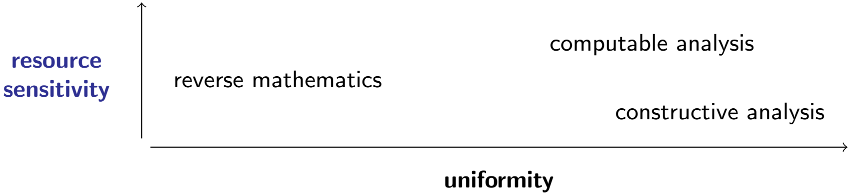

Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be problems.

- $g :\subseteq X \rightrightarrows Y$ solves $f :\subseteq X \rightrightarrows Y$, if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g(x) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$. We write $g \sqsubseteq f$.

- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $H\langle \mathrm{id}_X, gK \rangle \sqsubseteq f$.
- f is strongly Weihrauch reducible to g, $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_{\mathrm{W}} g$ and $f \equiv_{\mathrm{sW}} g$ are defined as usual.

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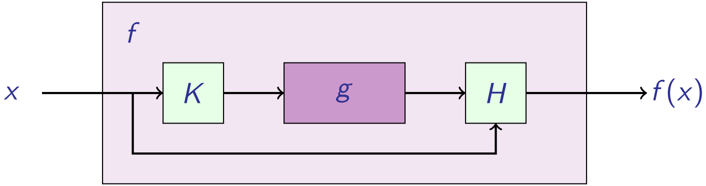

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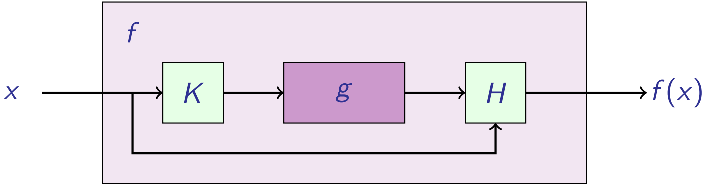

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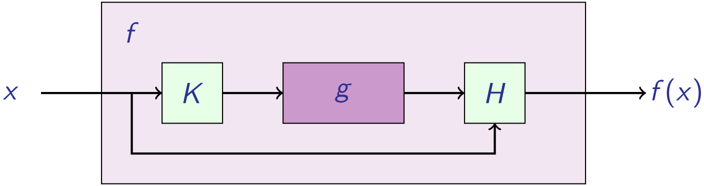

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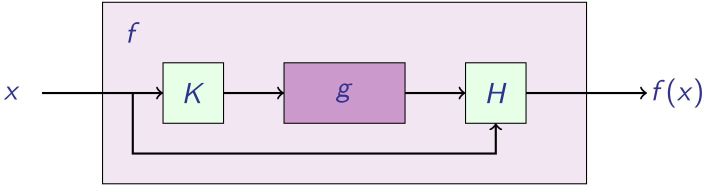

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Algebraic Operations in the Weihrauch Lattice

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$, $(x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$, $z \mapsto \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$, $(x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*$, $f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$, $\widehat{f} = \bigtimes_{i=0}^{\infty} f$ (Parallelization)
- $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f, g_0 \leq_{\mathrm{W}} g\}$ (Compos. product)

Theorem (B., Gherardi, Pauly)

- *Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum.*
- *Parallelization $\widehat{\ }$ and star operation * are closure operators in the Weihrauch lattice.*

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The Choice Problem

Definition

$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ with $\mathrm{dom}(\mathsf{C}_X) := \{A : A \neq \emptyset\}$ is called the choice problem of a computable metric space X.

We consider the following restrictions of choice:

- UC_X is C_X restricted to singletons (unique choice)
- CC_X is C_X restricted to connected sets (connected choice)
- XC_X is C_X restricted to convex sets (convex choice)

Theorem (B. and Gherardi)

- $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$,
- $\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \equiv_{\mathrm{sW}} \mathsf{WKL}$,
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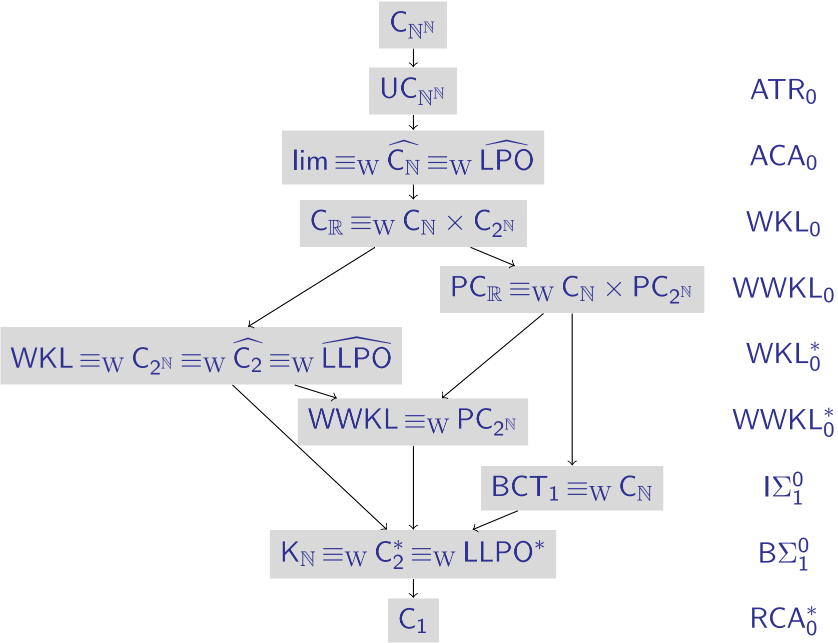

