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Diplomarbeit

The Geometric Phase in Quantum Electrodynamics

Dustin Lazarovici

betreut von Prof. Dr. Detlef Dürr,
Mathematisches Institut der Ludwig-Maximilians-Universität München

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Dustin Lazarovici

under supervision of Prof. Dr. Detlef Dürr,
Mathematisches Institut der Ludwig-Maximilians-Universität München

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Abstract

This work is concerned with the problem of second quantization of the Dirac time evolution in the external field setting in Quantum Electrodynamics with time-varying fields.
It refers mainly to two publications on this subject:

- D.-A. Deckert, D. Duerr, F. Merkl and M. Schottenloher: "Time Evolution of the External Field Problem in QED" (2010) [DeDuMeScho]

where the time evolution is realized as unitary transformations between time-varying Fock spaces, leaving the freedom of a complex phase. And

- E. Langmann, J. Mickelsson: "Scattering matrix in external field problems" (1996) [LaMi96]

where the authors construct a "renormalization" of the time evolution and propose a method to fix the phase of the second quantized scattering operator by parallel transport in the principle fibre bundle $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

In this work, I will inquire in what sense a time-evolution in the second-quantized Dirac theory can exist. Furthermore, I will present a systematic study of the second quantization by parallel transport as introduced in [LaMi96] and identify all freedoms contained in the proposed construction. To this end, I will develop the mathematical framework necessary for a rigorous treatment of the external field problem in QED. During my research I have found that the discrepancies between the mathematical and the physical treatment of the problem are dramatic. My hope is to make a little contribution to bridging this gap by adding some intuition to the abstract mathematical formalism.

What's new?
New results from this thesis include the following:

- It is shown that the construction of the fermionic Fock space as an "infinite wedge space" as developed in [DeDuMeScho] is equivalent to the more common construction of the Fock space from holomorphic sections in the dual of the determinant bundle over the infinite-dimensional Grassmann manifold (c.f.[PreSe]).
- I give an interpretation of the renormalizations introduced in [LaMi96] and show how they can be used to translate between the second quantization procedure on time-varying Fock spaces and the second quantization of the renormalized time evolution.
- I compute the holonomy group of the principle bundle $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ which turns out to equal the entire structure group U(1). It is argued that this corresponds to an additional freedom in the construction of the second quantized time evolution in [LaMi96] that might not have been fully appreciated in the original paper.
- I provide rigorous proof for the fact that the second quantization by parallel transport preserves causality. These findings seem to refute claims made in [Scha] that the phase of the second quantized S-matrix is essentially determined by the requirement of causality.
- I outline how the second quantization procedure could be made gauge-invariant by construction of a suitable renormalization.

*Mathematisches Institut, LMU München. Dustin.Lazarovici@mathematik.uni-muenchen.de

Contents

Notation and Mathematical Preliminaries

Throughout this work we use "natural units" in which $\hbar = c = 1$.
We use the Minkowski metric with signature $(+,-,-,-)$.

Furthermore, we introduce the following notations:

$\mathcal{H}$ a separable, complex Hilbert space.

$\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$.

$\mathrm{GL}(\mathcal{H})$ the space of bounded automorphisms of $\mathcal{H}$.

$\mathrm{U}(\mathcal{H})$ the group of unitary automorphisms of $\mathcal{H}$.

$I_p(\mathcal{H},\mathcal{H}')$ the *p-th Schatten class* of linear operators $T : \mathcal{H} \to \mathcal{H}'$ for which

$$(\|T\|_p)^p := \mathrm{tr}[(T^*T)^{p/2}] < \infty.$$

$\|\cdot\|_p$ is a norm that makes $I_p(\mathcal{H},\mathcal{H}')$ a Banach space. It satisfies $\|AT\|_p \leq \|A\|\,\|T\|_p$ and $\|TB\|_p \leq \|B\|\,\|T\|_p$, for $A \in \mathcal{B}(\mathcal{H}'), B \in \mathcal{B}(\mathcal{H})$. Thus, $I_p(\mathcal{H},\mathcal{H}) =: I_p(\mathcal{H})$ is a two-sided ideal in the algebra of bounded operators on $\mathcal{H}$.
If $T = \sum_{k\geq 0} \mu_k |f_k\rangle\langle e_k|$ is a singular-value decomposition, the p-th Schatten norm corresponds to the ℓ^p norm on the sequence $(\mu_k)_k$ of singular values.
If an operator T is in $I_p(\mathcal{H},\mathcal{H}')$ for some p, then T is compact.

In particular,

$I_1(\mathcal{H})$ the ideal of *trace-class operators* for which $\mathrm{tr}(T) := \sum_{k\geq 0} \langle e_k, Te_k\rangle$ is well-defined and independent of the Hilbert basis $(e_k)_{k\geq 0}$ of $\mathcal{H}$.

$I_2(\mathcal{H},\mathcal{H}')$ the class of *Hilbert-Schmidt operators* $\mathcal{H} \to \mathcal{H}'$. The product of two Hilbert-Schmidt operators is in the trace class with $\|ST\|_1 \leq \|S\|_2\,\|T\|_2$.

$Id + I_1(\mathcal{H}) = \{A = Id + T \mid T \in I_1(\mathcal{H})\}$ the set of operators for which the *Fredholm determinant* is well defined.
If $(e_k)_{k\geq 0}$ is on ONB of $\mathcal{H}$, then $\det(A) = \lim_{N\to\infty} \det\big(\langle e_i, Ae_j\rangle\big)_{i,j\leq N}$

$\mathrm{GL}^1(\mathcal{H}) := \mathrm{GL}(\mathcal{H}) \cap \big(Id + I_1(\mathcal{H})\big)$ the set of bdd. isomorphism that do have a determinant.

All further notations will be introduced in the course of the work.

Preface

Quantum Electrodynamics, or short: QED, is widely considered to be the most successful theory in entire physics. Indeed, its predictions have been confirmed time and time again with remarkable precision by various experiments in particle accelerators and laboratories all over the world. Probably the most famous and most spectacular demonstration of the potency of Quantum Electrodynamics is the prediction of the anomalous magnetic moment of the electron, known as "g - 2" in the physical literature. This electron g-factor has been measured with an accuracy of 7.6 parts in 10^{13}, i.e. with a stupendous precision of 12 decimal places and found to be in full agreement with the theoretical prediction (Odom et.al., Phys. Rev. Lett. 97, 030801 (2006)). Actually, we have to be more precise: since the fine structure constant α enters every QED-calculation as a free parameter, we have to gauge it by other experiments or, equivalently, express every QED-measurement as an independent measurement of α. In this sense, theory and experiment are in agreement up to 0.37 parts per billion i.e. to 10 decimal places in the determination of α (Hanneke et.al., Phys. Rev. Lett. 100, 120801 (2008)). This has often been called the best prediction in physics and whether this is factual or not, it is certainly very impressive.
In one of the standard textbooks on Quantum Field Theory it is even said that *"Quantum Electrodynamics (QED) is perhaps the best fundamental physical theory we have"*. (Peskin, Schröder, "An Introduction to Quantum Field Theory", 1995).

In this light it might seem surprising, that many of the brilliant minds that actually came up with the theory were not quite as enthusiastic about it. In a talk given in 1975, P.A.M. Dirac famously expressed:

> *"Most physicists are very satisfied with the situation. They say, Quantum electrodynamics is a good theory, and we do not have to worry about it any more. I must say that I am very dissatisfied with the situation, because this so-called good theory does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it just because it is infinitely great and you do not want it!"* [2]
>
> Cited after: H. Kragh, Dirac: A scientific biography, CUP 1990

And even in his Nobel lecture, where the occasion would have certainly excused some enthusiasm, Richard Feynman said:

[2] What I find most remarkable about this quote is that it takes such a great scientist to state something so obvious and be taken seriously.

> *"That is, I believe there is really no satisfactory quantum electrodynamics, but I'm not sure. [...] I don't think we have a completely satisfactory relativistic quantum-mechanical model, even one that doesn't agree with nature, but, at least, agrees with the logic that the sum of probability of all alternatives has to be 100%. Therefore, I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug. I am, of course, not sure of that."*
>
> R.P. Feynman: "The Development of the Space-Time View of Quantum Electrodynamics", Nobel Lecture (1965)[3]

Comparing these two statements with the assessment of Peskin and Schröder and really with the general spirit of the scientific community of today, one might think that great progress has been made on the foundations of QED ever since. Quite frankly, I don't see where. Of course, experimental success has steadily strengthened our trust in the usefulness of the framework of Quantum Field Theory, but I don't think that this was really the major concern of the Monsieurs Feynman and Dirac. Over the years we might have become desensitized to the problems of QED, but we haven't done a very good job at fixing them. So, what then is wrong with QED?

For once, more then 50 years after its development, Quantum Electrodynamics still lacks a rigorous mathematical formulation. It is well-known that QED (just as all realistic Quantum Field Theories) relies on different "renormalization" schemes to render its predictions (more or less) finite. And even after renormalization, the S-matrix expansion is widely believed to have *zero* radius of convergence in the coupling constant α. No matter how crafty physicists have gotten at manipulating infinities, this fact remains highly unsatisfying from a mathematical point of view.

It is also well known that the Wightman axioms, a minimal set of formal requirements that one would demand from a sensible field theory, are *not* satisfied by (3+1 - dimensional) QED (or any other known realistic Quantum field theory, for that matter). Interestingly enough, this realization had not so much shaken confidence in the theories themselves but rather ended the program of axiomatizing fundamental physics. A fact remarkably little known among physicist is that the mathematical deficiencies of QED do not start at the computational level but are really much more basic. We don't even know how to write down a theory that is mathematically meaningful, i.e. without objects that are intrinsically ill-defined. Most physicists seem either not to know or not to care about these kind of problems. This has created the somewhat tragicomical situation that nowadays, with very few exceptions, neither physicists nor mathematicians are working on the foundations of the theory, physicists because they find the theory so good that there's nothing left to do; mathematicians because they find the theory so bad that they don't know where to start.

But apart from all mathematical problems, QED (and really the entire Standard Model of particle physics) is incomplete in a very different sense: it's lacking an "ontology", a meaningful interpretation of the mathematical framework providing a clear and coherent picture of what the theory is really about.

[3] http://nobelprize.org/nobel_prizes/physics/laureates/1965/feynman-lecture.html

Naively spoken, a complete physical theory has to give us a list of all the physical objects that are assumed to be "real" as opposed to mere formal constructions or artifacts of the mathematical description. Then it has to tell us how these objects of our physical reality correspond to formal objects in the theory. Thinking about Quantum Electrodynamics, just try in all honesty to answer the question: what is it actually about? What are the fundamental physical objects of the theory?

Is QED essentially a theory about charged particles? At first glance, this seems like a good guess. However, the particle ontology was abandoned a long time ago by standard Quantum Theory. And indeed, taking a look into any textbook on Quantum Field Theory, we're going to find plenty of "particles" neatly listed in various tables or swirling around in funny little diagrams - yet, there aren't actually any particles in the theory. [4] So, maybe QED is a theory about "Quantum Fields"? The expression "Quantum Field Theory" might suggest that kind, but the answer is not too convincing as the role of the "fields" in the theory remains rather obscure. Mainly, they seem to appear as a formal device for setting up the perturbation expansions or for deriving the equations of motion from an action principle. I might be wrong about that. Even then, however, explaining what exactly "Quantum Fields" are supposed to be and how they constitute the physical world that we live in seems like a formidable task and not many people who invoke this answer like to engage in it.
Maybe QED is merely about "transition amplitudes". To my understanding, this very pragmatic standpoint was advocated by such distinguished scientists as Werner Heisenberg, for example, and it might very well be logically consistent, although I think it requires some mental gymnastics to avoid questions like *what* transitions from *what* into *what*?

One might call questions of this kind "metaphysical", but to me, they are as physical as it gets. And with such considerations in mind, the state of modern physics in general and of Quantum Electrodynamics in particular seems pretty bad. I have to repeat, though, that the vast majority of physicists does not share this kind of pessimism. I can only speculate about the reasons, of course, and a detailed discussion of this would certainly be beyond the scope of this introduction. However, I like the irony in the idea that the very genius of Richard Feynman might have to take a little of the blame for this. His ingenious method of visualizing the formal expansions of Quantum Field Theories by means of the famous diagrams that carry its name provided us with most of the intuition we have for the physical processes in Particle Physics and coined the way we use to think and talk about the theory. We are so used to talk about particles scattering from each other, about photons being emitted and absorbed or about pairs of virtual particles "screening" the charges and so on, that we tend to forget that *none of this is actually in the theory.* As my first teacher on Quantum Field Theory used to say: *"it's just a nice, cartoonish way to talk about these things."*

Above, I have suggested that physicists don't work on the foundations of QED anymore because they don't see any problems with the theory. This is just partly true. Actually, many physicists do acknowledge that the theory is deeply flawed, but have - de facto - given up on it. Instead they have adopted the point of view that QED (or rather the entire

[4] Indeed, in the Standard Model particles can have mass, charge, spin, even color, flavor or families, but no location in space-time. And therefore no substance as a physical object.

Standard Model) is indeed not a fundamental theory of nature but merely a low energy approximation of a fundamental theory (maybe a "theory of everything") still waiting to be discovered and that the problems we're facing today will vanish, once said theory is found. String Theory is usually considered to be the best candidate for such a fundamental theory of nature. Personally, I am sceptical whether the results of String Theory after 20 years of intensive research justify this kind of optimism, but that's a different debate. Anyways, claiming that QED will have to wait for the next big scientific revolution to resolve its various issues might be a valid standpoint. I just have two objections that I would like to mention.
First: it has never been like that in the history of physics. Classical Mechanics were perfectly well defined and well understood before Special Relativity and Quantum Mechanics came to extend the picture. The Maxwell-Lorentz theory of electromagnetism is a beautiful theory, both physically and mathematically, except for one little detail: the electron self-interaction. This problem was not solved but inherited by the "more fundamental" Quantum Electrodynamics, where it made quite a prominent career under the name "ultraviolet divergence".[5]

My second objection is this: even if the final answers *do* lie beyond QED, isn't it still important to understand as well as possible what exactly goes wrong and what can and cannot be done? Isn't it possible, even likely, that insights of this kind will lead the way to a new, better behaved, maybe more fundamental description? As John Bell put it [6]:

> *"Suppose that when formulation beyond FAPP* [for all practical purposes] *is attempted, we find an unmovable finger obstinately pointing outside the subject, to the mind of the observer, to the Hindu scriptures, to God, or even only Gravitation? Would not that be very, very interesting?"*
>
> J. Bell, "Against Measurement" in Speakable and Unspeakable in Quantum Mechanics

It is in this spirit that I wrote my thesis and that the research program started by my teachers and colleagues has to be understood. The goal is ambitious and very humble at the same time. We do not expect to "fix" QED or solve all the problems that have troubled so many greater physicists before us. We just hope to get a better understanding of the difficulties, approach them in a systematic way and see how far one can get with rigorous mathematics. The work I am presenting here is of rather technical nature and mostly concerned with the task of lifting the unitary time evolution to the fermionic Fock space ("second quantization") in the exterior field problem of QED. I hope that this will provide some insights into the fundamental difficulties of QED and of relativistic Quantum theory in general. To me, at least, it was a humbling realization to learn at what basic level the formalism already fails. Ultimately, though, my personal believe is that if we want to make significant progress towards a meaningful, well-defined, fundamental theory we need to think the formal aspects and the conceptual aspects together and return to a way of doing physics that takes rigorous mathematics seriously and takes ontology seriously.

[5] Above all, to this divergence, QED added the "infrared divergence" so things really got worse on that front.

[6] Actually, Bell wrote this about non-relativistic Quantum Mechanics, but his appeal seems even more urgent in the context of modern Quantum field theories

Chapter 1

Introduction

1.1 The Dirac equation

Our study of Quantum Electrodynamics starts with the one-particle Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^4)$ of square-integrable $\mathbb{C}^4$-valued functions. The fundamental equation of motion is the famous Dirac equation

$$\boxed{(i\partial\!\!\!/ - m)\Psi(t) = (i\gamma^\mu \partial_\mu - m)\Psi(t) = 0} \tag{1.1.1}$$

where $\Psi(t) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ for every fixed t. The gamma matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ form a 4-dimensional complex representation of the Clifford algebra $\mathcal{C}l(1,3)$, i.e. they satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}$ where $g^{\mu\nu}$ is the Minkowski metric tensor.

Using $(\gamma^0)^2 = \mathbb{1}$ we can rewrite the Dirac equation in Hamiltonian form as

$$\boxed{i\,\partial_t\,\Psi = D_0\,\Psi := \big(-i\underline{\alpha}\cdot\underline{\nabla} + m\beta\big)\Psi} \tag{1.1.2}$$

Here the notations $\beta = \gamma^0$ and $\alpha^\mu = \gamma^0\gamma^\mu$ are common.
In the presence of an electromagnetic field described by a vector potential $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\Phi, -\underline{\mathsf{A}})$, the partial derivative in the Dirac equation (1.1.1) is replaced by the covariant derivative $\partial_\mu \longrightarrow \partial_\mu + ieA_\mu$. This adds the interaction potential

$$\boxed{V(t) = e\,\alpha^\mu A_\mu = -e\,\underline{\alpha}\cdot\underline{A} + e\,\Phi} \tag{1.1.3}$$

to the Hamiltonian.

Just as a side-note we remark that a mathematically more sophisticated description would start with a space-time $M \times \mathbb{R}$, where M is a (3-dimensional) compact manifold with spin-structure and realize $\mathcal{H}$ as the space of L^2-sections in a spinor-bundle over M (cf. [LM]).

It is well known that the free Dirac Hamiltionan D_0 is unstable. It has the continuous spectrum $(-\infty, -m] \cup [+m, +\infty)$ which gives rise to a splitting of the one-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ into two spectral subspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Physical interpretation of the negative-energy free states is difficult and has troubled physicists for many years. In particular, as the Hamiltonian is unbounded from below, it would be possible to extract an arbitrary amount of energy from the system, which is unphysical. To deal with this problems, P.A.M. Dirac proposed the so called *Dirac Sea theory*:

> *"Admettons que dans l'Univers tel que nous le connaissons, les états d'energie négative soient presque tous occupés par des électrons, et que la distribution ainsi obtenue ne soit pas accessible à notre observation à cause de son uniformité dans toute l'etendue de l'espace. Dans ces conditions, tout état d'energie négative non occupé représentant une rupture de cette uniformité, doit se révévler à observation comme une sorte de lacune. Il es possible d'admettre que ces lacunes constituent les positrons."*
>
> P.A.M. Dirac, Théorie du Positron (1934), in Selected Papers on Quantum Electrodynamics, Ed. J. Schwinger, Dover Pub. (1958)

According to Dirac, the negative energy states are (almost entirely) occupied by an infinite number of electrons - the Dirac Sea - which due to its homogeneous distribution is hidden from physical observation. The Pauli exclusion principle will then prevent transition of positive energy electrons to negative energy states, which keeps the system stable. "Holes" in the otherwise homogeneously filled Dirac Sea will appear as particles of positive energy but opposite charge - we call them positrons.
Transition of an electron from the negative energy spectrum to the positive energy spectrum in the presence of an electromagnetic field will look like the simultaneous creation of an electron and a positron to the outside observer. Conversely, when a positive-energy electrons drops into an unoccupied state of negative energy, we see the annihilation of an electron/positron pair with energy being emitted in form of radiation.

Despite of the obvious peculiarities, that give us headaches, this is an ingenious picture which explains the most important phenomena of relativistic Quantum theory in a clear and elegant way. In todays physics, the Dirac Sea is usually not any more part of modern descriptions of Quantum Electrodynamics, but it seems to me that it hasn't been replaced with an equally compelling physical picture. Anyway, Dirac's theory provides a very good intuition for the difficulties of relativistic Quantum theory and will motivate most of the mathematically rigorous approaches to the external field problem in QED that we will present in this work.

1.2 An intuitive approach

From nonrelativistic Quantum Mechanics we are familiar with the fact that the n-Fermion Hilbert space is the n-fold exterior product $\bigwedge^n \mathcal{H}$ of the one-particle Hilbert space $\mathcal{H}$.
This space is spanned by decomposable states of the form $v_1 \wedge \cdots \wedge v_n$.
The Hermitian scalar product is given by:

$$\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n \rangle = \det(\langle v_i, w_j \rangle)_{i,j} \tag{1.2.1}$$

Under a unitary (Schrödinger) time evolution $U = U(t_1, t_0)$ the states evolve in the obvious way:

$$v_1 \wedge \cdots \wedge v_n \xrightarrow{U(t_1,t_0)} Uv_1 \wedge \cdots \wedge Uv_n \tag{1.2.2}$$

Projectively, (i.e. mod $\mathbb{C}$) such states are in one-to-one correspondence with n-dimensional subspaces of the Hilbert space $\mathcal{H}$, by

$$v_1 \wedge \cdots \wedge v_n \longmapsto \operatorname{span}(v_1, \ldots, v_n) =: V \subset \mathcal{H} \tag{1.2.3}$$

for if we take $w_1, \ldots, w_n \in \mathcal{H}$ with $\operatorname{span}(w_1, \ldots, w_n) = \operatorname{span}(v_1, \ldots v_n)$ (i.e. a different basis of V), then

$$w_1 \wedge \cdots \wedge w_n = \det(R)\, v_1 \wedge \cdots \wedge v_n \tag{1.2.4}$$

with R the matrix in $\mathrm{GL}_n(\mathbb{C})$ transforming the basis $(v_1, \ldots, v_n)$ into $(w_1, \ldots, w_n)$.
(Note that if $v_1, \ldots, v_n$ are not linearly independent, then $v_1 \wedge \cdots \wedge v_n = 0$).

In the setting of relativistic Quantum Electrodynamics, however, we have a *Dirac sea* containing *infinitely many* particles, which makes the situation more complicated. Projective decomposable states are now in correspondence with infinite dimensional subspaces of $\mathcal{H}$, so called *polarizations*, which will be precisely defined in Def. 2.1.1. For example, in the unperturbed Dirac sea (the ground state) all negative energy states are occupied and all positive energy states empty. So this state of the Dirac sea corresponds to the subspace $V := \mathcal{H}_- \subset \mathcal{H}$. Under a unitary transformation U, a time evolution $U = U(t_1, t_0)$ let's say, V evolves into $W := U(V) = UV$.

To give this some physical meaning, we would like to ask: "How many electrons and how many positrons (holes) were created?". Thanks to Diracs ingenious picture, we have a very simple intuition of how to answer this question: we just count! The negative energy states that remain occupied correspond to $W \cap \mathcal{H}_-$. Consequently, the number of holes is simply the codimension of $W \cap \mathcal{H}_-$ in $\mathcal{H}_-$, i.e. the dimension of the factor space $\mathcal{H}_-/(W \cap \mathcal{H}_-)$. Similarly, the Dirac sea picture tells us that we can think of the "parts" of W complementary to $W \cap \mathcal{H}_-$ as electrons that have been lifted from the Sea to positive energies.
In conclusion:

$$\#\text{electrons} \approx \dim\big(W/(W \cap \mathcal{H}_-)\big) \tag{1.2.5}$$

$$\#\text{holes} \approx \dim\big(\mathcal{H}_-/(W \cap \mathcal{H}_-)\big) \tag{1.2.6}$$

The *net-charge* of W is then:

$$\dim\big(W/(W \cap \mathcal{H}_-)\big) - \dim\big(\mathcal{H}_-/(W \cap \mathcal{H}_-)\big) \tag{1.2.7}$$

In order for all of this to make sense, we have to require that both (1.2.5) and (1.2.6) are finite. We call polarizations V and W satisfying

$$\dim\big(W/(W \cap V)\big) < \infty \text{ and } \dim\big(V/(W \cap V)\big) < \infty \tag{1.2.8}$$

commensurable.

There is a natural generalization of this (in fact corresponding to a closure in a topological sense): If P_V and P_W are the orthogonal projections onto V and W, respectively, we require that

$$P_V - P_W \text{ is a Hilbert-Schmidt operator}$$

In this case we will say that V and W belong to the same *polarization class*. At first glance, this looks nothing like the condition of "commensurabilty" formulated above. But note that $P_V - P_W$ being of Hilbert-Schmidt type just means that $(P_V - P_W)^*(P_V - P_W)$ is in the trace-class, i.e. (since orthogonal projections are self-adjoint)

$$\mathrm{tr}(P_V - P_W P_V + P_W - P_V P_W) < \infty \tag{1.2.9}$$

Aside we note that (1.2.9) can also be written as

$$\mathrm{tr}(P_{W^\perp} P_V + P_{V^\perp} P_W) < \infty \tag{1.2.10}$$

Now this starts to look more like what we're after. Orthogonal projections are self-adjoint operators with eigenvalues 1 (on the respective subspace) and zero (on the orthogonal complement). Thus, $\mathrm{tr}(P_V) = \dim(V)$ whenever this is finite. And if P_V and P_W commute (which in general they don't) $P_V P_W = P_W P_V = P_{V\cap W}$ and so $\mathrm{tr}(P_V - P_W P_V)$ really counts the dimension of the orthogonal complement of $W \cap V$ in V and so forth. Therefore it does indeed make sense to regard (1.2.9) as a generalization of "commensurablity" (1.2.8). The precise relationship between (1.2.9) and (1.2.8) is discussed in appendix A, but I hope that at this point the reader is convinced that polarization classes are an adequate concept.

Similarly, there's an abstract generalization of (1.2.7) counting the net-charges of polarizations : If V and W are in the same polarization class then $\mathrm{ind}(P_W|_{V\to W})$ is well defined and we will call this number the *relative charge* of V and W. Again, the motivation becomes more clear if we remember that

$$\begin{aligned}\mathrm{ind}(P_W|_{V\to W}) &= \dim\ker(P_W|_{V\to W})) - \dim\mathrm{coker}(P_W|_{V\to W}) \\ &= \dim\ker(P_W|_{V\to W})) - \dim\left(W/P_W(V)\right)\end{aligned}$$

This coincides with (1.2.7), whenever the latter is well defined (see appendix A).

Now let's go back to our initial setup with $V = \mathcal{H}_-$ and $W = UV$, for the unitary transformation $U \in \mathrm{U}(\mathcal{H})$. We denote by P_- the orthogonal projection onto $\mathcal{H}_-$ and by P_+ the orthogonal projection onto $\mathcal{H}_+$. In particular, $P_- + P_+ = \mathbb{1}$. As W results from $\mathcal{H}_-$ by the unitary transformation U, the orthogonal projection is $P_W = UP_-U^*$.
Therefore:

$$\begin{aligned} P_V - P_W \text{ Hilbert-Schmidt} &\iff P_- - UP_-U^* && \text{Hilbert-Schmidt} \\ &\iff P_-U - UP_- && \text{Hilbert-Schmidt} \\ &\iff P_-U(P_- + P_+) - (P_- + P_+)UP_- && \\ &= P_-UP_+ - P_+UP_- && \text{Hilbert-Schmidt} \end{aligned}$$

As P_-UP_+ and P_+UP_- map from and into complementary subspaces, this is satisfied if and only if both

$$U_{-+} := P_-UP_+|_{\mathcal{H}_+\to\mathcal{H}_-} \text{ and } U_{+-} := P_+UP_-|_{\mathcal{H}_-\to\mathcal{H}_+} \tag{1.2.11}$$

are of Hilbert-Schmidt type. This is known as the **Shale-Stinespring Condition**.

In the remainder of this work we will invoke lots of fancy mathematics to construct the fermionic Fock space and implement unitary transformations on it, but in the end, whatever path we take, it always comes down to this. The unitary time evolution has to fulfill the Shale-Stinespring condition in order to make sense on QED-states. Otherwise, the Dirac sea becomes too stormy: infinitely many particles are being created and we have no chance to compare initial and final state in a meaningful way. In other words: we cannot accommodate initial and final state in one and the same Fock space. There is no way around this, at least not in the usual formulations of the theory.

We have stumbled upon a serious difficulty, but it might not yet be clear how severe the problem really is as it's probably not obvious how restrictive the Shale-Stinespring criterion is. We might hope that in physically relevant situations everything turns out to be so nice that we don't have to worry about it. Actually, we shouldn't expect that. We should and would expect it if the difficulties were of merely technical nature. But this is not the case. The Shale-Stinespring criterion is not a mathematical prerequisite of the type "let f twice continuously differentiable". The Shale-Stinespring criterion reflects the problem of infinite-particle creation which seems to be deeply inherent to relativistic Quantum theory.
Finally, all remaining hopes that this might to out to be a minor hurdle are destroyed by the following theorem:

Theorem 1.2.1 (Ruijsenaars, 1976)**.**
Let $\mathsf{A} = (\mathsf{A}_0, -\underline{\mathsf{A}}) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ *be an external vector potential and* $U^{\mathsf{A}}(t,t')$ *the corresponding one-particle Dirac time evolution. Then* $U^{\mathsf{A}}(t,t')$ *fulfills the Shale-Stinespring criterion if and only if* $\underline{\mathsf{A}} = 0$*, i.e. iff the spatial part of the A-field vanishes identically.*[1]

I think that this result has not received the resonance that it deserves. It shows that typically the unitary time evolution can *not* be lifted to the Fock space.

Why only the spatial part of the vector potential ($\approx$ the magnetic field) is the evil-doer is not obvious. The following considerations might provide some intuition:
The free Dirac-Hamiltonian in momentum-space is

$$D_0(p) = \underline{\alpha} \cdot \underline{p} + \beta m \tag{1.2.12}$$

which reduces to $D_0(0) = \beta m$ for a particle at rest. In the so called Dirac representation,

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Obviously, the standard basis vectors represent eigenstates with energy $\pm m$.
In the presence of a vector potential $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3}$, the interaction term

$$V = e \sum_{\mu=0}^{3} \alpha^\mu \hat{A}_\mu$$

is added, where $\alpha^0 = \mathbb{1}$ and $\hat{A}_\mu$ is the Fourier transform of A_μ (acting as convolution operators). The electric potential $\Phi = A_0$ is harmless, it just shifts the energy of the particle. But the alpha-matrices satisfy $\beta\alpha_j = -\alpha_j\beta$ for $j = 1,2,3$ and therefore map the negative-energy eigenstates of D_0 to positive energy eigenstates (and vice-versa).
So, intuitively, the magnetic field "rotates" the Dirac sea into $\mathcal{H}_+$.

[1] To physicists it might seem suspicious that the condition $\mathsf{A} = 0$ is not gauge-invariant, obviously. This, however, just points to another deficiency of QED. Against all common intuition, the theory is not gauge-invariant in a trivial sense. Gauge-transformations (except constant ones) do not satisfy the Shale-Stinespring condition and therefore we have the very same problem as with the time evolution: they cannot be implemented as unitary transformations on the Fock space. See §6.2.1.

Note that so far, we have only acted with unitary transformations on "polarizations", which correspond to projective fermion states. If we have a unitary transformation U that *does* satisfy the Shale-Stinespring condition, a different problem arises when we try to act with it on proper (non-projective) fermion states: the action of U on the Fock space is defined only up to a phase. This is a well-known result e.g. from representation theory but our simplistic considerations are good enough to see why it must be so.

Let's again try to generalize the description of n-fermion systems represented in $\bigwedge^n \mathcal{H}$ to QED-states with infinitely many particles. We may think of such a state as in infinite wedge product, i.e. a formal expression

$$\Psi \mathrel{\widehat{=}} v_0 \wedge v_1 \wedge v_2 \wedge v_3 \wedge \dots$$

where the subspace spanned by the v_j is in the polarization class of $\mathcal{H}_-$. Now, let's consider the simplest case, where the v_j are all eigenstates of the operator U with eigenvalues $e^{i\varphi_j}$. On the n-fermion state $v_0 \wedge v_1 \wedge \ldots \wedge v_n$, U acts like

$$v_0 \wedge v_1 \wedge \ldots \wedge v_n \xrightarrow{U} e^{i\varphi_0} v_0 \wedge e^{i\varphi_1} v_1 \wedge \ldots \wedge e^{i\varphi_n} v_n \;=\; e^{i\left(\sum_{j=1}^{n} \varphi_j\right)} \left(v_0 \wedge v_1 \wedge \ldots \wedge v_n\right)$$

But again, things are difficult for infinitely many particles.
On $\Psi = v_0 \wedge v_1 \wedge v_2 \wedge \ldots$, the unitary transformation U would have to act like

$$v_0 \wedge v_1 \wedge v_2 \wedge \ldots \xrightarrow{U} Uv_0 \wedge Uv_1 \wedge Uv_2 \wedge \ldots \;=\; e^{i\varphi_0} v_0 \wedge e^{i\varphi_1} v_1 \wedge e^{i\varphi_2} v_2 \wedge \ldots$$

$$\text{“}=\text{“} \qquad e^{i(?)} \left(v_0 \wedge v_1 \wedge v_2 \wedge \ldots\right)$$

The product of infinitely many phases does not converge, in general and thus the phase of the "second quantization" of U, acting on the Fock space, is not well-defined.
This U(1)-freedom is also known as the *geometric phase* in QED. "Geometric" because we will identify it as the structure group of a principle fibre bundle over the Lie group of implementable unitary operators.

1.3 Dirac Sea versus electron-positron picture

Considering the problems described so far, the careful reader might say that all difficulties originate in the fact that we are dealing with an infinite number of particles. And some might go on to suggest that we can solve them by abandoning the Dirac sea with its infinitely many electrons and switch from the "electron-hole picture" to the "particle-antiparticle" picture which has been established in modern Quantum field theory[2]. The first statement is, of course, correct. The second one is not. To see why, we have to understand what all of this actually means.

What I'm referring to as the "particle-antiparticle-picture", or "electron-positron-picture", as opposed to the Dirac sea - or electron-hole- picture is the physical description in which the problematic negative spectrum of the Dirac Hamiltonian is fixed in a rather ad hoc way by mapping negative energy solutions to positive energy solutions with opposite charge and interpreting them as describing "antiparticles" (positrons). The Dirac sea is then omitted altogether as part of the physical description. This re-interpretation is also reflected in the mathematical formalism. In standard physics textbooks this is usually done in a more or less naive way :

[2]Indeed, most textbooks suggest, implicitly or explicitly, that this is a major improvement.

The formal quantization of the Dirac field yields the expression

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s_{\underline{p}} u^s(\underline{p}) e^{-ipx} + b^s_{\underline{p}}\ v^s(\underline{p}) e^{ipx}) \tag{1.3.1}$$

which leads to the second quantized Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (E(\underline{p})\ a^{s*}_{\underline{p}} a^s_{\underline{p}} - E(\underline{p})\ b^{s*}_{\underline{p}} b^s_{\underline{p}}) \tag{1.3.2}$$

This Hamiltonian is unstable, i.e. unbounded from below, as every particle of the type created by b^* decreases the total energy by, at least, its rest-mass m.
So one uses the canonical commutation relations

$$\{b^r_{\underline{p}}, b^{s*}_{\underline{q}}\} = b^r_{\underline{p}} b^{s*}_{\underline{q}} + b^{s*}_{\underline{q}} b^r_{\underline{p}} = (2\pi)^3 \delta^3(\underline{p} - \underline{q}) \delta^{r,s} \tag{1.3.3}$$

to write $-b^{s*}_{\underline{q}} b^s_{\underline{p}} = +b^s_{\underline{p}} b^{s*}_{\underline{q}} - (2\pi)^3\ \delta(0)$ and simply interchanges the roles of b and b^*.
After renaming the operators accordingly, one gets the stable Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (E(\underline{p})\ a^{s*}_{\underline{p}} a^s_{\underline{p}} + E(\underline{p})\ b^{s*}_{\underline{p}} b^s_{\underline{p}}) \tag{1.3.4}$$

plus an infinite "vacuum energy" that physicists boldly get right of by "shifting the energy" or, in other words, ignoring it.

A less playful approach would use the *charge conjugation operator* $\mathcal{C}$, an anti-unitary operator mapping negative-energy solutions to positive energy solutions of the Dirac equation with opposite charge. Then we construct the Fock space

$$\mathcal{F} = \bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}(\mathcal{H}_-) \tag{1.3.5}$$

and define the field operator Ψ which is a complex anti-linear map into the space $\mathcal{B}(\mathcal{F})$ of bounded operators on $\mathcal{F}$. Explicitely,

$$\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{F}),\ \Psi(f) = a(P_+ f) + b(P_- f) \tag{1.3.6}$$

where a is the annihilation operator on $\bigwedge \mathcal{H}_+$ and b^* the creation operator on $\bigwedge \mathcal{C}(\mathcal{H}_-)$ (i.e. for $g \in \mathcal{H}_-$, $b^*(g)$ creates the state $\mathcal{C}g$ in $\bigwedge \mathcal{C}(\mathcal{H}_-)$).
The idea behind all this is that $\bigwedge \mathcal{H}_+$ contains the electron-states and $\bigwedge \mathcal{C}(\mathcal{H}_-)$ the positron-states, all of positive energy. The vacuum state Ω is just $1 = 1 \otimes 1 \in \mathcal{F}$. If the reader excuses some physics jargon, we can say that the field operator Ψ "creates" antiparticles and "annihilates" particles. Consequently, Ψ^* "creates" particles and "annihilates" antiparticles.
This construction is carried out in more detail in the next chapter but clearly, as it involves only states of positive energy, it leads to a positive definite second quantized Hamiltonian on the Fock space.

We can relate this construction to the Dirac sea description in the following way:
In the Dirac sea description, the role of the vacuum is played by the unperturbed sea in which all the free, negative-energy states are occupied by electrons. Formally, we may think of this state as an "infinite-wedge product". We pick a basis $(e_k)_{k\in\mathbb{Z}}$ of $\mathcal{H}$, s.t. $(e_k)_{k\leq 0}$ is a basis of $\mathcal{H}_-$ and $(e_k)_{k>0}$ a basis of $\mathcal{H}_+$, and represent the "vacuum" by the formal expression

$$\Omega = e_0 \wedge e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \dots$$

Now we can define "field operators" Ψ and Ψ^* acting on Ω in the following:

$$\begin{aligned} &\Psi^*(e_k)\Omega = e_0 \wedge e_{-1} \wedge e_{-2} \wedge \dots \wedge e_{k+1} \wedge \cancel{e_k} \wedge e_{k-1} \wedge \dots && \text{for } k < 0 \\ &\Psi(e_{k'})\,\Omega = e_{k'} \wedge e_0 \wedge e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \dots && \text{for } k \in \mathbb{Z} \end{aligned} \tag{1.3.7}$$

We see that Ψ^* acting on Ω creates positive-energy states and Ψ annihilates negative energy states, i.e creates holes. In particular:

$$\Psi^*(g)\Omega = 0, \text{ for } g \in \mathcal{H}_-$$
$$\Psi(f)\Omega = 0, \text{ for } f \in \mathcal{H}_+$$

Acting successively with these operators on Ω we reach configurations of the Dirac sea (i.e. infinite-particle states) where finitely many positive energy states and almost all negative energy states are occupied. Formally, those are (linear combinations of) states of the form

$$\Phi = e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \dots \tag{1.3.8}$$

where $(i_0, i_1, i_2, ...)$ is a strictly decreasing sequence in $\mathbb{Z}$ with $i_{(k+1)} = i_k - 1$ for all large enough indices k.
On the formal level, the transition from the Dirac sea description to what we dubbed electron-positron description now just results in mapping (linear combinations of) states of the form (1.3.8) into the Fock-space $\mathcal{F} = \bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}(\mathcal{H}_-)$, such that the holes in the sea ($\sim$ the missing negative indices) are mapped to antiparticle-states and the positive energy states ($\sim$ positive indices) are mapped to the corresponding particle states. The idea is really very simple, although it's somewhat tedious to express it formally, so we will spare the reader the formal details until §5.4. But after introducing the appropriate Fock-space structure on Dirac seas, it is fairly easy to see that this assignments do indeed define an isomorphism and that under this isomorphism, the operator Ψ as defined in (1.3.7) acts just as the field operator in (1.3.6).

So it turns out that the two descriptions are really mathematically equivalent. If we carry out the whole process of second quantization in the electron-positron picture without any reference to the Dirac sea whatsoever (and we will do exactly that in the next chapter), we arrive at the very same obstacles. In particular, we will find the Shale-Stinespring condition for second quantization of unitary operators, although I would say that its meaning remains more obscure without infinite-particle states in mind. Indeed, certain features of the theory seem to indicate that the Dirac sea picture is really the more honest description. For example, in the formal quantization of the Dirac field, the vacuum-charge, just as the vacuum-energy above, appears as an infinite constant even with the renamed creators and annihilators. Nevertheless, my personal experience is that most physicists nowadays strongly favor the electron-positron description, often calling the idea of a Dirac sea an outdated concept that got obsolete as soon as the theory was properly understood.

On the other hand, my personal teachers strongly advocate Dirac's idea. What they ultimately have in mind is that the Dirac sea is just an approximate description of a universe consisting of a very large, yet finite, number of charged particles and that what we call electrons or holes are in fact only deviations from an equilibrium state in which the particles are so homogeneously distributed that the net-interaction is zero and therefore the sea "invisible". Until we have a consistent, fully-interacting theory of Quantum Electrodynamics, the choice between the two descriptions ultimately remains a matter of personal taste. However, it should be clear that at this level of description no serious problems can be solved by choosing one over the other.

Chapter 2

Polarization Classes and the Restricted Transformation Groups

In this chapter we give precise definitions for the concepts introduced in chapter 1 and study the *restricted* unitary- and general-linear group of automorphism that do satisfy the Shale-Stinespring condition (1.2.11) and can be implemented on the fermionic Fock space.

By $\mathcal{H}$ we will always denote an infinite-dimensional, complex, separable Hilbert space.

Note: In the light of the physical problem, we would think of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ as the spectral decomposition w.r.to the free Dirac Hamiltonian and think of $\mathcal{H}_-$ (projectively) as the "Dirac Sea". However, for some reason, mathematicians prefer to work with the subspace denoted by $\mathcal{H}_+$ and we will follow this convention in order to match the mathematical literature. Therefore, $\mathcal{H}_+$ will play the role of what one might think of as the Dirac sea.

2.1 Polarization Classes

Definition 2.1.1 (Polarizations)**.**

A *polarization* of $\mathcal{H}$ is an infinite dimensional, closed subspace $V \subset \mathcal{H}$ with infinite dimensional orthogonal complement $V^\perp$.

Accordingly, we will call $\mathcal{H}$ a *polarized Hilbert space* if we have a distinct splitting $\mathcal{H} = V \oplus V^\perp$ for a polarization V.

The set of all polarizations of $\mathcal{H}$ is denoted by $\texttt{Pol}(\mathcal{H})$.

By $P_V : \mathcal{H} \to \mathcal{H}$ we denote the orthogonal projection of $\mathcal{H}$ onto V, so $P_V + P_{V^\perp} = \mathbb{1}_{\mathcal{H}}$.

Definition 2.1.2 (Polarization Classes)**.**
On $\texttt{Pol}(\mathcal{H})$, we introduce the equivalence relation

$$V \approx W \iff P_V - P_W \in I_2(\mathcal{H})$$

The equivalence classes $C \in \texttt{Pol}(\mathcal{H})$ are called *polarization classes*.

Lemma 2.1.3 (Characterization of $\approx$)**.**
For $V, W \in \texttt{Pol}(\mathcal{H})$, the following are equivalent:

i) $V \approx W$

ii) $P_{W^\perp} P_V$, $P_W P_{V^\perp} \in I_2(\mathcal{H})$

iii) $P_W|_{V \to W}$ *is a Fredholm operator and* $P_{W^\perp}|_{V \to W^\perp} \in I_2(V)$

Proof.

i)⇒ii): If $V \approx W$, i.e. $P_V - P_W \in I_2(\mathcal{H})$ then

$$\begin{aligned} P_{W^\perp} P_V =& (Id_{\mathcal{H}} - P_W)P_V = (P_V - P_W)P_V \in I_2(\mathcal{H}) \\ P_W P_{V^\perp} =& P_W(Id_{\mathcal{H}} - P_V) = -P_W(P_V - P_W) \in I_2(\mathcal{H}) \end{aligned}$$

ii)⇒iii): We write the identity on $\mathcal{H}$ in matrix form as

$$Id_{\mathcal{H}} : V \oplus V^\perp \to W \oplus W^\perp = \begin{pmatrix} P_W|_{V\to W} & P_W|_{V^\perp \to W} \\ P_{W^\perp}|_{V\to W^\perp} & P_{W^\perp}|_{V^\perp \to W^\perp} \end{pmatrix}$$

By ii) the off-diagonal terms are of Hilbert-Schmidt type, so the operator

$$\begin{pmatrix} P_W|_{V\to W} & 0 \\ 0 & P_{W^\perp}|_{V^\perp \to W^\perp} \end{pmatrix}$$

is a compact perturbation of the identity. Consequently, $P_W|_{V\to W}$ and $P_{W^\perp}|_{V^\perp \to W^\perp}$ are Fredholm-operators.

ii)⇒i): If

$$\begin{aligned} P_{W^\perp} P_V &= (P_V - P_W)P_V \in I_2(\mathcal{H}), \\ P_W P_{V^\perp} &= -P_W(P_V - P_W) \in I_2(\mathcal{H}) \end{aligned}$$

then

$$(P_V - P_W) = (P_V - P_W)P_V + P_W(P_V - P_W) \in I_2(\mathcal{H})$$

iii)⇒ii): $P_W P_{V^\perp} \in I_2(\mathcal{H})$ by assumption and as $P_W|_{V\to W}$ is a Fredholm operator, the cokernel of $P_W P_V$ in W is finite-dimensional. Hence, $P_{W^\perp} P_V$ is a finite-rank operator, in particular Hilbert-Schmidt.