The Weihrauch Degree of the Hahn-Banach Theorem

$\mathrm{HBT}(X, Y, f) := \{g \in \mathcal{C}(X, \mathbb{R}) : g \text{ linear}, g|_Y = f, ||g|| = ||f||\}$.

Theorem (Gherardi and Marcone 2009)

$\mathrm{HBT} \equiv_{\mathrm{W}} \mathrm{WKL}$.

Proof. (Idea.)

- "$\leq_{\mathrm{W}}$": There is a computable version of the Banach-Alaoglu Theorem, that states that the unit ball in X^* is compact with respect to the weak* topology (B. 2008).
- The set of solutions $\mathrm{HBT}(X, Y, f)$ of the extension problem can be seen as a closed subset of this compact space.
- Hence finding an extension can be reduced to finding a point in a compact set, which is known to be reducible to WKL.
- "$\geq_{\mathrm{W}}$": the authors adapt a construction of Brown and Simpson (1990) that reduces the separation problem (equivalent to WKL) to HBT.
- This construction requires the construction of a Banach space that depends on the instance of the problem. □

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Theorem (Gherardi and Marcone 2009)

$\text{HBT} \equiv_{\mathrm{W}} \text{WKL}$.

We immediately obtain the following non-uniform counter example.

Corollary (Metakides, Nerode and Shore 1985)

There exists a computable normed space X with a computably separable closed linear subspace $Y \subseteq X$ and a computable linear $f : Y \to \mathbb{R}$ with computable norm $||f||$ such that every computable linear extension $g : X \to \mathbb{R}$ of f has norm $||g|| > ||f||$.

A set Y is called *computably separable closed* in X if there is a computable sequence $(x_n)_{n \in \mathbb{N}}$ in X with $Y = \overline{\{x_n : n \in \mathbb{N}\}}$.

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We immediately obtain the following non-uniform positive result.

Corollary

Let X be a computable normed space X with a computably separable closed linear subspace $Y \subseteq X$ and a computable linear $f : Y \to \mathbb{R}$ with computable norm $||f||$. Then f has a low bounded linear extension $g : X \to \mathbb{R}$ with $||g|| = ||f||$.

That $g : X \to \mathbb{R}$ is *low* means here that it is low as a point in $\mathcal{C}(X, \mathbb{R})$ which means that it has a low name $p \in \mathbb{N}^{\mathbb{N}}$.
A $p \in \mathbb{N}^{\mathbb{N}}$ is called *low* if $p' \leq_{\mathrm{T}} 0'$.

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Theorem (Gherardi and Marcone 2009)

$\mathsf{HBT} \equiv_{\mathrm{W}} \mathsf{WKL}$.

Let us denote by HBT_X the problem HBT for a fixed space X.

Question

Is there a computable Banach space X with $\mathsf{HBT}_X \equiv_{\mathrm{W}} \mathsf{WKL}$?

As an example we denote the Banach Inverse Mapping Theorem by

$$\mathsf{BIM}_X :\subseteq \mathcal{C}(X,Y) \to \mathcal{C}(Y,X), T \mapsto T^{-1},$$

i.e., BIM is defined for bijective linear T. We obtain:

- $\mathsf{BIM}_{X,Y} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ for all computable Banach spaces X,Y.
- $\mathsf{BIM}_{\ell_2,\ell_2} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.
- $\mathsf{BIM}_{X,Y}$ is computable for finite-dimensional computable Banach spaces X,Y.

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The Unique Case

Proposition (B. and Gherardi)

$\mathrm{UC}_{2^{\mathbb{N}}}$ *is computable.*

- $(Y, || \, ||)$ is called *strictly convex*, if $||x + y|| < ||x|| + ||y||$ holds for all linearly independent $x, y \in Y$.
- A normed spaces X has a strictly convex dual space X^* if and only if all linear bounded functionals $f : Y \to \mathbb{R}$ have unique extensions $g : X \to \mathbb{R}$ with $||f|| = ||g||$.

Corollary (B. 2008)

HBT_X *is computable for all computable normed spaces* X *with a strictly convex dual space* X^*.