□

By this Lemma, the following is well-defined:

Definition 2.1.4 (Relative Charge)**.**
For $V, W \in \mathtt{Pol}(\mathcal{H})$ with $V \approx W$, we define the *relative charge* of V, W to be the Fredholm index of $P_W|_{V\to W}$. I.e.:

$$\begin{aligned} \operatorname{charge}(V,W) :=& \operatorname{ind}(P_W|_{V\to W}) \\ =& \dim \ker(P_W|_{V\to W}) - \dim \operatorname{coker}(P_W|_{V\to W}) \\ =& \dim \ker(P_W|_{V\to W}) - \dim \ker\big((P_W|_{V\to W})^*\big) \end{aligned} \tag{2.1.1}$$

Lemma 2.1.5 (Properties of relative charge)**.**
The relative charge has the following intuitive properties:

$$\begin{aligned} &\operatorname{charge}(V,W) = -\operatorname{charge}(W,V) \ \textit{and} \\ &\operatorname{charge}(V,W) + \operatorname{charge}(W,X) = \operatorname{charge}(V,X) \end{aligned}$$

for $V \approx W \approx X \in \mathtt{Pol}(\mathcal{H})$.

Proof. For V, W, X from the same polarization class in $\mathtt{Pol}(\mathcal{H})/\approx$ we find

$$\begin{aligned} & \operatorname{charge}(V,W) + \operatorname{charge}(W,X) \\ =\ & \operatorname{ind}(P_W|_{V\to W}) + \operatorname{ind}(P_X|_{W\to X}) = \operatorname{ind}(P_X P_W|_{V\to X}) \\ =\ & \operatorname{ind}\big(P_X P_X + P_X(P_W - P_X)\big|_{V\to X}\big) = \operatorname{ind}(P_X|_{V\to X}) \\ =\ & \operatorname{charge}(V,X) \end{aligned}$$

The equality in the third line holds because $P_W - P_X$ is of Hilbert-Schmidt type, in particular a compact perturbation, and therefore doesn't change the index. As a special case we get $\text{charge}(V,W) + \text{charge}(W,V) = \text{charge}(V,V) = 0$ which proves the first identity.

□

It follows from the Lemma that we get a finer equivalence relation " $\approx_0$ ", by setting

$$V \approx_0 W :\Longleftrightarrow V \approx W \text{ and } \text{charge}(V,W) = 0 \tag{2.1.2}$$

For our purposes it is more convenient to work with these *equal charge classes*. Also, the physical principle of charge conservation tells us that the Dirac time evolution should preserve the relative charge.

In general, unitary transformations do NOT preserve the polarization class - they will map one polarization class into another. This is the source of all evil when we try to implement the unitary time evolution on Fock spaces. However, a unitary transformation induces at a well-defined map between polarization classes by $U[V] = [UV] \in \texttt{Pol}(\mathcal{H})/\approx$. This is true, because $P_{UV} - P_{UW} = UP_VU^* - UP_WU^* = U(P_V - P_W)U^*$ is of Hilbert-Schmidt type if and only $P_V - P_W$ is. The analogous map is also well-defined between equal charge classes, because unitary transformation do not change the index.

Definition 2.1.6 (Restricted Unitary Operators)**.**

For polarization classes $C, C' \in \texttt{Pol}(\mathcal{H})/\approx$ we define

$$\begin{aligned}\mathrm{U}_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C') &= \{U : \mathcal{H} \to \mathcal{H} \text{ unitary} \mid \forall\, V \in C :\, UV \in C'\} \\ &= \{U : \mathcal{H} \to \mathcal{H} \text{ unitary} \mid \exists\, V \in C \text{ with } UV \in C'\}\end{aligned}$$

as the set of unitary operators mapping the polarization class C into C'.
If we want to restrict to equal charge classes, we write $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C_0; \mathcal{H}, C_0')$ for $C_0, C_0' \in \texttt{Pol}(\mathcal{H})/\approx_0$ etc. The definition can be immediately generalized to unitary maps between different Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$.

Note that $\mathrm{U}_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C')$ is *not* a group, unless $C = C'$. However, they compose as

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}, C'; \mathcal{H}, C'')\mathrm{U}_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C') = \mathrm{U}_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C'') \tag{2.1.3}$$

The group $\mathrm{U}_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C)$ of unitarities preserving a fixed polarization class C will be of crucial importance to us. In particular, as argued in the introductory sections, it will turn out that a unitary transformation can be lifted to the (standard) Fock space if and only if it is compatible with the natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ in the sense that it preserves the polarization class $C = [\mathcal{H}_+]$.

2.2 The restricted unitary and general linear group

We want to study the *restricted unitary group* consisting of those operators which do preserve the polarization class $[\mathcal{H}_+]$ and will turn out to be implementable on the Fock space.

Definition 2.2.1 (Restricted Unitary Group)**.**
The group

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}) := \mathrm{U}_{\mathrm{res}}(\mathcal{H}, [\mathcal{H}_+]; \mathcal{H}, [\mathcal{H}_+])$$

is called the *restricted unitary group* on $\mathcal{H}$.

We introduce the notation $\epsilon = P_+ - P_-$ for the sign of the free Dirac-Hamiltonian.

Lemma 2.2.2 (Characterization of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$)**.**
For a unitary operator $U \in \mathrm{U}(\mathcal{H})$ the following statements are equivalent:

i) $U \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$

ii) $U_{+-} \in I_2(\mathcal{H}_-, \mathcal{H}_+)$ *and* $U_{-+} \in I_2(\mathcal{H}_+, \mathcal{H}_-)$

iii) $[\epsilon, U] \in I_2(\mathcal{H})$

Note that ii) is the Shale-Stinespring criterion we've been talking so much about. It appears very naturally in this setting.

Proof. Let $U \in \mathrm{U}(\mathcal{H})$. Then:

$$\begin{aligned} U \in \mathrm{U}_{\mathrm{res}}(\mathcal{H}) &\iff U\mathcal{H}_+ \approx \mathcal{H}_+ \in \mathtt{Pol}(\mathcal{H}) \iff P_+ - UP_+U^* \in I_2(\mathcal{H}) \\ &\iff P_+U - UP_+ \in I_2(\mathcal{H}) \\ &\iff P_+U(P_+ + P_-) - (P_+ + P_-)UP_- = P_+UP_- - P_-UP_+ \in I_2(\mathcal{H}) \\ &\iff U_{+-} \in I_2(\mathcal{H}_-, \mathcal{H}_+) \text{ and } U_{-+} \in I_2(\mathcal{H}_+, \mathcal{H}_-) \end{aligned}$$

where we have used that $P_{UV} = UP_VU^*$ for any $U \in \mathrm{U}(\mathcal{H})$ and any subspace $V \subset \mathcal{H}$. Furthermore we compute

$$\begin{aligned} [\epsilon, U] &= (P_+ - P_-)U - U(P_+ - P_-) \\ &= (P_+ - P_-)U(P_+ + P_-) - (P_+ + P_-)U(P_+ - P_-) \\ &= P_+UP_- - P_-UP_+ + P_+UP_- - P_-UP_+ \\ &= 2\left(P_+UP_- - P_-UP_+\right) \end{aligned}$$

And therefore[1]:

$$\begin{aligned} \frac{1}{4}\|[\epsilon, U]\|_2^2 =& \|(P_+UP_- - P_-UP_+)^*(P_+UP_- - P_-UP_+)\|_1 \\ =& \|(P_-U^*P_+ - P_+U^*P_-)(P_+UP_- - P_-UP_+)\|_1 \\ =& \|P_-U^*P_+UP_- + P_+U^*P_-UP_+\|_1 \\ =& \|U_{-+}^*U_{+-}\|_1 + \|U_{+-}^*U_{-+}\|_1 \\ =& \|U_{+-}\|_2^2 + \|U_{-+}\|_2^2 \end{aligned} \tag{2.2.1}$$

as the traces of the odd parts vanish. This finishes the proof. □

Thus, we can describe the restricted unitary group as

$$\begin{aligned} \mathrm{U}_{\mathrm{res}}(\mathcal{H}) &= \{U \in \mathrm{U}(\mathcal{H}) \mid [\epsilon, U] \in I_2(\mathcal{H})\} \\ &= \{U \in \mathrm{U}(\mathcal{H}) \mid U_{+-} \text{ and } U_{-+} \text{ are Hilbert-Schmidt operators}\} \end{aligned} \tag{2.2.2}$$

[1]Note that we use the notation U_{+-}^* for $(U^*)_{+-}$ and not for $(U_{+-})^* = (U^*)_{-+}$

This alternative definition extends immediately to arbitrary isomorphisms. The unitary case is the most relevant one for physical applications, but for the development of the mathematical framework it is very natural and convenient to study the more general case.

Definition 2.2.3 (Restricted General Linear Group)**.**
In analogy to (2.2.2) we define the group[2]

$$\begin{aligned}\mathrm{GL}_{\mathrm{res}}(\mathcal{H}) &:= \{A \in \mathrm{GL}(\mathcal{H}) \mid [\epsilon, A] \in I_2(\mathcal{H})\} \\ &= \{A \in \mathrm{GL}(\mathcal{H}) \mid U_{+-} \text{ and } U_{-+} \text{ are Hilbert-Schmidt operators}\}\end{aligned}$$

It is called the *restricted general linear group* of $\mathcal{H}$.

With respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ we can write any $A \in \mathrm{GL}(\mathcal{H})$ in matrix form as

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2.3}$$

Then A is in $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ if and only if the off-diagonal parts b and c are Hilbert-Schmidt operators. In this case, as the off-diagonal part is just a compact perturbation, it follows immediately that a and d are Fredholm operators with $\mathrm{ind}(a) = -\mathrm{ind}(d)$, since

$$0 = \mathrm{ind}(A) = \mathrm{ind}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{ind}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \mathrm{ind}(a) + \mathrm{ind}(d) \tag{2.2.4}$$

N.B. that for $A \in \mathrm{GL}(\mathcal{H})$, $A \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ is *not equivalent* to $[A\mathcal{H}_+] \approx [\mathcal{H}_+]$ in $\mathtt{Pol}(\mathcal{H})$, but the first implies the latter. Using Lemma, 2.1.3 iii), we see that for $[A\mathcal{H}_+] \approx [\mathcal{H}_+]$ it suffices that c is Hilbert-Schmidt and a a Fredholm operator, which for general isomorphisms is less restrictive than $A \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

We can regard $\mathrm{GL}_{\mathrm{res}}$ as the group of units in the algebra $\mathcal{B}_\epsilon(\mathcal{H})$ of all bounded operators $A : \mathcal{H} \to \mathcal{H}$ with $[\epsilon, A] \in I_2(\mathcal{H})$. $\mathcal{B}_\epsilon(\mathcal{H})$ is a Banach algebra when equipped with the norm $\|\cdot\|_\epsilon$, defined by

$$\|A\|_\epsilon := \|A\| + \|[\epsilon, A]\|_2 \tag{2.2.5}$$

Great news: with the topology induced by $\|\cdot\|_\epsilon$, $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ carries the structure of a complex Banach Lie group and $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ that of a real Banach Lie group.
The corresponding Lie algebras are

$$\mathfrak{gl}_1 := \{X \text{ bounded operator on } \mathcal{H} \mid [\epsilon, X] \in I_2(\mathcal{H})\} \tag{2.2.6}$$

and

$$\mathfrak{u}_{\mathrm{res}} := \{X \text{ Hermitian operator on } \mathcal{H} \mid [\epsilon, X] \in I_2(\mathcal{H})\}^3, \tag{2.2.7}$$

respectively. The Lie algebras are Banach spaces with respect to the norm (2.2.5).
It is readily checked that $\exp(X) \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ for $X \in \mathfrak{gl}_1$ and $\exp(iX) \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ for $X \in \mathfrak{u}_{\mathrm{res}}$, since it is true for every single term in the series expansion. Indeed, both Lie groups are *locally exponential*, i.e. the exponential map is diffeomorphic in a neighborhood of the identity. Geometrically, $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ can also be understood as the *complexification* of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. In particular, it inhibits $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ as a real subgroup and $\mathfrak{gl}_1 \cong \mathfrak{u}_{\mathrm{res}} \otimes_{\mathbb{R}} \mathbb{C}$.

[2]That $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ is indeed a subgroup of $\mathrm{GL}(\mathcal{H})$, i.e. closed under multiplication, can be easily checked using the fact that the Hilbert-Schmidt operators are a two-sided ideal in the space of bounded operators.

[3]For $\mathfrak{u}_{\mathrm{res}}$ we use the physicist convention with an additional factor of i^{-1} in front of the commutator and i in the exponential map.

Finally, two more results about the topology of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

Lemma 2.2.4 (Homotopy type of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$)**.**
The map

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a$$

from $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ *to the space* $\mathrm{Fred}(\mathcal{H}_+)$ *of Fredholm operators on* $\mathcal{H}_+$ *is a homotopy equivalence. In particular,* $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ *has infinitely many connected components indexed by* $\mathbb{Z}$*, corresponding to the index of the* $(++)$*-component* a *of* A*.*

The last conclusion is true, because ind : Fred $\to \mathbb{Z}$ is a continuous. We denote the n-th connected component by $\mathrm{GL}^n_{\mathrm{res}}(\mathcal{H})$. In particular, $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})$ denotes the identity component of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. Analogously for $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, of course.

The Fredholm-index of a has a very important physical interpretation.
For $V \approx \mathcal{H}_+ \in \mathtt{Pol}(\mathcal{H})$ and $A \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, we find that

$$\begin{aligned}
\mathrm{charge}(AV, \mathcal{H}_+) &= \mathrm{ind}(P_+|_{AV\to\mathcal{H}_+}) = \mathrm{ind}(P_+A|_{V\to\mathcal{H}_+}) \\
&= \mathrm{ind}\big(P_+A(P_+ + P_-)|_{V\to\mathcal{H}_+}\big) = \mathrm{ind}\big((P_+AP_+ + P_+AP_-)|_{AV\to\mathcal{H}_+}\big) \\
&= \mathrm{ind}(P_+AP_+|_{V\to\mathcal{H}_+}) = \mathrm{ind}(P_+AP_+ \circ P_+|_{V\to\mathcal{H}_+}) \\
&= \mathrm{ind}(P_+|_{V\to\mathcal{H}_+}) + \mathrm{ind}(P_+AP_+|\mathcal{H}_+ \to \mathcal{H}_+)
\end{aligned}$$

Thus:

$$\boxed{\mathrm{charge}(AV, \mathcal{H}_+) = \mathrm{charge}(V, \mathcal{H}_+) + \mathrm{ind}(a)} \tag{2.2.8}$$

So, physically, the index of the $(++)$-component corresponds to the net-charge that the transformation A "creates" from the vacuum ($\sim \mathcal{H}_+$, in this convention).

Lemma 2.2.5 (Homotopy groups of $\mathrm{GL}^0_{\mathrm{res}}$)**.**
The homotopy groups of the connected Lie group $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})$ *are for* $k \geq 0$

$$\pi_{2k+1}(\mathrm{GL}^0_{\mathrm{res}}) = \{0\} \text{ and } \pi_{2k+2}(\mathrm{GL}^0_{\mathrm{res}}) \cong \mathbb{Z}$$

In particular, $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})$ *and* $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H})$ *are simply-connected.*

For the proofs of the Lemmatas see [PreSe] §6 or [Wurz06],for a more complete version.

2.3 The restricted Grassmannian

To the spectral decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ corresponds the polarization class $[\mathcal{H}_+] \in \mathtt{Pol}(\mathcal{H})/_{\approx}$. This set, consisting of subspaces of $\mathcal{H}$, is known in the literature as the *(restricted) Grassmannian* of the Hilbert space $\mathcal{H}$ and denoted by $\mathrm{Gr}(\mathcal{H})$. [4] From the mathematical point of view it is a remarkably nice object. It can actually be given the structure of an infinite dimensional complex (Kähler) manifold ([PreSe]).

By Lemma 2.1.3 we can describe $\mathrm{Gr}(\mathcal{H})$ as the set of all closed subspaces W of H such that

i) the orthogonal projection $P_+ : W \to \mathcal{H}_+$ is a Fredholm operator

ii) the orthogonal projection $P_- : W \to \mathcal{H}_-$ is a Hilbert-Schmidt operator

This is the definition most commonly found in the mathematical literature (e.g. [PreSe], Def. 7.1.1). Another way of saying the same is thing is the following:
A subspace W belongs to $\mathrm{Gr}(\mathcal{H})$ if and only if it is the image of an operator $w : \mathcal{H}_+ \to \mathcal{H}$ with $P_+ \circ w \in \mathrm{Fred}(\mathcal{H}_+)$ and $P_- \circ w \in I_2(\mathcal{H}_+, \mathcal{H}_-)$.

We can cover $\mathrm{Gr}(\mathcal{H})$ by the sets $\big(U_W\big)_{W \in \mathrm{Gr}(\mathcal{H})}$, where

$$U_W := \{W' \in \mathrm{Gr}(\mathcal{H}) \mid P_W|_{W' \to W} \text{ is an isomorphism}\}. \tag{2.3.1}$$

The elements of U_W are "close" to W in the sense that they differ from W only by a Hilbert-Schmidt operator $T : W \to W^\perp$. We make this more precise:

Lemma 2.3.1 (Characterization of U_W)**.**
The set U_W consists of all the graphs of Hilbert-Schmidt operators $W \to W^\perp$.
There is a one-to-one correspondence between $U_W \subset \mathrm{Gr}(\mathcal{H})$ and $I_2(W, W^\perp)$.

Proof. Let $w : \mathcal{H}_+ \to W \subset \mathcal{H}$ an isomorphism with image W. Let $T : W \to W^\perp$ be a Hilbert-Schmidt operator. Then $\mathrm{Graph(T)} = \{(\mathrm{w}, T\,\mathrm{w}) \mid \mathrm{w} \in W\} = \mathrm{im}(w + T\,w)$.
$P_+(w + T \circ w) = P_+ w + P_+ T w$ is Fredholm and $P_-(w + T \circ w) = P_- w + P_- T w$ is of Hilbert-Schmidt type, therefore $\mathrm{Graph}(T) \in \mathrm{Gr}(\mathcal{H})$. Furthermore, the orthogonal projection onto W is obviously an isomorphism, so that $\mathrm{Graph}(T) \in U_W$.

Conversely, if $W' \in U_W$, it is the image of a map $w' = w + Tw : \mathcal{H}_+ \to \mathcal{H}$, where $T : W \to W^\perp$ is uniquely determined by w. Since $W' \approx W$ we know from Lemma 2.1.3 that $P_{W^\perp}|_{W' \to W^\perp}$ is Hilbert-Schmidt, hence $P_{W^\perp} w' = Tw$ is Hilbert-Schmidt. But because w is invertible, T itself is also of Hilbert-Schmidt type.

It is easy to see that the assignments $T \longleftrightarrow W' = \mathrm{Graph}(T)$ are inverses of each other, so the statement of the Lemma is proven. □

Proposition 2.3.2 (Manifold structure of $\mathrm{Gr}(\mathcal{H})$)**.**
$\mathrm{Gr}(\mathcal{H})$ *is a complex Hilbert manifold modelled on $I_2(\mathcal{H}_+, \mathcal{H}_-)$.*
The sets $U_W, W \in \mathrm{Gr}(\mathcal{H})$ form an open covering of $\mathrm{Gr}(\mathcal{H})$.

Proof. [PreSe] Prop. 7.1.2 □

By definition, a unitary transformation maps $\mathrm{Gr}(\mathcal{H})$ into $\mathrm{Gr}(\mathcal{H})$ iff it is in $\mathrm{U_{res}}(\mathcal{H})$. Conversely, for any two polarizations W, W' of $\mathcal{H}$ there is of course a unitary transformation mapping one to the other and if $W, W' \in \mathrm{Gr}(\mathcal{H})$, this transformation is necessarily in $\mathrm{U_{res}}(\mathcal{H})$. The same is true for general isomorphisms and $\mathrm{GL_{res}}(\mathcal{H})$.

[4] The notations $\mathrm{Gr_{res}}(\mathcal{H})$ or $\mathrm{Gr_1}(\mathcal{H})$ are also used.

In other words: $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ *act transitively* on $\mathrm{Gr}(\mathcal{H})$. This leads us to a different description of the Grassmann manifold as a *homogeneous space* under $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ or $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. The corresponding isotropy groups of $\mathcal{H}_+ \in \mathrm{Gr}(\mathcal{H})$ are

$$P := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \;\middle|\; c = 0 \right\}$$

respectively

$$Q := \left\{ U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}_{\mathrm{res}}(\mathcal{H}) \;\middle|\; b = c = 0 \right\}$$

This means:

$$\boxed{\mathrm{GL}_{\mathrm{res}}(\mathcal{H})/P \;\cong\; \mathrm{U}_{\mathrm{res}}(\mathcal{H})/Q \;\cong\; \mathrm{Gr}(\mathcal{H})} \tag{2.3.2}$$

In particular, it follows that just as $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ and $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, the Grassmannian $\mathrm{Gr}(\mathcal{H})$ has $\mathbb{Z}$ connected components corresponding to the relative charges

$$\mathrm{charge}(W, \mathcal{H}_+) = \mathrm{ind}(P_+|_{W \to \mathcal{H}_+}).$$

Alternatively, we could have noted that

$$\mathrm{charge}(W', \mathcal{H}_+) = \mathrm{charge}(W, \mathcal{H}_+),\ \forall\, W' \in U_W\, \forall\, W \in \mathrm{Gr}(\mathcal{H})$$

and hence the sets $\mathrm{Gr}^{(c)}(\mathcal{H}) := \{W \in \mathrm{Gr}(\mathcal{H}) \mid \mathrm{charge}(W, \mathcal{H}_+) = c \in \mathbb{Z}\}$ are open and disjoint. They are connected, because $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H})$ acts continuously and transitively on each of them. Consequently, they correspond to different connected components.

It is a nice feature of the mathematical structure that different charges are separated topologically. This does also reflect the physical intuition that a continuous time evolution preserves the total charge i.e. that particles and anti-particles are always created in pairs. We hope that the reader excuses that we won't always be too careful in distinguishing between $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gr}(\mathcal{H})$ and their identity components $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})/\mathrm{Gr}^0(\mathcal{H})$. Often, it is just the latter that we care for. Extension to arbitrary charges is usually unproblematic, but can become rather tedious. We will discuss it for example on §4.1.2.

Chapter 3

Projective Representations a Central Extensions

In this section we introduce the concept of *central extensions* of (Lie-)Groups, which is essential for the treatment of representations of Lie Groups in a Quantum Mechanical setting. Some of the results will go a little beyond what we actually need for our further discussion, but as we develop the formalism anyways, it would be a pity not to mention them. Again, we will start with a rather intuitive approach to motivate the concept.

3.1 Motivation

We have argued, and will state rigorous results, that a unitary operator can be lifted to the fermionic Fock space $\mathcal{F}$ if and only it satisfies the Shale-Stinespring condition i.e. if and only if it is in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. And even in this case, this lift is determined only up to a phase. [1] Suppose we choose any prescription for fixing this phase and denote by $\Gamma(U)$ the corresponding lift of $U \in \mathrm{U}_{\mathrm{res}}$ to an operator on the Fock space $\mathcal{F}$. This gives a map

$$\Gamma : \mathrm{U}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{U}(\mathcal{F}) \tag{3.1.1}$$

from $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ into the group of unitary automorphisms of the Fock space. What is - somewhat mysteriously - called a "second quantization" of unitary operators is really nothing more than such a map. But if we fix the phases of the lifts in some arbitrary way, we have no reason to hope that those lifts will preserve the group structure: for any $U, V \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$, $\Gamma(U)\Gamma(V)$ and $\Gamma(UV)$ are both implementations of the same unitary transformation but will, in general, differ by a complex phase. In other words, Γ will fail to be a *representation* of the restricted unitary group $\mathrm{U}_{\mathrm{res}}$ on $\mathcal{F}$. Instead, we just get a *projective representation* of $\mathrm{U}_{\mathrm{res}}$ on the projective Fock space $\mathbb{P}(\mathcal{F}) = \mathcal{F}\ mod\ \mathbb{C}$.

So the first question that comes to mind is: is there a way to fix the phases of the implementations that does preserve the group structure? In other words:

Is there a proper representation of $\mathrm{U}_{\mathrm{res}}$ *on the Fock space?*

To answer this question, a somewhat deeper study of the problem is required.

We can always write

$$\Gamma(U)\Gamma(V) = \chi(U,V)\Gamma(UV)\ ;\ U, V \in \mathrm{U}_{\mathrm{res}} \tag{3.1.2}$$

with $\chi(U,V) \in \mathrm{U}(1)$. This defines a map $\chi : \mathrm{U}_{\mathrm{res}} \times \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(1)$. It is reasonable to demand $\Gamma(\mathbb{1}) = \mathbb{1}_{\mathcal{F}}$, which implies $\chi(\mathbb{1}, \mathbb{1}) = 1$. Such a map χ is called a *2-cocycle*.

[1] We will make all this precise on in chapter 5 when we construct the Fock space and the action of $\mathrm{U}_{\mathrm{res}}$ on it in several different ways.

We can now define a group $\mathrm{U}(1) \times_\chi \mathrm{U}_{\mathrm{res}}$ as the direct product $\mathrm{U}(1) \times \mathrm{U}_{\mathrm{res}}$ with the multiplication

$$(a, U) \cdot_\chi (b, V) := \big(\chi(U, V)\, ab, UV\big) \tag{3.1.3}$$

and set $\widehat{\Gamma}((a, U)) := a\,\Gamma(U)$.
But the multiplication on $\mathrm{U}(1) \times_\chi \mathrm{U}_{\mathrm{res}}$ was just defined in such a way as to compensate the cocycle coming from Γ and make $\widehat{\Gamma}$ a homomorphism of groups:

$$\begin{aligned}\widehat{\Gamma}\big((a, U)(b, V)\big) &= \widehat{\Gamma}\big((\chi(U, V)\, ab, UV)\big) = ab\,\chi(U, V)\Gamma\big(UV\big) \\ &= ab\,\Gamma(U)\Gamma(V) = \widehat{\Gamma}\big((a, U)\big)\,\widehat{\Gamma}\big((b, V)\big)\end{aligned}$$

This group $\mathrm{U}(1) \times_\chi \mathrm{U}_{\mathrm{res}}$ is a *central extension of* $\mathrm{U}_{\mathrm{res}}$ *by* $\mathrm{U}(1)$.

This is quite nice. The study of such central extensions will help us to get a hold on the freedom we have in choosing the phases of the lifts and ultimately leads us to the answer of our question about the existence of a proper representation.

Of course, things aren't quite as simple as we have presented them so far. For once, we usually don't have the cocycle χ in our hands. If we take into consideration questions of continuity and differentiability that we have neglected so far, we will have reasonable phase-prescriptions and the corresponding cocycles only locally. Thus, a more general approach to the problem is necessary.

3.2 Central Extensions of Groups

Throughout this section let G be an arbitrary group and A an abelian group. The trivial group consisting of the neutral element only is denoted by 1. All the definitions and results apply naturally to Lie groups, if the respective structures on the groups and smoothness of the maps is implied.

Definition 3.2.1 (Central Extension)**.**
A central extension of G by A is a short exact sequence of group homomorphisms

$$1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$$

such that $\imath(A)$ is in the center of E, i.e commutes with all the elements of E.

Let's dissect this abstract definition. The sequence being *exact* means that the kernel of every map equals the image of the previous map. So, $\imath$ must be injective, π surjective and $\ker(\pi) = \mathrm{im}(\imath) \cong A$. Thus, E covers G with the preimage of every $g \in G$ being isomorphic to A. The requirement that $\imath(A)$ is central in E makes much sense in regard of studying representations of G or E, respectively. Because then, *Schurr's Lemma* ensures that in any irreducible representation the images of A in E will be constant multiples of the identity. In a certain sense, A will carry the information about the phases that lead to ambiguities when trying to lift a projective representation of G to a proper one.

Examples 3.2.2 (Trivial extension and universal covering group)**.**

1. A *trivial extension* has the form

   $$1 \longrightarrow A \xrightarrow{\imath} A \times G \xrightarrow{pr_2} G \longrightarrow 1$$

   where E is just the direct product of A and G and $i(a) = (a, 1),\ \forall a \in A$.

2. Let G be a connected topological group, $\hat{G}$ the universal covering group and $A = \pi_1(G) \cong \mathrm{Cov}(\hat{G}, G)$ the fundamental group of G. Then

$$1 \longrightarrow \pi_1(G) \xrightarrow{\imath} \hat{G} \xrightarrow{\pi} G \longrightarrow 1$$

is a central extension of G by its fundamental group, where π is the covering-homomorphism and $\imath$ is given by the action of the covering group $\mathrm{Cov}(\hat{G}, G) \cong \pi_1(G)$ on $1 \in \hat{G}$.

Proof. It is easily checked that the sequence is exact. Furthermore, the orbit of $1_{\hat{G}}$ under the action of $\mathrm{Cov}(\hat{G}, G)$ is by definition the preimage of 1_G under π. Thus: $i(\pi_1(G)) = \pi^{-1}(1_G) = \ker(\pi)$. The interesting part is to show that $\imath(\pi_1(G))$ is central in $\hat{G}$. To this end note that $\imath(\pi_1(G)) = \pi^{-1}(1_G)$ is a discrete set by definition of covering spaces. As the kernel of π, it is also a normal subgroup of $\hat{G}$. Now, for any fixed $a \in \imath(\pi_1(G))$ we can consider the map

$$\hat{G} \ni g \mapsto g^{-1}ag$$

This is a continuous map from $\hat{G}$ into $\imath(\pi_1(G))$ mapping 1 to a. As $\hat{G}$ is connected and $\imath(\pi_1(G))$ is discrete, the map must be constant. Thus, $g^{-1}ag = a \; \forall g \in \hat{G}$, i.e. a lies in the center of $\hat{G}$. □

3. As a special case of 3. , we can consider the well known covering

$$1 \longrightarrow \{\pm 1\} \to \mathrm{SU}(2) \to \mathrm{SO}(3) \longrightarrow 1$$

From the discussion of spin in non-relativistic Quantum Mechanics, we know that there is no irreducible 2-dimensional unitary representation of SO(3) but there is one of its universal covering group SU(2) (generated by the Pauli matrices).

Definition 3.2.3 (Equivalence of Central Extensions)**.**
Two central extension E and E' of a group G by A are *equivalent*, if there exists an isomorphism $\varphi : E \to E'$ compatible with the extensions i.e. such that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Id} & & \\ 1 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array} \tag{3.2.1}$$

Lemma 3.2.4 (Trivial Extensions)**.**

A central extension $1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$ is equivalent to the trivial extension if and only if there is a homomorphism $\sigma : G \to E$ with $\pi \circ \sigma = Id_G$.

In other words: σ is a section of G in E which is also a homomorphism of groups.
In this case, σ is also called a splitting map and the extension is said to split.

We emphasize that for topological groups or Lie groups the appropriate requirements on continuity/smoothness of the section are implied.

Proof. If the extension is trivial i.e. $E \cong A \times G$ set $\sigma(g) := (1, g) \in A \times G$.
Conversely, suppose there exists $\sigma : G \to E$ as above. Set $\varphi : A \times G \to E;\ (a, g) \mapsto \imath(a)\,\sigma(g)$. It is easily checked that this is a homomorphism compatible with the extension in the sense of (3.2.3). Furthermore, φ is bijective, since for every $g \in G$ and $\xi \in \pi^{-1}(g) \subset E$ there is one and only one $a \in A$ with $\xi = \imath(a)\,\sigma(g)$.

□

3.3 Projective Representations

Definition 3.3.1 (Projective Hilbert space)**.**
Let $\mathcal{H}$ be a complex Hilbert space of finite or infinite dimension.
The projective Hilbert space $\mathbb{P}(\mathcal{H})$ is the space of rays in $\mathcal{H}$, i.e.

$$\mathbb{P}(\mathcal{H}) := (\mathcal{H}\backslash\{0\})/\mathbb{C}^\times$$

where the equivalence relation is given by $\varphi \sim \lambda\,\varphi$, for $\lambda \in \mathbb{C}\backslash\{0\}$.
The topology on $\mathbb{P}(\mathcal{H})$ is the quotient topology induced by the quotient map $\gamma : \mathcal{H} \to \mathbb{P}(\mathcal{H})$.
We will also write $\hat{\varphi}$ for $\gamma(\varphi)$.

Obviously, the projective Hilbert space is no linear space any more. What is still well-defined, though, is what we one might call the *transition probability*. This is the map $\delta : \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0,1]$ defined by

$$\delta(\hat{\varphi}, \hat{\psi}) := \frac{|\langle\varphi, \psi\rangle|^2}{\|\varphi\|^2\|\psi\|^2} \tag{3.3.1}$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian scalar product on $\mathcal{H}$. Now, just as on the Hilbert space we are interested in unitary transformations that preserve the scalar product, on $\mathbb{P}(\mathcal{H})$ we are interested in transformations that preserve the transition probability δ.

Definition 3.3.2 (Projective Automorphisms)**.**
A projective automorphism is a bijective map $T : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ that preserves the transition probability i.e. that satisfies

$$\delta(T\hat{\varphi}, T\hat{\psi}) = \delta(\hat{\varphi}, \hat{\psi})\,,\ \forall\hat{\varphi}, \hat{\psi} \in \mathbb{P}(\mathcal{H})$$

We denote the set of all projective automorphisms by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$.

If U is a unitary map on $\mathcal{H}$, we can define $\hat{U}$ on $\mathbb{P}(\mathcal{H})$ by

$$\hat{U}(\hat{\varphi}) = \widehat{U(\varphi)} \tag{3.3.2}$$

This is clearly a projective automorphism. We thus get a group-homomorphism

$$\begin{aligned}\hat{\gamma} :\ &\mathrm{U}(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))\,,\\ &U \mapsto \hat{\gamma}(U) = \hat{U}\end{aligned}$$

As the projective Hilbert space doesn't care for multiplicative constants, this also works for *anti-unitary* maps U on $\mathcal{H}$ that are anti-linear (and satisfy $\langle Ux, Uy\rangle = \overline{\langle x, y\rangle}$ for $x, y \in \mathcal{H}$.)
In fact, every projective automorphism comes from a unitary or anti-unitary map on $\mathcal{H}$:

Theorem 3.3.3 (Wigner, 31)**.**
For every projective automorphism $T \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ there exists a unitary or an anti-unitary transformation U on $\mathcal{H}$ with $T = \hat{\gamma}(U)$.

The projective automorphisms coming from unitary transformations on $\mathcal{H}$ form a subgroup of $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ which we will denote by $\mathrm{U}(\mathbb{P}(\mathcal{H}))$.
Let's phrase our reasonings in the language of central extensions:

Lemma 3.3.4 (Unitary projective automorphism)**.**
The sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{U}(\mathcal{H}) \xrightarrow{\hat{\gamma}} \mathrm{U}(\mathbb{P}(\mathcal{H})) \longrightarrow 1 \tag{3.3.3}$$

defines a central extension of $\mathrm{U}(\mathbb{P}(\mathcal{H}))$ by $\mathrm{U}(1)$ which is non-trivial.

Proof. The only part in proving that the sequence is exact and central that might be non-trivial is to identify $\ker(\hat{\gamma})$ with $\mathrm{U}(1) \cdot Id \subset \mathrm{U}(\mathcal{H})$. "$\supseteq$" is clear. For "$\subseteq$" pick $U \in \ker(\hat{\gamma})$. Then for any $\varphi \in \mathcal{H}$: $\hat{\gamma}(U)(\gamma(\varphi)) = \gamma(U\varphi) = \gamma(\varphi) \Rightarrow \exists \lambda \in \mathbb{C} : U\varphi = \lambda\varphi$.
Since U is unitary, $|\lambda| = 1$ i.e. $\lambda \in \mathrm{U}(1)$. If we take any other $\psi \in \mathcal{H}$ which is not a multiple of φ it is by the previous consideration also an eigenvector to some eigenvalue $\lambda' \in \mathrm{U}(1)$ but so is the sum $\varphi + \psi$ (with eigenvalue μ). It follows that $U(\varphi + \psi) = \mu(\varphi + \psi) = \lambda\varphi + \lambda'\psi$ and thus, by linear independence, $\lambda' = \lambda = \mu$. Hence, $U = \lambda Id$.

To prove that the central extension is not trivial, we embed $\mathbb{C}^2$ in $\mathcal{H}$ be fixing any 2-dimensional subspace $V \subset \mathcal{H}$. Now consider the subgroup of all unitary operators on $\mathcal{H}$ leaving V invariant, i.e. $\{U \in \mathrm{U}(\mathcal{H}) \mid \mathrm{U}(V) = V\} =: \mathcal{V}$. On $\mathcal{V}$, we introduce an equivalence relation and identify two operators if they agree on V. Then $\mathcal{V}/\sim \cong U(V) \cong \mathrm{U}(2)$.
Now all the homomorphisms descend to a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{U}(2) \longrightarrow \mathrm{U}(\mathbb{P}(\mathbb{C}^2)) \longrightarrow 1$$

But $\mathbb{P}(\mathbb{C}^2) = \mathrm{CP}^1 \cong \mathrm{S}^1$, so its unitary group is just the isometry group of the sphere, i.e. $\mathrm{SO}(3)$. Also, $\mathrm{U}(1) \times \mathrm{SU}(2) \cong \mathrm{U}(2)$ by $(e^{i\phi}, U) \to e^{i\phi/2}U$. Thus, if the central extension (3.3.3) was trivial with splitting map σ, this σ would descend to a spitting map for

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{U}(1) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1$$

But then, the second component of the homomorphism $\sigma : \mathrm{SO}(3) \to \mathrm{U}(1) \times \mathrm{SU}(2)$ would be a splitting map for the universal covering

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1$$

which we now doesn't exist. This is a contradiction.

□

3.3.1 Lifting projective representations

Now we are able to formulate the problem of lifting projective representations in a more precise way: *Given a projective representation $\Gamma : G \to \mathrm{U}(\mathbb{P}(\mathcal{H}))$, is there a representation $\rho : G \to \mathrm{U}(\mathcal{H})$ such that $\hat{\gamma} \circ \rho = \Gamma$, i.e. such that the following diagram commutes?*

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \overset{\rho}{\swarrow} & \downarrow{\scriptstyle \Gamma} & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathcal{H}) & \xrightarrow{\hat{\gamma}} & \mathrm{U}(\mathbb{P}(\mathcal{H})) & \longrightarrow & 1 \end{array}$$

In general, the answer is NO. However, as we have suggested in the introducing remarks, there always exists a central extension $\widetilde{G}$ of G such that the projective representation of G lifts to a proper representation of $\widetilde{G}$.

Lemma 3.3.5 (Lifting projective representations)**.**
Let G be a group and $\Gamma : G \to \mathrm{U}(\mathbb{P}(\mathcal{H}))$ a homomorphism. There exists a central extension $\widetilde{G}$ of G by $\mathrm{U}(1)$ and a homomorphism $\widetilde{\Gamma} : \widetilde{G} \to \mathrm{U}(\mathcal{H})$, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \xrightarrow{\imath} & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \widetilde{\Gamma}} & & \downarrow{\scriptstyle \Gamma} & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathcal{H}) & \xrightarrow{\hat{\gamma}} & \mathrm{U}(\mathbb{P}(\mathcal{H})) & \longrightarrow & 1 \end{array} \tag{3.3.4}$$

$\widetilde{G}$ is sometimes called the deprojectivization *of G.*

Proof. We define

$$\widetilde{G} := \{(U, g) \in \mathrm{U}(\mathcal{H}) \times G \mid \hat{\gamma}(U) = \Gamma(g)\}$$

This is a subgroup of $\mathrm{U}(\mathcal{H}) \times G$. The inclusion $\mathrm{U}(1) \ni \lambda \overset{\imath}{\mapsto} (\lambda \cdot Id, 1)$ and the projection onto the second component $\pi = \mathrm{pr}_2 : \widetilde{G} \to G$ are homomorphisms that make the upper row of the diagram (3.3.5) a central extension. The projection onto the first component defines a representation $\widetilde{\Gamma} := \mathrm{pr}_1 : \widetilde{G} \to \mathrm{U}(\mathcal{H})$ which by construction satisfies $\hat{\gamma} \circ \widetilde{\Gamma} = \Gamma \circ \pi$.

□

At first, this is a pure algebraic statement and we have to see how it is compatible with a topological structure or a Lie group structure of G. Without going into too much detail, we summarize the main results [2]

- If G is a topological group, $\widetilde{G}$ can be given the structure of a topological group as a subgroup of $\mathrm{U}(\mathcal{H}) \times G$. Then, if Γ is continuous, so is $\widetilde{\Gamma}$.
- If G is a *finite-dimensional* Lie group, then $\widetilde{G}$ can be given the structure of a Lie group so that the upper sequence in (3.3.4) becomes a sequence of differentiable homomorphisms. If Γ is smooth (in a strong sense) so is $\widetilde{\Gamma}$.[3]
- If we have to deal with infinite-dimensional Lie groups - and we do- the Lie group structure of the central extension is not for free. Fortunately, things will work out nicely in the cases relevant to our discussion.

Now, as there exists the central extension $\widetilde{G}$ with has a representation $\widetilde{\Gamma}$ on $\mathcal{H}$, any attempt to lift the projective action of G to the Hilbert-space corresponds to a section $\sigma : G \to \widetilde{G}$, because then we just set $\rho := \widetilde{\Gamma} \circ \sigma : G \to \mathrm{U}(\mathcal{H})$. But this will define a (continuous/smooth) representation of G on $\mathcal{H}$ only if the section σ is a (continuous/smooth) homomorphism of groups i.e. if and only if the central extension is trivial.

We summarize this insight in the following Proposition.

Proposition 3.3.6 (Lifting projective representations)**.**

Let G be a topological group. A projective representation $\Gamma : G \to \mathrm{U}(\mathbb{P}(\mathcal{H}))$ can be lifted to a continuous unitary representation $\rho : G \to \mathrm{U}(\mathcal{H})$ with $\hat{\gamma} \circ \rho = \Gamma$ if and only if the central extension $1 \longrightarrow \mathrm{U}(1) \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$ splits by a continuous section i.e. is trivial.
If G is a finite-dimensional Lie group, smoothness of Γ implies smoothness of ρ.

Proof. By Lemma (3.2.4), the central extension is (algebraically) trivial if and only if there is a section $\sigma : G \to \widetilde{G}$ which is also a homomorphism of groups. In addition, require this section to be continuous. Then $\rho := \widetilde{\Gamma} \circ \sigma$ is the postulated representation with
$\hat{\gamma} \circ \rho = \hat{\gamma} \circ \widetilde{\Gamma} \circ \sigma = \Gamma \circ \pi \circ \sigma = \Gamma$. Conversely, if ρ is a unitary representation of G, then $\sigma(g) := (g, \rho(g)) \in \widetilde{G}$ is a continuous section in $\widetilde{G}$ and a homomorphism of groups and thus the desired splitting map.

□

[2] See e.g. [Scho] for more details

[3] This statement seems rather harmless, but it's quite the opposite. Indeed, it requires the solution of one of the famous "Hilbert problems": every topological group which is also a finite dimensional topological manifold is already a Lie group. This theorem was proven by Montgomory and Zippin in 1955.

3.4 Cocycles and 2nd Cohomology Group

We have seen in the last section that any prescription to lift a projective representation of a (Lie) group G to an action on the Hilbert-space $\mathcal{H}$ can be thought of as a section in a central extension of G. Therefore, we want to study this more thoroughly.

Let

$$1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of G by A. Let $\tau : G \to E$ be a map with

$$\pi \circ \tau = Id_G \text{ and } \tau(1) = 1 \tag{3.4.1}$$

The map might also just be defined in a neighborhood of the identity.
τ will in general fail to be a homomorphism. Nevertheless:

$$\pi(\tau(g)\tau(h)) = \pi(\tau(gh)) = gh, \ \forall g, h \in G$$

Therefore, there exists $\chi(g, h) \in A$ with

$$\tau(g)\tau(h) = \chi(g, h)\tau(gh) \tag{3.4.2}$$

This defines a map $\chi : G \times G \to A$.
χ satisfies

$$\chi(1, 1) = 1 \tag{3.4.3}$$

Furthermore: $\tau(x)\tau(y)\tau(z) = \chi(x, y)\tau(xyb)\tau(z) = \chi(x, y)\chi(xy, z)\tau(xyz)$
similarly: $\tau(x)\tau(y)\tau(z) = \tau(x)\chi(y, z)\tau(yz) = \chi(x, yz)\chi(y, z)\tau(xyz)$
We deduce:

$$\chi(x, y)\chi(xy, z) = \chi(x, yz)\chi(y, z), \ \forall \ x, y, z \in G \tag{3.4.4}$$

Now suppose we've just made a "bad" choice for τ and there really exists a (local) section τ' that is a homomorphism, i.e. for which the corresponding cocycle vanishes.
Let's fix this by writing $\tau'(x) = \tau(x)\lambda(x)$ with a function $\lambda : G \to A$.
Then: $\tau'(x)\tau'(y) = \tau(x)\tau(y)\lambda(x)\lambda(y) = \tau(xy)\chi(x, y)\lambda(x)\lambda(y)$.
Also: $\tau'(x)\tau'(y) = \tau'(xy) = \tau(xy)\lambda(xy)$.
And thus:

$$\lambda(xy) = \chi(x, y)\lambda(x)\lambda(y), \ \forall x, y, z \in G \tag{3.4.5}$$

This motivates the following definition:

Definition 3.4.1 (Cocycles and Second Cohomology Group)**.**

A map $\chi : G \times G \to A$ satisfying (3.4.3) and (3.4.4) is called a *factor set* or a *2-cocycle* on G with values in A.

A 2-cocycle $\chi : G \times G \to$ is called *trivial*, if there exists a map $\lambda : G \to A$ such that $\lambda(xy) = \chi(x, y)\lambda(x)\lambda(y)$, $\forall x, y, z \in G$.

Furthermore, we define the *second cohomology group* of G with coefficients in A as the set of all 2-cocycles on G with values in A modulo trivial cocycles, i.e.

$$\mathrm{H}^2(G, A) := \{\chi : G \times G \to A \mid \chi \text{ is a cocycle}\}/\sim \tag{3.4.6}$$

where $\chi_1 \sim \chi_2 :\iff \chi_1\chi_2^{-1}$ is trivial.

We have seen that given a central extension $1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$, every section $\tau : G \to E$ with $\pi \circ \tau = Id_G$ and $\tau(1) = 1$, defines a cocycle $\chi : G \times G \to A$. Different choices of τ lead to equivalent cocycles. Conversely, given a cocycle χ, we can define the group $A \times_\chi G$ as the direct product $A \times G$ with the multiplication

$$(a, x)(b, y) := (\chi(x, y)ab, xy) \tag{3.4.7}$$

This is a central extension of G by A with the cocycle χ coming from the obvious section $\tau(x) := (1, x)$. (Compare with the construction in the introduction of this chapter).

If the cocycle χ comes from a (global) section $\tau : G \to E$ as above, the central extensions

$$1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$$

and

$$1 \longrightarrow A \longrightarrow A \times_\chi G \xrightarrow{pr_2} G \longrightarrow 1$$

are equivalent by the isomorphism $\varphi : A \times_\chi G \to E,\ \varphi\big((a, c)\big) := \imath(a) \cdot \tau(x)$, because the group-multiplication in $A \times_\chi G$ is just so that in cancels with the cocycle that will come from τ on the right-hand-side. We leave to the reader the little joy of checking for himself how everything is designed to work out nicely. Now, it is somewhat exhausting, yet straightforward to show that for cocylces χ_1 and χ_2, the corresponding groups $A \times_{\chi_1} G$ and $A \times_{\chi_2} G$ are equivalent (isomorphic) as central extensions if and only if χ_1 and χ_2 are equivalent as cocycles.

We have derived the following theorem:

Theorem 3.4.2 (Central Extensions correspond to Cohomology Classes)**.**
There is a one-to-one correspondence between equivalence classes of central extension of G by A and the second cohomology classes of G with values in A.

Note that this is a pure algebraic statement! If we are dealing with Lie groups (or topological groups) and have to consider topological aspects, things aren't quite as easy and the arguments above will, in general, work only locally. In fact, there might not be any *continuous* section $\tau : G \to E$ with $\pi \circ \tau = Id_G$ and $\tau(1) = 1$. Therefore it is usually more convenient to discuss cocycles of the corresponding *Lie algebras*, which are somehow the infinitesimal version of the Lie group cocylces. They are already determined by a *local* section τ defined in some neighborhood of the identity of the Lie group G and therefore better behaved. However, the following is true:

Proposition 3.4.3 (Lie group extensions and local cocycles)**.**
Let $1 \to A \to E \to G \to 1$ be a central extension of a connected (finite- oder indinite-dimensional) Lie group G by the abelian Lie group A. Then, E carries the structure of a Lie group such that the central extension is smooth if and only if the central extension can be described by a cocycle $\chi : G \times G \to A$ which is smooth in a neighbourhood of $(e, e) \in G \times G$. For Banach Lie-groups, the statements applies also to non-connected G.

For the proof we refer to [Ne02] Prop.4.2 and [TW87] Prop.3.11 .

3.5 Central Extensions of Lie Algebras

Definition 3.5.1 (Central Extension of Lie Algebras)**.**
Let $\mathfrak{a}$ be an abelian Lie algebra and $\mathfrak{g}$ a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ (the dimensions may be infinite). A *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$ is an exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\mathfrak{i}} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

s.t. $\mathfrak{i}(\mathfrak{a}) \subset \mathfrak{h}$ is central in $\mathfrak{h}$, i.e. if $[\mathfrak{i}(X), Y] = 0$ for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{h}$.

Proposition 3.5.2 (C.E. of Lie groups induce C.E. of the Lie algebras)**.**
Let A, E and G finite dimensional Lie groups and

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

a central extension with differentiable homomorphisms $\imath$ and π.
Then

$$0 \longrightarrow \mathrm{Lie}(A) \xrightarrow{i} \mathrm{Lie}(E) \xrightarrow{\dot{\pi}} \mathrm{Lie}(G) \longrightarrow 0 \tag{3.5.1}$$

is a central extension of the corresponding Lie algebras. [4]

Now, for a central extension of a Lie algebras

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\mathfrak{i}} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

there always exists a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$ with $\pi \circ \beta = Id$.

Analogously to the central extension of groups, the central extension of Lie algebras is equivalent to the trivial extension $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ if and only if β can be chosen to be a Lie algebra homomorphism.

If the central extension of Lie algebras comes from a central extension of Lie groups any local section $\tau : G \to E$ for the Lie groups as in (3.4.1) defines such a Lie algebra section by $\beta := \dot{\tau}$. But as τ fails to be a Lie group homomorphism, β will fail to be a Lie algebra homomorphism. In general, how β fails to be a Lie algebra homomorphism can be expressed by the skew-symmetric map

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y]) \tag{3.5.2}$$

Θ is a map $\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(A)$ with the following properties:

$$\begin{aligned} &i)\ \ \Theta \text{ is bilinear and skew-symmetric} \\ &ii)\ \Theta\left(X, [Y, Z]\right) + \Theta\left(Y, [Z, X]\right) + \Theta\left(Z, [X, Y]\right) = 0 \end{aligned} \tag{3.5.3}$$

This motivates the following definition.

Definition 3.5.3 (Lie algebra cocycles)**.**
Let $\mathfrak{a}, \mathfrak{g}$ be two Lie algebras, $\mathfrak{a}$ abelian. A map $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ satisfying the conditions (3.5.3) is called a *Lie algebra 2-cocycle* or simply a cocycle.

[4] For a smooth map $\varphi : G \to H$ between two Lie groups, $\dot{\varphi}$ denotes the Lie algebra map corresponding to the derivative of φ at the identity under identification of the Lie algebras with the tangent space of the corresponding Lie groups at the identity: $D_e\varphi : T_eG \to T_eH \overset{\cong}{\leftrightarrow} \dot{F} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. If φ is a Lie group homomorphism, then $\dot{\varphi}$ is a Lie algebra homomorphism.

Proposition 3.5.4 (Computation of Lie algebra cocycles)**.**
If χ is a (local) Lie group cocycle coming from the (local) section τ, then the corresponding Lie algebra cocycle coming from $\dot\tau$ can be computed from χ as

$$\Theta(X,Y) \;=\; \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{sX},e^{tY}) \;-\; \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{tY},e^{sX}) \tag{3.5.4}$$

Proof. :

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{sX},e^{tY}) \;-\; \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{tY},e^{sX})\\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\tau(e^{sX})\tau(e^{tY})\tau(e^{sX}e^{tY})^{-1} \;-\; \Big|_{t=s=0}\tau(e^{tY})\tau(e^{sX})\tau(e^{tY}e^{sX})^{-1}\\
&= \dot\tau(X)\dot\tau(Y) \;-\; \dot\tau(XY) \;-\; \dot\tau(Y)\dot\tau(X) \;+\; \dot\tau(YX)\\
&= \dot\tau(X)\dot\tau(Y) - \dot\tau(Y)\dot\tau(X) - (\,\dot\tau(XY) - \dot\tau(YX)\,)\\
&= [\dot\tau(X),\dot\tau(Y)] - \dot\tau([X,Y]) \;= \Theta(X,Y)
\end{aligned}$$

□

Just as in the case of groups there is a correspondence between central extensions of Lie algebras and cocycles, which is also 1-to-1 modulo trivial extensions/cocycles. However, whereas the result was of pure algebraic nature for groups and the situation becomes unclear as soon as topology is involved, everything works out nicely for Lie algebras:

Given a Lie algebra cocycle $\Theta:\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$ consider the vector space $\mathfrak{h} := \mathfrak{g}\oplus\mathfrak{a}$ and define $[\cdot,\cdot]_\Theta$ on $\mathfrak{h}$ by

$$[X_1\oplus Y_1, X_2\oplus Y_2]_\Theta := [X_1,X_2]_\mathfrak{g} + \Theta(X_1,X_2) \tag{3.5.5}$$

for $X_1,X_2\in\mathfrak{g}, Y_1,Y_2\in\mathfrak{a}$. It is straight forward to check that this is a Lie bracket on $\mathfrak{h}$ if and only if Θ is a cocycle. In this case, $\mathfrak{h}$ becomes a Lie algebra and projection onto the first component makes

$$0\longrightarrow\mathfrak{a}\longrightarrow\mathfrak{h}\xrightarrow{pr_1}\mathfrak{g}\longrightarrow 0$$

a central extension of $\mathfrak{g}$ by $\mathfrak{a}$. Conversely, if Θ comes from a central extension and a linear map $\beta:\ \mathfrak{g}\to\mathfrak{h}$, $\mathfrak{h}\cong\mathfrak{g}\oplus\mathfrak{a}$ as vector spaces by the linear isomorphism

$$\mathfrak{F}\ :\mathfrak{g}\times\mathfrak{a}\to\mathfrak{h},\quad X\oplus Y=(X,Y)\mapsto\beta(X)+Y\,.$$

The Lie bracket (3.5.5) makes $\mathfrak{g}\oplus\mathfrak{a}$ a Lie algebra and $\mathfrak{F}$ a Lie algebra isomorphism:

$$\begin{aligned}
[\mathfrak{F}(X_1,Y_1),\mathfrak{F}(X_2,Y_2)] &= [\beta(X_1)+Y_1,\beta(X_2)+Y_2] = [\beta(X_1),\beta(X_2)]\\
&= \beta([X_1,X_2]) + \Theta(X_1,X_2) = \mathfrak{F}([X_1,X_2]) + \Theta(X_1,X_2))\\
&= \mathfrak{F}\left([(X_1,Y_1),(X_2,Y_2)]_\Theta\right)
\end{aligned}$$

We summarize the results in the following Lemma:

Lemma 3.5.5 (Central Extensions and Lie Algebra Cocycles)**.**
Every central extension of Lie algebras comes from a cocycle.
Conversely, every cocycle $\Theta:\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$ induces a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ as above.

Note that in all of these construction a particular choice of β was involved. What if we choose a different linear map $\beta':\ \mathfrak{g}\to\mathfrak{h}$ with $\dot\pi\circ\beta' = Id$? Then, since $\pi\circ(\beta'-\beta)=0$, the difference of β and β' is a linear map with values in $\mathfrak{a}$ (identified with the corresponding subalgebra in $\mathfrak{h}$), i.e.

$$\beta'-\beta=\mu:\mathfrak{g}\to\mathfrak{a}\cong\mathfrak{i}(\mathfrak{a})\subset\mathfrak{h}$$

For the corresponding cocycles, this means

$$\begin{aligned}\Theta'(X,Y) :&= [\beta'(X),\beta'(Y)] - \beta'([X,Y]) \\ &= [\beta(X)+\mu(Y),\beta(Y)+\mu(Y)] - \beta([X,Y]) - \mu([X,Y]) \\ &= [\beta(X),\beta(Y)] - \beta([X,Y]) - \mu([X,Y]) \\ &= \Theta(X,Y) - \mu([X,Y])\end{aligned}$$

since the image of μ is central in $\mathfrak{h}$. We deduce:

Theorem 3.5.6 (Triviality of Lie algebra extensions)**.**
Let Θ be a Lie algebra cocycle for the central extension $\mathfrak{h}$ of $\mathfrak{g}$.
Then, the following are equivalent:

- *There exists a section $\beta' : \mathfrak{g} \to \mathfrak{h}$ that is also a Lie algebra homomorphism*
- *The cocycle Θ' coming from β' vanishes*
- *There exists a linear map $\mu : \mathfrak{g} \to \mathfrak{a}$ with $\Theta(X,Y) \equiv \mu([X,Y])$*
- *The central extension is trivial, i.e. $\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{a}$ as Lie algebras*

In particular, if $\tau : G \to H$ is a splitting map for a central extension of Lie groups, then $\beta' := \dot{\tau}$ is a splitting map for the corresponding central extension of Lie algebras.

These considerations lead us straight to the mathematical concept of cohomology groups. This is basically only a reformulation of what we have just discussed.

Definition 3.5.7 (Second Cohomology Group)**.**
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an abelian Lie algebra.

Let $Z^2(\mathfrak{g},\mathfrak{a})$ be the set of all 2-cocycles on $\mathfrak{g}$ with values in $\mathfrak{a}$.

Let $B^2(\mathfrak{g},\mathfrak{a}) = \{\Theta \in Z^2(\mathfrak{g},\mathfrak{a}) \mid \exists \mu \in \mathrm{Hom}(\mathfrak{g},\mathfrak{a}) : \Theta(X,Y) = \mu([X,Y])\}$

Let $\mathrm{H}^2(\mathfrak{g},\mathfrak{a}) := Z^2(\mathfrak{g},\mathfrak{a})/B^2(\mathfrak{g},\mathfrak{a})$.

$\mathrm{H}^2(\mathfrak{g},\mathfrak{a})$ is the second cohomology group of $\mathfrak{g}$ with values in $\mathfrak{a}$.

Theorem 3.5.8 (Central Extensions correspond to Cohomology Classes)**.**
The cohomology group $\mathrm{H}^2(\mathfrak{g},\mathfrak{a})$ is in one-to-one correspondence with the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.

Let's summarize the insights we've got so far. We started with a Lie group G and a projective unitary representation Γ of G on a projective Hilbert space. In general, we can not expect to be able to lift the projective representation to a proper representation on the Hilbert space. However, there exists a central extension $\widetilde{G}$ of G by $\mathrm{U}(1)$ that does have a unitary representation $\widetilde{\Gamma}$ on the Hilbert space, with $\hat{\gamma} \circ \widetilde{\Gamma} = \Gamma$ (3.3.5). Any prescription to lift the projective representation Γ to $\mathrm{U}(\mathcal{H})$, i.e. any choice of phases for those lifts, corresponds to a section τ of G in $\widetilde{G}$, as for any section $\widetilde{\Gamma} \circ \tau$ is precisely such a lift. This will *not* be a representation, though, unless τ is a continuous homomorphism. The way in which it fails to be a representation, or equivalently, in which the section fails to be a homomorphism, is encoded in the corresponding *cocycle*. On the other hand, for the fortunate case that the we *can* lift the projective representation to a representation on $\mathcal{H}$ we have deduced that the following statements are equivalent

- There exists unitary representation $\rho : G \to \mathrm{U}(\mathcal{H})$ with $\hat{\gamma} \circ \rho = \Gamma$
- There exists a continuous section $\sigma : G \to \widetilde{G}$ that is also a homomorphism [Cor. 3.3.6]
- The central extension $1 \longrightarrow \mathrm{U}(1) \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$ is trivial i.e. splits by a continuous section [Lem. 3.2.4]
- Any Lie group cocycle corresponding to the central extension above is trivial i.e. corresponds to 0 in $\mathrm{H}^2(G,\mathrm{U}(1))$ [Thm.3.5.8]

The "infinitesimal version" of this, so to speak, is the corresponding central extension of Lie algebras (3.5.2) and its Lie algebra cocycle (3.5.2). We have suggested that it is more convenient to work with the Lie algebras, in general, but we haven't said yet, how this infinitesimal version is related to our original problem. In all benign cases, the relationship is basically 1-to-1 i.e. we can indeed reconstruct the Lie group extension from the Lie algebra extension. This is due to the following Lemma:

Lemma 3.5.9 (Lie group homomorphisms from Lie algebra homomorphisms)**.** [5]
Let G_1, G_2 two Lie groups with Lie algebra $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Assume that G_1 is connected and simply connected, and G_2 is locally exponential. Then, for every continuous Lie algebra homomorphism $\mathfrak{F} : \mathfrak{g}_1 \to \mathfrak{g}_2$ there exists a unique Lie algebra homomorphism $f : G_1 \to G_2$ with $\mathfrak{F} = \dot{f}$.

We conclude:

Proposition 3.5.10 (Triviality of Extensions)**.**
Let G be a connected and simply connected Lie group and $1 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow 1$ a central extension of G by A. Assume that E carries the structure of a locally exponential Lie group. Then, the central extension splits by a smooth section if and only if the corresponding central extension of Lie algebras (3.5.1) *splits by a continuous Lie algebra homomorphism.*

Proof. If the central extension of Lie groups splits by a differentiable section $\sigma : G \to E$, then $\beta = \dot{\sigma}$ is a Lie algebra homomorphism with $\dot{\pi} \circ \beta = Id$ and thus a splitting map for the LIe algebra extension which trivializes the central extension. Conversely, let $\beta : \mathrm{Lie}(G) \to \mathrm{Lie}(E)$ be a Lie algebra homomorphism with $\dot{\pi} \circ \beta = Id$. If the requirements of the previous Lemma are satisfied, we can deduce that there exists a Lie group homomorphism $\sigma : G \to E$ with $\beta = \dot{\sigma}$. Since $\dot{(\pi \circ \sigma)} = \dot{\pi}\,\beta = Id_{\mathfrak{g}} = \dot{Id}_G$, the uniqueness part of the Lemma yields $\pi \circ \sigma = Id_G$ and thus σ is a smooth splitting map for the central extension of Lie groups. □

Applying all these results to our discussion of projective representations, we have found several necessary and/or sufficient conditions for being able to lift a projective representation of a Lie group to a proper representation. In particular, we see that the *cocycles represent the obstruction* for lifting a projective representation to a proper representation.

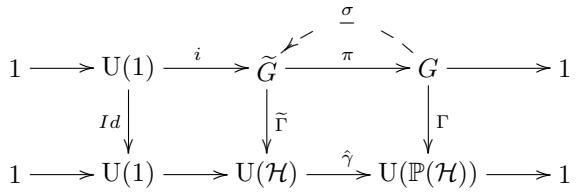
For completeness, we cite Bargmanns theorem, which is probably the main result of the theory of central extensions, and give a brief sketch of the proof. We will not make any further use of this result, but it is of great interest on its own. For the complete, original proof we refer to [Bar54].

Theorem 3.5.11 (Bargmann, 1954)**.**
Let G be a connected and simply connected finite-dimensional Lie group with

$$\mathrm{H}^2(\mathrm{Lie}(G), \mathbb{R}) = 0$$

Then, every projective unitary representation of G on $\mathbb{P}(\mathcal{H})$ can be lifted to a unitary representation on $\mathcal{H}$.

Proof. By Lemma(3.3.5) there exists a central extension $\widetilde{G}$ of G by U(1), such that the following diagram commutes:



[5] [Ne02] Thm. 3.16

Now, one has to show that $\widetilde{G}$ can be given the structure of a $(\dim(G)+1)$-dimensional Lie group. Thus, to this central extension of Lie groups corresponds a central extension of Lie algebras. Since $\mathrm{H}^2(\mathrm{Lie}(G),\mathbb{R}) = 0$, this central extension splits. By the previous Lemma, this implies that the central extension of Lie groups split, i.e. there is a differentiable homomorphism $\sigma : G \to \widetilde{G}$ with $\pi \circ \sigma = Id_G$. Then, $\widetilde{\Gamma} \circ \sigma$ is the postulated lift.

□

A final remark

Our discussion also shows, why in Quantum theory it makes sense to study representations not of the generic symmetry groups but of their *universal covering group*, which is (the unique) simply connected Lie group $\widehat{G} \xrightarrow{\pi} G$ covering G. Of course, every representation ρ of the original Lie group G can be lifted to a representation of the universal covering by $\hat{\rho} := \rho \circ \pi$, but the converse is not true. We have way better chances to find a unitary representation of the universal covering group. In fact, we will always be able to, if the conditions of Bargmans theorem are satisfied. For example, one might study the group $\mathrm{SL}(2,\mathbb{C})$, which is the universal covering group of the (proper orthochronous) Lorentz-group $L^+_\uparrow = \mathrm{SO}(1,3)$, or the *spin groups* $\mathrm{Spin}_n \xrightarrow{\pi} \mathrm{SO}(n)$ which can be realized as a subgroup of the *Clifford algebra* and lead to the spin representations.

Chapter 4

The central extensions $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ & $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$

We will construct a central extension of the restricted general linear group $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ by $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ which acts naturally on the fermionic Fock space $\mathcal{F}$ and restricts to a central extension of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ by $\mathrm{U}(1)$. One can think of the central extensions as a "bigger" group containing the information about the transformation itself plus all possible choices for the phase. An even better intuition is this: $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ (and $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$) act only projectively on Dirac Seas. The central extensions contains the additional information about how to rotate the basis of the Hilbert-space appropriatley, in order to make it a proper action on infinite-wedge-vectors representing the fermion states (cf. the discussion of the left- and right-action on infinite wedge-spaces in 5.2).

4.1 The central extension of $\mathrm{GL}_{\mathrm{res}}$

We have to construct the central extension of the identity component $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})^0 \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. As for the group $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ itself, the complete central extension is just a semi-direct product of the identity-component with $\mathbb{Z}$.

The construction is based on the standard polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, but application to a different (fixed) polarization is immediate.
With respect to this splitting, we can write every linear operator as

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.1.1}$$

We define

$$\mathcal{E} = \{(A, q) \in \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \times \mathrm{GL}(\mathcal{H}_+) \mid a - q \in I_1(\mathcal{H}_+)\}$$

Lemma 4.1.1 (Group structure of $\mathcal{E}$)**.**

$\mathcal{E}$ is a subgroup of $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \times \mathrm{GL}(\mathcal{H}_+)$.

Proof. : Clearly, $(Id_{\mathcal{H}}, Id_{\mathcal{H}_+}) \in \mathcal{E}$. Furthermore:

- If $(A_1, q_1), (A_2, q_2) \in \mathcal{E}$ with $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$,
  then $(A_1A_2)_{++} = a_1a_2 + b_1c_2$ and

  $$(A_1A_2)_{++} - q_1q_2 = a_1a_2 + b_1c_2 - q_1q_2 = \underbrace{(a_1 - q_1)}_{\in I_1(\mathcal{H}_+)} a_2 + q_1 \underbrace{(a_2 - q_2)}_{\in I_1(\mathcal{H}_+)} + \underbrace{b_1}_{\in I_2(\mathcal{H})} \underbrace{c_2}_{\in I_2(\mathcal{H})} \in I_1(\mathcal{H}_+)$$

  Thus: $(A_1A_2, q_1q_2) \in \mathcal{E}$.

- Let $(A, q) \in \mathcal{E}$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \;\Rightarrow\; A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})$$

We deduce: $a\alpha + b\beta = Id_{\mathcal{H}_+} \Rightarrow a\alpha \in Id + I_1(\mathcal{H}_+)$
Thus:

$$a\,(\alpha - q^{-1}) = \underbrace{a\alpha}_{\in Id + I_1(\mathcal{H}_+)} - \underbrace{aq^{-1}}_{\in Id + I_1(\mathcal{H}_+)} \in I_1(\mathcal{H}_+)$$

Since a has a pseudoinverse it follows that $(\alpha - q^{-1}) \in I_1(\mathcal{H}_+)$ and so $(A^{-1}, q^{-1}) \in \mathcal{E}$.

□

We give $\mathcal{E}$ not the subgroup topology, but the topology induced by the embedding

$$(A, q) \mapsto (A, a - q) \in \mathrm{GL}_{\mathrm{res}} \times I_1(\mathcal{H}_+)$$

It is then a Banach Lie group ([PreSe]).

We make a few simple but crucial observations:

i) $a - q \in I_1(\mathcal{H}_+)$ is equivalent to aq^{-1} having a determinant.

ii) For all $A \in \mathrm{GL}_{\mathrm{res}}$ $\exists\, q \in \mathrm{GL}(\mathcal{H}_+)$ such that $(A, q) \in \mathcal{E}$.

iii) If aq^{-1} has a determinant then aq'^{-1} has a determinant too, if and only if $q'q^{-1}$ has a determinant.

From all this it follows that we can think of $\mathcal{E}$ as an extension (not a central one!) of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ by $\mathrm{GL}^1(\mathcal{H}_+)$:

$$1 \longrightarrow \mathrm{GL}^1(\mathcal{H}_+) \longrightarrow \mathcal{E} \xrightarrow{pr_2} \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1$$

ii) implies that the projection pr_2 is surjective onto $\mathrm{GL}^0_{\mathrm{res}}$ and iii) implies that two preimages differ (multiplicatively) by an element in $\mathrm{GL}^1(\mathcal{H}_+)$.

Proof of the statements.

i) $a - q \in I_1(\mathcal{H}_+) \iff (a - q)q^{-1} = aq^{-1} - Id \in I_1(\mathcal{H}_+) \iff aq^{-1} \in Id + I_1(\mathcal{H}_+)$

ii) $U \in \mathrm{GL}^0_{\mathrm{res}} \Rightarrow U_{++} =: a$ is a Fredholm operator with index 0. Thus, there exists a finite rank operator $t : \mathcal{H}_+ \to \mathcal{H}_+$ such that $q := a - t$ is invertible.
But then, $a - q = t \in I_1(\mathcal{H}_+)$.

iii) If aq^{-1} and aq'^{-1} both have determinants then so does $(aq'^{-1})^{-1}(aq^{-1}) = q'q^{-1}$.
Conversely, if $aq^{-1}, q'q^{-1}$ have determinants so does $(aq^{-1})(q'q^{-1})^{-1} = aq'^{-1}$.

□

Definition 4.1.2 (Central extension of $\mathrm{GL}_{\mathrm{res}}$)**.**
On $\mathcal{E}$ we introduce the equivalence relation:

$$(A_1, q_1) \sim (A_2, q_2) : \iff\; A_1 = A_2 \text{ and } \det(q_2\, q_1^{-1}) = 1$$

Now, we define

$$\widetilde{\mathrm{GL}}^0_{\mathrm{res}}(\mathcal{H}) \;:=\; \mathcal{E}/\sim$$

We will show that this is indeed a central extension of $\mathrm{GL}^0_{\mathrm{res}}$ by $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Lemma 4.1.3 (Central extension of $\mathrm{GL}_{\mathrm{res}}$)**.**
The extension

$$1 \longrightarrow \mathrm{GL}^1(\mathcal{H}_+) \longrightarrow \mathcal{E} \xrightarrow{pr_2} \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1$$

descends to a central extension

$$\boxed{1 \longrightarrow \mathbb{C}^\times \xrightarrow{\imath} \widetilde{\mathrm{GL}}^0_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}^0_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1} \tag{4.1.2}$$

with $\pi : [(A,q)] \mapsto A$ *and* $\imath : c \mapsto [(Id,t)]$ *for any* $t \in \mathrm{GL}^1(\mathcal{H}_+)$ *with* $\det(t) = c$.

Proof. :

- Multiplication is well-defined on $\mathcal{E}/\sim$:
  If $(A_1,q_1) \sim (A_1,r_1)$ and $(A_2,q_2) \sim (A_2,r_2)$ i.e. $\det(q_1 r_1^{-1}) = 1 = \det(q_1 r_1^{-1})$ then $\det(r_1 r_2 q_2^{-1} q_1^{-1}) = \det(r_2 q_2^{-1} q_1^{-1} r_1) = \det(r_2 q_2^{-1}) \det(q_1^{-1} r_1) = 1$,
  i.e. $(A_1 A_2, q_1 q_2) \sim (A_1 A_2, r_1 r_2)$
- Now it is easy to see that π and $\mathfrak{i}$ are well-defined homomorphisms of groups and that π surjective and $\mathfrak{i}$ injective.
- We show $\ker(\pi) = \mathrm{im}(\mathfrak{i})$:
  $\ker(\pi) = \{(\mathbb{1}, q)\} \subset \mathcal{E}$. All such elements satisfy $\mathbb{1}_{++} - q = \mathbb{1}_{\mathcal{H}_+} - q \in I_1(\mathcal{H}_+)$ and thus $q \in \mathrm{GL}(\mathcal{H}_+) \cap (Id + I_1(\mathcal{H}_+)) = \mathrm{GL}^1(\mathcal{H}_+)$. Hence, $\ker(\pi) = \mathrm{im}(\mathfrak{i})$.
- The extension is central:
  For all $(A,q) \in \mathcal{E}$ and $t \in \mathrm{GL}^1(\mathcal{H}_+)$, we find $(A,qt) \sim (A,tq)$, since:
  $\det(tq\,(qt)^{-1}) = \det(t\,q\,t^{-1}q^{-1}) = \det(t)\det(q\,t^{-1}q^{-1}) = \det(t)\ \det(t^{-1}) = 1$.
  Thus, $[(A,q)] \cdot [(\mathbb{1},t)] = [(\mathbb{1},t)] \cdot [(A,q)]\,,\ \forall\, [(A,q)] \in \widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ and $[(\mathbb{1},t)] \in \mathrm{im}(\mathfrak{i})$.

□

In fact, $\pi : \widetilde{\mathrm{GL}}^0_{\mathrm{res}} \to \mathrm{GL}^1(\mathcal{H}_+)^0_{\mathrm{res}}$ is not only a central extension but also carries the structure of a *principle fibre bundle*. On principle-bundles, local trivializations are given by *local sections*. Since we are dealing we Lie groups, the bundle structure is already defined by a continuous section in a neighborhood of the identity. This can then be translated to arbitrary points on the manifold simply by the group action. Indeed, there is a very natural local section around the identity in $\mathrm{GL}^0_{\mathrm{res}}(\mathcal{H})$, which will be of great importance for our further discussion.

Consider

$$W := \{A \in \mathrm{GL}_{\mathrm{res}} \mid a = A_{++} \in \mathrm{GL}(\mathcal{H}_+)\} \subset \mathrm{GL}^0_{\mathrm{res}}$$

Claim: W is a dense, open subset of $\mathrm{GL}^0_{\mathrm{res}}$.

Proof. Recall that the topology in $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ is given by the norm $\|\cdot\|_\epsilon = \|\cdot\|_\infty + \|[\epsilon,\cdot]\|_2$.
Clearly, $A \mapsto a = P_+ A P_+$ is continuous w.r.to this norm, therefore W is open in $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ because $\mathrm{GL}(\mathcal{H}_+)$ is open in the space of bounded operators on $\mathcal{H}_+$: For $a \in \mathrm{GL}(\mathcal{H}_+)$ and $\|k\|$ small enough, $a - k$ is also invertible with $(a-k)^{-1} = \big[\sum_{\nu=o}^{\infty} (a^{-1}k)^\nu\big] a^{-1}$.
Furthermore, W is dense in $\mathrm{GL}_{\mathrm{res}}$ because for any Fredholm operator a with index 0 (such as $a = A_{++}$ for $A \in \mathrm{GL}_{\mathrm{res}}$), there exists a compact operator k of arbitrary small norm so that $a + k$ is invertible.

□

On W we can define a smooth section

$$\boxed{\tau : W \longrightarrow \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}); \quad A \longmapsto [(A, A_{++})]} \tag{4.1.3}$$

which satisfies $\tau(\mathbb{1}) = \mathbb{1}$, of course.
This section induces the local trivialization

$$\boxed{\begin{array}{c}\phi : \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \supseteq W \longrightarrow \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \times \mathbb{C}^{\times}; \\ [(A, q)] \longmapsto \big(A\,, \det(a^{-1}q)\big)\end{array}} \tag{4.1.4}$$

by $\phi^{-1}(A, \lambda) := \tau(A) \cdot \lambda \in \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.

Proposition 4.1.4 (Lie group cocycle)**.**
On W we get the continuous 2-cocycle

$$\tau(A)\tau(B) = \chi(A, B)\,\tau(AB)$$

with

$$\chi(A, B) = \det[A_{++}B_{++}(AB_{++})^{-1}] \tag{4.1.5}$$

Proof.

$$\begin{aligned}\tau(AB) =& [(AB, (AB)_{++})] \\ \tau(A) \cdot \tau(B) =& [(A, A_{++})]\,[(B, B_{++})] = [(AB, A_{++}B_{++})] \\ =& [(\mathbb{1}, A_{++}B_{++}((AB)_{++})^{-1})]\,\tau(AB) \\ =& \det[A_{++}B_{++}((AB)_{++})^{-1}] \cdot \tau(AB)\end{aligned}$$

□

The central extension of Lie groups

$$1 \longrightarrow \mathbb{C}^{\times} \xrightarrow{\imath} \widetilde{\mathrm{GL}}^{0}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}^{0}_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1 \tag{4.1.6}$$

induces a central extension

$$0 \longrightarrow \mathbb{C} \xrightarrow{i_*} \tilde{\mathfrak{g}}_1 \xrightarrow{\pi_*} \mathfrak{g}_1 \longrightarrow 0 \tag{4.1.7}$$

of the corresponding Lie algebras. We compute the cocycle of this central extension.

Proposition 4.1.5 (Lie algebra cocycle)**.**
The Lie algebra cocycle for to the central extension (4.1.7) *is*

$$\boxed{\begin{aligned}\Theta(X, Y) = c(X, Y) &= \mathrm{tr}(X_{-+}Y_{+-} - Y_{-+}X_{+-}) \\ &= \frac{1}{4}\,\mathrm{tr}(\epsilon\,[\epsilon, X][\epsilon, Y])\end{aligned}} \tag{4.1.8}$$

This is known as the *Schwinger cocycle* or *Schwinger term.*

Proof. Using the formula (3.5.4) we find:

$$\begin{aligned}\Theta(X, Y) =& \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0} \det\big[e^{sX}_{++}\, e^{tY}_{++}((e^{sX}e^{tY})_{++})^{-1}\big] - \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0} \det\big[e^{tY}_{++}\, e^{sX}_{++}((e^{tY}e^{sX})_{++})^{-1}\big] \\ =& \,\mathrm{tr}\big(X_{++}Y_{++} - (XY)_{++}\big) - \mathrm{tr}\big(Y_{++}X_{++} - (YX)_{++}\big) \\ =& \,\mathrm{tr}\big(X_{++}Y_{++} - X_{++}Y_{++} - X_{+-}Y_{-+}\big) - \mathrm{tr}\big(Y_{++}X_{++} - Y_{++}X_{++} - Y_{+-}X_{+-}\big) \\ =& \,\mathrm{tr}\big(-X_{+-}Y_{-+}\big) + \mathrm{tr}\big(Y_{+-}X_{-+}\big)\end{aligned}$$

The traces converge individually and put together we get

$$\Theta(X,Y) \;= \mathrm{tr}\big(X_{-+}Y_{+-} - Y_{-+}X_{+-}\big)$$

A straight forward computation shows that this can also be expressed as

$$\Theta(X,Y) \;= \frac{1}{4}\;\mathrm{tr}(\epsilon\,[\epsilon,X][\epsilon,Y])$$

□

4.1.1 The central extension of $\mathrm{U}_{\mathrm{res}}$ and its local trivialization

Having defined the central extension $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, we set

$$\boxed{\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) := \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \cap \big(\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H}_+)\big)} \tag{4.1.9}$$

This is a central extension of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ by $\mathrm{U}(1)$ and its complexification is $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.

Just as $\widetilde{\mathrm{GL}}_{\mathrm{res}}$, $\widetilde{\mathrm{U}}_{\mathrm{res}}$ is a principle bundle, but the section τ defined above does not restrict to a section in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) = \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \cap \big(\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H}_+)\big)$ because for unitary U, U_{++} need not be unitary, even if it's invertible. We can however use a polar decomposition to write $U_{++} = V_U|U_{++}|$ with $|U_{++}| = \sqrt{(U_{++}^* U_{++})}$ and $V_U \in \mathrm{U}(\mathcal{H}_+)$ unitary.

Lemma 4.1.6 (Local Section of $\widetilde{\mathrm{U}}_{\mathrm{res}}$)**.**
On $W \cap \mathrm{U}_{\mathrm{res}}(\mathcal{H})$, the map

$$\sigma : U \longmapsto [(U, V_U)] \tag{4.1.10}$$

defines a local section in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$.
The local trivialization $\phi_{\mathrm{U}_{\mathrm{res}}}$ induced by this section equals ϕ up to normalization.

Proof. For $U \in \mathrm{U}_{\mathrm{res}}$, unitarity implies $U_{++}^* U_{++} + U_{+-}^* U_{+-} = \mathbb{1}_{\mathcal{H}_+}$. Since the off-diagonal terms are Hilbert-Schmidt operators, $U_{+-}^* U_{+-}$ is trace-class and thus $U_{++}^* U_{++} \in Id_{\mathcal{H}_+} + I_1(\mathcal{H}_+)$ has a determinant. This is only possible if $\sqrt{(U_{++}^* U_{++})} = |U_{++}|$ is also in $Id_{\mathcal{H}_+} + I_1(\mathcal{H}_+)$. We conclude

$$U_{++} - V_U = V_U\big(|U_{++}| - \mathbb{1}\big) \in I_1(\mathcal{H}_+)$$

and so $[(U, V_U)] \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ for all $U \in \mathrm{U}_{\mathrm{res}}$.

Furthermore, for $[(U,r)]$ we find $\phi\big([(U,r)]\big) = (U,\lambda)$ with

$$\begin{aligned}
\lambda &= \det(U_{++}^{-1}\, r) = \det(|U_{++}|)^{-1} \det(V_U^*\, r) \\
&= \det\Big(\sqrt{(U_{++}^{-1}\, r)(U_{++}^{-1}\, r)^*}\Big) \det(V_U^*\, r) \\
&= \sqrt{\det(U_{++}^{-1}\, r)\,\overline{\det(U_{++}^{-1}\, r)}}\;\det(V_U^*\, r) = |\lambda| \det(V_U^*\, r)
\end{aligned}$$

But $\det(V_U^*\, r)$ is just the $\mathrm{U}(1)$-component w.r.to the loc. trivialization defined by σ. Hence, we read off:

$$\begin{array}{ccc}
 & \mathfrak{U} \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) & \\
{\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\phi_{\mathrm{U}_{\mathrm{res}}}} \\
\mathrm{U}_{\mathrm{res}} \times \mathbb{C}^\times \ni (U,\lambda) & & \big(U, \tfrac{\lambda}{|\lambda|}\big) \in \mathrm{U}_{\mathrm{res}} \times \mathrm{U}(1)
\end{array} \tag{4.1.11}$$

This finishes the proof. □

In the local trivialization of $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$, the U(1)-component corresponds to the information about the phase of the lift of the unitary transformation. In this sense, we can think of the sections as "gauging" the phases of the lifts by picking out a reference lift.

It can be readily computed that the cocycle for the section σ is, as expected, just the normalized version of the $\mathrm{GL}_{\mathrm{res}}$-cocycle χ in (4.1.5). For the corresponding central extension of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathfrak{u}}_{\mathrm{res}} \longrightarrow \mathfrak{u}_{\mathrm{res}} \longrightarrow 0 \tag{4.1.12}$$

we compute the cocycle

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{sX},e^{tY})\,|\chi(e^{sX},e^{tY})|^{-1} - \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\chi(e^{tY},e^{sX})\,|\chi(e^{tY},e^{sX})|^{-1}\\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{t=s=0}\Big[\chi(e^{sX},e^{tY})\big(\chi(e^{sX},e^{tY})\overline{\chi(e^{sX},e^{tY})}\,\big)^{-\frac{1}{2}} - \chi(e^{tY},e^{sX})\big(\chi(e^{tY},e^{sX})\overline{\chi(e^{tY},e^{sX})}\,\big)^{-\frac{1}{2}}\Big]\\
&= \Theta(X,Y) - \tfrac{1}{2}\big(\Theta(X,Y) + \overline{\Theta(X,Y)}\big) = i\,\mathfrak{Im}\big(\Theta(X,Y)\big), \quad \text{for } X,Y \in \mathfrak{u}_{\mathrm{res}}
\end{aligned}$$

But as all $X, Y \in \mathfrak{u}_{\mathrm{res}}$ are (anti-)Hermitian, we find

$$\overline{\Theta(X,Y)} = \Theta(Y^*,X^*) = \Theta(Y,X) = -\Theta(X,Y),$$

which means $i\,\mathfrak{Im}\big(\Theta(X,Y)\big) = \Theta(X,Y)$ on $\mathfrak{u}_{\mathrm{res}}$.
Thus, the Lie algebra cocycle on $\mathfrak{u}_{\mathrm{res}}$ corresponding to the local section σ is just the Schwinger cocycle restricted to the subalgebra $\mathfrak{u}_{\mathrm{res}} \subset \mathfrak{g}_1$.

4.1.2 The complete $\widetilde{\mathrm{GL}}_{\mathrm{res}}$

So far we have constructed the identity component $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ of the central extension, corresponding to transformations preserving the net-charge. Generalization to the other connected components indexed by the relative charge $q = \mathrm{charge}(W, AW)$ for $A \in \mathrm{GL}_{\mathrm{res}}, W \in \mathrm{Gr}(\mathcal{H})$, can be understand by concatenating charge-preserving transformations with a shift of a suitably chosen Hilbert-basis.

Let $(e_k)_{k\in\mathbb{Z}}$ be a basis of $\mathcal{H}$ such that $(e_k)_{k\geq 0}$ is a basis of $\mathcal{H}_+$ and $(e_k)_{k<0}$ a basis of $\mathcal{H}_-$. We need a $\sigma \in \mathrm{GL}_{\mathrm{res}}$ with $\mathrm{ind}(\sigma_{++}) = \pm 1$. Conveniently, we choose σ defined by $e_k \longmapsto e_{k+1}$. Then σ is unitary with $\mathrm{ind}(\sigma_{++}) = -1$. σ acts on $\mathrm{GL}^0_{\mathrm{res}}$ by $A \mapsto \sigma A \sigma^{-1}$.

We define a semi-direct product on $\mathbb{Z} \ltimes \mathrm{GL}^0_{\mathrm{res}}$ by

$$(n, A) \cdot_{\ltimes} (m, A') = (n + m, A\sigma^n A' \sigma^{-n}) \tag{4.1.13}$$

with this group-structure, the obvious map

$$\mathbb{Z} \ltimes \mathrm{GL}^0_{\mathrm{res}} \to \mathrm{GL}_{\mathrm{res}},\ (A, n) \longmapsto A\,\sigma^n \tag{4.1.14}$$

becomes an isomorphism of groups. [1]
So we can express describe the restricted general group as a semi-direct product of the identity component $\mathrm{GL}^0_{\mathrm{res}}$ with $\mathbb{Z}$, the $\mathbb{Z}$-component corresponding to $(-1)\times$ the index of the $++$ component, i.e. the relative charge.

[1] $(n, A)\cdot(m, A') = (n + m, A\sigma^n A'\sigma^{-n}) \longmapsto A\sigma^n A'\sigma^{-n}\sigma^{n+m} = A\sigma^n A'\sigma^m$

The action of σ on $\mathrm{GL}^0_{\mathrm{res}}$ (by conjugation) is covered by an endomorphism $\tilde{\sigma} : \mathcal{E} \to \mathcal{E}$, $\tilde{\sigma}((A, q)) = (\sigma A \sigma^{-1}, q_\sigma)$ where

$$q_\sigma \;=\; \begin{cases} \sigma q \sigma^{-1} & ;\text{ on } \sigma(\mathcal{H}_+) = \mathcal{H}_{\geq 1} \\ \\ \mathrm{Id} & ;\text{ on } \sigma(\mathcal{H}_+)^\perp = \mathrm{span}(e_0) \end{cases}$$

This is not an automorphism, but it descends to one on $\widetilde{\mathrm{GL}}^{\,0}_{\mathrm{res}}$, which we also denote by $\tilde{\sigma}$.

Definition 4.1.7 (The complete $\widetilde{\mathrm{GL}}_{\mathrm{res}}$)**.**
We define

$$\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) := \mathbb{Z} \ltimes \widetilde{\mathrm{GL}}^{\,0}_{\mathrm{res}}(\mathcal{H})$$

where the action of $\mathbb{Z}$ is generated by $\tilde{\sigma}$.
Then

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1$$

is a central extension of $\mathrm{GL}_{\mathrm{res}}$ by $\mathbb{C}^\times$. Over the identity components of the Lie groups it restricts to (4.1.2). The unitary version works analogously.

4.2 Non-Triviality of the central extensions

Theorem 4.2.1 (Non-triviality of the central extension)**.**

The Lie algebra extensions $\tilde{\mathfrak{g}}_1 \to \mathfrak{g}_1$ *as well as* $\widetilde{\mathfrak{u}}_{\mathrm{res}} \to \mathfrak{u}_{\mathrm{res}}$ *are not trivial.*

In particular, this implies that there exists no differentiable [2] *section* $\Gamma : \mathrm{U}_{\mathrm{res}} \to \widetilde{\mathrm{U}}_{\mathrm{res}}$ *which is also a homomorphism of groups.*

We will see that the central extension $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ or $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ are the groups with a natural irreducible representation on the fermionic Fock space. So, the theorem tells us in particular that there exists no (smooth) representation of the entire group $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ on the Fock space. Other results from the classification theory of principle bundles prove that there is not even a continuous global section of $\mathrm{U}_{\mathrm{res}}$ in $\widetilde{\mathrm{U}}_{\mathrm{res}}$[3].
One might ask how the central extension $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ is relates to the abstractly defined "de-projetivization" $\widehat{\mathrm{GL}}_{\mathrm{res}}$ defined in [Lem. 3.3.5]. In [Wurz01] is it shown that $\widehat{\mathrm{GL}}_{\mathrm{res}}$ can be indeed given the structure of a Banach Lie group, s.t. it is diffeomorphic to $\widetilde{\mathrm{GL}}_{\mathrm{res}}$.