- Examples of strictly convex spaces are ℓ_p for $1 < p < \infty$.
- All Hilbert spaces are strictly convex.
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Finite Dimensional Extensions

Lemma (Folklore)

Let $(X, ||\ ||)$ *be a normed space,* $Y \subseteq X$ *a linear subspace,* $x \in X$ *and* Z *the linear subspace generated by* $Y \cup \{x\}$. *Let* $f : Y \to \mathbb{R}$ *be a linear functional with* $||f|| = 1$. *A linear* $g : Z \to \mathbb{R}$ *with* $g|_Y = f|_Y$ *extends* f *with* $||g|| = 1$, *if and only if*

$$\sup_{u \in Y}(f(u) - ||x - u||) \leq g(x) \leq \inf_{v \in Y}(f(v) + ||x - v||).$$

By HBT_n we denote the Hahn-Banach Theorem HBT restricted to subspaces Y of codimension $\leq n$.

Corollary

$$\mathrm{HBT}_n \leq_{\mathrm{W}} \underbrace{\mathrm{CC}_{[0,1]} * \ldots * \mathrm{CC}_{[0,1]}}_{n-times}.$$

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$$\mathsf{HBT}_n \leq_{\mathrm{W}} \underbrace{\mathsf{CC}_{[0,1]} * \ldots * \mathsf{CC}_{[0,1]}}_{n\text{–}times}.$$

$\mathsf{CC}_{[0,1]}$ is non-uniformly computable, i.e., every connected co-c.e. closed subset $A \subseteq [0,1]$ contains a computable point.

Corollary (Metakides and Nerode 1985)

Let X be a finite-dimensional computable Banach space with some closed linear subspace $Y \subseteq X$. For any computable linear functional $f : Y \to \mathbb{R}$ with computable norm $||f||$ there exists a computable linear extension $g : X \to \mathbb{R}$ with $||g|| = ||f||$.

Note however, that $\mathsf{LLPO} \equiv_{\mathrm{sW}} \mathsf{C}_2 \leq_{\mathrm{sW}} \mathsf{HBT}_2$ (this can be proved using ideas of Ishihara). Hence, HBT_n is not computable.

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A Space of Maximal Complexity

Question

Is there a computable Banach space X with $\mathrm{HBT}_X \equiv_{\mathrm{W}} \mathrm{WKL}$?

Some negative results on a possible X:

- X cannot have a strictly convex dual space X^*,
- X cannot be a Hilbert space,
- X cannot be a space ℓ_p for $1 < p < \infty$,
- X cannot be finite-dimensional.

We recall $||x||_{\ell_p} = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$.

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Is $\mathrm{HBT}_{\ell_1} \equiv_{\mathrm{W}} \mathrm{WKL}$?

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A Classification of the Finite Dimensional Case

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Can $\text{HBT}_n \leq_W \text{CC}_{[0,1]} * \dots * \text{CC}_{[0,1]}$ *be improved?*

A plausible candidate is the following:

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Is $\text{HBT}_n \leq_W \text{XC}_{[0,1]^n}$ *or even* $\text{HBT}_n \equiv_W \text{XC}_{[0,1]^n}$*?*

More specifically we can even ask:

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Is $\text{HBT}_{\ell_1(n)} \equiv_W \text{XC}_{[0,1]^n}$*?*

Such a result would be nice since we have:

Theorem (Le Roux and Pauly)

$\text{XC}_{[0,1]^n} <_W \text{XC}_{[0,1]^{n+1}}$ *for all* $n \in \mathbb{N}$.

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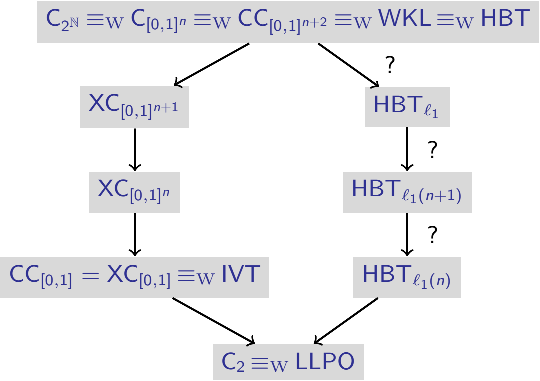
The Hahn-Banach Theorem in the Weihrauch Lattice



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A Survey on Weihrauch Complexity

Mathematics > Logic

Weihrauch Complexity in Computable Analysis

Vasco Brattka, Guido Gherardi, Arno Pauly

(Submitted on 11 Jul 2017)

We provide a self-contained introduction into Weihrauch complexity and its applications to computable analysis. This includes a survey on some classification results and a discussion of the relation to other approaches.

Comments: 49 pages plus 10 pages appendix
Subjects: **Logic (math.LO)**; Logic in Computer Science (cs.LO)
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