Proof of the Theorem. [4] We formulate the proof for the unitary case, although it applies almost word-for-word to $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ as well. However, non-triviality of $\widetilde{\mathrm{U}}_{\mathrm{res}}$ implies non-triviality of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ anyways. From our discussion of central extensions we know that it suffices to show that the Schwinger cocycle c is not trivial, i.e. that there is no linear map $\mu : \mathfrak{u}_{\mathrm{res}} \to \mathbb{R}$ with $\mu([X,Y]) = c(X,Y),\ \forall X, Y \in \mathfrak{u}_{\mathrm{res}}$. This is proven, for example, if we can find $X, Y \in \mathfrak{u}_{\mathrm{res}}$ with $[X,Y] = 0$ but $c(X,Y) \neq 0$. For completeness, we recall the argument why this proves that a (differentiable) section in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ cannot exist.

In Lemma 4.1.6 we have constructed a section $\sigma : \mathrm{U}_{\mathrm{res}} \to \widetilde{\mathrm{U}}_{\mathrm{res}}$. However, this section fails to be a homomorphism of groups, i.e. we get $\sigma(U)\sigma(V) = \kappa(U,V)\sigma(UV)$ with a Lie group cocycle $\kappa : \mathrm{U}_{\mathrm{res}} \times \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(1)$. Consequently, its derivative (at the identity) $\dot{\sigma} : \mathfrak{u}_{\mathrm{res}} \to \tilde{\mathfrak{u}}_{\mathrm{res}}$ fails to be a Lie algebra homomorphism. This is expressed by the *Schwinger cocycle*

$$c(X,Y) := [\dot{\sigma}(X), \dot{\sigma}(Y)] - \dot{\sigma}([X,Y]) \text{ for } X, Y \in \mathfrak{u}_{\mathrm{res}}$$

Now suppose the section σ is just a bad choice and there was in fact a different section $\Gamma : \mathrm{U}_{\mathrm{res}} \to \widetilde{U}_{\mathrm{res}}$ which is also a homomorphism of groups. Since two elements in the same fibre in $\widetilde{U}_{\mathrm{res}}$ differ only by a complex phase, the "good" section differs from σ only by a map $\lambda : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(1)$. As Γ is a Lie group homomorphism, the corresponding Lie algebra map $\dot{\Gamma} = \dot{\lambda} + \dot{\sigma}$ is a Lie algebra homomorphism. This means:

$$0 = [\dot{\lambda}(X) + \dot{\sigma}(X), \dot{\lambda}(Y) + \dot{\sigma}(Y)] = [\dot{\sigma}(X), \dot{\sigma}(Y)] = \dot{\lambda}([X,Y]) + \dot{\sigma}([X,Y])$$

$$\Rightarrow \dot{\lambda}([X,Y]) = c(X,Y),\ \forall X, Y \in \mathfrak{u}_{\mathrm{res}}$$

For the first equality we have used that $\dot{\lambda} : \mathfrak{u}_{\mathrm{res}} \to \mathrm{Lie}(\mathrm{U}(1)) = \mathbb{R}$ maps into the center of $\tilde{\mathfrak{u}}_{\mathrm{res}}$ and so all the commutators with $\dot{\lambda}$ vanish.
We observe: if a lift $\Gamma : \mathrm{U}_{\mathrm{res}} \to \widetilde{\mathrm{U}}_{\mathrm{res}}$ preserving the group structure exists then there must be a linear map $\mu\,(= \dot{\lambda}) : \mathfrak{u}_{\mathrm{res}} \to \mathbb{R}$ with $\mu([X,Y]) = c(X,Y),\ \forall X, Y \in \mathfrak{u}_{\mathrm{res}}$.
Therefore, to prove that such a lift does *not* exist it suffices to find $X, Y \in \mathfrak{u}_{\mathrm{res}}$ with $[X,Y] = 0$ but $c(X,Y) \neq 0$.
The Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with the polarization given by the sign of the free Dirac Hamiltonian is somewhat difficult to handle. Fortunately, by unitary equivalence we can just as well consider any other polarized (separable, infinite-dimensional, complex) Hilbert space.

[2] Actually, for the proof it suffices to assume differentiability at the identity so that the section in the Lie group extension induces a Lie algebra homomorphism.

[3] [PreSe], see also [Scho]§3.2 for an outline of the argument.

[4] The proof follows [Wurz01], where it is presented in a very nice and complete way. However, we believe that the abstract mathematical language might conceal the ultimately simple nature of the proof, at least from a physicists point of view. We have therefore tried to rephrase it in more elementary terms without any reference to cohomology or the like.

For our purpose it is nice to work with the Hilbert space $\mathcal{K} := L^2(S^1, \mathbb{C})$ with the polarization given by separation into Fourier components with positive and negative frequencies. The natural Hilbert-basis on $\mathcal{K}$ is the Fourier-Basis $(e_k)_{k\in\mathbb{Z}}$ where $e_k(t) := e^{i2\pi kt} \in L^2(S^1, \mathbb{C})$.

Writing $\mathcal{K} \ni f = \sum_{k\in\mathbb{Z}} f_k e_k$ we have:

$$P_+ f := \sum_{k\geq 0} f_k e_k\,, \quad P_- f := \sum_{k<0} f_k e_k \tag{4.2.1}$$

On $\mathcal{K}$ we consider the class of operators given by multiplication with smooth functions. For $g \in C^\infty(S^1, \mathbb{C})$ and $f \in L^2(S^1, \mathbb{C})$ we write $M_g(f) = g \cdot f$.
In Fourier space multiplication corresponds to convolution.
So if $g = \sum_{k\in\mathbb{Z}} g_k\, e_k$ then

$$M_g(e_l) = \sum_{k\in\mathbb{Z}} g_{k-l}\, e_k \tag{4.2.2}$$

Be careful not to make the easy mistake to confuse the Fourier components of g with those of the multiplication operator $M_g : \mathcal{K} \to \mathcal{K}$.
For $g, h \in C^\infty(S^1, \mathbb{C})$ we compute:

$$\begin{aligned}
\mathrm{tr}(M_{h-+}M_{g+-}) &= \sum_{l<0}\langle e_l, M_h \sum_{k\geq 0} g_{k-l}e_k\rangle \\
&= \sum_{l<0}\langle e_l, \sum_{k\geq 0}\sum_{m<0} h_{m-k}\, g_{k-l}\, e_m\rangle \\
&= \sum_{l<0}\sum_{k\geq 0} h_{l-k}\, g_{k-l} = -\sum_{l<0} l\, h_l\, g_{-l}
\end{aligned} \tag{4.2.3}$$

Similarly

$$\mathrm{tr}(M_{g-+}M_{h+-}) = \sum_{l>0} l\, h_l\, g_{-l} \tag{4.2.4}$$

In particular we conclude that

$$\begin{aligned}
\|[\epsilon, M_g]\|_2^2 =& 4 \cdot \mathrm{tr}\Big((M_{g+-})(M_{g+-})^* + (M_{g-+})(M_{g-+})^*\Big) \\
&= 4 \cdot \Big(\sum_{l\geq 0} l|g_l|^2 + \sum_{l<0}(-l)|g_l|^2\Big) \\
&= 4 \cdot \Big(\sum_{l\in\mathbb{Z}} |l||g_l|^2\Big) < \infty \text{ for smooth g.}
\end{aligned}$$

So all operators of this type have off-diagonal components in the Hilbert-Schmidt class, meaning that indeed

$$\Big\{M_g \mid |g(t)|^2 \equiv 1\Big\} \subset \mathrm{U}_{\mathrm{res}}(\mathcal{K})$$
$$\Big\{M_g \mid g(t) \in \mathbb{R}\Big\} \subset \mathfrak{u}_{\mathrm{res}}(\mathcal{K})$$

From (4.2.3) and (4.2.4) we also read off that the Schwinger cocycle is well-defined on operators of this type with

$$c(M_h, M_g) = \mathrm{tr}(M_{h-+}M_{g+-} - M_{g-+}M_{h+-}) = -\sum_{l\in\mathbb{Z}} l h_l g_{-l}$$

$$\boxed{= \frac{1}{2\pi i}\int_0^1 h(t)\dot{g}(t)\,\mathrm{d}t} \tag{4.2.5}$$

The rest is easy. Just take any two functions $g, h \in C^\infty(S^1, \mathbb{R})$ with $\int_0^1 h(t)\dot{g}(t)\,\mathrm{d}t \neq 0$. For instance, consider $g(t) = \cos(2\pi t)$ and $h(t) = \sin(2\pi t)$. Then we got $M_g, M_h \in \mathfrak{u}_{\mathrm{res}}(h)$ with $[M_h, M_g] = 0$ but

$$c(M_h, M_g) = \frac{1}{2\pi i}\int_0^1 h(t)\dot{g}(t)\,\mathrm{d}t = (-1)\int_0^1 \sin^2(t)\,\mathrm{d}t \neq 0$$

This shows that the Schwinger cocycle c is non-trivial and completes our proof.

□

Remark 4.2.2. (Embedding of Loop Groups)
In the related mathematical literature, the multiplication operators and the cocycle (4.2.5) considered on the proof of the previous theorem arise in the very abstract context of the embeddings of *loop groups* into $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. If K is a d-dimensional (compact) Lie group, then $C^\infty(S^1, K)$ is an infinite-dimensional Lie group, called a *loop group* and usually denoted by LK or $Map(S^1, K)$. Any (faithful) representation $\rho : K \to \mathrm{GL}(\mathbb{C}^d)$ then provides an action of the loop group on the Hilbert space $\mathcal{K} = L^2(S^1, \mathbb{C})$ via

$$\begin{aligned} C^\infty(S^1, K) \times L^2(S^1, \mathbb{C}) = LK \times \mathcal{K} &\to \mathcal{K} \\ (\varphi, f) &\longmapsto \rho(\varphi(t)) \cdot f(t) \end{aligned}$$

Just as we did above (for d=1), this provides an embedding of the loop group LK into $\mathrm{GL}_{\mathrm{res}}(\mathcal{K}) \cong \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. The central extension $\tilde{\mathrm{GL}}_{\mathrm{res}}$ of $\mathrm{GL}_{\mathrm{res}}$ induces a central of LK and the corresponding Lie algebra cocycle takes the form analogous to (4.2.5). The simple multiplication in $\mathbb{C}$ is just replaced by matrix-multiplication and taking the trace in $Mat(d\times d, \mathbb{C})$. In our proof we have explicitly avoided any reference to loop groups or cohomology theory as usually found in the mathematical literature, since I feel that they unnecessarily obscure the otherwise simple nature of the proof.

Chapter 5

Three routes to the Fock space

5.1 CAR Algebras and Representations

In this section we carry out the Quantization of the Dirac Field in the spirit of what I have called the "electron-positron picture". Thereby, we will mostly follow the discussion in [Tha]. I will assume that the reader is somewhat familiar with creation/annihilation operators and the standard construction of the Fock space, so we will review them just briefly. If the reader is well familiar with the subject as a whole, he or she should feel free to skip this section altogether. Adjacent is a brief discussion of abstract CAR algebras. Fock spaces will then arise as representation spaces of irreducible representations of the algebra. This is quite an abstract mathematical machinery, but I think it's worth going into it for different reasons:

1. It is probably the most common and most developed mathematical description of Fock spaces and "second quantization".
2. It constitutes, at least rudimentary, a rigorous mathematical formalization of what physicists usually try to say.
3. It provides a language in which the mathematical problems can be formulated very nicely and thereby opens up a perspective which can be fruitful from time to time.

What do I mean by a "fruitful perspective"? For instance, it is my impression that as a physicist with some training in (non-relativistic) Quantum theory one is used to think of "the" Fock space as a fundamental object ("the space of all physical states"). This makes it hard to grasp some of the issues we're facing in relativistic Quantum Field Theory, like the fact that the time evolution is "leaving" the Fock space (where else would it go?). Thus, I do find it helpful sometimes to think of the CAR-algebra as the more fundamental object and of Fock spaces only in terms of its representations. The danger however is that this easily becomes a vain exercise in abstract mathematics, detached from the physical problems.

5.1.1 The Field Operator

On the one-particle Hilbert-space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ we have a *charge conjugation* operator $\mathcal{C}$ mapping negative energy solutions of the free Dirac equation to positive energy solutions with opposite charge. The charge-conjugation is anti-unitary, in particular *anti-linear*. The exact form of the operator depends on the representation of the Dirac algebra. In the so called "Majorana representation", charge conjugation just corresponds to complex conjugation. In Appendix A.2 we give an intuitive, yet general derivation of the charge-conjugation operator.

Definition 5.1.1 (Fermionic Fock space)**.**
Let $\mathcal{F}_+ := \mathcal{H}_+$ and $\mathcal{F}_- := \mathcal{C}\mathcal{H}_-$. We define the *fermionic Fock space* as [1]

$$\mathcal{F} := \bigoplus_{n,m=0}^{\infty} \mathcal{F}^{(n,m)}; \;\; \mathcal{F}^{(n,m)} := \bigwedge^n \mathcal{F}_+ \otimes \bigwedge^m \mathcal{F}_- \tag{5.1.1}$$

We can also split the Fock space into the different *charge sectors*

$$\mathcal{F} := \bigoplus_{c=0}^{\infty} \mathcal{F}^{(c)}; \;\; \mathcal{F}^{(c)} := \bigoplus_{n-m=c} \mathcal{F}^{(n,m)} \tag{5.1.2}$$

The state

$$\Omega := 1 \otimes 1 \in \mathcal{F}^{(0,0)} = \mathbb{C} \otimes \mathbb{C} \subset \mathcal{F}$$

is called the *vacuum state*

Recall the definitions of the **creation and annihilation operators**.
On the "particle-sector" $\bigwedge \mathcal{F}_+$:

$$\begin{aligned} &a(f) : \mathcal{F}^{(n+1,m)} \longrightarrow \mathcal{F}^{(n,m)}, \\ &a(f) f_0 \wedge \cdots \wedge f_n := \sum_{k=0}^{n} (-1)^k \langle f, f_k \rangle f_0 \wedge \cdots \wedge \widehat{f_k} \wedge \cdots \wedge f_n \qquad (5.1.3) \\ &a^*(f) : \mathcal{F}^{(n-1,m)} \longrightarrow \mathcal{F}^{(n,m)}, \\ &a^*(f) f_1 \wedge \cdots \wedge f_n - 1 = f \wedge f_1 \wedge \cdots \wedge f_{n-1} \qquad (5.1.4) \end{aligned}$$

On the "anti-particle sector" $\bigwedge \mathcal{F}_-$:

$$\begin{aligned} &b(g) : \mathcal{F}^{(n,m+1)} \longrightarrow \mathcal{F}^{(n,m)}, \\ &b(g)\, \mathcal{C}g_0 \wedge \cdots \wedge \mathcal{C}g_n := (-1)^n \sum_{k=0}^{n} (-1)^k \langle \mathcal{C}g, \mathcal{C}g_k \rangle \mathcal{C}g_0 \wedge \cdots \wedge \widehat{\mathcal{C}g_k} \wedge \cdots \wedge \mathcal{C}g_n \qquad (5.1.5) \\ &b^*(g) : \mathcal{F}^{(n,m-1)} \longrightarrow \mathcal{F}^{(n,m)}, \\ &b^*(g)\, \mathcal{C}g_1 \wedge \cdots \wedge \mathcal{C}g_n - 1 = \mathcal{C}g \wedge \mathcal{C}g_1 \wedge \cdots \wedge \mathcal{C}g_{n-1} \qquad (5.1.6) \end{aligned}$$

The reader is probably familiar with the fact that a and a^*, as well as b and b^*, are formal adjoints of each other and satisfy the *canonical anti-commutation relations*

$$\begin{aligned} &\{a(f_1), a^*(f_2)\} = a(f_1)a^*(f_2) + a^*(f_2)a^(f_1) = \langle f_1, f_2 \rangle_{\mathcal{H}} \cdot \mathbb{1}, \quad \forall f_1, f_2 \in \mathcal{H}_+ \\ &\{b(g_1), b^*(g_2)\} = \langle \mathcal{C}g_1, \mathcal{C}g_2 \rangle_{\mathcal{H}} \cdot \mathbb{1} = \overline{\langle g_1, g_2 \rangle} \cdot \mathbb{1} = \langle g_2, g_1 \rangle_{\mathcal{H}} \cdot \mathbb{1}, \; \forall g_1, g_2 \in \mathcal{H}_- \end{aligned} \tag{5.1.7}$$

and all other possible combination *anti-commute*.

If $(f_j)_{j\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ are ONB's of $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively, the elements of the form

$$a^*(f_{j_1})a^*(f_{j_2}) \ldots a^*(f_{j_n}) b^*(g_{k_1}) b^*(g_{k_2}) \ldots b^*(g_{k_m}) \Omega \in \mathcal{F}^{(n,m)} \subset \mathcal{F} \tag{5.1.8}$$

for $j_1 < \cdots < j_n; k_1 < \cdots < k_m$ and $n, m = 0, 1, 2, 3, \ldots$ form an ONB of the Fock space $\mathcal{F}$.

So, the creation operators acting on the vacuum generate the dense subspace

$$\mathcal{D} := \Big\{ \text{finite linear combinations of vectors of the form (5.1.8)} \Big\}$$

which is the usual domain for the second quantization of bounded operators.

[1] By the direct sums we implicitly understand the completion w.r.to the induced scalar product

Definition 5.1.2 (Field Operator)**.**

For any $f \in \mathcal{H}$ we define the *field operator* $\Psi(f)$ on $\mathcal{F}$ by

$$\begin{aligned} \Psi(f) &:= a(P_+ f) + b^*(P_- f) \\ \Psi^*(f) &= a^*(P_+ f) + b(P_- f) \end{aligned} \tag{5.1.9}$$

We can view the field operator as an *anti-linear* map $\Psi : \mathcal{H} \longrightarrow \mathcal{B}(\mathcal{F})$.

It satisfies the *canonical anti-commutation relations* (CAR)

$$\boxed{\begin{aligned} \big\{\Psi(f), \Psi^*(g)\big\} &= \langle f, g\rangle \cdot \mathbb{1} \\ \big\{\Psi(f), \Psi(g)\big\} &= \big\{\Psi(f)^*, \Psi^*(g)\big\} = 0 \end{aligned}} \tag{5.1.10}$$

Second Quantization of Unitary Operators

Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary transformation on the one-particle Hilbert space. We want to lift it to a unitary transformation $\Gamma(U)$ on the Fock space $\mathcal{F}$.
In non-relativistic Quantum mechanics, we would lift an operator U to the n-particle Hilbert-space $\bigwedge^n \mathcal{H}$ product-wise by

$$U \otimes U \otimes \cdots \otimes U \tag{5.1.11}$$

But the naive generalization

$$f_1 \wedge \cdots \wedge f_n \otimes \mathcal{C}g_1 \wedge \cdots \wedge \mathcal{C}g_m \longmapsto Uf_1 \wedge \cdots \wedge Uf_n \otimes \mathcal{C}Ug_1 \wedge \cdots \wedge \mathcal{C}Ug_m$$

does only make sense if U preserves the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ i.e. only if $U_{+-} = U_{-+} = 0$. In general, this is not the case and U will mix positive and negative energy states. Physically this leads to the phenomenon we call pair creation. Mathematically this leads to trouble.

While it is difficult to say how the unitary transformation is supposed to act on the Fock space, it acts very naturally on the field operator Ψ, namely by

$$\begin{aligned} \Psi \longmapsto \beta_u(\Psi) &:= \Psi \circ U, \\ \textit{i.e. } \beta_u(\Psi)(f) &= \Psi(Uf) = a(P_+ Uf) + b^*(P_- Uf),\ \forall f \in \mathcal{H} \end{aligned} \tag{5.1.12}$$

β_U is called a *Bogoljubov transformation.*

It is easy to see that $\tilde{\Psi} := \beta_U(\Psi)$ is still an antilinear map $\mathcal{H} \to \mathcal{B}(\mathcal{F})$ satisfying the CAR (5.1.10). Mathematically this means that $\tilde{\Psi}$ is also a field operator, inducing another *representation of the CAR-algebra* with the new annihilation operators *defined* as

$$\begin{aligned} c(f) &:= \tilde{\Psi}(f), \text{ for } f \in \mathcal{H}_+ \\ d(f) &:= \tilde{\Psi}^*(g), \text{ for } g \in \mathcal{H}_- \end{aligned} \tag{5.1.13}$$

Definition 5.1.3 (Implementability of unitary transformations)**.**
The unitary transformation $U \in \mathrm{U}(\mathcal{H})$ is *implementable* on the Fock space $\mathcal{F}$ if there exists a unitary map $\Gamma(U) : \mathcal{F} \to \mathcal{F}$ with

$$\Gamma(U)\Psi(f)\Gamma(U)^* = \beta_U(\Psi)(f) = \Psi(Uf),\ \forall f \in \mathcal{H} \tag{5.1.14}$$

If U is implementable, the implementation is unique up to a phase.

That the implementation can be determined uniquely only up to a phase is obvious, because if $\Gamma(U)$ is an implementation of $U \in \mathrm{U}(\mathcal{H})$, so is $e^{i\varphi}$ for all $e^{i\varphi} \in \mathrm{U}(1)$.

Supposed that $\Gamma(U)$ is an implementation of U we note that by (5.1.14) it acts on a basis vector of the form

$$\begin{aligned} &a^*(f_1)a^*(f_2)\dots a^*(f_n)b^*(g_1)b^*(g_2)\dots b^*(g_m)\,\Omega \\ =\,&\Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\Psi(g_2)\dots\Psi(g_m)\,\Omega \end{aligned}$$

as

$$\begin{aligned} &\Gamma(U)\left(\Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\,\Omega\right) \\ =&\,\Gamma(U)\Psi^*(f_1)\Gamma(U)^*\Gamma(U)\Psi^*(f_2)\Gamma(U)^*\dots\Gamma(U)\Psi(g_{m-1})\Gamma(U)^*\Gamma(U)\Psi(g_m)\Gamma(U)^*\Gamma(U)\,\Omega \\ =&\,\Psi^*(Uf_1)\Psi^*(Uf_2)\dots\Psi^*(Uf_n)\Psi(Ug_1)\dots\Psi(Ug_m)\,\left[\Gamma(U)\,\Omega\right] \\ =&\,c^*(f_1)c^*(f_2)\dots c^*(f_n)d^*(g_1)d^*(g_2)\dots d^*(g_m)\,\widetilde{\Omega} \end{aligned}$$

with $\widetilde{\Omega}=\Gamma(U)\Omega$. This simple consideration gives us a pretty interesting result

Proposition 5.1.4 (New Vacuum)**.**
A unitary transformation $U\in\mathrm{U}(\mathcal{H})$ is implementable, if and only if there exists a vacuum for the new annihilation operators (5.1.13) *coming from $\beta_U(\Psi)$.*
That is, if there exists a normalized state $\widetilde{\Omega}$ with

$$c(f)\widetilde{\Omega}=\beta_U(\Psi)(f)\,\widetilde{\Omega}=0,\ \ \forall f\in\mathcal{H}_+ \tag{5.1.15}$$

$$d(g)\widetilde{\Omega}=\beta_U(\Psi)^*(g)\,\widetilde{\Omega}=0,\ \ \forall g\in\mathcal{H}_- \tag{5.1.16}$$

Proof. If $\Gamma(U)$ is an implementer of U set $\widetilde{\Omega}=\Gamma(U)\Omega$.
Then,

$$\beta_U(\Psi)(f)\widetilde{\Omega}=\Gamma(U)\Psi(f)\Gamma(U)^*\Gamma(U)\Omega=\Gamma(U)\Psi(f)\Omega=0,\ \ \forall f\in\mathcal{H}_+ \tag{5.1.17}$$

$$\beta_U(\Psi)^*(g)\widetilde{\Omega}=\Gamma(U)\Psi(g)\Gamma(U)^*\Gamma(U)\Omega=\Gamma(U)\Psi(g)\Omega=0,\ \ \forall g\in\mathcal{H}_- \tag{5.1.18}$$

so $\widetilde{\Omega}$ is a vacuum for $\beta_U(\Psi)$. Conversely, if we have a vacuum $\widetilde{\Omega}$ for $\beta_U(\Psi)$, we can define a unitary transformation on $\mathcal{F}$ by

$$\begin{aligned} &a^*(f_1)a^*(f_2)\dots a^*(f_n)b^*(g_1)b^*(g_2)\dots b^*(g_m)\,\Omega \\ \longmapsto\,&e^{i\phi}c^*(f_1)c^*(f_2)\dots c^*(f_n)d^*(g_1)d^*(g_2)\dots d^*(g_m)\,\widetilde{\Omega} \end{aligned}$$

with an arbitrary phase $e^{i\phi}$ and the computation above shows that this is an implementation of U on $\mathcal{F}$.

□

In the simplest case when U leaves the positive and negative energy subspace invariant, we note that $c(f)=a(Uf),\ \forall f\in\mathcal{H}_+$ and $d(g)=b(Ug),\ \forall g\in\mathcal{H}_-$, so that $\widetilde{\Omega}=e^{i\varphi}\Omega$. Of course, the phase is then usually chosen to be 1.
Consequently, the implementation is nothing else than

$$\begin{aligned} &f_1\wedge\dots\wedge f_n\otimes\mathcal{C}g_1\wedge\dots\wedge\mathcal{C}g_m=a^*(f_1)\dots a^*(f_n)b^*(g_1)\dots b^*(g_m)\,\Omega \\ \overset{\Gamma(U)}{\longmapsto}\,&c^*(f_1)\dots c^*(f_n)d^*(g_1)\dots d^*(g_m)\,\Omega=Uf_1\wedge\dots\wedge Uf_n\otimes\mathcal{C}Ug_1\wedge\dots\wedge\mathcal{C}Ug_m \end{aligned} \tag{5.1.19}$$

This shows us that the whole business with the Bogoljubov transformations and its implementation (that we hope to exist) is indeed the natural generalization of the product-wise lift of U to the Fock-space.

Almost sneakily, we have established a dual way of thinking about the implementation problem that is quite exciting. We can either put the problem as one about lifting a unitary transformation U on $\mathcal{H}$ to the Fock space $\mathcal{F}$ which was our initial motivation. Or, we can ask the equivalent question whether the two representations of the CAR-algebra induced by Ψ and $\beta_U(\Psi)$, respectively, are equivalent. So far, we have but hinted at this fact. A more detailed discussion will follow in the next section.

We can finally state our first version of the Shale-Stinespring theorem.

Theorem 5.1.5 (Shale-Stinespring, explicit version)**.** [2]
A unitary operator $U \in \mathrm{U}(\mathcal{H})$ is unitarily implementable on $\mathcal{F}$ if and only if the operators U_{+-} and U_{-+} are Hilbert-Schmidt. In this case, an implementation $\Gamma(U)$ acts on the vacuum as

$$\boxed{\Gamma(U)\,\Omega := N e^{i\phi} \prod_{l=1}^{L} a^*(f_l) \prod_{m=1}^{M} b^*(g_m)\, e^{Aa^*b^*}\,\Omega} \tag{5.1.20}$$

where

- $e^{\phi} \in \mathrm{U}(1)$ *is an arbitrary phase*
- $N = \sqrt{\det(1 - U_{+-}U_{-+}^*)} = \sqrt{\det(1 - U_{-+}U_{+-}^*)}$ *is a normalization constant*[3]
- $A := (U_{+-})(U_{--})^{-1}$ *is Hilbert-Schmidt*
- $\{f_1, \ldots, f_L\}$ *is an ONB of* $\ker U_{++}^*$ *and* $\{g_1, \ldots, g_M\}$ *an ONB of* $\ker U_{--}^*$

Recall that U_{++} and U_{--} are Fredholm-operators and so $\ker U_{++}^*$ and $\ker U_{--}^*$ are indeed finite-dimensional subspaces. Also, $(U_{--})^{-1}$ is well-defined and bounded on $\operatorname{im} U_{--} = (\ker U_{--}^*)^{\perp}$ and we extend it to 0 on $\ker U_{--}^*$. This is how $A := (U_{+-})(U_{--})^{-1}$ has to be understood.

In the explicit form of the transformed vacuum also our result (2.2.8) about the net-charge "created" by U becomes manifest. From (5.1.20) we can directly read of that $\Gamma(U)\Omega$ contains $M = \dim\ker(U_{--}^*)$ particles of negative charge and $L = \dim\ker(U_{++}^*)$ particles of positive charge. Hence, the net charge is

$$\begin{aligned} L - M &= \dim\ker(U_{++}^*) - \dim\ker(U_{--}^*) \\ &= \dim\ker(U_{--}) - \dim\ker(U_{--}^*) \\ &= \operatorname{ind}(U_{--}) = -\operatorname{ind}(U_{++})^{4} \end{aligned} \tag{5.1.21}$$

The different sign in comparison to (2.2.8) stems from the unfortunate fact that the roles of $\mathcal{H}_+$ and $\mathcal{H}_-$ are opposite in the two chapters.

Second Quantization of Hermitian Operators

Now, let's look at the second quantization of an arbitrary bounded operator $A : \mathcal{H} \to \mathcal{H}$. We will denote it by $\mathrm{d}\Gamma(A)$, in the hope that it does really exists in a meaningful way. Of particular interest are self-adjoint operators, of course.
To $\bigwedge^n \mathcal{H}$, we would simply lift the operator additively as:

$$A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes A \otimes \mathbb{1} \ldots \mathbb{1} \;+\; \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A \tag{5.1.22}$$

[2] cf. [Tha], Thms. 10.6 & 10.7
[3] Note that, in contrast to [Tha], we use the notation $U_{-+}^* = (U^*)_{-+}$ and not $(U_{-+})^* = (U^*)_{+-}$
[4] For the last equality we need a Lemma, which is proved in Appendix A.3.

But again, the immediate generalization to $\mathcal{F} = \bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}(\mathcal{H}_-)$ makes sense only if A respects the polarization. So let's again turn to the field operator and use it to formulate a generalization of the additive lifting prescription.
We will demand the following conditions for the "second quantization":

$$\begin{aligned} [\mathrm{d}\Gamma(A), \Psi^*(f)] &= \Psi^*(Af) \\ [\mathrm{d}\Gamma(A), \Psi(f)] &= -\Psi(A^*f) \end{aligned} \tag{5.1.23}$$

The minus-sign in the second line might look ad-hoc, but will be of great importance.
Note that if $\mathrm{d}\Gamma(A)$ satisfies (5.1.23) so does $\mathrm{d}\Gamma(A) + c\,\mathbb{1}_{\mathcal{F}}$ for every constant $c \in \mathbb{C}$.
Therefore, the implementation of a Hermitian operator is also well defined only up to a constant at best.

An operator $\mathrm{d}\Gamma(A)$ satisfying (5.1.23) then acts on a basis state (5.1.8) as

$$\begin{aligned} &\mathrm{d}\Gamma(U)\Big(\Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Omega\Big) \\ =&\Psi^*(Af_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Omega + \Psi^*(f_1)\mathrm{d}\Gamma(A)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Omega \\ =&\cdots = \\ =&\Psi^*(Af_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Omega + \Psi^*(f_1)\Psi^*(Af_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Omega + \dots \\ &+ \Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(Af_n)\Psi(g_1)\dots\Psi(g_m)\Omega - \Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(A^*g_1)\dots\Psi(g_m)\Omega - \dots \\ &- \Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(A^*g_m)\Omega \\ &+ \Psi^*(f_1)\Psi^*(f_2)\dots\Psi^*(f_n)\Psi(g_1)\dots\Psi(g_m)\Big(\mathrm{d}\Gamma(A)\Omega\Big) \end{aligned}$$

We see that $\mathrm{d}\Gamma(A)$ is well-defined on the dense domain $\mathcal{D}$ if and only if the last summand, i.e. $\mathrm{d}\Gamma(A)\Omega$ is well-defined. Then, this would really provide a sensible generalization of the additive-lift (5.1.22).
Let $(e_k)_{k\in\mathbb{Z}}$ be a basis of $\mathcal{H}$, s.t. $(e_k)_{k\leq 0}$ is a basis of $\mathcal{H}_-$ and $(e_k)_{k>0}$ a basis of $\mathcal{H}_+$.
A first educated guess for the second quantization of a bounded operator A would be $\mathrm{d}\Gamma'(A) := A\Psi^*\Psi$, where $A\Psi^*\Psi$ is short-hand notation for

$$\sum_{i,j\in\mathbb{Z}} \langle e_i, Ae_j\rangle \Psi^*(e_j)\Psi(e_i)$$

[5] This is independent of the chosen basis for $\mathcal{H}$. We can decompose this into the creation and annihilation operators and write $A\Psi^*\Psi = Aa^*a + Aa^*b^* + Aba + Abb^*$.
The vacuum expectation value of this operator can then be computed as

$$\begin{aligned} \langle\Omega, \mathrm{d}\Gamma'(A)\Omega\rangle &= \langle\Omega, A\Psi^*\Psi\Omega\rangle = \langle\Omega, Abb^*\Omega\rangle \\ &= \sum_{i,j<0} \langle e_i, Ae_j\rangle \langle\Omega, b(e_i)b^*(e_j)\Omega\rangle \\ &= \sum_{i,j<0} \langle e_i, Ae_j\rangle\, \delta_{ij} = \mathrm{tr}(A_{--}) \end{aligned}$$

So, unless A_{--} is trace-class, not even the vacuum state is in the domain of $A\Psi^*\Psi$!
A more successful attempt is

$$\mathrm{d}\Gamma(A) =: A\Psi^*\Psi : = Aa^*a + Aa^*b^* + Aba - Ab^*b \tag{5.1.24}$$

which also satisfies (5.1.23). Physicists call this *normal ordering* and denote it by two colons embracing expressions like $A\Psi^*\Psi$. With this prescription, $: A\Psi^*\Psi : \Omega = Aa^*b^*\Omega$ and therefore

$$\langle\Omega, \mathrm{d}\Gamma(A)\,\Omega\rangle = \langle\Omega, : A\Psi^*\Psi : \Omega\rangle = 0 \tag{5.1.25}$$

[5] Consult e.g. [Tha] §10 for more details

We note that this implies

$$: A\Psi^*\Psi : = A\Psi^*\Psi - \mathrm{tr}(A_{--}) \cdot \mathbb{1} \tag{5.1.26}$$

whenever $\mathrm{tr}(A_{--}) < \infty$. So in this case, $A\Psi^*\Psi$ and $: A\Psi^*\Psi :$ are both second quantizations of A differing by a complex constant.
Assuming that $A_{+-} : \mathcal{H}_- \to \mathcal{H}_+$ is compact, there exists an ONB $(u_j)_j$ of $\mathcal{H}_+$ and $(v_j)_j$ of $\mathcal{H}_-$ and principle values $\lambda_j \geq 0$ such that $A_{+-} = \sum_{j=0}^{\infty} \lambda_j \langle v_j, \cdot \rangle u_j$. [6]
Hence, $Aa^*b^* = \sum_j \lambda_j a^*(u_j) b^*(v_j)$ and we compute

$$\begin{aligned}
\|Aa^*b^*\Omega\|^2 &= \sum_{k,j} \lambda_k \lambda_j \langle \Omega, b(v_k) a(u_k) a^*(u_j) b^*(v_j) \Omega \rangle \\
&= \sum_{k,j} \lambda_k \lambda_j \langle \Omega, b(v_k) \big(\{a(u_k), a^*(u_j)\} - a^*(u_j) a(u_k) \big) b^*(v_j) \Omega \rangle \\
&= \sum_{k,j} \lambda_k \lambda_j \Big(\delta_{jk} \langle \Omega, \big(\{b(v_k), b^*(v_j)\} - b^*(v_j) b(v_k) \big) \Omega \rangle - \langle \Omega, b(v_k) a^*(u_j) b^*(v_j) a(u_k) \Omega \rangle \Big) \\
&= \sum_{k,j} \lambda_k \lambda_j \delta_{jk} = \sum_k \lambda_k^2 = \|A_{+-}\|_2^2
\end{aligned}$$

This shows that $: A\Psi^*\Psi :$ is well-defined on the dense domain $\mathcal{D}$ if and only if A_{+-} is of Hilbert-Schmidt type. For a Hermitian operator, the same must be true for the adjoint so we require also A_{-+} to be Hilbert-Schmidt.

How does this relate to the discussion in the previous section? A link between between unitary and Hermitian operators is given by *Stone's Theorem*, saying that every strongly continuous, one-parameter group of unitary operators is generated by a unique Hermitian operator. We are in particular interested in the free time evolution $U_t = e^{-itD_0}$.
Let $\Psi_t := \beta_{U_t}(\Psi)$ be the time-evolving field operator. Then, for any fixed f in the domain of D_0, $\Psi_t(f)$ is even norm-differentiable in t [7] with

$$\frac{d}{dt} \Psi_t(f) = \Psi_t(-iD_0\, f) = i\Psi_t(D_0\, f) \tag{5.1.27}$$

The same is true for Ψ_t^*, but note that this is linear and not anti-linear in f.
On a basis state of the form (5.1.8) we therefore compute

$$\begin{aligned}
& i\frac{d}{dt}\Big|_{t=0} \Big[\Gamma(U_t) \Big(\Psi^*(f_1)\Psi^*(f_2) \dots \Psi^*(f_n) \Psi(g_1) \dots \Psi(g_m)\, \Omega \Big) \Big] \\
&= i\frac{d}{dt}\Big|_{t=0} \Big[\Psi_t^*(f_1)\Psi_t^*(f_2) \dots \Psi_t^*(f_n) \Psi_t(g_1) \dots \Psi_t(g_m)\, \Omega \Big] \\
&= \Psi^*(D_0 f_1)\Psi^*(f_2) \dots \Psi^*(f_n)\Psi(g_1) \dots \Psi(g_m)\Omega + \Psi^*(f_1)\Psi^*(D_0 f_2) \dots \Psi^*(f_n)\Psi(g_1) \dots \Psi(g_m)\Omega + \cdots \\
&+ \Psi^*(f_1)\Psi^*(f_2) \dots \Psi^*(D_0 f_n)\Psi(g_1) \dots \Psi(g_m)\Omega - \Psi^*(f_1)\Psi^*(f_2) \dots \Psi^*(f_n)\Psi(D_0 g_1) \dots \Psi(g_m)\Omega - \cdots \\
&- \Psi^*(f_1)\Psi^*(f_2) \dots \Psi^*(f_n)\Psi(g_1) \dots \Psi(D_0 g_m)\Omega \\
&= \mathrm{d}\Gamma(D_0) \Big(\Psi^*(f_1)\Psi^*(f_2) \dots \Psi^*(f_n)\Psi(g_1) \dots \Psi(g_m)\, \Omega \Big)
\end{aligned}$$

This means:

$$i\frac{d}{dt}\Gamma(e^{-iD_0 t}) \;=\; \mathrm{d}\Gamma(D_0) \;=: D_0 \Psi^*\Psi : \tag{5.1.28}$$

on the dense domain where both sides are well-defined.

[6] λ_j^2 are the eigenvalues of the positive operator $(A_{+-})^*(A_{+-})$

[7] because: $\lim_{h\to 0} \|\frac{1}{h}[\Psi_{t+h}(f) - \Psi_t(f)] - \Psi_t(-iD_0\, f)\| = \lim_{h\to 0} \|\Psi\big(e^{iD_0 t}[\frac{1}{h}(e^{iD_0 h} - \mathbb{1}) + iD_0]f\big)\| = \lim_{h\to 0} \|[\frac{1}{h}(e^{iD_0 h} - \mathbb{1}) + iD_0]f\| = 0$

$\mathrm{d}\Gamma(D_0)$ is the **second quantization of the free Dirac Hamiltonian.**

If $(f_k)_{k\in\mathbb{N}}$ is a basis of $\mathcal{H}_+$ and $(g_k)_{k\in\mathbb{N}}$ a basis of $\mathcal{H}_-$ then

$$\boxed{\begin{aligned}\mathrm{d}\Gamma(D_0) &=: D_0\Psi^*\Psi :\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Big(\langle f_i, D_0 f_j\rangle a^*(f_i)a(f_j) - \langle g_j, D_0 g_i\rangle b^*(g_i)b(g_j)\Big)\end{aligned}} \tag{5.1.29}$$

which is obviously semi-bounded from below because $-D_0$ is positive definite on $\mathcal{H}_-$.
Note that the anti-linearity of the charge-conjugation and thus of the Fock-space sectors $\bigwedge\mathcal{F}_-$ makes all the difference. It is also the origin of the minus-sign in (5.1.23). Similarly, the global gauge-transformations $e^{-it\mathbb{1}}$ are generated by the *charge-operator*

$$Q := \mathrm{d}\Gamma(\mathbb{1}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Big(a^*(f_i)a(f_j) - b^*(g_i)b(g_j)\Big) \tag{5.1.30}$$

which is just the difference of the number operators on $\bigwedge\mathcal{F}_+$ and $\bigwedge\mathcal{F}_-$.
Of course, the generators of the free time evolution and of the global gauge-transformations are easy to compute because both operators are diagonal w.r.to the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In particular, we can set $\Gamma(U_t)\Omega = \Omega$.
However, the suggested relationship between Γ and $\mathrm{d}\Gamma$ is true in the general case:

Theorem 5.1.6 (Generators of unitary groups)**.**
Let A be a self-adjoint operator on $\mathcal{H}$ with $[\epsilon, A]$ Hilbert-Schmidt. Then $\mathrm{d}\Gamma(A) =: A\Psi^*\Psi :$ as defined in (5.1.24) *is essentially self-adjoint on $\mathcal{F}$ and for its (unique) self-adjoint extension on the Fock space $\mathcal{F}$ it is true that $e^{i:A\Psi^*\Psi:}$ is an implementation of e^{iA}.*
In particular, we can choose phases for the implementations such that

$$\Gamma(e^{iA}) = e^{i\,\mathrm{d}\Gamma(A)} = e^{i:A\Psi^*\Psi:} \tag{5.1.31}$$

Proof. [CaRu87] Proposition 2.1 ff. seems to be the first complete proof of this theorem. □

Note that (5.1.31) fixes the phase of the implementation $\Gamma(e^{iA})$ only in a neighborhood of the identity where the exponential map is 1:1 from the Lie algebra on to the Lie group.

In physics *normal ordering* is often introduced just formally by simply "writing all the annihilators on the right of the creation operators" so that it gives zero when acting on the vacuum. To me, it always seems to get something mysterious or even dishonest that way. The last theorem tells us that the second quantization in normal order just defines an action of the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ on $\mathcal{F}$. No big mystery there.
For A in the trace-class, $A\Psi^*\Psi$ and $:A\Psi^*\Psi:$ differ only by a constant multiple of the identity. In fact, they just correspond to different "lifts" of A to the universal covering $\tilde{\mathfrak{u}}_{\mathrm{res}} \cong \mathfrak{u}_{\mathrm{res}} \oplus \mathbb{C}$ which is actually the algebra acting on the Fock space. From the discussion of central extensions in chapter 3 we also know that Γ is only a *projective* representation of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and that, for this reason, we get a *commutator anomaly* in the representation of the Lie algebra in form of a Lie algebra 2-cocycle. It should come as no surprise that we just encounter the Schwinger-cocycle (4.1.8) again.

Proposition 5.1.7. *(Schwinger cocycle)*
For $A, B \in \mathfrak{u}_{\mathrm{res}}$ we find

$$[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)] = \mathrm{d}\Gamma([A,B]) + c(A,B)\mathbb{1} \tag{5.1.32}$$

with the Schwinger cocycle $c(A,B) = \mathrm{tr}(A_{-+}B_{+-} - B_{-+}A_{+-})$ computed in (4.1.8).

Proof. Because $[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)]$ and $\mathrm{d}\Gamma([A,B])$ are both second quantizations of $[A,B]$, they differ only by a constant multiple c_{AB} of the identity [8], i.e.

$$[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)] = \mathrm{d}\Gamma([A,B]) + c_{AB} \cdot \mathbb{1}$$

Now $\mathrm{d}\Gamma([A,B])$ is defined precisely in such a way that the vacuum expectation value $\langle \Omega, \mathrm{d}\Gamma([A,B])\Omega\rangle$ vanishes. Therefore,

$$c_{AB} = \langle \Omega, [\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)]\Omega\rangle$$

We take bases $(f_j)_{j\in\mathbb{N}}$ of $\mathcal{H}_+$ and $(g_k)_{k\in\mathbb{N}}$ of $\mathcal{H}_-$ and compute:

$$\begin{aligned}
&\langle \Omega, \mathrm{d}\Gamma(A)\mathrm{d}\Gamma(B)\,\Omega\rangle = \langle \Omega, Aba\,Ba^*b^*\,\Omega\rangle \\
=&\sum_{j,k}\sum_{l,m}(g_k, Af_j)(f_l, Bg_m)\langle \Omega, b(g_k)a_(f_j)a^*(f_l)b^*(g_m)\Omega\rangle \\
=&\sum_{j,k}(g_k, Af_j)(f_j, Bg_k)\langle \Omega, b(g_k)a(f_j)a^*(f_j)b^*(g_k)\Omega\rangle \\
=&\sum_{j,k}(g_k, Af_j)(f_j, Bg_k) = \mathrm{tr}(A_{-+}B_{+-})
\end{aligned}$$

Similarly, $\langle \Omega, \mathrm{d}\Gamma(B)\mathrm{d}\Gamma(A)\Omega\rangle = \mathrm{tr}(B_{-+}A_{+-})$. And therefore

$$c_{AB} = c(A,B) = \mathrm{tr}(A_{-+}B_{+-} - B_{-+}A_{+-})$$

□

Note that on $\mathfrak{u}_{\mathrm{res}} \cap I_1(\mathcal{H})$ the proof simplifies considerably. We can use $:A\Psi^*\Psi: = A\Psi^*\Psi - \mathrm{tr}(A_{--})\mathbb{1}$ and it is easy to check that $[A\Psi^*\Psi, B\Psi^*\Psi] = [A,B]\Psi^*\Psi$. Hence, we find

$$[:A\Psi^*\Psi:, :B\Psi^*\Psi:] = [A,B]\Psi^*\Psi = :[A,B]\Psi^*\Psi: + \mathrm{tr}([A,B]_{--})\cdot\mathbb{1}$$

and

$$\begin{aligned}
\mathrm{tr}([A.B]_{--}) &= \mathrm{tr}\big(A_{-+}B_{+-} + A_{--}B_{--} - B_{-+}A_{+-} - B_{--}A_{--}\big) \\
&= \mathrm{tr}\big(A_{-+}B_{+-} - B_{-+}A_{+-}\big) = c(A,B)
\end{aligned}$$

In Conclusion:

We have sketched the quantization of the Dirac theory in the most established (rigorous) way. I want to emphasize that at no point in this description does an infinite number of particles appear. Still, we have encountered all the difficulties that I have hinted at in the introductory chapter and that were suggested by the Dirac Sea picture:

- Unitary transformations are not implementable as unitary operators on the Fock space unless they satisfy the Shale-Stinespring condition
- Even for implementable operators, the implementation is well-defined only up to a complex phase
- Second quantization of Hermitian operators has similar difficulties. Where it does exist, it gives rise to a commutator anomaly in form of the Schwinger term

These facts make a second quantization of the time-evolution, in general, impossible. The asymptotic case will turn out to be somewhat better behaved. But even then, at least the phase of the second quantized scattering operator $\mathbf{S}$ is ill-defined.

[8] In the language of the previous chapters: $[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)]$ and $\mathrm{d}\Gamma([A,B])$ are just different lifts of $[A,B] \in \mathfrak{u}_{\mathrm{res}}$ to $\tilde{\mathfrak{u}}_{\mathrm{res}} \cong \mathfrak{u}_{\mathrm{res}} \oplus \mathbb{C}$.

5.1.2 C^* - and CAR- Algebras

We take a quick look at Fock spaces and implementations of unitary transformations from the perspective of representation theory. This approach is very abstract but will reveal its beauty along the way.

Definition 5.1.8 (C^*-algebra)**.**
A *Banach algebra* (over $\mathbb{C}$) is a complex, associative algebra B with 1, equipped with a norm $\|\cdot\|$ that makes it a complex Banach space and has the properties

$$i)\ \|1\| = 1 \qquad\qquad ii)\ \|ab\| \leq \|a\| \cdot \|b\|,\ \forall a, b \in B$$

A C^*-*algebra* is a complex Banach algebra A together with an anti-linear involution $* : A \to A$ satisfying:

$$iii)\ (a^*)^* = a \qquad iv)\ (ab)^* = b^*a^* \qquad v)\ \|a^*a\| = \|a\|^2$$

$\forall a, b \in A$. Properties $ii) - v)$ together imply $\|a^*\| = \|a\|$.

Definition 5.1.9 (C^*-homomorphism)**.**
A C^*-homomorphism is an algebra homomorphism $h : A \to B$ between two C^*-algebras A and B satisfying $h(a^*) = h(a)^*,\ \forall a \in A$.
It can be shown that any such homomorphism is continuous with norm ≤ 1.

Examples 5.1.10 (C^*-algebra of bounded operators)**.**
Consider the Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.
On $\mathcal{B}(\mathcal{H})$ we have a $*$-involution $T \mapsto T^*$ by taking the Hermitian conjugate of bounded operators. From the familiar properties of Hermitian conjugation and of the operator norm it follows that $\big(\mathcal{B}(\mathcal{H}), \|\cdot\|, *\big)$ is a C^*-algebra.
In fact, *every* C^*-algebra is isomorphic to a sub-algebra of $\mathcal{B}(\mathcal{H})$ for suitable $\mathcal{H}$.

Definition 5.1.11 (States of a C^*-algebra)**.**
A *state* of a C^*-algebra A is a complex-linear map $\omega : A \to \mathbb{C}$ satisfying $\omega(1) = 1$ and $\omega(a^*a) \geq 0,\ \forall a \in A$.

Examples 5.1.12 (Quantum States)**.**
For the C^*-algebra $A = \mathcal{B}(\mathcal{H})$ every $\Psi \in \mathcal{H}$ defines a state ω_Ψ by

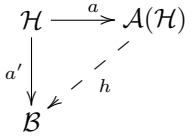
$$\omega_\Psi(T) := \frac{\langle \Psi, T\,\Psi \rangle}{\langle \Psi, \Psi \rangle},\ \forall T \in \mathcal{B}(\mathcal{H})$$

In the language of Quantum Theory we would say that the *state* represented by the Hilbert-space vector $\Psi \in \mathcal{H}$ is characterized by all "expectation values".

Definition 5.1.13 (CAR-algebra)**.**
A CAR-map into a C*-algebra A is an anti-linear map $a : \mathcal{H} \to A$ satisfying the CAR-relations

$$a(x)a(y)^* + a(y)a(x)^* = \{a(x), a^*(y)\} = \langle x, y \rangle \cdot 1 \tag{5.1.33}$$
$$a(x)a(y)\ +\ a(y)a(x)\ =\ \{a(x), a(y)\} = 0 \tag{5.1.34}$$

A CAR algebra over $\mathcal{H}$ is a C*-algebra $\mathcal{A}(\mathcal{H})$ together with a CAR-map $a : \mathcal{H} \to \mathcal{A}(\mathcal{H})$ having the following universal property: for every CAR-map $a' : \mathcal{H} \to \mathcal{B}$ into a C*-algebra $\mathcal{B}$ there is a unique C*-homomorphism $h : \mathcal{A}(\mathcal{H}) \to \mathcal{B}$ s.t. $a' = h \circ a$.



(5.1.35)

Such a CAR-algebra does exist and by the universal property it is unique up to canonical isomorphism.

It is probably more intuitive to think of the CAR Algebra as the free associative algebra generated by $\mathcal{H}$ modulo the canonical anti-commutation relations (5.1.10). Mathematicians, however, like to define it by the universal property as above which is rather abstract, but very elegant to work with.
We might get a hint of how powerful the universal property is, by clarifying the relationship between "field operators" as introduced in the last section and representations of the abstract CAR-algebra defined above. What we called a field operator was a map $\Psi : \mathcal{H} \to B(\mathcal{F})$ from the Hilbert-space into the C^*-algebra of bounded operators on the Fock-Space, satisfying the CAR (5.1.10). By the universal property of the CAR-algebra there exists a C*- homomorphism $\pi : \mathcal{A}(\mathcal{H}) \to B(\mathcal{F})$ *s.t.* $\Psi = \pi \circ a$. This homomorphism π is then a representation of the CAR-algebra $\mathcal{A}(\mathcal{H})$ on $\mathcal{F}$.

Proposition 5.1.14 (Fock Representation)**.**
The field operator Ψ *defined in* (5.1.9) *defines an irreducible*[9]*. representation of the CAR-algebra of* $\mathcal{H}$ *on the Fock space* $\mathcal{F} = \bigwedge \mathcal{H}_+ \oplus \bigwedge \overline{\mathcal{H}_-}$*, called the* Fock representation*.*

5.1.3 Representations of the CAR Algebra

Mathematically, a "Fock space" is often understood as a Hilbert space, arising as the representation space of an irreducible representation of an abstract CAR-algebra (in the fermionic case), or a CCR-algebra (in the bosonic case). There are various ways to get irreducible representations of a CAR-algebra and they are generally not equivalent to each other.

Definition 5.1.15 (Equivalent Representations)**.**
Consider the CAR algebra $\mathcal{A}(\mathcal{H})$ ober $\mathcal{H}$. Two representations π and π' of $\mathcal{A}(\mathcal{H})$ on Fock spaces $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent* if there exists a unitary transformation $T : \mathcal{G} \to \mathcal{G}'$ with $T\,\pi(a(f)) = \pi'(a(f))\,T,\ \forall f \in \mathcal{H}$.

We generalize the notion of "implementability" introduced in Def. 5.1.3 to arbitrary representations of the CAR-algebra.

Definition 5.1.16 (Implementation of unitary transformation)**.**
Let $U \in \mathrm{U}(\mathcal{H})$ a unitary operator. It is easily checked that if $a : \mathcal{H} \to \mathcal{A}(\mathcal{H})$ is a CAR-map, so is $a \circ U$. By the universal property of the CAR-algebra there exists a unique C^*-homomorphism $\beta_U : \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ with $\beta_U \circ a = a \circ U$.

β_U is the *Bogoljubov transformation* corresponding to U.

Let $\pi : \mathcal{A}(\mathcal{H}) \to F$ be a representation of the CAR-algebra on the Fock space F.
U (or β_U) is called *implementable* on F if there exists a unitary operator $\tilde{U}$ on F such that $\tilde{U}\pi(a(f)) = \pi(a(Uf))\tilde{U},\ \forall f \in \mathcal{H}$.

$\tilde{U}$ is then called an *implementation* of U on F.

In other words: U is implementable on the Fock space F if and only if the two CAR-representations π and $\pi \circ \beta_U$ are unitarily equivalent.
So the "implementation problem" and the "equivalence problem" are closely related.

We have explicitly constructed the Fock representation on $\mathcal{F} = \bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}\mathcal{H}_-$, by defining creation and annihilation operators an then the field operator Ψ (5.1.9). But this was just a very particular example. The seminal paper of Shale and Stinespring is actually formulated for spin-representations of the infinite dimensional Clifford algebra of $\mathcal{H}$ which is closely related to the CAR-algebra $\mathcal{A}(\mathcal{H})$ ([SS65], see [G-BVa94] for a more detailed construction.)

[9] cf. [Tha], Thm.10.2

The most common way to get representations of a CAR algebra are the so called *GNS constructions.*[10]
Given a state ω on $\mathcal{A}(\mathcal{H})$, the GNS-construction yields a Hilbert space $\mathcal{H}_\omega$, a representation π_ω on $\mathcal{H}_\omega$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ (the "GNS-vacuum"), such that

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\,\Omega_\omega \rangle\,,\ \forall a \in \mathcal{A}(\mathcal{H}) \tag{5.1.36}$$

and the action of $\pi_\omega(\mathcal{A}(\mathcal{H}))$ on Ω_ω generates a dense subset of $\mathcal{H}_\omega$.

In particular, given an orthogonal projection Q, i.e. a self-adjoint operator on $\mathcal{H}$ with $Q^2 = Q$, we can *define* a state ω_Q by setting $\omega_Q(1) = 1$ and fixing the "two-point functions"

$$\omega_Q(a^*(f)a(g)) := \langle g, Qf \rangle_{\mathcal{H}} \tag{5.1.37}$$

where $f, g \in \mathcal{H}$ and a was the CAR-map defining $\mathcal{A}(\mathcal{H})$.
Then, the corresponding GNS-construction $(\mathcal{F}_Q, \pi_Q, \Omega_Q)$ satisfies

$$\langle \Omega_Q, \pi_V(a^*(g)a(f))\Omega_Q \rangle = \langle g, Qf \rangle_{\mathcal{H}} = \langle Qf, Qg \rangle_{\mathcal{H}} \tag{5.1.38}$$

and, using $a(g)a^*(f) = \langle g, f \rangle_{\mathcal{H}} \mathbb{1} - a^*(f)a(g)$,

$$\langle \Omega_Q, \pi_Q(a(g)a^*(f))\Omega_Q \rangle = \langle f, g \rangle_{\mathcal{H}} - \langle Qf, Qg \rangle_{\mathcal{H}} = \langle (\mathbb{1} - Q)f, (\mathbb{1} - Q)g \rangle_{\mathcal{H}} \tag{5.1.39}$$

In particular, we read off

$$\begin{aligned} \pi_Q(a(g))\Omega_Q = 0,\ \text{for } g \in \ker(Q) \\ \pi_Q(a^*(f))\Omega_Q = 0,\ \text{for } f \in \mathrm{im}(Q) \end{aligned} \tag{5.1.40}$$

so that $\pi_Q \circ a : \mathcal{H} \to \mathcal{B}(\mathcal{F}_Q)$ acts like a field-operator on the vacuum Ω_Q.
Indeed, we observe that for $Q = P_-$, the two-point functions of the GNS-construction are the same as the two-point functions of the field operator Ψ on $\bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}\mathcal{H}_-$, because

$$\langle \Omega, \Psi^*(f)\Psi(g)\Omega \rangle = \langle \mathcal{C}P_-f, \mathcal{C}P_-g \rangle = \langle g, P_-f \rangle_{\mathcal{H}}$$

This implies that the two representations are actually equivalent. The same is true, of course, for the Fock- and GNS representation w.r.to a different polarization of $\mathcal{H}$.

Now, in the language of representation theory we can formulate another, very general version of the Shale-Stinespring theorem, also known as the theorem of Powers and Størmer.

Theorem 5.1.17 (Powers, Størmer 1969)**.**
The GNS-representations $(\mathcal{F}_P, \pi_P, \Omega_P)$ and $(\mathcal{F}_Q, \pi_Q, \Omega_Q)$ corresponding to orthogonal projections P and Q on $\mathcal{H}$ are unitary equivalent if and only if $P - Q$ is a Hilbert-Schmidt.

Proof. [PoSt70] □

Applying this result to the "implementation problem" we find ones more that $U \in \mathrm{U}(\mathcal{H})$ is implementable in $\mathcal{F} = \bigwedge \mathcal{H}_+ \otimes \bigwedge \mathcal{C}\mathcal{H}_-$ if and only if $P_- - UP_-U^*$ is Hilbert-Schmidt i.e. iff U_{+-} and U_{-+} are Hilbert-Schmidt i.e. iff $U \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

[10] after I.M. Gelfand, M.A. Neumark and I.E. Siegel.

5.2 The Infinite Wedge Space Construction

In the light of realization that the unitary time-evolution typically can not be implemented on a fixed Fock space, my esteemed teachers and colleagues Dirk Deckert, Detlef Dürr, Franz Merkl and Martin Schottenloher developed a formulation of the external field problem in QED on *time-varying* Fock spaces. These Fock spaces are realized as "infinite wedge spaces" constructed from "Dirac seas" over a chosen polarization class. This construction of the Fock spaces is very nice for several reasons:

1. It is very down to earth and close to the physical intuition
2. It is designed to highlight all the ambiguities and choices involved in the construction rather than hiding them
3. The formalism is very flexible and well-suited for the treatment of different polarization classes and the implementation of the time evolution on varying Fock spaces

Of course, Deckert et.al. are consistently ignoring the geometric structure that - as we will see- is somehow present in the construction. But by doing so, it just becomes more clear that this is not essential for the understanding of the problems. Whether it is helpful in solving them is a different question that we will discuss in later chapters.

The basic idea of the infinite-wedge-space construction is this:

- For a fixed (equal charge) polarization class, Dirac seas will be defined as certain maps from an index-space ℓ into the Hilbert space $\mathcal{H}$ with image in the respective polarization class.
- If we fix a basis $(e_k)_{k\in\mathbb{N}}$ in ℓ, we can think of a Dirac sea $\Phi : \ell \to \mathcal{H}$ as the infinite exterior product $\Phi(e_0) \wedge \Phi(e_1) \wedge \Phi(e_2) \wedge \dots$
- We generalize the finite-dimensional scalar product (1.2.1) (sometimes refered to as Slater-Determinant) by the infinite-dimensional Fredholm determinant i.e. by

$$\langle \Phi, \Psi \rangle = \det(\Phi^* \Psi) = \lim_{N\to\infty} \det\big(\langle \Phi(e_i), \Psi(e_j)\rangle\big)_{i,j\leq N} \tag{5.2.1}$$

  In order for this to make sense, we need to restrict to a class of Dirac seas so that the Determinant is well-defined.
- Finally, we get a linear space by taking the algebraic dual of such a Dirac sea class and take the completion with respect to before mentioned scalar product. This will be the Fock space realized as an infinite wedge-space over the chosen Dirac sea class.

Now, we make this more precise:
For the remainder of this chapter, let ℓ be an infinite-dimensional, separable, complex Hilbert-space. ℓ will serve as an index space. A convenient choice is therefore $\ell = \ell^2$.

Definition 5.2.1 (Dirac Seas)**.**

Let $\texttt{Seas}(\mathcal{H})$ be the set of all bounded, linear maps $\Phi : \ell \to \mathcal{H}$ such that $\mathrm{im}(\Phi) \in \texttt{Pol}(\mathcal{H})$ and $\Phi^*\Phi : \ell \to \ell$ has a determinant, i.e. $\Phi^*\Phi \in Id + I_1(\ell)$.

Let $\texttt{Seas}^\perp(\mathcal{H})$ denote the subset of all $\Phi \in \texttt{Seas}(\mathcal{H})$ that are also isometries.

For any equal-charge polarization class $C \in \texttt{Pol}(\mathcal{H})/\approx_0$ let $\texttt{Ocean}(C)$ be the set of all $\Phi \in \texttt{Seas}^\perp(\mathcal{H})$ with $\mathrm{im}(\Phi) \in C$.

Definition 5.2.2 (Dirac Sea Classes)**.**
For $\Phi, \Psi \in \texttt{Seas}(\mathcal{H})$ we introduce the relation

$$\Phi \sim \Psi \iff \Phi^*\Psi \in Id + I_1(\ell)$$

This defines an equivalence relation on $\texttt{Seas}(\mathcal{H})$ ([DeDuMeScho], Cor. II.9) . The equivalence classes are called *Dirac sea classes* and denoted by $\mathcal{S}(\Phi) \in \texttt{Seas}(\mathcal{H})/\sim$.

Lemma 5.2.3 (Connection between $\sim$ and $\approx_0$)**.**
Given $C \in \mathtt{Pol}(\mathcal{H})/\approx_0$ *and* $\Phi \in \mathtt{Ocean}(C)$ *we have*

$$C = \{\mathrm{im}(\Psi) \mid \Psi \in \mathtt{Seas}^{\perp}(\mathcal{H}) \text{ such that } \Psi \sim \Phi\}.$$

In other words: $\Psi \sim \Phi$ *in* $\mathtt{Seas}^{\perp}(\mathcal{H}) \Rightarrow \mathrm{im}(\Psi) \approx_0 \mathrm{im}(\Phi)$ *in* $\mathtt{Pol}(\mathcal{H})$
and $W \in C = [\mathrm{im}(\Phi)]_{\approx_0} \Rightarrow \exists\, \Psi \in \mathtt{Seas}^{\perp}(\mathcal{H}) : \Psi \sim \Phi \wedge \mathrm{im}(\Psi) = W$.

Proof. See Lemma II.12 in [DeDuMeScho] or note that the statement follows as a reformulation of our Lemma 5.3.2 ("Existence of admissible basis") and the remark following Definition 5.3.1("Admissible Basis"). A polar decomposition can be used to make the Dirac Sea/admissible basis isometric. □

Construction 5.2.4 (Formal Linear Combinations)**.**

1. For any set $\mathcal{S}$, let $\mathbb{C}^{(\mathcal{S})}$ denote the set of all maps $\alpha : \mathcal{S} \to \mathbb{C}$ for which the support $\{\Phi \in \mathcal{S} \mid \alpha(\Phi) \neq 0\}$ is finite. For $\Phi \in \mathcal{S}$, let $[\Phi] \in \mathbb{C}^{(\mathcal{S})}$ denote the algebraic dual, i.e. the map fulfilling $[\Phi](\Phi) = 1$ and $[\Phi](\Psi) = 0$ for $\Phi \neq \Psi \in \mathcal{S}$. Thus, $\mathbb{C}^{(\mathcal{S})}$ consists of all finite formal linear combinations $\alpha = \sum_{\Psi \in \mathcal{S}} \alpha(\Psi)[\Psi]$ of elements of $\mathcal{S}$ with complex coefficients.

2. Now, let $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$ be a Dirac Sea Class.
We define the map $\langle \cdot, \cdot \rangle : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$, $(\Phi, \Psi) \mapsto \langle \Phi, \Psi \rangle := \det(\Phi^* \Psi)$.
The determinant exists and is finite by definition of $\sim$.

3. For $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$, let $\langle \cdot, \cdot \rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}$ denote the sesquilinear extension of $\langle \cdot, \cdot \rangle : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ to the linear space $\mathbb{C}^{(\mathcal{S})}$ defined as follows:
For $\alpha, \beta \in \mathbb{C}^{(\mathcal{S})}$,
$$\langle \alpha, \beta \rangle := \sum_{\Phi \in \mathcal{S}} \sum_{\Psi \in \mathcal{S}} \overline{\alpha(\Phi)} \beta(\Psi) \det(\Phi^* \Psi). \tag{5.2.2}$$
The bar denotes the complex conjugate. Note that the sums consist of at most finitely many nonzero summands. In particular we have $\langle [\Phi], [\Psi] \rangle = \langle \Phi, \Psi \rangle$ for $\Phi, \Psi \in \mathcal{S}$.

The sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{(\mathcal{S})}$ is Hermitian and positive-semidefinite[11]. Therefore, it defines a semi-norm on $\mathbb{C}^{(\mathcal{S})}$ by $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$.

Definition 5.2.5 (Infinite Wedge Space)**.**
Let $\mathcal{F}_{\mathcal{S}}$ be the completion of $\mathbb{C}^{(\mathcal{S})}$ with respect to the semi-norm $\|\cdot\|$.
$\mathcal{F}_{\mathcal{S}}$ is an infinite-dimensional, separable, complex Hilbert space.
We will refer to it as the *Infinite Wedge Space* over the Dirac Sea class $\mathcal{S}$.

By $\bigwedge : \mathcal{S} \to \mathcal{F}_{\mathcal{S}}$ we denote the canonical map $\bigwedge \Phi := [\Phi]$ coming from the inclusions

$$\mathcal{S} \hookrightarrow \mathbb{C}^{\mathcal{S}} \hookrightarrow \mathcal{F}_{\mathcal{S}}.$$

Note, that the null-space $N_{\mathcal{S}} := \{\alpha \in \mathbb{C}^{(\mathcal{S})} \mid \|\alpha\| = 0\}$ is factored out in the process of completion.

Construction 5.2.6 (The Left Operation)**.**

$\mathrm{U}(\mathcal{H})$ acts on $\mathtt{Seas}(\mathcal{H})$ from the left by $(U, \Phi) \to U\Phi$.
This extends to a well-defined map

$$\mathcal{S} \xrightarrow{U} U\mathcal{S} := \{U\Phi \mid \Phi \in \mathcal{S}\}$$

between Dirac Sea classes.

[11] see[DeDuMeScho], Lemma II.14 or our Proposition 5.3.11 "Hermitian Form".

For $U \in \mathrm{U}(\mathcal{H})$, the induced left operation $\mathcal{L}_U : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(U\mathcal{S})}$, given by

$$\mathcal{L}_U \left(\sum_{\Phi \in \mathcal{S}} \alpha(\Phi)[\Phi] \right) = \sum_{\Phi \in \mathcal{S}} \alpha(\Phi)[U\Phi],$$

is an isometry with respect to the Hermitian forms $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{(\mathcal{S})}$ and $\mathbb{C}^{(U\mathcal{S})}$.
Consequently, it extends to a unitary map $\mathcal{L}_U : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{U\mathcal{S}}$ between infinite wedge spaces, characterized by $\mathcal{L}_U(\bigwedge \Phi) = \bigwedge(U\Phi)$.

This extends immediately to unitary maps between different Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$.

Construction 5.2.7 (The Right Operation)**.**

Let $\mathrm{Gl}_-(\ell) := \{R \in \mathrm{Gl}(\ell) \mid R^*R \in Id + I_1(\ell)\}$.
$\mathrm{Gl}_-(\ell)$ acts on $\mathtt{Seas}(\mathcal{H})$ from the right by $(\Phi, R) \to \Phi R$.
This extends to a well-defined map

$$\mathcal{S} \xrightarrow{R} \mathcal{S}R := \{\Phi R \mid \Phi \in \mathcal{S}\}$$

between Dirac Sea classes.

For $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\!\sim$ and $R \in \mathrm{GL}_-(\ell)$ we have an induced operation from the right $\mathcal{R}_R : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(\mathcal{S}R)}$ given by

$$\mathcal{R}_R \left(\sum_{\Phi \in \mathcal{S}} \alpha(\Phi)[\Phi] \right) = \sum_{\Phi \in \mathcal{S}} \alpha(\Phi)[\Phi R],$$

W.r.to the Hermitian forms, this is an isometry up to scaling, since

$$\det\big((\Phi R)^*(\Psi R)\big) = \det(R^*\Phi\Psi R) = \det(R^*R)\det(\Phi^*\Psi)$$

$\forall R \in Gl_-(\ell)$ and $\Phi, \Psi \in \mathcal{S}$ and therefore, for all $\alpha, \beta \in \mathbb{C}^{(\mathcal{S})}$:

$$\langle \mathcal{R}_R\alpha, \mathcal{R}_R\beta \rangle = \det(R^*R)\,\langle \alpha, \beta \rangle\,.$$

In particular one has $\mathcal{R}_R[N_{\mathcal{S}}] \subseteq N_{\mathcal{S}R}$.

It follows that for every $R \in \mathrm{Gl}_-(\ell)$, the operation $\mathcal{R}_R : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(\mathcal{S}R)}$ induces a bounded linear map $\mathcal{R}_R : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{\mathcal{S}R}$ between the infinite wedge spaces, characterized by $\mathcal{R}_R(\Lambda\Phi) = \Lambda(\Phi R)$ for $\Phi \in \mathcal{S}$. This map is unitary, up to scaling.

The definition extends immediately to maps between different index spaces ℓ and ℓ'.

We can think of the right action as basis transformations on the Hilbert space or, with a somewhat greater leap of imagination, as "rotations" of the Dirac Seas. The Dirac Seas will stay in the same Dirac Sea class if and only if the transformations are "small" in the sense that they vary from the identity only by a trace-class operator or, in other words, have a determinant. Pictorially speaking, we "rotate" a Dirac sea just a little if we don't stir too much deep down in the sea. This is expressed in the following Lemma:

Lemma 5.2.8 (Uniqueness up to a Phase)**.**
Let $R \in \mathrm{Gl}_-(\ell)$ and $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\!\sim$ a Dirac Sea class. Then, $\mathcal{S}R = \mathcal{S}$ if and only if R has a determinant, i.e. iff $R \in \mathrm{GL}^1(\ell)$.
In this case, the right operation $\mathcal{R}_R : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{\mathcal{S}}$ corresponds to multiplication with $\det(R)$ on $\mathcal{F}_{\mathcal{S}}$. In particular, if $R \in \mathrm{U}(\ell) \cap \mathrm{GL}^1(\ell)$, the right operation $\mathcal{R}_R$ corresponds to multiplication by a complex phase $\in \mathrm{U}(1)$.

Since the right-action is a well-defined map between Dirac Sea classes, it follows immediately for another operator $Q \in \mathrm{GL}_-(\ell)$ that $\mathcal{S}R = \mathcal{S}Q$ if and only if $Q^{-1}R$ has a determinant. In this case, $\mathcal{R}_R = \det(Q^{-1}R)\mathcal{R}_Q$ on $\mathcal{F}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$. For any $\Phi \in \mathcal{S}$, $\Phi^*(\Phi R)$ has a determinant if and only if R has a determinant, since $\Phi^*\Phi \in Id + I_1(\ell)$ by definition. Thus:

$$\Phi \sim \Phi R \iff \mathcal{S} = \mathcal{S}R \iff R \in \mathrm{Gl}^1(\ell)$$

Now, on $\mathbb{C}^{(\mathcal{S})}$ with the semi-norm defined by $\langle \cdot, \cdot \rangle$:

$$\begin{aligned} \|[\Phi R] - (\det R)[\Phi]\|^2 = &\det((\Phi R)^*(\Phi R)) - (\det R)\det((\Phi R)^*\Phi) \\ &- \overline{\det R}\det(\Phi^*\Phi R) + |\det R|^2 \det(\Phi^*\Phi) \\ &= 2|\det R|^2 \det(\Phi^*\Phi) - 2|\det R|^2 \det(\Phi^*\Phi) = 0. \end{aligned} \tag{5.2.3}$$

Therefore, $\mathcal{R}_R = \det(R) \cdot \mathrm{Id}$ on $\mathcal{F}_\mathcal{S}$.

□

Recall, that the construction of the Fock space as an infinite wedge space involves two consecutive choices. The first one (usually determined by the physics) is the choice of a polarization class $C \in \mathtt{Pol}(\mathcal{H})$. Afterwards, we have a more or less arbitrary choice of a polarization class $\mathcal{S} \in \mathtt{Ocean}(C)/\sim$. As this is an equivalence class, it is uniquely determined by any "reference polarization" $\Phi \in \mathtt{Ocean}(C)$. This duality is reflected in the duality of left- and right- operations. The operations from the left are transformations between polarization classes. The unitary transformation induced by $U \in \mathrm{U}(\mathcal{H})$ stays in the same polarization class if and only if it satisfies the Shale-Stinespring condition (1.2.11).
It preserves the charge, if and only if the $++-$component w.r.t the corresponding polarization has index 0.

The operations from the right, on the other hand, are transformations between different Dirac Sea classes within the same ocean i.e. between Dirac seas with image in the same polarization class. We see how the mathematical structure at hand gives us a very natural way to handle unitary transformations.

The left-action alone would correspond to the product-wise lift

$$\Phi(e_0) \wedge \Phi(e_1) \wedge \Phi(e_2) \wedge \cdots \xrightarrow{\mathcal{L}_U} U\Phi(e_0) \wedge U\Phi(e_1) \wedge U\Phi(e_2) \wedge \cdots$$

But this alone is not very helpful, in general. In addition, we need a suitable right-operation $\mathcal{R}_R$ to rotate the seas back into the desired Dirac sea class. If U preserves an equal-charge polarization class $C \in \mathtt{Pol}(\mathcal{H})/\approx_0$, i.e. $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C)$ a right-operation can be chosen in such a way as to make $\mathcal{R}_R\mathcal{L}_U$ a unitary transformation on the Fock space $\mathcal{F}_\mathcal{S}$ for a fixed $\mathcal{S} \in \mathtt{Ocean}(C)/\sim$. If, however, U maps one polarization class C into a different one, i.e. $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C')$ with $C' \neq C$, the best we can do is to implement U as a unitary transformation between *different* Fock spaces $\mathcal{F}_\mathcal{S}$ and $\mathcal{F}_{\mathcal{S}'}$ for $\mathcal{S} \in \mathtt{Ocean}(C)/\sim$ and $\mathcal{S}' \in \mathtt{Ocean}(C')/\sim$. The right-operation then has to be such that $U\mathcal{S}R = \mathcal{S}'$.

The previous Lemma tells us that the implementations are uniquely determined up to a complex phase only. So, the group that acts on the Fock space contains an additional U(1)-freedom besides the information contained in $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C) = \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$. Indeed, it corresponds to the central extension $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ by U(1) studied in chapter 4. [12]

All these considerations lie at the heart of the following, crucial theorem.

[12] cf. [Cor.5.4.3]

Theorem 5.2.9 (Lift of unitary transformations, [DeDuMeScho] Thm. II.26)**.**
For given polarization classes $C, C' \in \mathtt{Pol}(\mathcal{H})/\!\approx_0$ let $\mathcal{S} \in \mathtt{Ocean}(C)/\!\sim$ and $\mathcal{S}' \in \mathtt{Ocean}(C')/\!\sim$. Then, for any unitary map $U : \mathcal{H} \to \mathcal{H}$, the following are equivalent:

i) $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C')$.

ii) There is $R \in \mathrm{U}(\ell)$ such that $U\mathcal{S}R = \mathcal{S}'$, and hence $\mathcal{R}_R\mathcal{L}_U$ maps $\mathcal{F}_{\mathcal{S}}$ to $\mathcal{F}_{\mathcal{S}'}$.
In this case, if $R' \in \mathrm{U}(\ell)$ is another map with $U\mathcal{S}R' = \mathcal{S}'$, then $\mathcal{R}'_R\mathcal{L}_U = \det(R^*R')\,\mathcal{R}_R\mathcal{L}_U$ with $\det(R^*R') \in \mathrm{U}(1)$.

This theorem is a generalization of the Shale-Stinespring theorem in the language of the infinite wedge spaces. Setting $C = C' = [\mathcal{H}_+]$ it reproduces the well-known result that a unitary operator $U \in \mathrm{U}(\mathcal{H})$ is implementable on a fixed Fock space $\mathcal{F}$ over the polarization class $[\mathcal{H}_+]$ if and only if $U \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

In any case, it follows from Lemma 5.2.8 that the implementation of a unitary operator (on a fixed Fock space or as a map between different Fock spaces) is unique up to a complex phase. So again, we have encountered the infamous geometric phase of QED.

Corollary 5.2.10 ($\mathrm{U}(\ell)$ acts transitively on oceans)**.**
Setting $U = \mathbb{1}_{\mathcal{H}}$ in the last theorem it follows immediately that

$$\mathtt{Ocean}(C)/\!\sim \;= \{\mathcal{S}R \mid R \in \mathrm{U}(\ell)\}.$$

for any given $C \in \mathtt{Pol}(\mathcal{H})/\!\approx_0$ and $\mathcal{S} \in \mathtt{Ocean}(C)/\!\sim$.

In other words: $\mathrm{U}(\ell)$ *acts transitively on* $\mathtt{Ocean}(C)/\!\sim$ *from the right.*

5.3 The Geometric Construction

In this section we are going to follow the construction of the fermionic Fock space as presented, for example, in [PreSe] or [Mi]. The Fock space will be constructed from holomorphic sections in the dual of the determinant line-bundle over the complex (restricted) Grassmannian manifold. We will refer to this as the *geometric construction* of the Fock space. It is very nice for several reasons

- It reveals and exploits a remarkably elegant geometric structure that appears naturally in our mathematical framework
- Polarizations, Dirac Seas and their basis transformations are embraced by the geometric description, were they appear as base-manifold, fibre bundle and structure group. This also provides a very nice picture for illustrating the relationships between them
- The central extensions $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ and $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ constructed in Ch. 4 act most naturally on this Fock space.

Note: Unfortunately, the mathematical convention does not follow the physical motivation. Instead, mathematicians prefer to write $\mathcal{H}_+$ instead of $\mathcal{H}_-$ for the polarizations which -intuitively- plays the role of the "Dirac sea" in the geometric construction of the Fock space. We will follow this convention in order to match the mathematical literature and hope that it doesn't cause too much confusion. Anyways, note that the convention used in this chapter (and most of the work) is in conflict with the one used in §5.1 .

We will follow the pertinant literature in constructing the Fock space for the standard polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. This will serve as the exemplar for the general case. The construction can be easily applied to arbitrary polarizations (and polarization classes), matching the generality of the infinite wedge-space construction.

We have already argued that polarizations, which are points on the Grassmanian manifold $\mathrm{Gr}(\mathcal{H})$, can be thought of as "projective states" of infinitely many Fermions. But of course, we are not satisfied with a projective description, but aiming for a Hilbert space structure. In particular, we want to generalize the scalar product

$$\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n \rangle = \det(\langle v_i, w_j \rangle)_{i,j}$$

of finite exterior products to infinite particle states. But while this expression is always well-defined and independent of the choice of (positively oriented) orthonormal bases, the infinite-dimensional limes will only exist if the Hilbert-bases of different polarizations are chosen in a sensible and compatible way. In the infinite wedge space construction this is done by identifying "Dirac sea classes" on which the Hermitian form is well-defined. In the geometric construction the related concept is that of *admissible bases*.

Definition 5.3.1 (Admissible basis)**.**
Let $W \in \mathrm{Gr}^0(\mathcal{H})$. An *admissible basis* for W is a bounded isomorphism $w : \mathcal{H}_+ \to W$ with the property that $P_+ \circ w$ has a determinant.

It might be helpful to note:

- $W = \mathrm{im}(w) \in \mathrm{Gr}(\mathcal{H})$ iff $P_+ w$ is Fredholm and $P_- w$ Hilbert-Schmidt
- $W = \mathrm{im}(w) \in \mathrm{Gr}^0(\mathcal{H})$ iff in addition $\mathrm{ind}(P_+ w) = 0$
- w is admissible basis of $W \in \mathrm{Gr}^0(\mathcal{H})$ iff in addition $P_+ w \in Id + I_1(\mathcal{H}_+)$

Lemma 5.3.2 (Existence of admissible bases)**.**

Every $W \in \mathrm{Gr}^0(\mathcal{H})$ has an admissible basis.

Proof. Let $W \in \mathrm{Gr}^0(\mathcal{H})$. Set $\tilde{w} := P_W|_{\mathcal{H}_+} \colon \mathcal{H}_+ \to W$.
$P_+\tilde{w}$ has a determinant, because $\mathbb{1}_{\mathcal{H}_+} - P_+\tilde{w} = \big(P_+ - P_+P_WP_+\big)\big|_{\mathcal{H}_+}$ and
$P_+ - P_+P_WP_+ = P_+P_{W^\perp}P_+ = (P_{W^\perp}P_+)^*(P_{W^\perp}P_+) \in I_1(\mathcal{H})$ (cf. Lemma 2.1.3). We also know that $\tilde{w}$ is a Fredholm operator of index 0 (bc. $\mathrm{charge}(W, \mathcal{H}_+) = \mathrm{ind}(P_W|_{\mathcal{H}_+}) = 0$) which means that the kernel of $\tilde{w}$ and its cokernel in W are of the same, finite dimension. Therefore, we can define $w : \mathcal{H}_+ \to W$ by setting $w = \tilde{w}$ on $\ker(\tilde{w})^\perp$ and extending it to the whole $\mathcal{H}_+$ in such a way that it maps $\ker(\tilde{w})$ to a complement of $\mathrm{im}(\tilde{w})$ in W. Thus, by construction, $\mathrm{im}(w) = W$ and $P_+ w$ has a determinant as it differs from $P_+\tilde{w}$ by a finite-rank operator only. Using a polar decomposition, we could even make w unitary.

□

Lemma 5.3.3 (Relationship between admissible bases)**.**
w' and w are two admissible bases for $W \in \mathrm{Gr}^0(\mathcal{H})$ if and only if $w' = w \circ L$ for an $L \in \mathrm{GL}^1(\mathcal{H}_+)$.

Proof. If $w' = w \circ L$ then w' is an isomorphism from H_+ to W and $P_+w' = (P_+w)\,L$ has a determinant because P_+w and L do. Therefore, w' is an admissible basis for W.
Conversely, if w', w are two admissible bases for W set $L := w^{-1} \circ w' : \mathcal{H}_+ \to \mathcal{H}_+$.
Clearly L is invertible and $P_+wL = P_+w' \in Id + I_1(\mathcal{H}_+)$ which, because of
$P_+w \in Id + I_1(\mathcal{H}_+)$, implies $L \in Id + I_1(\mathcal{H}_+)$. □

Arbitrary charges

As we already know, the Grassmanian $\mathrm{Gr}(\mathcal{H})$ has $\mathbb{Z}$ connected components corresponding to the relative charges of the polarizations w.r.to $\mathcal{H}_+$. Usually, we are just interested in the connected component $\mathrm{Gr}^0(\mathcal{H})$ which is that of the initial vacuum $\approx \mathcal{H}_+$. The geometric construction over $\mathrm{Gr}^0(\mathcal{H})$ will therefore lead to the zero-charge sector of the fermionic Fock space. Note that the infinite wedge space construction also gives "Fock spaces" of constant charge. We can easily extend the construction to a "full" Fock space by taking the direct (orthogonal) sum of infinite wedge-spaces over polarization classes with arbitrary relative charges. Such a construction is straight forward, yet not canonical as it involves a choice of a Dirac sea class for ever such charge sector. So it should come as no surprise that the same is true for the geometric construction as well. It inherits arbitrary charges very naturally, however the construction is not canonical. It involves a choice of a (polarized) basis.

For the rest of this section we fix an orthonormal basis $\{(e_k)_{k\in\mathbb{Z}}\}$ of $\mathcal{H}$ such that $(e_k)_{k\leq 0}$ is an ONB of $\mathcal{H}_-$ and $(e_k)_{k\geq 0}$ is an ONB of $\mathcal{H}_+$. For any $n \in \mathbb{Z}$ we write

$$\mathcal{H}_{\{\geq -n\}} := \mathrm{span}(\{e_k \,|\, k \geq -n\})$$

By $P_+^{(n)}$ we denote the orthogonal projection $P_{\mathcal{H}_{\{\geq -n\}}}$.
From the construction of the complete $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ in section 4.1.2 we recall the shift-operator σ defined by $\sigma(e_k) := e_{k+1}$. For any $d, n \in \mathbb{Z}$, σ^d maps $\mathcal{H}_{\{\geq -n\}}$ to $\mathcal{H}_{\{\geq -n+d\}}$.[13]

Definition 5.3.4 (Admissible basis for arbitrary charges)**.**
Let $W \in \mathrm{Gr}(\mathcal{H})$ with $\mathrm{charge}(W, \mathcal{H}_+) = n$, i.e. W lies in the connected component $\mathrm{Gr}^n(\mathcal{H})$ of $\mathrm{Gr}(\mathcal{H})$. An *admissible basis* for $W \in \mathrm{Gr}^n(\mathcal{H})$ is a bounded linear map $w : \mathcal{H}_{\{\geq -n\}} \to H$ with $\mathrm{im}(w) = W$ s.t. $w : \mathcal{H}_{\{\geq -n\}} \to W$ is an isomorphism and $P_+^{(n)} \circ w$ has a determinant.

[13] If we seek more generality we can choose for each $n \in \mathbb{Z}$ an arbitrary subspace $\mathcal{H}_{(n)} \in \mathrm{Gr}(\mathcal{H})$ with $\mathrm{charge}(\mathcal{H}_{(n)}, \mathcal{H}_+) = n$ and then a one-parameter group of unitary operators translating between them.

We denote the set of all admissible bases by St and the subset of admissible bases for $\mathrm{Gr}^n(\mathcal{H}), n \in \mathbb{Z}$ by $\mathrm{St}^{(n)}$. On $\mathrm{St}^{(n)}$ we let $\mathrm{GL}^1(\mathcal{H}_+)$ act from the right by

$$w \xmapsto{L} w \circ \sigma^{-n} \circ L^{-1} \circ \sigma^n$$

With a little abuse of notation we will just write $w \circ L^{-1}$ for the right action and P_+ instead of $P_+^{(n)}$ for the orthogonal projection whenever the specific index is clear or irrelevant. With this sneaky notation we can forget about the different charges for the most parts. The construction will look the same over any connected component.

Remark 5.3.5 (Stiefel manifold)**.**
The set St of all admissible bases for the polarizations in $\mathrm{Gr}(\mathcal{H})$ carries the structure of an infinite-dimensional manifold, called the (restricted) *Stiefel manifold* St. The topology is given by the metric $\mathrm{d}(w, w') = \|P_+(w - w')\|_1 + \|P_-(w - w')\|_2$. The Stiefel manifold has $\mathbb{Z}$-connected components corresponding to those of $\mathrm{Gr}(\mathcal{H})$. Actually, it follows from the previous discussion that St is naturally a principle $\mathrm{GL}^1(\mathcal{H}_+)$-bundle over $\mathrm{Gr}(\mathcal{H})$. The fibre over a basepoint $W \in \mathrm{Gr}(\mathcal{H})$ is just the set of admissible bases for W and $\mathrm{GL}^1(\mathcal{H}_+)$ acts on these fibers by right composition as above.

Definition 5.3.6 (Determinant Line Bundle)**.**
Consider the set $\mathrm{St} \times \mathbb{C}$ written as

$$\{(W, w, \lambda) \mid \lambda \in \mathbb{C}, W \in \mathrm{Gr}(\mathcal{H}),\ \text{w admissible basis for W}\,\}$$

Introduce the equivalence relation

$$[W, w, \lambda] \sim [W', w', \lambda] : \Longleftrightarrow\ W' = W,\ w' = w \circ L \text{ and } \lambda' = \det(L)^{-1}\lambda$$

i.e.

$$[W, w, \det(L)\lambda] \sim [W, w \circ L, \lambda],\ \text{for } L \in \mathrm{GL}^1(\mathcal{H}_+) \tag{5.3.1}$$

The set of equivalence classes $[W, w, \lambda]$ together with the projection π onto the first factor forms a holomorphic line bundle[4]

$$\big(\mathrm{St} \times \mathbb{C}\big)/\,\mathrm{GL}^1 =: \mathrm{DET} \xrightarrow{\pi} \mathrm{Gr}(\mathcal{H})$$

called the *determinant bundle* DET.
The connected component over $\mathrm{St}^{(n)}$ will be denoted by DET_n.

Further on we will often drop the first entry and write $[w, \lambda]$ for $[\mathrm{im}(w), w, \lambda]$.

We can get some intuition for this construction by thinking of the fibre over $W \in \mathrm{Gr}(\mathcal{H})$ in DET as consisting of the complex multiples of a formal expression

$$\pi^{-1}(W) \ni [W, w, 1] = [w, 1] \ \simeq\ w_0 \wedge w_1 \wedge w_2 \wedge w_3 \wedge \dots \tag{5.3.2}$$

where $\{w_j\}_{j \geq 0}$ is a basis of W.

In this sense, the Determinant bundle over $\mathrm{Gr}(\mathcal{H})$ contains precisely the information that we expect to be encoded in the fermionic Fock space. But so for we don't even have a linear structure, except of course for the $\mathbb{C}^1$-structure on every fibre. There's no meaningful way of "adding" points in different fibers of the bundle. Therefore, the idea is to consider *sections* of the Determinant line bundle which do naturally form an (infinite dimensional) complex vector space $\Gamma(\mathrm{DET})$. Which section would then correspond to a state of the form (5.3.2), though? We might possibly think of the section that picks out the point $[w, 1]$ over $W \in \mathrm{Gr}(\mathcal{H}$ and is zero everywhere else. But such a section is not even continuous and wouldn't do justice to the beautiful geometric structure we have so far.

We can however take sections in the *dual bundle* DET^* and identify $w \in \mathrm{DET}$ with $\xi_w \in \Gamma(\mathrm{DET}^*)$ defined by $\xi_w([z,\lambda]) := \lambda \det(w^* z)$. It turns out that such a section is not just continuous or smooth but even *holomorphic* if the right structure is introduced[14]This might not be essential to the physical discussion but it's a great treat for mathematicians.

Remark 5.3.7 (Holomorphic sections)**.**
We denote by DET^* the *dual bundle* of the Determinant bundle DET and by $\Gamma(\mathrm{DET}^*)$ the space of holomorphic sections in DET^*. A holomorphic section Ψ of DET^* is a holomorphic map $\mathrm{DET} \to \mathbb{C}$ which is linear in every fibre. This corresponds to a holomorphic map

$$\psi : \mathrm{St} \to \mathbb{C} \text{ with } \psi(z \circ L) = \det(L) \cdot \psi(z),\ \forall L \in \mathrm{GL}^1(\mathcal{H}_+) \tag{5.3.3}$$

Because then and only then is $\Psi([z,\lambda]) := \lambda \cdot \psi(z)$ a well defined map $\mathrm{DET} \to \mathbb{C}$, as $\Psi([z \circ L, \lambda/\det(L)]) = \frac{\lambda}{\det(L)}\, \psi(z \circ L) = \lambda\, \psi(z) = \Psi([z,\lambda])$.

Similarly, a holomorphic section Φ of DET would correspond to a holomorphic map

$$\phi : \mathrm{St} \to \mathbb{C} \text{ with } \phi(z \circ L) = \det(L)^{-1}\, \phi(z)\,,\ \forall L \in \mathrm{GL}^1(\mathcal{H}_+) \tag{5.3.4}$$

Due to the factor of $\det(L)^{-1}$ on the right-hand-side of (5.3.4), points arbitrarily close in St can be mapped to points arbitrarily far in $\mathbb{C}$ which contradicts the existence of a bounded derivative. Hence, there are *no* holomorphic sections of DET, except the zero section.

Construction 5.3.8 (Embedding of DET in $\Gamma(\mathrm{DET}^*)$)**.**
Consider the map $\Phi : \mathrm{St} \times \mathrm{St} \to \mathbb{C}$,

$$\Phi(z,w) \;=\; \begin{cases} \det(z^* w) & ;\text{ for } \operatorname{ind}(P_+ z) = \operatorname{ind}(P_+ w) \\ \\ 0 & ;\text{ else} \end{cases}$$

This is well defined since for $\operatorname{ind}(P_+ z) = \operatorname{ind}(P_+ w) = n$:

$$z^* w = z^* P_+^{(n)} w + z^* P_-^{(n)} w =$$

$$\underbrace{(P_+^{(n)} z)^*}_{\in Id + I_1(\mathcal{H}_{\geq -n})} \quad \underbrace{(P_+^{(n)} w)}_{\in Id + I_1(\mathcal{H}_{\geq -n})} \;+\; \underbrace{(P_-^{(n)} z)^*}_{\in I_2(\mathcal{H}_{<-n}, \mathcal{H}_{\geq -n})} \quad \underbrace{(P_-^{(n)} w)}_{\in I_2(\mathcal{H}_{\geq -n}, \mathcal{H}_{<-n})} \quad \in Id + I_1(\mathcal{H}_{\{\geq -n\}})$$

Now for any fixed $z \in \mathrm{St}$, the map

$$w \longmapsto \Phi(z,w) =: \xi_z(w)$$

is holomorphic with $\Phi(z, w \circ L) = \det(z^* w L) = \det(z^* w)\det(L)$ for $L \in \mathrm{GL}^1(\mathcal{H}_+)$ and hence descends to a holomorphic section of DET^* which we denote by the same symbol $\xi_z \in \Gamma(\mathrm{DET}^*)$. This defines an *anti-linear* map $\xi : \mathrm{DET} \to \Gamma(\mathrm{DET}^*)$ by

$$\xi([z,\lambda]) := \bar{\lambda} \xi_z = \bar{\lambda} \Phi(z, \cdot) \tag{5.3.5}$$

Note that

$$\xi_{z \circ L} = \overline{\det(L)}\, \xi_z,\ \forall L \in \mathrm{GL}^1(\mathcal{H}_+) \tag{5.3.6}$$

thus ξ has to be anti-linear, as $[z \circ L, \lambda] = [z, \det(L)\, \lambda] \in \mathrm{DET}$.

Deleting the zero-section from DET, we get an injection $\xi : \mathrm{DET}^\times \to \Gamma(\mathrm{DET}^*)$.

This construction contains almost everything we need. We will construct the Fock space from the space of holomorphic sections of DET^* with the Hermitian scalar product given by the determinant. But first we need good coordinates to handle this.

[14] cf. [PreSe]

Definition 5.3.9 (Pflücker coordinates)**.**
Let $\{(e_j)_{j\in\mathbb{Z}}\}$ be the ONB of $\mathcal{H}$ chosen above.

i) We denote by $\mathcal{S}$ the set of increasing sequences $S = (i_0, i_1, i_2.i_3, \dots)$ s.t. $i_{k+1} = i_k + 1$ for large enough k, i.e. the sequences $S \in \mathcal{S}$ contain only finitely many negative indices and all but finitely many positive indices.

ii) For $S \in \mathcal{S}$ we define the charge c(S) to be the unique number $c \in \mathbb{Z}$ with $i_k = k - c$ for all k large enough.

iii) For $S = (i_0, i_1, i_2.i_3, \dots) \in \mathcal{S}$ with $c(S) = n$ we define

$$H_S := \mathrm{span}\big(\{e_{i_k} \mid i_k \in S\}\big) = \mathrm{span}(e_{i_0}, e_{i_1}, e_{i_2}, \dots)$$

and

$$w_S : \mathcal{H}_{\{\geq -n\}} \to \mathcal{H}\,,\ w_S(e_k) := e_{i_{(k+n)}}$$

This is an admissible basis for H_S since it differs from Id by a finite rank matrix only. We denote by Ψ_S the section $\xi_{w_S} \in \Gamma(\mathrm{DET}^*)$.

Examples 5.3.10 (Pflücker coordinates)**.**

- $S_0 = \mathbb{N} = (0, 1, 2, 3, \dots) \Rightarrow c(S_0) = 0$.
  We call the corresponding state
  $$\xi_{w_{S_0}} =: \Psi_0 \approx e_0 \wedge e_1 \wedge e_2 \wedge e_3 \dots$$
  the *vacuum state*.
- $S_1 = (-2, -1, 0, 1, 2, \dots) \Rightarrow c(S_1) = 2$.
  This corresponds to the state $e_{-2} \wedge e_{-1} \wedge e_0 \wedge e_1 \wedge \dots$ where 2 negative energy states (e_{-2} and e_{-1}) are occupied and all positive energy states $e_{j\geq 0}$ are occupied.
- $S_2 = (-5, -1, 1, 3, 4, 5, \dots) \Rightarrow c(S_2) = 0$.
  This corresponds to the state $e_{-5} \wedge e_{-1} \wedge e_1 \wedge e_3 \wedge e_4 \dots$ with two negative energy states occupied (e_{-5} and e_{-1}) and two "holes" in the positive spectrum (e_0 and e_2). Therefore, the net-charge of the state is zero.

The role of the basis $\{(e_j)_{j\in\mathbb{Z}}\}$ and the sequences defined above is pretty intuitive. We just choose a complete set of one-particle states that will characterize the particle-content of the Fock-space states ($\approx$ Dirac seas). Ideally, this choice is determined or at least motivated by the physics e.g. as the spectral decomposition of a (or several commuting) self-adjoint operator(s) on $\mathcal{H}$. Anyways, the sections $(\Psi_S)_{S\in\mathcal{S}}$ will make a good basis for the Fock space.

Proposition 5.3.11 (Hermitian Form)**.**
Let $V \subseteq \Gamma(\mathrm{DET}^*)$ be the complex vector space spanned by the sections $\{\xi_z \in \Gamma(\mathrm{DET}^*) \mid z = [(z, 1)] \in \mathrm{DET}\}$. Then

$$\langle \xi_z, \xi_w \rangle := \xi_z(w) := \Phi(z, w) \tag{5.3.7}$$

defines a Hermitian form on V, antilinear in the second *component.*
This form is positive semi-definite.
The sections Ψ_S, $S \in \mathcal{S}$ form a complete orthonormal set in V w.r.to $\langle \cdot, \cdot \rangle$.

Proof. First we note that $\langle \xi_z, \xi_w \rangle = \Phi(z, w) = \det(z^* w) = \overline{\det(w^* z)} = \overline{\langle \xi_w, \xi_z \rangle}$.
Then we recall that $\Phi(\cdot, \cdot)$ is $\mathbb{C}$-linear in the second entry and anti-linear in the first entry, but the mapping $w \mapsto \xi_w$ is $\mathbb{C}$-anti-linear. In conclusion, $\langle \cdot, \cdot \rangle$ defines a sesquilinear form anti-linear in the second entry.

It is easy to see that for $S, S' \in \mathcal{S}$

$$\langle \Psi_S, \Psi'_S \rangle = \Psi_S(w'_S) = \det(w_S^* w'_S) = \delta_{SS'}$$

because $w_S^* w_S = \mathbb{1}_{\mathcal{H}_{\{\geq -n\}}}$, whereas $w_S^* w'_S$ has non-trivial kernel if $S \neq S'$.

Now, in finite-dimensions if A is a $n \times m$ matrix and B a $m \times n$ matrix with $n \leq m$ then

$$\det(AB) = \sum_{(i)=(1\leq i_1\leq \ldots \leq i_n \leq m)} \det(AP_{(i)}) \det(P_{(i)}B)$$

where $P_{(i)}$ denotes the projection onto $\mathrm{span}(e_{i_1}, \ldots e_{i_n})$. As the Fredholm determinant of an operator is the limes of the determinants of its restriction to n-dimensional subspaces as $n \to \infty$, the above formula extends to the infinite-dimensional case yielding

$$\begin{aligned}\langle \xi_z, \xi_w \rangle = \Phi(z,w) = \det(z^* w) &= \sum_{S\in\mathcal{S},\, c(S)=d} \det((P_{\mathcal{H}_S} z)^*) \det(P_{\mathcal{H}_S} w) \\ &= \sum_S \det((w_S^* z)^*) \det(w_S^* w) = \sum_S \overline{\det((w_S^* z))} \det(w_S^* w) \\ &= \sum_{S\in\mathcal{S}} \overline{\Psi_S(z)} \Psi_S(w) = \sum_{S\in\mathcal{S}} \langle \xi_z, \Psi_S\rangle \langle \Psi_S, \xi_w\rangle\end{aligned}$$

for $\mathrm{ind}(P_+ z) = \mathrm{ind}(P_+ w) = d$ and 0 else.
This shows that $\{\Psi\}_{S\in\mathcal{S}}$ is indeed a complete, orthonormal set in V.

Finally, for general $\Psi = \sum_{\text{finite}} \alpha_n \xi_{z_n} \in V$ we compute

$$\begin{aligned}\langle \Psi, \Psi\rangle &= \sum_{n,m} \alpha_n \overline{\alpha_m} \langle \xi_{z_n}, \xi_{z_m}\rangle = \sum_{n,m} \sum_{S\in\mathcal{S}} \alpha_n \overline{\alpha_m} \langle \xi_{z_n}, \Psi_S\rangle \langle \Psi_S, \xi_{z_m}\rangle \\ &= \sum_{S\in\mathcal{S}} \Big(\sum_n \alpha_n \langle \xi_{z_n}, \Psi_S\rangle \Big) \Big(\sum_m \overline{\alpha_m} \langle \Psi_S, \xi_{z_m}\rangle \Big) \\ &= \sum_{S\in\mathcal{S}} \Big| \sum_n \alpha_n \langle \xi_{z_n}, \Psi_S\rangle \Big|^2 \geq 0\end{aligned}$$

This finishes the proof.

□

Definition 5.3.12 (Fermionic Fock space)**.**
Let $V_0 := \{v \in V \mid \langle v, v\rangle = 0\}$ be the null-space of $(V, \langle\cdot,\cdot\rangle)$.

We define the *fermionic Fock space* $\mathcal{F}$ to be the completion of V/V_0 w.r.to $\langle\cdot,\cdot\rangle$.
$\mathcal{F}$ is an infinite-dimensional, complex, separable Hilbert space.

It can be written as the direct sum $\mathcal{F} = \bigoplus_{c\in\mathbb{Z}} \mathcal{F}^{(c)}$ of $\mathbb{Z}$ "charge-sectors" built from holomorphic sections with support in the connected component of DET^* over $\mathrm{Gr}^{(c)}(\mathcal{H})$.

The sections Ψ_S, $S \in \mathcal{S}$ are a Fock basis of $\mathcal{F}$. They define the *Pflücker coordinates*

$$\begin{aligned}&\mathcal{F} \longrightarrow \ell^2, \\ &\xi_z \longmapsto (\xi_z(w_S))_{S\in\mathcal{S}} = (\langle \xi_z, \Psi_S\rangle)_{S\in\mathcal{S}}\end{aligned} \tag{5.3.8}$$

Definition 5.3.13 (Pflücker embedding)**.**
We have a natural embedding of $\mathrm{Gr}(\mathcal{H})$ into the projective Fock space $\mathbb{P}(\mathcal{F})$, given by

$$\begin{aligned}&\mathrm{Gr}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{F}), \\ &W \longmapsto \mathbb{C} \cdot \xi_w = \mathbb{C} \cdot \xi([w,1])\end{aligned} \tag{5.3.9}$$

where w is an admissible basis for W, called the *Pflücker embedding.*
This gives precise meaning to our often employed intuition that polarizations (in $[\mathcal{H}_+]$) correspond to projective, decomposable states of infinitely many fermions.

Proposition 5.3.14 (Action of $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ on DET_0)**.**
$\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ *has a natural action on* DET *defined by*

$$\begin{aligned}\mu_0 : \widetilde{\mathrm{GL}}^0_{\mathrm{res}} \times \mathrm{DET}_0 &\longrightarrow \mathrm{DET}_0,\\ ([U,R],[W,w,\lambda]) &\longmapsto [UW, UwR^{-1}, \lambda]\end{aligned} \tag{5.3.10}$$

Proof. We have to show that the action μ_0 is well-defined.

i) DET_0 *is closed under the action of* $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$:
We know that $W \in \mathrm{Gr}(\mathcal{H}) \Rightarrow UW \in \mathrm{Gr}(\mathcal{H})$ for $U \in \mathrm{GL}_{\mathrm{res}}$. We still have to show:
w admissible basis for $W \Rightarrow UwR^{-1}$ admissible basis for UW.
Obviously, $\mathrm{im}(UwR^{-1}) = UW$. Furthermore:

$$\begin{aligned}P_+UwR^{-1} &= P_+UP_+wR^{-1} + P_+UP_-wR^{-1}\\ &= P_+UP_+P_+wR^{-1} + P_+UP_-P_-wR^{-1}\\ &= \underbrace{P_+UP_+R^{-1}}_{\in Id+I_1(\mathcal{H}_+)}\ \underbrace{RP_+wR^{-1}}_{\in Id+I_1(\mathcal{H}_+)} + \underbrace{P_+UP_-}_{\in I_2(\mathcal{H}_+)}\ \underbrace{P_-wR^{-1}}_{\in I_2(\mathcal{H}_+)}\\ &\in Id + I_1(\mathcal{H}_+)\end{aligned}$$

since the product of two Hilbert-Schmidt operators is trace-class.
Thus, UwR^{-1} is an admissible basis for UW.

ii) *The definition is independent of the representative of* $[W,w,\lambda]$:
Let $(W,w,\lambda) \sim (W,w',\lambda') = (W, w\circ L, \lambda/\det(L))$.
We need $(UW, UwR^{-1},\lambda) \sim (UW, UwLR^{-1}, \lambda/\det(L))$, for U and R as above.
This is true, since $UwLR^{-1} = UwR^{-1}(RLR^{-1}) = UwR^{-1}L'$, with
$L' = RLR^{-1} \in \mathrm{GL}^1(\mathcal{H}_+)$ and $\det(L') = \det(RLR^{-1}) = \det(L)$.

iii) *The definition is independent of the representative of* $[(U,R)] \in \widetilde{\mathrm{GL}}^0_{\mathrm{res}}$:
Let $[(U,R)] = [(U,R')] \in \widetilde{\mathrm{GL}}^0_{\mathrm{res}}$. This means $\det(RR'^{-1}) = 1$. Therefore
$(UW, UwR^{-1},\lambda) \sim (UW, UwR'^{-1},\lambda)$, since $UwR'^{-1} = UwR^{-1}RR'^{-1} =: UwR^{-1}L$,
with $\det(L) = \det(RR'^{-1}) = 1$.

□

Construction 5.3.15 (Arbitrary charges and the complete $\widetilde{\mathrm{GL}}_{\mathrm{res}}$)**.**
So far we have defined the action of $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ on DET_0, which is in fact all we need. Extending the action to arbitrary "charges" is somehow tedious but pretty much straightforward, nevertheless. Again, we use the "shift"-operator σ and the structure of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ as a semi-direct product $\mathbb{Z} \ltimes \widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ with $\mathbb{Z}$ generated by the action of $\tilde{\sigma}$ (see §4.1.2).

We define a $\mathbb{Z}-$action on DET by

$$\vartheta(n)([W,w,\lambda]) := [\sigma^n(W), \sigma^n w \sigma^{-n}, \lambda] \tag{5.3.11}$$

$\vartheta(n)$ maps DET_d to DET_{d-n}.

Now we extend the action μ_0 defined above to an action
$\mu : \mathbb{Z} \ltimes \widetilde{\mathrm{GL}}^0_{\mathrm{res}} \times \mathrm{DET} \to \mathrm{DET}$ by

$$\mu\{(n,[A,R])\}\big|_{\mathrm{DET}_d \to \mathrm{DET}_{d-n}} := \ \vartheta(n-d) \circ \mu_0(\tilde{\sigma}^{d-n}([A,R])) \circ \vartheta(d) \tag{5.3.12}$$

for any $d \in \mathbb{Z}$.
In particular, $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ acts on DET_d by

$$\mu\{[A,R]\} := \vartheta(-d) \circ \mu_0(\tilde{\sigma}^d[A,R]) \circ \vartheta(d) \tag{5.3.13}$$

This is almost the same as the action of $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ on $\mathrm{St}^{(d)}$ descending to DET only with A acting from the left and $(R_{\sigma^d})^{-1}$ (instead of R^{-1}) acting from the right.

Admittedly, this is not very pretty. Therefore, we will be so bold to drop the $\mathbb{Z}-$indices for the remainder of this chapter and use the simple notation for $\widetilde{\mathrm{GL}}^0_{\mathrm{res}}$ acting on the "zero-charge sector", while still stating the results for the whole transformation group and the whole line bundle. If needed, the full expressions for arbitrary charges can be worked out in detail using the scheme we've just outlined.

Finally, we can say how the central extension $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ (and $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$) defined in chapter 4 act on the fermionic Fock space and everything fits together nicely.

Proposition 5.3.16 (Action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on $\mathcal{F}$)**.**
The action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ on $\mathcal{F}$ is given by the adjoint action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on DET *as defined in Prop. 5.3.14. This action restricts to a unitary representation of $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ on the Fock space. Explicitely, the action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on $\mathcal{F}$ is defined by*

$$\begin{aligned} \mu^* : \widetilde{\mathrm{GL}}_{\mathrm{res}} \times \mathcal{F} &\longrightarrow \mathcal{F}, \\ ([U,R],\xi_z) &\longmapsto \mu^*_{[U,R]}\xi_z = \xi_z \circ \mu^{-1}_{[U,R]} \end{aligned} \tag{5.3.14}$$

i.e. for $w \in$ DET:

$$\begin{aligned} \mu^*_{[U,R]}\xi_z(w) &= \xi_z(\mu^{-1}_{[U,R]}(w)) = \xi_z(U^{-1}wR) \\ &= \det(z^*U^{-1}wR) = \det(Rz^*U^{-1}wRR^{-1}) \\ &= \det((U^{-1*}zR^*)^*w) =: \xi_{z'}(w) \end{aligned}$$

with $z' = (U^*)^{-1}zR^*$.
Thus, for $[(U,R)] \in \widetilde{\mathrm{U}}_{\mathrm{res}} \subset \widetilde{\mathrm{GL}}_{\mathrm{res}}$, i.e. $U \in \mathrm{U}(\mathcal{H}), R \in \mathrm{U}(\mathcal{H}_+)$,

$$\xi_z \mapsto \mu^*_{[U,R]}(\xi_z) = \xi_{z'}, \text{ with } z' = UzR^* \tag{5.3.15}$$

Clearly, this defines a representation of $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ which is indeed unitary, because

$$\langle \mu^*_{[U,R]}\xi_z, \mu^*_{[U,R]}\xi_w \rangle = \det\big((UzR^*)^*(UwR^*)\big) = \det(Rz^*wR^{-1}) = \det(z^*w) = \langle \xi_z, \xi_w \rangle.$$

5.4 Equivalence of Fock space constructions

We have already pointed to the fact that the infinite wedge space construction of Deckert, Dürr, Merkl and Schottenloher is closely related to the geometric construction of the Fock space as the space of holomorphic section in the dual of the determinant bundle. Deckert et.al. are just ignoring the underlying geometry and focus on the mathematical structure that seems more essential to the physical problems. In this section we prove that the two constructions lead to equivalent descriptions of the fermionic Fock space. We will focus on the polarization class of $[\mathcal{H}_+]$, i.e. on $\mathrm{Gr}(\mathcal{H})$ but the results can be easily generalized to any other polarization class.

It has probably become apparent that the "admissible bases" are for the geometric construction what the Dirac Seas are for the infinite wedge spaces. It might be interesting to note an important difference, though. In the construction of the infinite wedge space we start with a given polarization class and then have the freedom of choice of a Dirac Sea class. In the geometric construction we have to fix a polarization in the polarization class (e.g. $\mathcal{H}_+$). With this selection then comes a preferred choice for what corresponds to the Dirac sea class induced by the identity map on that subspace.
If we focus on the standard Fock space, where with $\mathcal{H}_+$ we have a natural choice for the polarization, too, this freedom that is so manifest in the infinite wedge space construction is rather hidden in the geometric construction. Nevertheless, it's still there.

We formulate the precise relationship between Dirac Sea classes and admissible bases in the following Lemma:

Lemma 5.4.1 (Dirac Seas vs. Admissible Bases)**.**
Let $(e_k)_{k\in\mathbb{Z}}$ be a polarized basis of $\mathcal{H}$ as above.

Any isometric Dirac sea $\Phi_0 \in \mathtt{Seas}^\perp(\mathcal{H})$ with $\mathrm{im}(\Phi_0) = \mathcal{H}_{\{\geq -n\}}$ gives rise to an isomorphism

$$\left(\mathtt{Seas}^\perp(\mathcal{H})/\sim\right) \ni S(\Phi_0) \xrightarrow{\cong} \mathrm{St}^{(n)}$$

between its Dirac Sea class and the n-th connected component of the Stiefel manifold.

Proof. Given a $\Phi_0 : \ell \to \mathcal{H}_{\{\geq -n\}} \subset \mathcal{H}$ as above, consider the map

$$\mathrm{St}^{(n)} \to S(\Phi_0);\ w \mapsto w \circ \Phi_0 \tag{5.4.1}$$

This is a well-defined map into $S(\Phi_0)$: For an admissible basis w, $w \circ \Phi_0$ is a map from ℓ to $\mathcal{H}$ and indeed $w \circ \Phi_0 \sim \Phi_0$, because

$$\begin{aligned} P_+ w = \Phi_0\Phi_0^* w \text{ has a determinant} \\ \Rightarrow\ \Phi_0^*\, w\, \Phi_0 : \ell \to \ell \text{ has a determinant} \\ \Rightarrow\ w \circ \Phi_0 \sim \Phi_0 \text{ in } \mathtt{Seas}^\perp(\mathcal{H}) \end{aligned}$$

It remains to show, that (5.4.1) is bijective.
Let $\Phi \sim \Phi_0 \in \mathtt{Seas}(\mathcal{H})$. Define $w := \Phi \circ \Phi_0^*\big|_{\mathcal{H}_{\{\geq -n\}}}$.
w is an isomorphism $\mathcal{H}_{\{\geq -n\}} \to \mathcal{H}$ with $\mathrm{im}(w) = \mathrm{im}(\Phi) \in \mathrm{Gr}(\mathcal{H})$ and clearly $w \circ \Phi_0 = \Phi$.
Finally, w is indeed an admissible basis i.e. an element of $\mathrm{St}^{(n)}$, since
$P_+ w = \Phi_0\Phi_0^*\,\Phi\Phi_0^*$ has a determinant $\iff$ $\Phi_0^*\Phi$ has a determinant $\iff$ $\Phi \sim \Phi_0$.

□

Theorem 5.4.2 (Anti-Isomorphism of Fock spaces)**.**
Let $\Phi_0 \in \mathtt{Seas}^\perp(\mathcal{H})$ with $\mathrm{im}(\Phi_0) = \mathcal{H}_{\{\geq -n\}}$.
Then the infinite wedge space $\mathcal{F}_{S(\Phi_0)}$ is naturally anti-unitary *equivalent to $\mathcal{F}^{(n)}$, the charge-n-sector of the fermionic Fock space constructed from $\Gamma(\mathrm{DET}_n^*)$.*

Proof. Consider the map $\mathfrak{f}:\ \mathcal{F}^{(n)} \longrightarrow \mathcal{F}_{S(\Phi_0)}$, defined by

$$\xi_w \longmapsto \bigwedge\big(w \circ \Phi_0\big) \tag{5.4.2}$$

for $w \in \mathrm{St}^{(n)}$ and *anti-linear* extension.
This is well defined, since for $L \in \mathrm{GL}^1(\mathcal{H}_+)$, $\xi_{z\circ L} = \overline{\det(L)}\,\xi_z$ by (5.3.6) and

$$\bigwedge\big(w \circ L \circ \Phi_0\big) = \bigwedge\big(w \circ \Phi_0\Phi_0^* L \circ \Phi_0\big) = \det(\Phi_0^* L \circ \Phi_0) \bigwedge\big(w \circ \Phi_0\big) = \det(L) \bigwedge\big(w \circ \Phi_0\big)$$

It is also an (anti-) isometry, since

$$\big\langle \bigwedge\big(z \circ \Phi_0\big), \bigwedge\big(w \circ \Phi_0\big)\big\rangle_{\mathcal{F}_{S(\Phi_0)}} = \det(\Phi_0^* z^* w \Phi_0) = \det(z^* w) = \big\langle \xi_z, \xi_w \big\rangle_{\mathcal{F}}$$

By construction of the infinite wedge space and by the previous Lemma, Dirac Seas of the form $w \circ \Phi_0,\ w \in \mathrm{St}^{(n)}$ span the entire Fock space $\mathcal{F}_{S(\Phi_0)}$.
We conclude that the two Fock spaces are anti-unitary equivalent. □

Corollary 5.4.3 (Action of $\widetilde{\mathrm{U}}^0_{\mathrm{res}}$ on infinite wedge spaces)**.**

Let Φ_0 and $\mathcal{F}_{S(\Phi_0)}$ as above.

Then we have a natural representation of $\widetilde{\mathrm{U}}^0_{\mathrm{res}}(\mathcal{H})$ on $\mathcal{F}_{S(\Phi_0)}$ given by

$$\widetilde{\mathrm{U}}^0_{\mathrm{res}}(\mathcal{H}) \ni [U, R] \longmapsto \mathcal{L}_U \mathcal{R}_{R^*_{\Phi_0}} \tag{5.4.3}$$

with $R_{\Phi_0} := \Phi_0^*\, R\, \Phi_0 \in \mathrm{U}(\ell)$, i.e. $\bigwedge\Phi \xrightarrow{[U,R]} \bigwedge\big(U\,\Phi\,\Phi_0^* R^* \Phi_0\big)$.
Of course, this is the representation induced by the that on $\mathcal{F}$ via the anti-automorphism $\mathfrak{f}$:

$$\begin{array}{ccc} \mathcal{F}^{(n)} & \xrightarrow{\mu^*_{[U,R]}} & \mathcal{F}^{(n)} \\ \big\downarrow \mathfrak{f} & & \big\downarrow \mathfrak{f} \\ \mathcal{F}_{S(\Phi_0)} & \xrightarrow{\mathcal{L}_U \mathcal{R}_{R^*_{\Phi_0}}} & \mathcal{F}_{S(\Phi_0)} \end{array} \tag{5.4.4}$$

Proof. It suffices to note that

$$\mathfrak{f} \circ \mu^*_{[U,R]}\,(\xi_w) = \mathfrak{f}\,(\xi_{(UwR^*)}) = \bigwedge\big(UwR^* \circ \Phi_0\big) = \bigwedge\big(Uw\,\Phi_0\,\Phi_0^*\,R^*\Phi_0\big) = \mathcal{L}_U \mathcal{R}_{R^*_{\Phi_0}} \circ \mathfrak{f}\,(\xi_w).$$

□

5.5 Relationship to CAR representations

In this section we clarify the relationship between the fermionic Fock space constructed from Dirac Seas or from holomorphic section in the $DET*$-bundle and the fermionic Fock space studied in §5.1., which can be thought of as a representation-space of the abstract CAR-algebra generated by creation and annihilation operators. This is of particular interest because this description of the Fock space is arguably the one most familiar to physicists. Basically, we will give rigorous proof of our initial argument that descriptions invoking "particles" and "antiparticles" are equivalent to descriptions that take Diracs idea of infinitely many particles seriously.

Unfortunately, at this point we will pay the price for bowing to different, contradictory conventions and using $\mathcal{H}_+$ for the unperturbed Dirac Sea in §5.2 but $\mathcal{H}_-$ for the anti-particle states in §5.1 that are supposed to correspond to "holes" in that sea.

We will circumvent this difficulty simply by setting $\mathcal{F}_+ := \mathcal{H}_-$ and $\mathcal{F}_- := \mathcal{C}\mathcal{H}_+$ in the definition of the Fock space which was

$$\mathcal{F} := \bigoplus_c^\infty \mathcal{F}^{(c)};\;\; \mathcal{F}^{(c)} := \bigoplus_{n-m=c} \bigwedge^n \mathcal{F}_+ \otimes \bigwedge^m \mathcal{F}_-$$

So compared to section 5.1, the roles of $\mathcal{H}_+$ and $\mathcal{H}_-$ are interchanged. I hope the reader will excuse this little blemish.

As $\mathcal{F}$ in this section is reserved for the Fock space just defined, we will denote the Fock space obtained from the geometric construction by $\mathcal{F}_{\text{geom}}$.

Construction 5.5.1 (Fock space isomorphism and field operator on Dirac seas)**.**

Fix a basis $\{(e_k)_{k\in\mathbb{Z}}\}$ of $\mathcal{H}$ such that $(e_k)_{k\leq 0}$ is ONB of $\mathcal{H}_-$ and $(e_k)_{k\geq 0}$ ONB of $\mathcal{H}_+$. Let $\mathcal{S}$ be the set sequences defined in [Def. 5.3.9 i)]. It consists of increasing sequences $S = (i_0, i_1, i_2, \ldots)$ containing only finitely many negative integers and all but finitely many positive integers. Recall that by proposition 5.3.11, the holomorphic sections $(\Psi_S)_{s\in\mathcal{S}}$ defined [Def. 5.3.9 iii)] form an orthonormal basis of the Fock space $\mathcal{F}_{\text{geom}}$.

We define an isomorphism between $\mathcal{F}_{\text{geom}}$ and $\mathcal{F}$ by

$$\Psi_S \longmapsto \left(e_{i_0} \wedge \ldots \wedge e_{i_{n-1}}\right) \otimes \left(\mathcal{C}e_{j_0} \wedge \ldots \wedge \mathcal{C}e_{j_{m-1}}\right) \in \bigwedge^n \mathcal{F}_+ \otimes \bigwedge^m \mathcal{F}_- \tag{5.5.1}$$

for $\{i_0 < i_1 < \ldots < i_{n-1}\} = S \cap \mathbb{Z}^-$ and $\{j_0 < j_1 < \ldots < j_{m-1}\} = \mathbb{N} \setminus S$.
In particular, for $S = (0, 1, 2, \ldots), \Psi_S = \Psi_0$ is mapped to the vacuum in $\mathcal{F}$.

Intuitively, the states indexed by the negative integers in S are mapped to "electron states" in $\mathcal{F}_+$ and the states indexed by the positive integers *missing* in S (the "holes") are charge-conjugated and mapped to "positron states" in $\mathcal{F}_-$.
Obviously, the assignment is 1-to-1 and isometric. It also preserves the charge in the sense that the states Ψ_S with $c(S) = c$, spanning the charge-c-sector of $\mathcal{F}_{\text{geom}}$ are mapped into $\mathcal{F}^{(c)}$, the charge-c-sector of $\mathcal{F}$.

Now we introduce the equivalent of creation- and annihilation operators on $\mathcal{F}_{\text{geom}}$.
It is convenient to use the intuitive notation

$$\Psi_S = e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots\,, \text{ for } S = (i_0, i_1, i_2, i_3 \ldots)$$

Then, we define the *field operator* (or rather its hermitian conjugate) Ψ^* on $\mathcal{F}_{\text{geom}}$ by

$$\Psi^*(e_k) :\; e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots \longmapsto e_k \wedge e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots \tag{5.5.2}$$

That is, $\Psi^*(e_k)$ maps Ψ_S with $S = (i_0, i_1, i_2, i_3 \ldots)$ to zero, if k is a member of S and otherwise to $(-1)^j \Psi_{S'}$ with $S' = (i_0, i_1, \ldots i_{j-1}, k, i_j, \ldots) \in \mathcal{S}$.
By linear extension in the argument if $\Psi^*(\cdot)$ we get a linear map $\Psi^* : \mathcal{H} \to \mathcal{B}(\mathcal{F}_{\text{geom}})$.

Obviously, under the isomorphism defined above, this field operator acts just as the usual field operator defined in (5.1.9) from creation- and annihilation operators.
The same is true for the formal adjoint $\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{F}_{\text{geom}})$, acting as

$$\Psi(e_k)\; e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots \;=\; \begin{cases} (-1)^j\, e_{i_0} \wedge e_{i_1} \ldots \wedge e_{i_{j-1}} \wedge \cancel{e_{i_j}} \wedge e_{i_{j+1}} \wedge \ldots & ;\text{ if } k = i_j \\ 0 & ;\text{ if } k \notin S \end{cases}$$

In particular, they define a representation of the CAR-algebra on the Fock space $\mathcal{F}_{\text{geom}}$ which is equivalent to the Fock representation on $\mathcal{F}$.

Chapter 6

Time-Varying Fock spaces

We have seen that a unitary operator on $\mathcal{H}$ can be implemented on the fermionic Fock space if and only if it satisfies the Shale-Stinespring condition $U_{+-}, U_{-+} \in I_2$ i.e. if and only if it is in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$. In the previous chapter this result appeared in different disguises depending on the construction of the Fock space, but there was no way around the fact itself. The drastic conclusion seems to be that there is no time-evolution in the external field problem of QED. By the theorem of Ruijsenaars (1.2.1), the Dirac-time evolution for an external field A will not be in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$, unless the spatial component $\underline{\mathsf{A}}$ of the vector potential vanishes. Physically, the reason is *infinite particle creation*. Mathematically, this is reflected in the fact that a unitary transformation that doesn't satisfy the Shale-Stinespring condition will leave the polarization class $[\mathcal{H}_+] = \mathrm{Gr}(\mathcal{H})$, over which the Fock space is constructed. Following an idea of Scharf and Fierz ([FS79]) Deckert et.al. concluded that, given these facts, the best that can be done is to realize the unitary time-evolution as unitary transformations between *different* Fock spaces over varying polarization classes.

Remark: In this work we do not focus on finding the weakest regularity condition on the external vector potentials. For simplicity, we always assume smooth fields with compact support in space and time. The respective publications that we are referring to might inform about bigger classes of interactions for which the results hold.

6.1 Identification of polarization classes

Given an external field in a form of 4-vector potential

$$\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\underline{\mathsf{A}}) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$$

let $U^{\mathsf{A}} = U^{\mathsf{A}}(t, t'),\ t, t' \in \mathbb{R}$ be the unitary Dirac time evolution determined by the Dirac Hamiltonian $H = D_0 + e \sum_{\mu=0}^{3} \alpha^\mu \mathsf{A}_\mu$.

Now, the question is this: How can we determine polarization classes $C(t)$ such that

$$U^{\mathsf{A}}(t_1, t_0) \in \mathrm{U}_{\mathrm{res}}\big(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1)\big) \tag{6.1.1}$$

for all $t_1, t_0 \in \mathbb{R}$?
The answer is a generalization of Ruijsenaars theorem and was provided in a very nice and systematic way by Deckert et. al. in [DeDuMeScho]. We summarize their main results:

Recall that the Dirac equation has the form

$$i\partial_t \Psi(t) = H^{\mathsf{A}}\psi(t) = \big(D_0 + V^{\mathsf{A}}(t)\big)\Psi(t) \tag{6.1.2}$$

with

$$V^{\mathsf{A}} = e\sum_{\mu=0}^{3} \alpha^\mu A_\mu, \tag{6.1.3}$$

In momentum representation, i.e. taking the Fourier transform of the equation, D_0 acts as a multiplication operator with the energy $E(p) = \sqrt{|\underline{p}|^2 + m^2}$ and we write the interaction term as

$$iZ^A = e\sum_{\mu=0}^{3} \alpha^\mu \widehat{A}_\mu, \tag{6.1.4}$$

where $\widehat{A}_\mu$ now act as convolution operators

$$(\widehat{A}_\mu\psi)(p) = \int_{\mathbb{R}^3} \widehat{A}_\mu(p-q)\psi(q)\,dq, \quad p \in \mathbb{R}^3, \tag{6.1.5}$$

for $\psi \in \mathcal{H}$ and $\widehat{A}_\mu$ the Fourier transform of A_μ.
Deckert et. al. define the integral operator $Q^A : \mathcal{H} \mapsto \mathcal{H}$ with kernel

$$Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{6.1.6}$$

Here, A is understood as the vector potential at a fixed time.
The operators Q^A are bounded and skew-adjoint and thus the operators e^{Q^A} are unitary.
Now, we have the following theorem:

Theorem 6.1.1 (Identification of Polarization classes)**.**
Let $\mathsf{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$. The operators $e^{Q^{\mathsf{A}(t)}}$ have the following properties:

i) Setting $C(\mathsf{A}(t)) := [e^{Q^{\mathsf{A}(t)}}\mathcal{H}_+]$ it is true that

$$U^{\mathsf{A}}(t_1,t_0) \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, C(\mathsf{A}(t_0)); \mathcal{H}, C(\mathsf{A}(t_1))\big) \tag{6.1.7}$$

for all $t_0, t_1 \in \mathbb{R}$.

ii) For two potentials $\mathsf{A} = (\mathsf{A}_0, -\underline{\mathsf{A}})$ and $\mathsf{A}' = (\mathsf{A}'_0, -\underline{\mathsf{A}'}) \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$ we find

$$[e^{Q^{\mathsf{A}}}\mathcal{H}_+]_{\approx_0} = [e^{Q^{\mathsf{A}'}}\mathcal{H}_+]_{\approx_0} \iff \underline{\mathsf{A}} = \underline{\mathsf{A}'} \tag{6.1.8}$$

It follows that the polarization class is completely determined by the spatial component of the vector potential A at any fixed time.

In more physical terms, the theorem says that the polarization classes $C(t) = C(\mathsf{A}(t))$

- depend only on the magnetic part of the interaction
- depend on A instantaneous in time and **not** on the history of the system

Note that Ruijsenaars Theorem (Thm.1.2.1) follows immediately as a special case of (6.1.8).

We can also easily derive the following important result:

Theorem 6.1.2 (Implementability of the S-matrix)**.**
Let $\mathsf{A} \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ *and* U_I^{A} *the corresponding unitary time evolution in the interaction picture (cf. §8.1.2). Then, the scattering matrix*

$$S := \lim_{t\to\infty} e^{itD_0} U^{\mathsf{A}}(t,-t) e^{itD_0} = \lim_{t\to\infty} U_I^{\mathsf{A}}(t,-t) \tag{6.1.9}$$

is in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ *and can be implemented as a unitary operator on the (standard) Fock space.*

Proof. Since the interaction potential has compact support, especially also compact support in time, we have $\mathsf{A}(t) = 0$ and hence $e^{Q^{\mathsf{A}(t)}} = \mathbb{1}$ for all $|t|$ large enough. In particular, $[e^{Q^{\mathsf{A}}(t)}\mathcal{H}_+]_{\approx_0} = [\mathcal{H}_+]_{\approx_0}$ for all $|t|$ large enough.
By the previous theorem it follows that $U_I^{\mathsf{A}}(T,-T) = U^0(0,T)U^{\mathsf{A}}(T,-T)U^0(-T,0) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},[\mathcal{H}_+];\mathcal{H},[\mathcal{H}_+]) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ for all T large enough, and hence $S \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

□

6.2 Second Quantization on time-varying Fock spaces

Here is the recipe for the second quantization of the time-evolution on time-varying Fock spaces, following [DeDuMeScho]. The main ingredient is Theorem 5.2.9, the abstract version of "Shale-Stinespring". We will present the method mainly in the language of infinite-wedge-spaces, which is best suited for this task. However, with some modifications it can be applied to the geometric constructions.

Let $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\underline{\mathsf{A}}) \in C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$ be an external field and $U^{\mathsf{A}}(t,t')$ the corresponding Dirac time evolution.

- Let $C(t) = C(\mathsf{A}(t)) \in \mathtt{Pol}(\mathcal{H})/\approx_0$ be the polarization classes identified above. We know that this polarization class is uniquely determined by the spatial component of the vector potential at that time. Then

  $$U^{\mathsf{A}}(t,t') \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, C(t');\mathcal{H},C(t)\big),\ \forall t,t' \in \mathbb{R}$$

- For every $t \in \mathbb{R}$ choose a Dirac Sea class $\mathcal{S}(t) \in \mathtt{Ocean}(C(t))/\sim$.
  We should demand that $\mathcal{S}(t)$ depends only on $\mathsf{A}(t)$ or even just $\underline{\mathsf{A}}(t)$. In particular, we want at least to stay in the initial Fock space when the interaction is turned off.

  For the geometric construction we would have to choose a polarization $W(t) \in C(t)$ for every $t \in \mathbb{R}$.

- We construct a family of Fock spaces $(\mathcal{F}_{\mathcal{S}(t)})_t$ as infinite wedge spaces over the Dirac Sea classes $\mathcal{S}(t)$. Equivalently, we can use the geometric construction where admissible bases are those compatible with the identity map on $W(t)$.

- By Theorem 5.2.9 we can implement $U^{\mathsf{A}}(t_1,t_0)$ as a unitary map between the Fock spaces $\mathcal{F}_{\mathcal{S}(t_0)}$ and $\mathcal{F}_{\mathcal{S}(t_1)}$. I.e. there exists $R \in \mathrm{U}(\ell)$ with $U^{\mathsf{A}}(t_1,t_0)\mathcal{S}(t_0)R = \mathcal{S}(t_1)$ and therefore

  $$\mathcal{L}_{U^{\mathsf{A}}(t_1,t_0)}\mathcal{R}_R : \mathcal{F}_{\mathcal{S}(t_0)} \to \mathcal{F}_{\mathcal{S}(t_1)} \tag{6.2.1}$$

  is a unitary map between the Fock spaces $\mathcal{F}_{\mathcal{S}(t_0)}$ and $\mathcal{F}_{\mathcal{S}(t_1)}$

- By Lemma 5.2.8, two such right operations implementing $U^{\mathsf{A}}(t_1,t_0)$ differ by an operator in $\mathrm{U}(\ell) \cap Id + I_1(\ell)$. The induced transformation between $\mathcal{F}_{\mathcal{S}(t_0)}$ and $\mathcal{F}_{\mathcal{S}(t_1)}$ can differ by the determinant of such an operator, i.e. by a complex phase $\in \mathrm{U}(1)$.
  However, transition probabilities are well-defined: if we have an "in-state" $\Psi^{in} \in \mathcal{F}_{\mathcal{S}(t_0)}$ and an "out-state" $\Psi^{out} \in \mathcal{F}_{\mathcal{S}(t_1)}$, the transition probability

  $$|\langle \Psi^{out}, \mathcal{L}_{U^{\mathsf{A}}(t_1,t_0)}\mathcal{R}_R\, \Psi^{in}\rangle|^2 \tag{6.2.2}$$

  is independent of the specific choice of $R \in \mathrm{U}(\ell)$.

So far, the choices of the right-operations for different time-intervals must not be "compatible" with each other and the transitions amplitudes are also well-defined independent of that. However, it is possible to choose the implementations so that they concatenate and preserve the semi-group structure of the time-evolution:

Proposition 6.2.1 (Semi-Group Structure)**.**
Let U^{A} be a unitary time evolution for the interaction potential A with time-support in $[T,-T]$. For every t we choose a Dirac Sea class $\mathcal{S}_t \in \mathtt{Ocean}(U^{\mathsf{A}}(t,-T)\mathcal{H}_+)$ and want to implement the time evolution on the family of Fock spaces $\mathcal{F}_{\mathcal{S}_t}, t \in \mathbb{R}$. By [Thm.5.2.9] there exists for every t an operator $R_t \in \mathrm{U}(\ell)$ s.t. $\mathcal{L}_{U(t,-T)}\mathcal{R}_{R_t} : \mathcal{F}_{\mathcal{S}_{-T}} \to \mathcal{F}_{\mathcal{S}_t}$.
Setting

$$\widetilde{U}(t_1,t_0) := \mathcal{L}_{U(t_1,t_0)}\mathcal{R}_{(R_{t_0}^{-1}R_{t_1})}$$

we get a two-parameter family of unitary transformations satisfying

$$\widetilde{U}(t_1,t_0) : \mathcal{F}_{\mathcal{S}_{t_0}} \to \mathcal{F}_{\mathcal{S}_{t_1}} \text{ and } \widetilde{U}(t_2,t_1)\,\widetilde{U}(t_1,t_0) = \widetilde{U}(t_2,t_0)$$

for $t_2 \geq t_1 \geq t_0 \in \mathbb{R} \cup \{\pm\infty\}$.

Proof. First, we need to show that $U(t_1,t_0)\Phi_0 R_{t_0}^{-1} R_{t_1} \sim \Phi_1$ for all $\Phi_0 \in \mathcal{S}_0,\ \Phi_1 \in \mathcal{S}_1$ i.e. that $\Phi_1^* U(t_1,t_0)\Phi_0 R_{t_0}^* R_{t_1}$ has a determinant. This is true because

$$\begin{aligned}
&U^*(t_i,-T)\,\Phi_i\,R_{t_i}^* \in \mathcal{S}_{-T}, \text{ for } i=0,1\\
\Rightarrow\ &\big(U^*(t_1,-T)\Phi_1 R_{t_1}^*\big)^*\big(U^*(t_0,-T)\Phi_0 R_{t_0}^*\big) = \big(R_{t_1}(\Phi_1)^* U(t_1,-T)U^*(t_0,-T)\Phi_0 R_{t_0}^*\big)\\
=\ &R_{t_1}\Phi_1^*\,U(t_1,t_0)\Phi_0 R_{t_0}^* \text{ has a determinant} \Rightarrow\ \Phi_1^*\,U(t_1,t_0)\Phi_0\,R_{t_0}^* R_{t_1} \text{ has a determinant.}
\end{aligned}$$

The composition property follows immediately from

$$\mathcal{L}_{U(t_2,t_1)}\mathcal{R}_{(R_{t_1}^{-1}R_{t_2})}\mathcal{L}_{U(t_1,t_0)}\mathcal{R}_{(R_{t_0}^{-1}R_{t_1})} = \mathcal{L}_{U(t_2,t_1)U(t_1,t_0)}\mathcal{R}_{(R_{t_0}^{-1}R_{t_1}R_{t_1}^{-1}R_{t_2})} = \mathcal{L}_{U(t_2,t_0)}\mathcal{R}_{(R_{t_0}^{-1}R_{t_2})}$$

using that left- and right- operation commute.

□

Note, however, that this result is not fully satisfying, because the phase of the implementation of the time-evolution $U(t_1,t_0)$ between times $t_0 < t_1$ depends on the one-particle time-evolution not only between t_0 and t_1, but between $t=-T$ and $t=t_1$ (cf. our discussion of causality in §8.2.1).

In the way we have chosen to present the construction, we exploited the whole freedom of choosing a family of Fock spaces for a fixed time-evolution $U^{\mathsf{A}}(t,t')$. We have merely suggested to choose the same Fock space over $[\mathcal{H}_+]$ outside the time-support of the interaction in order to implement at least the S-matrix on a fixed Fock space. In practice, one would rather make a global choice of the Fock spaces, depending only on the external potential, locally in time. For example, starting with any Dirac sea $\Phi \in \mathtt{Ocean}(\mathcal{H}_+)$ we can set

$$\mathcal{S}(\mathsf{A}(t)) := [e^{Q^{\mathsf{A}}(t)}\Phi] \in \mathtt{Ocean}\big(C(\mathsf{A}(t))\big)/\sim \tag{6.2.3}$$

which depends only on $\mathsf{A}(t)$ at the fixed time t. By [Thm. 6.1.1], $\mathcal{F}_{\mathcal{S}(\mathsf{A}(t))}$ is a suitable family of Fock spaces for the time-evolution U^{A}, for any field A. Note, however, that this choice is merely convenient and not motivated by any physical insight.

The Dirac Sea classes $\mathcal{S}(t)$ alone are not enough structure to define how the Dirac Sea is filled i.e. how many electrons and positrons a given state contains. For this, we would have to distinguish instantaneous "vacua" to compare our states to. In the geometric construction, the Dirac Sea classes are paralleled by the equivalence classes of *admissible bases*. But if we fix the admissible bases by choice of polarizations $\mathcal{H} = W(t) \oplus W(t)^\perp$ for every time t, the "vacuum state" is already distinguished by the identity map on $W(t)$.

We try to formalize this in the following Definition.

Definition 6.2.2 (Family of Vacua)**.**

We fix a basis $(e_k)_{k\geq 0}$ of $\mathcal{H}_+ \subset \mathcal{H}-$. We choose $\Phi_0 \in \mathtt{Ocean}(\mathcal{H}_+)$ to be the Dirac sea mapping the standard basis in $\ell = \ell^2$ to the basis of $\mathcal{H}_+$. Formally:

$$\Phi_0 \mathrel{\widehat{=}} \Phi(l_0) \wedge \Phi(l_1) \wedge \Phi(l_2) \wedge \ldots \mathrel{\widehat{=}} e_0 \wedge e_1 \wedge e_2 \wedge \ldots \tag{6.2.4}$$

we call $\bigwedge\Phi_0$ the *free vacuum state* in $\mathcal{F}_{\mathcal{S}(\Phi_0)}$. With a little abuse of the expression we will also refer to Φ_0 itself as the *free vacuum*.

Let U^{A} be the unitary time evolution for an interaction potential $\mathsf{A} \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$.
We call a family $(\Phi^{(t)})_t$ of isometric Dirac Seas a *family of vacua* for the time evolution if $\Phi^{(t)} \in \mathtt{Ocean}^\perp(C(t))$ for all $t \in \mathbb{R}$ and if for $|t|$ large enough they become the free vacuum Φ_0. Obviously, such a family of vacua determines a family of Dirac Sea classes $\mathcal{S}(t) \in \mathtt{Ocean}(C(t))/\sim$ for the time-evolution by $\mathcal{S}(t) := \mathcal{S}(\Phi^{(t)}) = [\Phi^{(t)}] \in \mathtt{Seas}^\perp(\mathcal{H})/\sim$.

We are knowingly overselling the Dirac seas $\Phi^{(t)}$ by calling them "vacua". But if we use such reference states to specify the Dirac sea classes, the states themselves may or may not be physically distinguished from others.

The second quantization of the time evolution on varying Fock spaces seems to be in a way the best that can be done within the current framework. However, the physical content of this procedure remains doubtful. To make sense of it we would have to say what *physical* quantities characterize the *physical* states represented by vectors in different Fock spaces. Right now it seems rather unlikely that this is possible.

6.2.1 Gauge Transformations

The polarization classes $C(t)$ are determined by the spatial components of the A-field defining the interaction potential. But this quantity is not gauge-invariant in the most obvious way. Therefore, neither the condition $U^{\mathsf{A}}(t,t') \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ for implementability on the standard Fock space, nor the procedure of second quantization on time-varying Fock spaces is gauge-invariant. This might seem highly suspicious, at first. But the right conclusion to draw here is that the (second quantized) theory is just not gauge-invariant in the naive sense. Actually, the problem with gauge-transformations is the same as with the time evolution. It turns out that smooth gauge transformations

$$\mathcal{G} \ni g : \Psi(x) \to e^{i\,\Lambda_g(x)}\Psi(x),\ \Lambda_g \in C_c^\infty(\mathbb{R}^3, \mathbb{R}) \tag{6.2.5}$$

do *not* satisfy the Shale-Stinespring condition $[\epsilon, g] \in I_2(\mathcal{H})$, unless they are constant, and are therefore not implementable on a fixed Fock space [MiRa88].

More precisely, the following is true:

Theorem 6.2.3 (Gauge-Transformation, [DeDuMeScho]Thm. III.11)**.**

Let $\mathcal{G}$ denote the space of smooth gauge-transformations as in (6.2.5)

By [Thm. 6.1.1, ii)] it is justified to write $[e^{Q^{\mathsf{A}}}\mathcal{H}_+] =: C(\underline{\mathsf{A}})$ etc. for the polarization classes, as they are completely determined by the spatial part of the A-field.
Then, for all $g \in \mathcal{G}$ it is true that

$$\boxed{g = e^{i\,\Lambda_g} \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, C(\underline{\mathsf{A}})\,;\,\mathcal{H}, C(\underline{\mathsf{A}} + \nabla\Lambda_g)\big)} \tag{6.2.6}$$

for any $\underline{\mathsf{A}} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Of course, this is what *must* be true in order to be consistent with our previous results.

Following the concept of varying Fock spaces, we can therefore realize gauge-transformation not as unitary operations on a Fock space, but as a transformation of the Fock spaces themselves. In this sense, the theory might become "gauge-covariant" rather than "gauge-invariant" in the usual sense.

We will come back to the problem of gauge-invariance in §8.4.

Remark: The Fock bundle

It is possible to fit the concept of time-varying Fock spaces into quite an elaborated geometric structure as well, by introducing a manifold structure on $\mathcal{A}$, the space of (time-independent) vector potentials, and constructing the *Fock bundle* over $\mathcal{A}$. This is a principle fibre bundle, where the fibre over any $A \in \mathcal{A}$ is an entire Fock space. $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ is the structure group acting transitively on every fibre (i.e. on the Fock spaces themselves) whereas the time evolution and gauge-transformations act as unitary bundle-maps between different fibres. For an outline of the construction consult e.g. [Mi] or [CaMiMu00].

Results from the theory of bundle gerbes suggest that the Fock bundle can be pushed down to a bundle over $\mathcal{A}/\mathcal{G}_e$ where $\mathcal{G}_e$ is the space of based, smooth, compactly supported gauge-transformations ([CaMiMu97]). In this sense, the description could be made gauge-invariant. This does seem promising indeed, but the mathematical treatment is quite abstract and not very physicists-friendly. We therefore suggest that one should try to reformulate the construction in terms closer to the physical intuition and compare it to the infinite wedge space constructions presented before. Also, the choices involved in the construction of the Fock bundle should be worked out in a systematic way.

Chapter 7

The Parallel Transport of Langmann & Mickelsson

We have seen that the central extensions $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ and $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ also carry the structure of principle fibre bundles. This allows us to equip them with a *bundle-connection* as an additional geometric structure. Our aim is to use parallel transport with respect to this connection to lift the unitary time evolution to the Fock space, or at least to fix the phase of the second-quantized S-matrix in a well-defined manner.
We will follow [LaMi96] and define a suitable connection not on $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ but on its complexification $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$. However, the construction restricts easily to the unitary case.

Recall that a connection Γ on a principle bundle is a distribution in its tangent bundle, distinguishing the tangent vectors that will be called *horizontal* (to the base-manifold). This distribution is complementary to the space of *vertical* vector fields, which is always defined as the kernel of $\mathrm{D}\pi$, where π is the projection onto the base-manifold. That is, vertical vectors are vectors along the fibres of the bundle. A connection is a geometric structure which allows us to lift paths from the base-manifold (here: $\mathrm{U}_{\mathrm{res}}$ or $\mathrm{GL}_{\mathrm{res}}$) to the bundle in a unique way by demanding that the tangent vectors of the lifted path are always horizontal. These *horizontal lifts* define parallel transport in the principle bundle.[1] Also recall that connections are in one-to-one correspondence with *connection one-forms* which are certain one forms on the principle bundle with values in the Lie-algebra of its structure group. For every connection Γ there exists one and only one connection one-form Φ with $\Gamma = \ker(\Phi)$.

A nice treatment of connections and parallel transport on principle bundles can be found in [KoNo], for example .

Throughout this chapter a little care is required with the fact that we're working on infinite dimensional manifolds. But Banach manifolds, as the ones we're dealing with, are generally pretty well-behaved. One essential point is that on Banach manifolds (as opposed to manifolds modeled on infinite dimensional Fréchet spaces) the Implicit Mapping Theorem holds and thus the local calculus works almost exactly as in the finite dimensional case.
Note however, that we avoid all expansions in local coordinate frames, as are often used for the analogous proofs in the finite dimensional case. Fundamentals of Differential Geometry on Banach manifolds are presented for example in [Lang].

[1] If $\pi : P \to M$ is a principle bundle over M equipped with a connection, the parallel transport of $p \in P$ along a curve γ in M starting in $\pi(p)$ is the end-point of the horizontal lift of γ with starting point $p \in P$.

7.1 The Maurer-Cartan forms

Let G be a Lie group. The group operation on G define smooth actions of G on itself by multiplication from the left or from the right.
For $g \in G$ we denote by L_g the action from the left, i.e.

$$L_g : G \to G,\ L_g(h) := g \cdot h \tag{7.1.1}$$

and by R_g the action from the right:

$$R_g : G \to G,\ R_g(h) := h \cdot g \tag{7.1.2}$$

The differentials of these maps (push forwards) as maps between tangent-spaces are then isomorphisms:

$$\begin{aligned} (L_g)_* &: T_hG \to T_{gh}G \\ (R_g)_* &: T_hG \to T_{hg}G \end{aligned} \tag{7.1.3}$$

We will also need the corresponding pull-backs $(L_g)^*, (R_g)^*$ on one-forms.
Recall that a vector field $X \in \mathfrak{X}(G)$ is called left-invariant if $(L_g)_* X_h = X_{gh},\ \forall g, h \in G$ and right-invariant if $(R_g)_* X_h = X_{hg},\ \forall g, h \in G$.
G can also act on itself by conjugation:

$$c_g : G \to G,\ c_g(h) := L_g R_{g^{-1}} h = ghg^{-1} \tag{7.1.4}$$

The map

$$Ad : G \to \mathrm{GL}(\mathrm{Lie}(G)),\ g \mapsto \dot{c_g} \tag{7.1.5}$$

yields a natural representation of G on it's own Lie-algebra called the adjoint representation. The kernel of Ad equals the center of G. [2]

In the following, we identify the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ of G with the tangent space above the identity, i.e. $\mathfrak{g} :=\cong T_eG$, for e, the neutral element in G.

Definition 7.1.1 (The Maurer-Cartan Forms)**.**
The left Maurer-Cartan form, also called the canonical left-invariant one-form, is a one-form on G with values in the Lie algebra. It is defined by:

$$\begin{aligned} \omega_L(g)\ :\ T_gG &\longrightarrow T_eG \cong \mathfrak{g}\,; \\ V &\longmapsto (L_{g^{-1}})_*\, V \end{aligned} \tag{7.1.6}$$

Analogously, we define the right Maurer-Cartan form by:

$$\begin{aligned} \omega_R(g)\ :\ T_gG &\longrightarrow T_eG \cong \mathfrak{g}\,; \\ V &\longmapsto (R_{g^{-1}})_*\, V \end{aligned} \tag{7.1.7}$$

We will often drop the subscript "L" or "R" if it is clear which form we're referring to.

In a matrix representation, the Maurer-Cartan forms can be simply written as

$$\omega_L = g^{-1}\mathrm{d}g \ \text{ and } \ \omega_R = \mathrm{d}g\, g^{-1} \tag{7.1.8}$$

respectively. Obviously, ω_L/ω_R is constant on left- / right- invariant vector fields.
Furthermore, $\omega_L(e) = \omega_R(e) = Id_{T_eG}$.
We state a few less obvious properties of the Maurer-Cartan forms.

[2] In a Matrix representation, $G \hookrightarrow \mathrm{GL}(V), \mathrm{Lie}(G) \hookrightarrow End(V)$ and $Ad_g(X) = gXg^{-1}$. Thus, one can think of the adjoint representation in the following way: If the elements of a vector space V (or a vector bundle with fibre V) transform under a symmetry $g \in G$ via an action of G on V then the endomorphisms of V transform by the corresponding adjoint action.

Lemma 7.1.2 (Properties of the left Maurer-Cartan form)**.**
The left Maurer-Cartan form ω_L has the following properties

i) ω_L *is left-invariant, i.e.* $(L_g)^*\omega_L = \omega_L$

ii) $(R_g)^*\omega_L = Ad_{g^{-1}}\,\omega_L,\ \forall g \in G$

iii) $d\omega_L + \frac{1}{2}[\omega_L \wedge \omega_L] = 0$

The right Maurer-Cartan form ω_R has the following properties

I) ω_R *is right-invariant, i.e.* $(R_g)^*\omega_R = \omega_R$

II) $(L_g)^*\omega_R = Ad_g\,\omega_R,\ \forall g \in G$

III) $d\omega_R - \frac{1}{2}[\omega_R \wedge \omega_R] = 0$

Proof. i) For $V \in T_hG$ and $g \in G$ we have

$$(L_g^*\,\omega_{gh})(V) = \omega_{gh}((L_g)_*V) = L_{(gh)^{-1}*}L_{g*}V = L_{h^{-1}*}L_{g^{-1}*}L_{g*}V = L_{h^{-1}*}(V) = \omega_h(V)$$

ii) $(R_g^*\,\omega_{hg})(V) = \omega_h(R_{g*}V) = L_{(hg)^{-1}*}R_{g*}V = L_{g^{-1}*}L_{h*}R_{g*}V = Ad_{g^{-1}}\,\omega_h(V)$

iii) $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[X,Y]$, for $X, Y \in \mathfrak{X}(G)$.
So, on left-invariant vector-fields the identity is true, because
$d\omega(X,Y) = -\omega[X,Y] = -[\omega(X),\omega(Y)] = -\frac{1}{2}[\omega(X) \wedge \omega(Y)]$. But the expression is bilinear in smooth functions (tensorial) and thus depends on the vectors only pointwise. Therefore, the identity holds for all vector fields.

The proofs for the right-invariant form work analogously. The different sign in III) as compared to iii) comes from the fact that the commutator on right invariant vector fields equals *minus* the commutator on the corresponding left-invariant vector fields which by convention defines the Lie bracket. □

Usually, the Lie algebra of a Lie group G is defined as the vector space of left-invariant vector fields on G with the Lie bracket given by the commutator of vector fields. Therefore, the left Maurer-Cartan form is the more canonical object. However, it will turn out that the right-invariant form is better suited to our purposes.

7.2 The Langmann-Mickelsson Connection

We have gathered everything we need to define a connection on the principle $\mathbb{C}^\times$-bundle $\widetilde{\mathrm{GL}}_{\mathrm{res}} \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}$. Recall that the central extension of Lie groups (4.1.2)

$$1 \longrightarrow \mathbb{C}^\times \xrightarrow{\imath} \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \longrightarrow 1$$

induces a central extension of the corresponding Lie algebras

$$0 \longrightarrow \mathbb{C} \xrightarrow{i} \tilde{\mathfrak{g}}_1 \xrightarrow{\dot{\pi}} \mathfrak{g}_1 \longrightarrow 0 \tag{7.2.1}$$

Furthermore, every (local) trivialization of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ about the identity defines an isomorphism of vector spaces $\tilde{\mathfrak{g}}_1 \cong \mathfrak{g}_1 \oplus \mathbb{C}$. In particular, we have the local trivialization ϕ defined in (4.1.4) as, arguably, the most natural choice and we use this trivialization to identify $\tilde{\mathfrak{g}}_1$ with $\mathfrak{g}_1 \oplus \mathbb{C}$.

With all this in our hands, a quite natural connection on $\widetilde{\mathrm{GL}}_{\mathrm{res}} \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}$ can be readily defined. The (left) Maurer-Cartan form ω_L on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ is a one-form on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ with values in the Lie-algebra $\tilde{\mathfrak{g}}_1 = \mathrm{Lie}(\widetilde{\mathrm{GL}}_{\mathrm{res}})$. A connection, however, is defined by a one-form with values in the Lie algebra of the structure group. In our case this is just $\mathrm{Lie}(\mathbb{C}^\times) = \mathbb{C}$. Thus, we simply set $\Phi_{LM} := \mathrm{pr}_{\mathbb{C}} \circ \omega_L$, where $\mathrm{pr}_{\mathbb{C}}$ is the projection onto the $\mathbb{C}$-component with respect to the splitting $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}$. This is indeed a connection one-form:

Definition 7.2.1 (Langman-Mickelsson-connection)**.**
Let ω_L be the left Maurer-Cartan form on the Lie group $\widetilde{\mathrm{GL}}_{\mathrm{res}}$.
The one-form

$$\Phi_{LM} := \mathrm{pr}_{\mathbb{C}} \circ \omega_L \in \Omega^1(\widetilde{\mathrm{GL}}_{\mathrm{res}}; \mathbb{C}) \tag{7.2.2}$$

is a connection one-form on $\pi : \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. We will call it the *Langmann-Mickelsson-connection*, because it was appearently first studied in [LaMi96].

Langmann and Mickelsson introduce this connection in [LaMi96] and use it to fix the phase of the S-matrix by parallel transport. We want to propose a slightly modified connection using the right-invariant Maurer-Cartan form instead of the left-invariant form. The advantage of this choice will become clear when we carry out the lift of the time-evolution. We prove that the so constructed one-form indeed defines a connection. The proof for the Langmann-Mickelsson connection is almost identical.

Proposition 7.2.2 (Connection on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$)**.**
Let ω_R be the right Maurer-Cartan form on the Lie group $\widetilde{\mathrm{GL}}_{\mathrm{res}}$.
Then,

$$\Phi := \mathrm{pr}_{\mathbb{C}} \circ \omega_R \in \Omega^1(\widetilde{\mathrm{GL}}_{\mathrm{res}}; \mathbb{C}) \tag{7.2.3}$$

defines a connection one-form for the principle $\mathbb{C}^\times$ bundle $\pi : \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

Proof. We need to show[3]

i) $\Phi(x^*) = x$ for all x in the Lie algebra $\mathbb{C}$ of $\mathbb{C}^\times$.

ii) $R_\lambda^* \Phi = Ad_{\lambda^{-1}} \Phi = \Phi$ for all λ in the structure group $\mathbb{C}^\times$
(which in our case happens to be abelian, thus $Ad_{\lambda^{-1}} \equiv Id$).

Here, x^* denotes the fundamental vector field on the principle bundle, generated by the Lie algebra element x via its exponential action on the Lie group.
ii) is trivial because ω_R is invariant even under the right-action of the entire $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on itself, in particular under the right-action of $\imath(\mathbb{C}^\times) \subset \widetilde{\mathrm{GL}}_{\mathrm{res}}$. Thus:

$$R_\lambda^* \, \Phi = \mathrm{pr}_{\mathbb{C}} \circ R_\lambda^* \, \omega_R = \mathrm{pr}_{\mathbb{C}} \circ \omega_R = \Phi = Ad_{\lambda^{-1}} \Phi, \ \forall \lambda \in \mathbb{C}^\times.$$

For the first property take $x \in \mathbb{C}$. Then:

$$\begin{aligned} x^*(p) = & \left. \frac{d}{dt} \right|_{t=0} p \cdot \imath(\exp(tx)) = (L_p)_* \imath_*(x) \\ \Rightarrow \omega_R(x^*)|_p &= (R_{p^{-1}})_* (L_p)_* \, \imath_* \, x = \imath_* \, x \\ \Rightarrow \Phi(x^*)|_p &= \mathrm{pr}_{\mathbb{C}} \circ \imath_* \, x = x, \ \forall \, p \in \widetilde{\mathrm{GL}}_{\mathrm{res}} \end{aligned}$$

where we have used that $\imath$ maps into the center of $\widetilde{\mathrm{GL}}_{\mathrm{res}})\mathcal{H})$ and also that $\phi \circ \imath(c) = (\mathbb{1}, c) \in \mathrm{GL}_{\mathrm{res}} \times \mathbb{C}^\times$, which implies $\mathrm{pr}_{\mathbb{C}} \circ (\imath_*)|_e = Id_{\mathfrak{g}_1}$.

□

Thus, Φ is a connection one-form and defines the connection

$$\Gamma_\Phi := \ker \Phi \subset \mathrm{T}\widetilde{\mathrm{GL}}_{\mathrm{res}} \tag{7.2.4}$$

on the principle bundle $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.

Obviously, we could have defined the analogous connection on $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{U}_{\mathrm{res}}(\mathcal{H})$, however we follow [LaMi96] in discussing the connection on the complexification $\widetilde{\mathrm{GL}}_{\mathrm{res}} \to \mathrm{GL}_{\mathrm{res}}$. Apart from being more general, this has the advantage that the the local section in $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ is nicer which makes computations in local coordinates easier. Anyway, the construction restricts immediately to $\widetilde{\mathrm{U}}_{\mathrm{res}} \subset \widetilde{\mathrm{GL}}_{\mathrm{res}}$.

[3] cf. [KoNo], Prop. 1.1

Proposition 7.2.3 (Curvature of the Connection)**.**
The curvature Ω of the connection Φ corresponds to the Schwinger cocycle (4.1.8) *in the following sense: If $\widetilde{X}, \widetilde{Y}$ are right-invariant vector fields on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ identified with elements of the Lie algebra $\tilde{\mathfrak{g}}_1$ of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ then*

$$\Omega(\widetilde{X}, \widetilde{Y}) = c(X, Y) \tag{7.2.5}$$

for $X := \mathrm{pr}_{\mathfrak{g}}(\widetilde{X}), Y := \mathrm{pr}_{\mathfrak{g}}(\widetilde{Y}) \in \mathfrak{g}_1$. [4]
Since the curvature is a two-form, it is completely determined by this equality.

Proof. Let $\widetilde{X}, \widetilde{Y}$ be left-invariant vectorfields corresponding to $(X, \lambda_1), (Y, \lambda_2) \in \mathfrak{g}_1 \oplus \mathbb{C} \cong \tilde{\mathfrak{g}}_1$, respectively. The curvature 2-form can be computed as $\Omega = \mathrm{d}\Phi + \frac{1}{2}[\Phi \wedge \Phi]$. Since Φ takes values in the abelian Lie algebra $\mathbb{C}$, the second summand is zero and Ω equals $\mathrm{d}\Phi$. Also recall that under the isomorphism $\tilde{\mathfrak{g}_1} \cong \mathfrak{g}_1 \oplus \mathbb{C}$ induced by the local trivialization ϕ, the Lie bracket on $\tilde{\mathfrak{g}}_1$ is the Lie bracket on $\mathfrak{g}_1$ plus the Schwinger-cocycle (4.1.8). Thus we get:

$$\begin{aligned}\Omega(\widetilde{X}, \widetilde{Y}) =& \mathrm{d}\Phi(\widetilde{X}, \widetilde{Y}) = \mathrm{pr}_{\mathbb{C}}(\mathrm{d}\omega(\widetilde{X}, \widetilde{Y})) = \mathrm{pr}_{\mathbb{C}}([\omega(\widetilde{X}), \omega(\widetilde{Y})]) \\ =& \mathrm{pr}_{\mathbb{C}}([\widetilde{X}, \widetilde{Y}]) = \mathrm{pr}_{\mathbb{C}}\big(([X, Y], c(X, Y)\big) = c(X, Y)\end{aligned}$$

□

7.2.1 Classification of Connections

Obviously, the connection is an additional structure on the bundle and as such constitutes a particular choice. We demand that the connection is invariant under the right-action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on itself, because, as we will see, this assures that the horizontal lift of the time evolution preserves the semi-group structure. Note that a connection is always demanded to be invariant under the (fibre-preserving) right-action of the structure group, but we demand much more: that it is invariant under the right action of the entire Lie group $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on itself. But even then, the connection is not at all unique. For example, a different isomorphism $\tilde{\mathfrak{g}}_1 \to \mathfrak{g}_1 \oplus \mathbb{C}$, coming from a different local section in $\widetilde{\mathrm{GL}}_{\mathrm{res}}$, would also yield a different right-invariant connection.

Generally, every local section $\upsilon : \mathrm{GL}_{\mathrm{res}} \to \widetilde{\mathrm{GL}}_{\mathrm{res}}$ around the identity with $\upsilon(e) = e$ (e being the neutral element in the group, i.e. the identity) induces the isomorphism $D_e\upsilon : \tilde{\mathfrak{g}}_1 \to \mathfrak{g}_1 \oplus \mathbb{C}$ so that $\mathrm{pr}^{\upsilon}_{\mathbb{C}} \circ \omega_R$ is a right-invariant connection one-form.

Conversely, every right-invariant connection on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ comes from a one-form of this kind. To see this, we recall the geometric meaning of a connection. A connection is a distribution Γ in the tangent-bundle $\mathrm{T}\widetilde{\mathrm{GL}}_{\mathrm{res}}$ which is complementary to the kernel of the projection $D\phi$ and invariant under the right-action of the structure group on every fibre. Additionally, we demand that the connection be invariant under the right-action of the entire Lie group $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on iteslf. Altogether, this means

$$\begin{aligned}\mathrm{T}_p\widetilde{\mathrm{GL}}_{\mathrm{res}} &= \ker(D_p\pi) \oplus \Gamma_p, \\ \text{and } \ (R_g)_*\Gamma_p &= \Gamma_{pg}, \ \forall g \in \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})\end{aligned} \tag{7.2.6}$$

at every point $p \in \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$. From this we see that to define such a connection we have the freedom to choose a subspace Γ_e complementary to $\ker D_e\pi$ in $\mathrm{T}_e\widetilde{\mathrm{GL}}_{\mathrm{res}}$. Then, the connection is completely determined by $\Gamma_p = (R_p)_*\,\Gamma_e$ for all $p \in \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$. But such a subspace Γ_e can always be realized as the image of $\mathrm{T}_e\mathrm{GL}_{\mathrm{res}}$ under $D_e\upsilon : \mathrm{T}_e\mathrm{GL}_{\mathrm{res}} \to \mathrm{T}_e\widetilde{\mathrm{GL}}_{\mathrm{res}}$ for a suitable local section υ as above. We conclude:

[4] The analogous computation for Φ_{LM} would yield *minus* the Schwinger cocycle. In [LaMi96] there might be a sign error or a different convention for the cocycle might be used.

Lemma 7.2.4 (Right-invariant Connections and local sections)**.**
Let Γ be a connection on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$, invariant under the right-action of the Lie group on itself. Let $\upsilon : \mathrm{GL}_{\mathrm{res}} \to \widetilde{\mathrm{GL}}_{\mathrm{res}}$ be a local section around the identity with $\upsilon(e) = e$ and $D_e\upsilon(\mathrm{T}_e\mathrm{GL}_{\mathrm{res}}) = \Gamma_e$. Then Γ corresponds to the connection one-form $\Phi_\upsilon := \mathrm{pr}^\upsilon_{\mathbb{C}} \circ \omega_R$.

Proof. Note that $\mathrm{im}(D_e\upsilon) = \ker(\mathrm{pr}^\upsilon_{\mathbb{C}})$. By assumption, $X \in \mathrm{T}_e\widetilde{\mathrm{GL}}_{\mathrm{res}}$ is horizontal, i.e. $X \in \Gamma_e$, if and only if $X \in \mathrm{im}(D_e\upsilon)$, which then is true if and only if $X \in \ker\big(\mathrm{pr}^\upsilon_{\mathbb{C}}\big) = \ker\big(\mathrm{pr}^\upsilon_{\mathbb{C}} \circ \omega_R(e)\big)$. Since both, the distribution Γ and the kernel of $\mathrm{pr}^\upsilon_{\mathbb{C}} \circ \omega_R$ are invariant under the right-action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ on itself, the identity holds everywhere. □

Theorem 7.2.5 (Uniqueness of the connection)**.**
The connection Γ_Φ defined in (7.2.3) *is the unique connection on $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ which is invariant under the right-action of the Lie group on itself and whose curvature equals the Schwinger cocycle c in the sense of Prop. 7.2.3.*

Proof. Recall that the connection Γ_Φ comes from the local trivialization defined by the section τ (4.1.3). If we take a connection Γ', different from Γ_Φ but also invariant under the right-action of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$, the previous Lemma tells us that it comes from a local section υ with $D_e\upsilon \neq D_e\tau$. From the discussion of central extensions of Lie algebras in §3.4. we know that this means that the Lie algebra cocycle Θ corresponding to υ differs from the Schwinger cocycle (corresponding to τ) by a homomorphism[5] $\mu = \dot\upsilon - \dot\tau = D_e\upsilon - D_e\tau : \mathfrak{g}_1 \to \mathbb{C}$ i.e. $\Theta(X, Y) = c(X, Y) - \mu([X, Y])$ for $X, Y \in \mathfrak{g}_1$. Therefore, in the sense of Prop. 7.2.3, the curvature of the connection Γ' differs from the Schwinger cocycle by $\mu([\cdot, \cdot])$.
□

In this sense, the connection we have defined is "unique". Of course, it's in no way necessary to demand that the curvature of the connection equals the Schwinger cocycle. We can just agree that it's nice if it does. Also, and this might be the more relevant point, comparison with (5.1.32) tells us that horizontal lift from $\mathfrak{g}_1 \cong \mathrm{T}_e\mathrm{GL}_{\mathrm{res}}$ to $\tilde{\mathfrak{g}}_1 \cong \mathrm{T}_e\widetilde{\mathrm{GL}}_{\mathrm{res}}$ with respect to the connection Γ_Φ corresponds in the Fock representation to the second quantization prescription $\mathrm{d}\Gamma$ defined by normal ordering.

Note that the arguments used in the proof of [Thm. 7.2.5] together with the non-triviality of the Schwinger-cocycle prove that there is not flat right-invariant connection on $\mathrm{GL}_{\mathrm{res}}$.

7.2.2 Local Formula

Let's compute an explicit formula for the connection w.r.to the coordinates defined by the local section (4.1.3): Let $\gamma : t \mapsto [(g(t), q(t))],\ t \in [-\epsilon, \epsilon]$ be a C^1-curve in $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.
We write $[(g, q)]$ for $\gamma(0) = [(g(0), q(0)]$ and ϕ for the local trivialization (4.1.4) on $\pi^{-1}(U)$. Note that Φ can be expressed as:

$$\Phi_g = \mathrm{pr}_{\mathbb{C}}(\mathrm{d}g\, g^{-1}) = \mathrm{pr}_{\mathbb{C}} \circ DR_g^{-1} \circ Id_{T_g\widetilde{\mathrm{GL}}_{\mathrm{res}}}$$

Thus, we compute:

$$\begin{aligned}
\Phi(\dot\gamma(0)) =& \frac{d}{dt}\Big|_{t=0} \mathrm{pr}_{\mathbb{C}} \circ \phi\big([g(t)g^{-1}, q(t)q^{-1}]\big) \\
=& \frac{d}{dt}\Big|_{t=0} \det\big[(g(t)g^{-1})^{-1}_{++}(q(t)q^{-1})\big] \\
=& -\operatorname{tr}\big[(\dot g(0)g^{-1})_{++} - \dot q(0)q^{-1}\big]
\end{aligned}$$

[5] a "coboundary" in the sense of cohomology

Writing

$$g(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \text{ and } g^{-1}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$$

w.r.t. the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, the formula becomes

$$\boxed{\Phi \;=\; -\operatorname{tr}\big[\mathrm{d}a\,\alpha + \mathrm{d}b\,\gamma - \mathrm{d}q\,q^{-1}\big]} \tag{7.2.7}$$

The corresponding expression for the Mickelsson-Langmann connection can be computed analogously to be

$$\boxed{\Phi_{LM} \;=\; -\operatorname{tr}\big[\alpha\,\mathrm{d}a + \beta\,\mathrm{d}c \;- q^{-1}\,\mathrm{d}q\big]} \tag{7.2.8}$$

7.3 Parallel Transport in the $\widetilde{\mathrm{GL}}_{\mathrm{res}}-$bundle

We are actually interested in *parallel transport* in the $\widetilde{\mathrm{GL}}_{\mathrm{res}}-$bundle. The bundle-econnection defines a "horizontal" distribution in the tangent bundle of $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ which gives us the notion of a *horizontal lift* of paths in $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. A horizontal lift is a path in $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ that projects to the original path in the base-manifold $\mathrm{GL}_{\mathrm{res}}$ and whose tangent vector is always "horizontal" to the base-manifold. So, if we think of the unitary time evolution as a differentiable path in $\mathrm{U}_{\mathrm{res}} \subset \mathrm{GL}_{\mathrm{res}}$, the horizontal lift of that path will be a continuous (even differentiable) choice of implementations on the fermionic Fock space.

We can use the local expression (7.2.7) for the connection form to compute an explicit formula for parallel transport inside the domain U of the local section (4.1.3).
Let $g(t)$ be a path in $U \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, $-T \le t \le T$, with $g(-T) = \mathbb{1}$.
The lift $\tilde{g}(t) = [(g(t), q(t))]$ in $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ is horizontal if and only if

$$\operatorname{tr}\big[\dot{a}(t)\,\alpha(t) + \dot{b}(t)\,\gamma(t) - \dot{q}(t)\,q^{-1}(t)\big] \equiv 0$$

Formally, this implies

$$\operatorname{tr}\big(\dot{q}(t)\,q^{-1}(t)\big) = \operatorname{tr}\big(\dot{a}(t)\,\alpha(t) + \dot{b}(t)\,\gamma(t)\big)$$

Identifying the LHS as the logarithmic derivative of $\det(q(t))$ we can write

$$\det(q(T)) = \exp\Big[\int_{-T}^{T} \operatorname{tr}(\dot{a}(t)\,\alpha(t) + \dot{b}(t)\,\gamma(t))\,\mathrm{d}t\Big]$$

Also, formally: $\det(a(T)) = \exp\Big[\int_{-T}^{T} \operatorname{tr}(a^{-1}(t)\dot{a}(t))\,\mathrm{d}t\Big]$

Individually, the traces do not converge but put together the trace converges and gives:

$$\boxed{\det\big[a^{-1}(T)q(T)\big] = \exp\Big[-\int_{-T}^{T} \operatorname{tr}\big[\dot{a}(t)\,(a^{-1}(t) - \alpha(t)) + \dot{b}(t)\,\gamma(t)\big]\,\mathrm{d}t\Big]} \tag{7.3.1}$$

In the local trivialization ϕ on U, this is precisely the $\mathbb{C}$-component of the horizontal lift $\tilde{g}(T)$ (4.1.4). In other words, parallel transport in the $\widetilde{\mathrm{GL}}_{\mathrm{res}}-$bundle corresponds to multiplication by the right-hand-side of (7.3.1) in local coordinates.
Note, that the expression (7.2.7) is valid everywhere, while (7.3.1) makes sense only in the neighborhood U of the identity, where a is invertible.

The corresponding expression for the Mickelsson-Langmann connection is

$$\det\big[a^{-1}(T)\,q(T)\big] = \exp\Big[-\int_{-T}^{T}\mathrm{tr}\big[(\alpha(t)-a^{-1}(t))\dot{a}(t)+\beta(t)\dot{c}(t)\big]dt\Big] \tag{7.3.2}$$

For a unitary path, these factor corresponds to the phase of the lift of $g(T)$, up to normalization (cf. §4.1.1). We have said that connection and parallel transport defined on $\widetilde{\mathrm{GL}}_{\mathrm{res}}$ restricts simply to the unitary case. Still, in case someone suspects hand-waving here, we show by explicit computation that the lift of a unitary path doesn't leave $\widetilde{\mathrm{U}}_{\mathrm{res}} = \widetilde{\mathrm{GL}}_{\mathrm{res}} \cap \big(\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H}_+)\big)$.

Lemma 7.3.1 (Parallel transport stays in $\widetilde{\mathrm{U}}_{\mathrm{res}}$)**.**
Let $u(t)\,, t\in I$ be a (piecewise C^1) path in $\mathrm{U}_{\mathrm{res}}\subset\mathrm{GL}_{\mathrm{res}}$ and $\widetilde{u}(t)$ a horizontal lift of $u(t)$ with initial conditions $\widetilde{u}(0)=[U,r]\in\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})\subset\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.
Then $\widetilde{u}(t)\in\widetilde{\mathrm{U}}_{\mathrm{res}}\ \forall t\in I$, i.e. the horizontal lift stays in $\widetilde{\mathrm{U}}_{\mathrm{res}}$.

Proof. If $\widetilde{u}(t)=[u(t),q(t)]$ is the horizontal lift, consider the path

$$\hat{u}(t) := [u(t),p(t)],\ \text{with } p(t) := (q^*(t))^{-1}$$

Clearly, $\pi\circ\hat{u}(t)=u(t)$ and $\hat{u}(0)=\widetilde{u}(0)=[U,r]\in\widetilde{\mathrm{U}}_{\mathrm{res}}$. Furthermore, by (7.2.7), using the fact that $u(t)$ is unitary:

$$\begin{aligned}
0 &= \mathrm{tr}\big[(\dot{u}(t)u^{-1}(t))_{++}-\dot{q}(t)q^{-1}(t)\big]\\
&= \mathrm{tr}\big[(u(t)\dot{u^*}(t))_{++}-q^{*-1}(t)\dot{q^*}(t)\big]^{c.c.}\\
&= \mathrm{tr}\big[(u(t)\dot{u^*}(t))_{++}-p(t)\dot{p^{-1}}(t)\big]^{c.c.}
\end{aligned}$$

And using

$$\begin{aligned}
0 &= \frac{d}{dt}\big(u(t)u^*(t)\big) = \dot{u}(t)u^*(t)+u(t)\dot{u^*}(t)\\
0 &= \frac{d}{dt}(p(t)p^{-1}(t)) = \dot{p}(t)p^{-1}(t)+p(t)\dot{p^{-1}}(t)
\end{aligned}$$

we derive

$$\mathrm{tr}\big[(\dot{u}(t)u^{-1}(t))_{++}-\dot{p}(t)p^{-1}(t)\big]=0$$

Thus, $\hat{u}$ is also a horizontal lift of $u(t)$ satisfying the same initial condition.
By uniqueness of the horizontal lift it follows that $\hat{u}(t)\equiv\widetilde{u}(t)$ and thus
$\widetilde{u}(t)\in\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})\cap\big(\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H}_+)\big)=\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}),\ \forall t\in I$.

□

Chapter 8

Geometric Second Quantization

Parallel Transport was first suggested by E.Langmann and J.Mickelsson as a method to fix the phase of the second quantized scattering matrix. We will call this method *Geometric Second Quantization*. The bundle connection allows us to lift - in a unique way - paths from the base manifold $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ to the central extension $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) \subset \widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ which carries the information about the geometric phase. It is important that these horizontal lifts are determined by the connection, i.e. the geometric structure of the bundle only. We will prove that this fact ensures that the geometric second quantization is *causal*, i.e. preserves the causal structure of the one-particle Dirac theory.

We present the method of geometric second quantization in a slightly more general setting as a method of second quantization of the unitary time evolution. However, the time-evolution will always require renormalization and the physical significance of the renormalized time evolution is unclear. We will briefly address this question in the final chapter of this work.

The phase of the second quantized S-matrix, however, is certainly of great practical relevance for the further formulation of the theory, mainly for the following reason: It is well known that the *vacuum polarization* in QED is ill-defined and requires various renormalizations to be made finite. [1] One reason for this is that the current density, usually defined as

$$j^\mu(x) \ = e\overline{\Psi}(x)\gamma^\mu\Psi(x) \tag{8.0.1}$$

is not a well-defined object in the second-quantized theory.
The best definition for the current-density can be given in terms of the second quantized scattering operator $\mathbf{S}$ by (c.f.[Scha], §2.10)

$$\boxed{j^\mu(x) := i\mathbf{S}^* \frac{\delta}{\delta \mathsf{A}_\mu(x)} \mathbf{S}[\mathsf{A}]} \tag{8.0.2}$$

which equals $e : \overline{\Psi}(x)\gamma^\mu\Psi(x) :$ in first order perturbation theory. Here, $\mathbf{S}[\mathsf{A}]$ is meant as the map sending the external field $\mathsf{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ to the corresponding S-operator in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$. The current-density itself has to be understood as an *operator-valued distribution*. The **vacuum-polarization** is then well-defined as the vacuum expectation value

$$\boxed{\left\langle \Omega, j^\mu(x)\Omega \right\rangle = i \left\langle \mathbf{S}\,\Omega, \frac{\delta \mathbf{S}}{\delta A_\mu(x)}\,\Omega \right\rangle}$$

[1] See [Dys51] for a very honest computation.

or, more precisely, as a distribution evaluated at the test-function A_1,

$$\begin{aligned}\big\langle \Omega, j[A](A_1)\Omega\big\rangle =& i\,\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\big\langle \Omega, S^{-1}(A)S(A+\epsilon A_1)\Omega\big\rangle \\ =& i\,\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\log\big\langle \Omega, S^{-1}(A)S(A+\epsilon A_1)\Omega\big\rangle\end{aligned}$$

Here, the phase of the S-operator becomes practically relevant. If we separate the phase-freedom, we find

$$\mathbf{S}[\mathsf{A}] = \tilde{\mathbf{S}}[\mathsf{A}]e^{i\varphi[\mathsf{A}]} \Rightarrow \frac{\delta\mathbf{S}}{\delta A_\mu(x)} = i\frac{\delta\varphi}{\delta A_\mu(x)}\mathbf{S} + e^{i\varphi}\frac{\delta\tilde{\mathbf{S}}}{\delta A_\mu(x)} \tag{8.0.3}$$

and for the vacuum-polarization:

$$\big\langle \Omega, j^\mu(x)\Omega\big\rangle = i\Big[i\frac{\delta\,\varphi[\mathsf{A}]}{\delta A_\mu(x)} + \big\langle \tilde{\mathbf{S}}\,\Omega, \frac{\delta\tilde{\mathbf{S}}}{\delta A_\mu(x)}\,\Omega\big\rangle\Big] \tag{8.0.4}$$

The second term on the right-hand side is okay, but the first term does obviously require a well-defined prescription for the phase of the S-matrix. We will try to provide this now.

8.1 Renormalization of the Time Evolution

Our motivation for studying the parallel transport was the second quantization of the Dirac time-evolution. That is, given an external field

$$\mathsf{A} = (\mathsf{A}_0, -\underline{\mathsf{A}}) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$$

and the corresponding Dirac time-evolution $U^{\mathsf{A}}(t_1, t_0)$ between t_0 and $t_1 \in \mathbb{R}$ we want to lift the path

$$s \mapsto U^{\mathsf{A}}(t_1 + s, t_1),\ s \in [0, t_2 - t_1] \tag{8.1.1}$$

to the group $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ that acts on the Fock space. This would provide a well-defined prescription for the implementation of the time evolution on the Fock space, including phase. In particular, since the interaction has time-support contained in some interval $[-T, T]$ for T large enough,

$$t \mapsto U_I^{\mathsf{A}}(t, -T),\ t \in [-T, T] \tag{8.1.2}$$

is a path from the identity $Id_{\mathcal{H}}$ to the S-Matrix $S = U_I^{\mathsf{A}}(T, -T) = U_I^{\mathsf{A}}(\infty, -\infty)$.

However, as the theorem of Ruijsenaars (see 1.2.1 and especially 6.1.1) tells us, these paths are typically NOT in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, in fact they will leave $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ as soon as the spatial component $\underline{\mathsf{A}}$ of the interaction potential becomes non-zero. To handle this problem and be able to apply the method of parallel-transport, Langmann and Mickelsson introduced a *renormalization* of the time evolution, such that the renormalized time evolution $U_{ren}^{\mathsf{A}}(t, t')$ stays in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ $\forall t, t' \in \mathbb{R}$ and $U_{ren}^{\mathsf{A}}(T, -T) = U_I^{\mathsf{A}}(T, T) = S$.

Concretely, they prove the following:

Theorem 8.1.1 (Langmann,Mickelsson 1996)**.**
Let $\mathsf{A} \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ a 4-vector potential and $U(t, s)$ the corresponding Dirac time evolution. There is a family of unitary time evolutions $\mathrm{T}_t(\mathsf{A})$, $t \in \mathbb{R}$ such that the modified time evolution

$$\mathrm{T}_t^{-1}(\mathsf{A})\, U(t, t')\, \mathrm{T}_{t'}$$

stays in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. [2]

Moreover, $\mathrm{T}(\mathsf{A})$ can be chosen such that $\mathrm{T}_t(\mathsf{A}) = \mathbb{1}$ if $\mathsf{A}(t) = 0$ and $\partial_t\mathsf{A}(t) = 0$.

[2] Note that the roles of T and T^{-1} are interchanged in out convention compared to [LaMi96].

An explicit expression for the operators $\mathrm{T}_t(\mathsf{A})$ is given in [Mi98]. With the abbreviations $\not{\mathsf{A}} = \sum_\mu \alpha^\mu A_\mu$ and $\not{E} := \partial_t \not{\mathsf{A}} - \not{\partial}\mathsf{A}_0 + [\mathsf{A}_0, \not{\mathsf{A}}]$ it reads

$$\mathrm{T}(\mathsf{A}) = \exp\Big(\frac{1}{4}\big[D_0^{-1}, \not{\mathsf{A}}\big] - \frac{1}{8}\big[D_0^{-1}\not{\mathsf{A}}D_0^{-1}, \not{\mathsf{A}}\big] - \frac{i}{4} D_0^{-1}\not{E} D_0^{-1}\Big) \tag{8.1.3}$$

In the original paper of Langmann and Mickelsson the renormalization appears as a mere technical tool - the meaning of the unitary transformations $\mathrm{T}_t(\mathsf{A})$ is not discussed. Apart from the question of differentiability which will be the focus of the next section this meaning becomes more clear by the following considerations:

Let $U(t,t')$ by the unitary evolution for a fixed external field A and let $\mathrm{T}(t)$, $t \in \mathbb{R}$ a family of unitary operators such that $\mathrm{T}(t)^{-1} U(t,t')\,\mathrm{T}(t') \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H})$, $\forall t, t' \in \mathbb{R}$ and $\mathrm{T}(t) = \mathbb{1}$ for $|t|$ large enough. In particular, for any $r \ll 0$ outside the time-support of A this means

$$\begin{aligned} & \mathrm{T}^*(t)\, U(t,r)[\mathcal{H}_+] = [\mathcal{H}_+] \in \mathtt{Pol}(\mathcal{H})/\approx_0 \\ \iff\ & U(t,r)[\mathcal{H}_+] = \mathrm{T}(t)[\mathcal{H}_+] \in \mathtt{Pol}(\mathcal{H})/\approx_0 \\ \iff\ & U(t,r) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, [\mathcal{H}_+]; \mathcal{H}, \mathrm{T}(t)[\mathcal{H}_+]) \end{aligned}$$

Hence, by the composition property (2.1.3):

$$U(t,t') = U(t,r)U(r,t') \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, \mathrm{T}(t')[\mathcal{H}_+]; \mathcal{H}, \mathrm{T}(t)[\mathcal{H}_+])$$

We see that we can either use the T_t-operators to "renormalize" the unitary time evolution, i.e. transform it back to $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and implement it on a fixed Fock space. Or we can use them to identify the right polarization class at any time t and implement the time evolution as unitary transformations between time-varying Fock spaces as in Ch. 6 .
Recall, that the construction of a Fock space does not only involve the choice of a particular polarization class, but also the choice of a Dirac sea class (or a class of admissible bases in the geometric construction). We can use the unitary operators $\mathrm{T}(t)$ to pick out one reference state ($\approx$ vacuum) in the right polarization class at every time. In the language of the infinite wedge space construction:

Let $\Phi_0 : \ell \to \mathcal{H}_+ \in \mathtt{Ocean}(\mathcal{H}_+)$ be the Dirac sea corresponding to the vacuum state in the initial Fock space $\mathcal{F}_{\mathcal{S}_0}$ with $\mathcal{S}_0 = \mathcal{S}(\Phi_0)$. Then $\mathrm{T}(t)\Phi_0$ is a Dirac sea with image in the polarization class $[U(t,r)\mathcal{H}_+]_{\approx_0}$ and we can implement $U(t,r)$ as a unitary map between the Fock spaces $\mathcal{F}_{\mathcal{S}_0}$ and $\mathcal{F}_{\mathcal{S}_t}$, where $\mathcal{S}_t = \mathcal{S}(\mathrm{T}(t)\Phi_0)$.

The dual point of view corresponding to a "renormalization of the time evolution" would be that the T_t-operators "rotate" the Dirac seas back into the original Fock space in a well-defined manner. We express this duality in the following proposition:

Proposition 8.1.2 (Interpretation of the renormalization)**.**
Let $U^{\mathsf{A}}(t,t')$ be the unitary time evolution for an interaction potential $\mathsf{A} \in \mathcal{C}^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$. Then, every family of unitary operators $\mathrm{T}(t), t \in \mathbb{R}$ satisfying

i) $\mathrm{T}(t)^*\, U^{\mathsf{A}}(t,t')\ \mathrm{T}(t') \in \mathrm{U}_{\mathrm{res}}(\mathcal{H}), \forall t, t'$

ii) $\mathrm{T}(t)$ *has compact support in time, i.e.* $\mathrm{T}(t) = \mathbb{1}$ *for all* $|t|$ *large enough.*

determines a family of vacua for the time-evolution as in [Def. 6.2.2].
Conversely, every family of vacua determines a family of unitary operators satisfying i) and ii), modulo the equivalence relation $\mathrm{T} \sim \mathrm{T}' :\iff \mathrm{T}\,|_{\mathcal{H}_+} = \mathrm{T}'\,|_{\mathcal{H}_+}$

Proof. Let Φ_0 be the free vacuum state as in [Def. 6.2.2] (or any other fixed Dirac Sea in the initial class). Given a family of unitary operators as above, we set $\Phi^{(t)} := \mathrm{T}(t)\Phi_0$. This defines a family of vacua for the time-evolution because $\Phi^{(t)} \in \mathtt{Ocean}(C(\mathsf{A}(t)))$, $\forall t \in \mathbb{R}$ and $\Phi^{(t)} = \Phi_0$ for $|t|$ large enough.

Conversely, given a family of vacua for the time evolution we note that by definition $\Phi^{(t)} \circ \Phi_0^*(\mathcal{H}_+) \in C(\mathsf{A}(t))$, $\forall t$ and $\Phi^{(t)} \circ \Phi_0^* = \Phi_0 \circ \Phi_0^* = \mathrm{Id}_{\mathcal{H}_+}$ for $|t|$ large enough. Therefore, the operators $\Phi^{(t)} \circ \Phi_0^*$ can be extended to the whole Hilbert space by any choice of unitary maps $\mathcal{H}_- \to \mathrm{im}\big(\Phi^{(t)} \circ \Phi_0^*(\mathcal{H}_+)\big)^\perp$ and will satisfy the requirements i) and ii).
Obviously, the assignments are inverse to each other. □

With this understanding it becomes clear that such a renormalization is not at all unique, but represents a very particular choice. It seems to me that this point was not evident to Langmann and Mickelsson by the time of their 96' publication, although they pick up the issue in a later publication [Mi98]. In [LaMi96], however, they do not discuss if and how their results depend on the particular choice of the renormalization. The bad news is that it turns out that the full freedom of the geometric phase is now contained in this freedom of choosing a renormalization. It merely gets a different name: geometrically, it is described by the *holonomy group* of the principle bundle. We will make this more precise in §8.3.

To study the renormalizations more properly, we propose a general definition:

Definition 8.1.3 (Space of Vector Potentials)**.**

Let $\mathcal{A}$ be the space of 4-vector potentials equipped with a suitable topology and differentiable structure.

In our context, $\mathcal{A} = C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and we write $\mathcal{A} \ni \mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\underline{\mathsf{A}})$.

On a general space-time $\mathbb{R} \times M$ with M a compact manifold without boundary $\mathcal{A}$ corresponds to the space $\mathcal{A} = \Omega^1(\mathbb{R} \times M, \mathbb{R})$ of smooth connection one-forms[3]

By $\mathsf{A}(t)$ we always mean the function $\mathsf{A}(t, \cdot) \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$ for fixed $t \in \mathbb{R}$.

Definition 8.1.4 (Renormalization)**.**

We call a mapping $\mathrm{T} : \mathbb{R} \times \mathcal{A} \to \mathrm{U}(\mathcal{H})$ a *quasi-renormalization* if it satisfies

i) $\mathrm{T}_t(0) \equiv \mathbb{1}$

ii) $\mathrm{T}(t, \mathsf{A}) = \mathrm{T}_t(\mathsf{A}) \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, [\mathcal{H}_+], \mathcal{H}, C(\mathsf{A}(t))\big)$, $\forall t \in \mathbb{R}$

iii) $\mathrm{T}_t(\mathsf{A})$ depends only on $\mathsf{A}(t)$ and $\partial^k \mathsf{A}(t)$ for $k = 0, 1, ..., n$ and some $n \in \mathbb{N}$

We call T a *renormalization* if it has the additional property that

iv) For any $\mathsf{A} \in \mathcal{A}$, the *renormalized (interaction picture) time evolution*

$$\boxed{U^{\mathsf{A}}_{ren}(t, s) = e^{itD_0}\, \mathrm{T}^*(t)\, U^{\mathsf{A}}(t, s)\; \mathrm{T}(s) e^{-isD_0}} \tag{8.1.4}$$

is continuously differentiable in t w.r.to the differentiable structure of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

iii) formulates a requirement of *causality*. i) and iii) together imply that $\mathrm{T}_t(\mathsf{A}) = \mathbb{1}$ whenever A vanishes in some open time-interval around t. In particular, this assures that a (quasi-)renormalization doesn't alter the S-operator for compactly supported interactions.

Note that by our definition, the renormalized time-evolution is an interaction-picture time-evolution.

Examples 8.1.5 (Quasi-/Renormalizations)**.**

- The renormalization $\mathrm{T}(\mathsf{A})$ constructed in [LaMi96] is a renormalization in the sense of Def. 8.1.4 (with n=2).
- The operators $e^{Q^{\mathsf{A}(t)}}$ introduced in §6.1. provide a quasi-renormalization (with n=0), but not a renormalization (c.f. Cor. 8.1.11).

[3] more generally, the one-forms take values in the Lie algebra of the gauge group which for QED is just $\mathrm{Lie}(\mathrm{U}(1)) = \mathbb{R}$.

Unfortunately, to our knowledge these are the only two examples that have been explicitly constructed so far.

In the sense of 8.1.2, we can also interpret a quasi-renormalization as a choice of vacua or a choice of Fock spaces depending only on the A-field locally in time. In particular, if the quasi-renormalization depends only on $\mathsf{A}(t)$ and not on time-derivatives it makes sense to regard it as determining one Fock space over every polarization class $C(\mathsf{A})$.
Note that this is a slightly different perspective than choosing a family of Fock spaces for a *fixed* time-evolution.

Remark 8.1.6 (Furry picture)**.**
The most obvious way to transform the time evolution back to $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ would be simply to set $\mathrm{T}_t(\mathsf{A}) = U^{\mathsf{A}}(t, -\infty)$. Then, the renormalized time-evolution is [4]

$$U_I^*(t, -\infty)U_I(t, t')U_I(t', -\infty) = \mathbb{1},\ \forall t, t' \in \mathbb{R}$$

This approach is also known as "Furry picture". The only problem with the Furry picture: nothing's happening. The polarization, i.e what we call "particles" and "antiparticles" evolves in just the same way as the states themselves:

$$\mathcal{H} = U(t, -\infty)\mathcal{H}_+ \oplus U(t, -\infty)\mathcal{H}_- \tag{8.1.5}$$

for each instantaneous time $t \in \mathbb{R}$. What we end up caling the "vacuum" at time t is exactly the state into which the original ($t \to -\infty$) vacuum has evolved. There is no particle creation or annihilation, an empty universe remains empty. For the obvious reasons, we are not satisfied with that. By theorem 6.1.2 we can assure that the S-matrix is in $\mathrm{U}_{\mathrm{res}}$ an hence make sense of the particle/antiparticle - picture at least asymptotically. Before the interaction is switched on and after it is switched off, we can determine the particle content with respect to the same vacuum and the question of how many particles and anti-particles were created has a well-defined answer. Therefore, we will demand from the renormalization to allow us to do this, at least.

8.1.1 Renormalization - Differential Form

Now we shift our focus to the question of differentiability, which makes all the difference between a quasi-renormalization and a proper renormalization. In order to apply the method of parallel transport, we need the time-evolution to be *differentiable* - notably with respect to the differentiable structure of $\mathrm{GL}_{\mathrm{res}}$ (or $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, respectively) which is induced by the norm $\|\cdot\|_\epsilon = \|\cdot\| + \|[\epsilon, \cdot]\|_2$ on the Banach-algebra $\mathcal{B}_\epsilon(\mathcal{H})$. This requirement is not harmless. The interaction picture time-evolution is in general differentiable in the operator norm, but in addition we will need to control the Hilbert-Schmidt norms.

Renormalization of the Interaction Hamiltonians

Recall that if $U^{\mathsf{A}} = U^{\mathsf{A}}(t, t')$ is a solution of

$$i\partial_t\, U^{\mathsf{A}}(t, t') = H^{\mathsf{A}}(t)\, U^{\mathsf{A}}(t, t'),\ \ U^{\mathsf{A}}(t', t') = \mathbb{1} \tag{8.1.6}$$

with the Hamiltonian $H^{\mathsf{A}} = D_0 + e\sum_{\mu=0}^{3} \alpha^\mu A_\mu = D_0 + V^{\mathsf{A}}(t)$,
then $U_I^{\mathsf{A}}(t, t') = e^{itD_0}U^{\mathsf{A}}(t, t')e^{-it'D_0}$ solves the equivalent equation

$$i\partial_t\, U_I^{\mathsf{A}}(t, t') = V_I(t)\, U_I^{\mathsf{A}}(t, t'),\ \ U_I^{\mathsf{A}}(t', t') = \mathbb{1} \tag{8.1.7}$$

with $V_I(t) = e^{itD_0}V^{\mathsf{A}}(t)e^{-itD_0}$. This is called the *interaction picture*.

[4] here, $t = -\infty$ is understood as a large negative t outside the time-support of the interaction.

The basic idea is that in the interaction picture, the free time evolution is absorbed in the states so that the remaining time evolution is generated by the interaction-part of the Hamiltonian only. One advantage of this procedure is that $V_I(t)$, in contrast to $H^{\mathsf{A}}(t)$, is a bounded operator and thus the solution of (8.1.7) is given for all finite times by the Dyson series

$$\begin{aligned} U_I^{\mathsf{A}}(t,t') &= \sum_{n=0}^{\infty} U_n(t,t'), \\ U_0(t,t') \equiv \mathbb{1}, \quad U_{n+1}(t,t') &= -i \int_{t'}^{t} h^{\mathsf{A}}(s) U_n(s,t') \mathrm{d}s \end{aligned} \tag{8.1.8}$$

Now let $\mathrm{T}_t = \mathrm{T}_t(\mathsf{A})$ be a renormalization for this time evolution and consider the renormalized Schrödinger-picture time-evolution

$$U'(t,t') = \mathrm{T}_t^* \; U^{\mathsf{A}}(t,t') \; \mathrm{T}_{t'} = e^{-itD_0} U_{ren}^{\mathsf{A}}(t,t') e^{it'D_0}$$

Differentiation with respect to t yields:

$$\begin{aligned} i\,\partial_t U'(t,t') &= i\,(\partial_t \; \mathrm{T}_t^*)\, U'(t,t') \; \mathrm{T}_{t'} + \mathrm{T}_t^* \; H^{\mathsf{A}}(t)\, U'(t,t') \; \mathrm{T}_{t'} \\ &= \left[i\,(\partial_t \; \mathrm{T}_t^*) \; \mathrm{T}_t + \mathrm{T}_t^* \; H^{\mathsf{A}}(t) \; \mathrm{T}_t \right] \; \mathrm{T}_t^* \; U'(t,t') \; \mathrm{T}_{t'} \\ &= \left[-i \; \mathrm{T}_t^* \, (\partial_t \; \mathrm{T}_t) + \mathrm{T}_t^* \; H^{\mathsf{A}}(t) \; \mathrm{T}_t \right] U'(t,t') \end{aligned}$$

And we can rewrite

$$\left[-i \; \mathrm{T}_t^* (\partial_t \, \mathrm{T}_t) + \mathrm{T}_t^* \, H^{\mathsf{A}}(t) \, \mathrm{T}_t \right] =: \left(D_0 + i V_{ren}^{\mathsf{A}} \right)$$

with

$$\boxed{V_{ren}^{\mathsf{A}} = \left[\mathrm{T}_t^* \, V^{\mathsf{A}} \, \mathrm{T}_t + \mathrm{T}_t^* [D_0, \mathrm{T}_t] - i \, \mathrm{T}_t^* (\partial_t \, \mathrm{T}_t) \right]} \tag{8.1.9}$$

So the renormalized time evolution is generated by the Hamiltonian $H_{ren}^{\mathsf{A}} = D_0 + V_{ren}^{\mathsf{A}}$ with the "renormalized" interaction (8.1.9).

Theorem 8.1.7 (Generators of renormalized time evolution)**.**
Let $V_{(ren)}(t)$ be a (renormalized) interaction potential . We set $h(t) = e^{itD_0} V(t) e^{-itD_0}$. Let $U(t,t')(= U_{ren}(t,t'))$ be a solution of

$$\begin{cases} i\,\partial_t U(t,t') = \; h_(t)\, U(t,t') \\ U(t',t') \; = \; \mathbb{1} \end{cases}$$

in $\mathrm{U}(\mathcal{H})$ given by the norm-convergent Dyson-series (8.1.8)*.*

If $U(t,t')$ is a solution in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ w.r.to the differentiable structure induced by the norm $\|\cdot\|_\epsilon$ then $[\epsilon, V(t)] \in I_2(\mathcal{H}), \forall t \in \mathbb{R}$ i.e. $V(t)$ and $h(t)$ lie in the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ of $\mathrm{U}_{\mathrm{res}}$.

Conversely, if $[\epsilon, V(t)] \in I_2(\mathcal{H}), \forall t \in \mathbb{R}$, then $U(t,t')$ is a solution in $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ w.r.to the norm $\|\cdot\|_\epsilon$ if additionally we assume that

$$\int_{\mathbb{R}} \|[\epsilon, V(t)]\|_2 \, \mathrm{d}t < \infty \tag{8.1.10}$$

What is actually proven in [LaMi96] for the renormalization constructed by Langmann and Mickelsson is that the *renormalized interaction* stays in $\mathfrak{u}_{\mathrm{res}}$. It follows that the unitary transformations they construct are indeed a renormalization in the sense of Def. 8.1.4.

Note that for $U(t,t') \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ it would suffice that

$$\int_{t'}^{t} \big[\epsilon\,,\, h(t)\big]\mathrm{d}t \,=\, \int_{t'}^{t} \big[\epsilon\,,\, e^{iD_0 t}V(t)e^{-iD_0 t}\big]\mathrm{d}t < \infty,\ \forall t,t' \tag{8.1.11}$$

which is much less restrictive than (8.1.10).

The proof of the theorem requires some estimates.

Lemma 8.1.8 (Estimates)**.**
For the terms in the Dyson series (8.1.8) *we get the norm-estimates*

$$\|U_n(t,t')\| \leq \frac{1}{n!}\left(\int_{t'}^{t}\|V(s)\|\mathrm{d}s\right)^n,\ \forall n \geq 0 \tag{8.1.12}$$

and

$$\begin{aligned} \|[\epsilon, U_1(t,t')]\|_2 &\leq \int_{t'}^{t}\|[\epsilon,V(s)]\|_2\,\mathrm{d}s \\ \|[\epsilon, U_n(t,t')]\|_2 &\leq \frac{1}{(n-2)!}\int_{t'}^{t}\|[\epsilon,V(s)]\|_2\,\mathrm{d}s\left(\int_{t'}^{t}\|V(r)\|\mathrm{d}r\right)^{n-1},\ \forall n \geq 1 \end{aligned} \tag{8.1.13}$$

Proof. See appendix A3. □

Proof of the Theorem. First we note that $V(t)$ is Hermitian with $[\epsilon, V(t)] \in I_2(\mathcal{H})$ if and only if $V_I(t) = e^{itD_0}V(t)e^{-itD_0}$ is, because e^{itD_0} is unitary and diagonal w.r.to the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Now, suppose $U(t,t')$ is in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ for all t,t' and is differentiable in the norm $\|\cdot\|_\epsilon$, solving $i\,\partial_t\,U(t,t') = \,V_I(t)\,U(t,t')$.
Then, for any fixed t,

$$-i\,V_I(t) = \,\frac{d}{ds}\Big|_{s=0} U(t+s,t')U^*(t,t') \in T_e\mathrm{U}_{\mathrm{res}} \cong (-i)\cdot\mathfrak{u}_{\mathrm{res}}$$

Conversely, if $\int_{\mathbb{R}}\|[\epsilon,V(t)]\|_2\,\mathrm{d}t < \infty$ the Hilbert-Schmidt norm estimates in the previous Lemma show that $U(t,t') \in \mathrm{U}_{\mathrm{res}}(\mathcal{H}),\ \forall t,t'$ and

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} V_I(s)\mathrm{d}s + \mathcal{O}(|t-t'|^2) \text{ in } \mathrm{U}_{\mathrm{res}}(\mathcal{H})$$

so that indeed $i\,\partial_t\,U(t,t') = V_I(t)\,U(t,t')$ as long as $s \mapsto V_I(s)$ is continuous in $\mathfrak{u}_{\mathrm{res}}$.

□

Corollary 8.1.9 (Time Evolution always requires renormalization)**.**
Let $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\phi, -\underline{A}) \in \mathcal{A}$. *The interaction potential*

$$V(t) = e\,\alpha^\mu A_\mu = -e\,\underline{\alpha}\cdot\underline{A} + e\,\phi \tag{8.1.14}$$

is not in $\mathfrak{u}_{\mathrm{res}}$ *unless* $\underline{\mathsf{A}} \equiv 0$ *and* $\underline{\nabla}\phi \equiv 0$.
Consequently, $U_I^{\mathsf{A}}(t,t')$ *is not differentiable in the* $\|\cdot\|_\epsilon$*-norm, except for those cases.*

Proof. We apply Theorem 8.1.7: We know that $U_I^{\mathsf{A}}(t,t')$ is not even in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, unless $\underline{\mathsf{A}} \equiv 0$, and hence $V(t)$ cannot be in $\mathfrak{u}_{\mathrm{res}}$ unless $\underline{\mathsf{A}} \equiv 0$.
Now consider a purely electric potential $V(t) = e\,\phi$ where $\phi(x) = \phi(x) \cdot \mathbb{1}_{\mathbb{C}^4}$ has to be understood as the multiplication operator in $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$. Suppose ϕ is in $\mathfrak{u}_{\mathrm{res}}$, i.e. $\|[\epsilon, \phi]\|_2 \in I_2(\mathcal{H})$. This would imply that its exponential, $e^{i\phi}$ is in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. But this is a gauge-transformation which by Thm. 6.2.3 is in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ if and only if $\underline{\nabla}\phi(t,\underline{x}) \equiv 0$.
Remark: In general, it is merely true that $\int_{t'}^{t} [\epsilon, e^{iD_0 t}\phi(t,\underline{x})e^{-iD_0 t}] < \infty$ ([Ruij77]) which implies $U^{\phi}(t,t') \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ but not the required differentiability.

□

This is very bad news. We see:
Even if the time-evolution stays in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$*, it will nevertheless - except for the most trivial cases - require renormalization to be made differentiable as required for parallel transport.*

8.1.2 Outlook: On Renormalizations

Obviously, the renormalizations will play a crucial role in our further discussion and we should try to understand them a little better.

For the quasi-renormalizations, we have already understood that their main role is to identify for any external field $\mathsf{A} \in \mathcal{A}$ the right polarization classes $C(t)$ into which the time-evolution $U^{\mathsf{A}}(t,-T)$ is mapping. We also know that these polarization classes depend only on the spatial part of the electromagnetic potential at fixed time t, i.e $C(t) = C(\underline{\mathsf{A}}(t))$. Therefore, the operators $e^{Q^{\mathsf{A}(t)}}$ actually require too much information. It should be possible to construct a quasi-renormalization that depends only the spatial part $\underline{\mathsf{A}}(t)$ of the vector potentials and only at fixed times t. One could call such a quasi-renormalization *minimal* because it requires only the minimal amount of information from the A-field.

Such a quasi-renormalization would be ideal to use for a "global" choice of Dirac Sea classes (i.e. Fock spaces) for the second quantization on time-varying Fock spaces, for the "minimality" of the quasi-renormalization assures that we would change Fock spaces only if necessary, i.e. only if the polarization class really changes.

We formalize this in the following Conjecture:

Conjecture (Minimal Quasi-Renormalization)**.**
Let $\mathcal{A}_3 = C_c^{\infty}(\mathbb{R}^3, \mathbb{R}^3)$ (or $\mathcal{A}_3 = \Omega^1(M, \mathbb{R})$) be the space of time-independent space-like vector potentials. There exists a map $\mathrm{T} : \mathcal{A} \to \mathrm{U}(\mathcal{H})$ satisfying

i) $\mathrm{T}(0) = \mathbb{1}$

ii) $\mathrm{T}(\underline{\mathsf{A}}) \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, [\mathcal{H}_+], \mathcal{H}, C(\underline{\mathsf{A}})\big),\ \forall \underline{\mathsf{A}} \in \mathcal{A}_3$

This would then be a quasi-renormalization which depends only on the spatial part of the external field A, locally in time. If $\Phi_0 \in \mathtt{Ocean}(\mathcal{H}_+)$ is the free vacuum state,
$C(\underline{\mathsf{A}}) \longmapsto \mathcal{S}\big(\mathrm{T}(\underline{\mathsf{A}})\Phi_0\big)$ would define a section in the formal bundle of $\mathtt{Oceans}/\sim$ over $\mathtt{Pol}(\mathcal{H})$.

Could it be possible to define an actual renormalization which is "minimal" in this sense? The answer is **no**. And the reason is Cor. 8.1.9 saying that even for a purely electric potential the time evolution requires renormalization to become differentiable in $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

However, we could hope that there exists a renormalization which depends only on the A-field itself, locally in time, and not on its time-derivatives. Then, the renormalized time-evolution would depend only on $\mathsf{A}(t)$ and the renormalized Hamiltonian on $\mathsf{A}(t)$ and $\dot{\mathsf{A}}(t)$.

For such a renormalization we derrive:

Lemma 8.1.10 (Strictly Causal Renormalizations)**.**
Suppose $\mathrm{T} = \mathrm{T}_t(\mathsf{A})$ *was a renormalization in the sense of [Def.8.1.4] that depends only on* $\mathsf{A}(t)$ *and not an any time-derivatives. Then:*

$$\left[\, \epsilon \,, \mathrm{T}^*(\mathsf{A}(t))\, \partial_t\, \mathrm{T}(\mathsf{A}(t))\right] \in I_2(\mathcal{H}) \tag{8.1.15}$$

for all $\mathsf{A} \in \mathcal{A}$ *and all* $t \in \mathbb{R}$.

Proof. Given $\mathsf{A} \in \mathcal{A}$ we can write $\mathrm{T}_t := \mathrm{T}_t(\mathsf{A}) = \mathrm{T}(\mathsf{A}(t))$, since $\mathrm{T}_t(\mathsf{A})$ depends only on $\mathsf{A}(t)$. As T is supposed to be a renormalization, we know from Thm. 8.1.7(Generators of the renormalized time evolution) and the differential form of the renormalization (8.1.9), that it must satisfy

$$\left[\mathrm{T}_t^*\, V^{\mathsf{A}(t)}\, \mathrm{T}_t + \mathrm{T}_t^*[D_0, \mathrm{T}_t] - i\, \mathrm{T}_t^*(\partial_t\, \mathrm{T}_t)\right] \in \mathfrak{u}_{\mathrm{res}} \tag{8.1.16}$$

forall $\mathsf{A} \in \mathcal{A}$ and all $t \in \mathbb{R}$.
Now, for any fixed t we can take a vector potential $\mathsf{A}' \in \mathcal{A}$ which is constantly equal to $\mathsf{A}(t)$ in a time-intervall around t, i.e. $\mathsf{A}'(s) = \mathsf{A}(t),\ \forall\, s \in (t-\epsilon, t+\epsilon)$. But then for A', the last term in (8.1.16) vanishes at t and the first two terms must agree with those for A, because T depends only on $\mathsf{A}(t)$.
We conclude that already $\mathrm{T}^*(\mathsf{A}(t))V^{\mathsf{A}}\, \mathrm{T}(\mathsf{A}(t)) + \mathrm{T}^*(\mathsf{A}(t))[D_0, \mathrm{T}(\mathsf{A}(t))] \in \mathfrak{u}_{\mathrm{res}}$.
Thus, it must also be true that $-i\ \mathrm{T}^*(\mathsf{A}(t))\, \partial_t\, \mathrm{T}(\mathsf{A}(t)) \in \mathfrak{u}_{\mathrm{res}}$.

□

Corollary 8.1.11 ($e^{Q^{\mathsf{A}(t)}}$ is not a renormalization)**.**
The quasi-renormalization $e^{Q^{\mathsf{A}(t)}}$ *constructed in [DeDuMeScho] is not a renormalization.*

Proof. $\mathrm{T}(\mathsf{A}(t)) := e^{Q^{\mathsf{A}(t)}}$ is a quasi-renormalization which, by construction, depends only on $\mathsf{A}(t)$ and not on time-derivatives. Suppose it was not only a quasi-renormalization, but a renormalization. Then, the previous Lemma implies $[\epsilon\,, \dot{Q}(\mathsf{A}(t))] \in I_2(\mathcal{H})$.
But this cannot be true. In [DeDuMeScho] Lemma III.7 it is proven that $\dot{Q}Q$ is always Hilbert-Schmidt, but $[\epsilon, Q^{\mathsf{A}}] \in I_2(\mathcal{H})$ if and only if $\mathsf{A} = 0$.Therefore, $[\epsilon, \dot{Q}]$ is not Hilbert-Schmidt unless $\mathsf{A} \equiv 0$.[5]

□

[5] Actually, the operators Q and $\dot{Q}$ are odd w.r.to $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, so its superfluous to take the commutator with ϵ.

8.2 Geometric Second Quantization

Finally, we define the method of second quantization by parallel transport in the $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ bundle w.r.to the connection Γ_Φ defined by the one-form (7.2.3).

In the following let $\mathsf{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and $U_I^{\mathsf{A}}(t,t'),\ t,t',\in \mathbb{R}\cup\{\pm\infty\}$ the unitary (interaction picture) time evolution for the external field A. After applying a suitable renormalization T as discussed in the last section, the renormalized time-evolution $U_{ren}^{\mathsf{A}}(t,t')$ stays in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and is continuously differentiable in t with respect to the differentiable structure of $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$.

We define the lift of the renormalized time evolution to the group $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ acting on the Fock space by the following prescription:

For $t_1 \geq t_0 \in \mathbb{R}$ we define the time evolution

$$\mathfrak{U}(t,t_0),\ t \in [t_0,t_1] \tag{8.2.1}$$

between t_1 and t_0 on the Fock space as the Γ_Φ-horizontal lift of the renormalized one-particle time evolution $U_{ren}^{\mathsf{A}}(t,t_0),\ t\in[t_0,t_1]$ to $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ with initial condition $\mathfrak{U}(t_0,t_0) = \mathbb{1}_{\widetilde{\mathrm{U}}_{\mathrm{res}}}$

In other words: for $t_1 \geq t_0 \in \mathbb{R}\cup\{\pm\infty\}$ we define the lift $\mathfrak{U}(t_1,t_0) \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ of $U_{ren}^{\mathsf{A}}(t_1,t_0)$ as the parallel transport of $\mathbb{1} \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ along the path

$$s \to U_{ren}^{\mathsf{A}}(t_0+s,t_0),\ s\in[0,t_1-t_0]$$

in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ w.r.to the connection determined by Φ.

This procedure gives a well-defined lift of the renormalized time evolution to the fermionic Fock space. In particular, it fixes the geometric phase of the implementations in a smooth way.

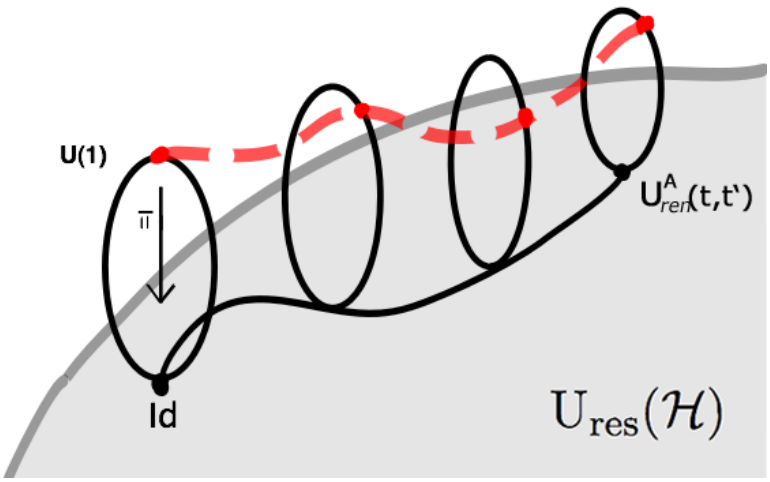


Figure 8.1: Lifting the time evolution by parallel transport.

Second Quantization of the S-matrix

Since A has compact support in time, so do $U_I^{\mathsf{A}}(\cdot,\cdot)$ and the renormalization $\mathrm{T}_t(\mathsf{A})$. Therefore, there exists $T > 0$ such that $U_{ren}^{\mathsf{A}}(t_1, t_0) = U_I^{\mathsf{A}}(T, -T)$ whenever $t_1 \geq T, t_0 \leq -T$.
In particular, for *any* such T

$$S = U_I^{\mathsf{A}}(\infty, -\infty) = U_{ren}^{\mathsf{A}}(T, -T)$$

and the S-matrix is unaltered by the renormalization. Therefore,

$$\gamma : s \longrightarrow U_{ren}^{\mathsf{A}}(-T + s, -T),\ s \in [0, 2T] \tag{8.2.2}$$

is a differentiable path from $\mathbb{1}$ to S in $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. The second quantization $\mathbf{S}$ of the scattering operator S as then defined as the parallel transport of $\mathbb{1} \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ along γ.

This procedure determines the second quantization in a well-defined manner. However, as we will see, the phase of the second quantized S-matrix will depend on the choice of the renormalization. Therefore, we can write $\mathbf{S} = \mathbf{S}[\mathsf{A}, \mathrm{T}]$.

Second Quantization of the Generators

A connection on the principle bundle allows us to lift vector fields from the base-manifold to the principle bundle in a unique way. In our setting, this means that we can lift the *renormalized interaction Hamiltonians* in a unique way to the universal covering of the Lie algebra. One might call this a second quantization of the Hamiltonians.

Theorem 8.2.1 (Second Quantization of the Hamiltonian)**.**

The connection Γ_Φ on $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ defines a unique (horizontal) lift of the renormalized interaction Hamiltonians

$$h(t) = h[\mathsf{A}, T](t) = e^{iD_0 t} V_{\mathrm{ren}}^{\mathsf{A}}(t) e^{-iD_0 t} \tag{8.2.3}$$

from $\mathfrak{u}_{\mathrm{res}}$ to $\tilde{\mathfrak{u}}_{\mathrm{res}}$. $V_{\mathrm{ren}}^{\mathsf{A}}$ is the renormalized interaction (8.1.9)*.*
These lifts generate the time evolution $\mathfrak{U}(t, t')$ on the Fock space.

Diagrammatically, if Γ denotes second quantization of unitary operators and $\mathrm{d}\Gamma$ second quantization of Hermitian operators we have

$$\begin{array}{ccc} \mathfrak{U}(t,t') & \xrightarrow{\mathrm{d}} & \mathfrak{h}(t) \\ {\scriptstyle\Gamma}\uparrow\ \downarrow{\scriptstyle\pi} & & {\scriptstyle\mathrm{d}\Gamma}\uparrow\ \downarrow{\scriptstyle\hat{\pi}} \\ U^{\mathsf{A}}(t,t') & \xrightarrow{\mathrm{d}} & h^{\mathsf{A}}(t) \end{array}$$

What is essential here (and not expressed by the diagram) is that the lift of the Hamiltonians are completely determined by the geometric structure alone.

Proof of the Theorem. We identify the Lie algebras $\mathfrak{u}_{\mathrm{res}}$ of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and $\tilde{\mathfrak{u}}_{\mathrm{res}}$ of $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ with $(-i$ times) their tangent spaces at the identity and define $\mathfrak{h}(t) \in \tilde{\mathfrak{u}}_{\mathrm{res}}$ as the horizontal lift of $h(t) \in \mathfrak{u}_{\mathrm{res}}$ with respect to the connection Γ_Φ.
Now, for fixed t', $U_{ren}^{\mathsf{A}}(t, t')$ satisfies $i\,\partial_t\, U_{ren}^{\mathsf{A}}(t, t') = h(t) U_{ren}^{\mathsf{A}}(t, t')$.
By construction, $t \mapsto \mathfrak{U}(t, t')$, $t \in [t', \infty)$ is the integral curve to the horizontal lift of the vector field $-i\, h^{\mathsf{A}}(t)\, U^{\mathsf{A}}(t, t')$ along $U^{\mathsf{A}}(t, t'), t \in [t', \infty)$. But, since the connection is right-invariant, $-i\, \mathfrak{h}(t)\, \mathfrak{U}(t, t')$ is horizontal for all t and it is obviously a lift of $-i\, h(t)\, U_{ren}^{\mathsf{A}}(t, t')$ to the tangent space of $\widetilde{\mathrm{U}}_{\mathrm{res}}$. It follows that $\mathfrak{U}(t, t')$ satisfies

$$i\,\partial_t\, \mathfrak{U}(t, t') = \mathfrak{h}(t)\, \mathfrak{U}(t, t')$$

for all $t, t' \in \mathbb{R}$ as was claimed.

□

8.2.1 Causality

For the second quantized time-evolution $\mathfrak{U}(t,t')$ we can derive the following important result:

Theorem 8.2.2 (Semigroup-structure of the time evolution)**.**
The lifted time evolution $\mathfrak{U}(t,t')$ in $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ defined by horizontal lifts as above preserves the semigroup-structure of the time evolution i.e. satisfies

$$\begin{cases} \mathfrak{U}(t,t) = \mathbb{1}_{\widetilde{\mathrm{U}}_{\mathrm{res}}} & \forall\, t \in \mathbb{R} \\ \\ \mathfrak{U}(t_2,t_1)\,\mathfrak{U}(t_1,t_0) = \mathfrak{U}(t_2,t_0) & \forall\, t_0 < t_1 < t_2 \in \mathbb{R} \cup \{\pm\infty\} \end{cases} \tag{8.2.4}$$

These properties justify its denomination as a "time evolution" on the Fock space.

Proof. $\mathfrak{U}(t,t) = \mathbb{1}, \forall\, t \in \mathbb{R}$ holds by construction. For the composition property, note that $\mathfrak{U}(t_2,t_0)$ is the end-point of the horizontal lift of

$$s \to U^{\mathsf{A}}_{ren}(t_1+s,t_0),\ s \in [0, t_2-t_1]$$

with starting point $\mathfrak{U}(t_1,t_0)$. On the other hand, consider the curve

$$s \to \mathfrak{U}(s,t_1)\,\mathfrak{U}(t_1,t_0),\ s \in [0, t_2-t_1]$$

It has the same starting point $\mathfrak{U}(t_1,t_0) \in \widetilde{\mathrm{U}}_{\mathrm{res}}$ and projects down to $\pi\big(\mathfrak{U}(s,t_1)\,\mathfrak{U}(t_1,t_0)\big) = U^{\mathsf{A}}_{ren}(s,t_1)\,U^{\mathsf{A}}_{ren}(t_1,t_0) = U^{\mathsf{A}}_{ren}(s,t_0)$ in $\mathrm{U}_{\mathrm{res}}$. Furthermore, $\frac{d}{ds}\,\mathfrak{U}(s,t_1)\,\mathfrak{U}(t_1,t_0) = (R_{\mathfrak{U}(t_1,t_0)})_*\,\dot{\mathfrak{U}}(s,t_1)$ is horizontal, because $X_{(s)} := \dot{\mathfrak{U}}(s,t_1)$ is horizontal by construction of the lift and the connection one form Φ is invariant under right-action of $\widetilde{\mathrm{U}}_{\mathrm{res}}$ i.e.

$$\Phi\big((R_{\mathfrak{U}(t_1,t_0)})_* X_{(s)}\big) = (R_{\mathfrak{U}(t_1,t_0)})^*\Phi\,(X_{(s)}) = \Phi(X_{(s)}) = 0$$

By uniqueness of the parallel transport it follows that both curves are actually the same. In particular $\mathfrak{U}(t_2,t_1)\,\mathfrak{U}(t_1,t_0) = \mathfrak{U}(t_2,t_0)$.

□

Note that with the left-invariant Langmann-Mickelsson connection, the same argument wouldn't work. We could obtain the analogous result by lifting the unitary evolution "backwards" in time, i.e. lifting the path

$$s \to U^{\mathsf{A}}_{ren}(t_1,t_1-s),\ s \in [0, t_1-t_0]$$

This seems more artificial from a physicists point of view. Therefore, we proposed the right-invariant connection. This convention fits better with the form of the time evolution where subsequent time-steps correspond to unitary operators multiplied from the left.

Theorem 8.2.2 might suggest that the additional structure of a connection on the $\widetilde{\mathrm{U}}_{\mathrm{res}}$-bundle and the technique of parallel transport are required to lift the semi-group structure. This is not quite the case, as the following proposition shows:

Proposition 8.2.3 (Lifting the semi-group structure)**.**
Let $U(t,t'), t,t' \in \mathbb{R}$ a two-parameter semi-group in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ with compact support in time.

For all $t \in \mathbb{R}$ choose any lift $\tilde{U}(t,-\infty)$ of $U(t,-\infty)$ to $\widetilde{\mathrm{U}}_{\mathrm{res}}$ and set

$$\tilde{U}(t_1,t_0) := \tilde{U}(t_1,-\infty)\,\tilde{U}(t_0,-\infty)^{-1}$$

for $t_1 \geq t_0 \in \mathbb{R} \cup \{\pm\infty\}$. The so defined lift has the semigroup properties (8.2.4)*.*
If $t \to \tilde{U}(t,-\infty)$ is continuous/differentiable then $\tilde{U}(t,t')$ is continuous/differentiable in t.

Proof. For $t_2 \geq t_1 \geq t_0 \in \mathbb{R}$ we find

$$\begin{aligned}\tilde{U}(t_2,t_1)\,\tilde{U}(t_1,t_0) &= \tilde{U}(t_2,-\infty)\,\tilde{U}(t_1,-\infty)^{-1}\,\tilde{U}(t_1-\infty)\,\tilde{U}(t_0,-\infty)^{-1}\\ &= \tilde{U}(t_2,-\infty)\,\tilde{U}(t_0,-\infty)^{-1} = \tilde{U}(t_2,t_0)\end{aligned}$$

And of course $\tilde{U}(t,t) = \tilde{U}(t,-\infty)\,\tilde{U}(t,-\infty)^{-1} = \mathbb{1}, \forall t \in \mathbb{R}$.

□

This construction is just the analogue of [Prop. 6.2.1] on a fixed Fock-space.

To fully appreciate the virtue of the geometric construction of the time evolution we need to understand the difference between the lift constructed in the previous proposition and the lift obtained by parallel-transport. Obviously, the semi-group structure, as it is so surprisingly cheap, cannot be the whole point.

The composition property

$$\tilde{U}(t_2,t_1)\,\tilde{U}(t_1,t_0) = \tilde{U}(t_2,t_0), \forall t_0 < t_1 < t_2 \in \mathbb{R} \tag{8.2.5}$$

is often related to the physical principle of *causality*. But so far, this is actually too big a word. The algebraic relationships by themselves are merely a question of consistency. They are what justifies at all the title "time evolution" for a two-parameter group of unitary operators. To relate the semi-group structure to something like a causal structure of the physical theory, we have to take a closer look at the construction of the (second quantized) time evolution and what it involves.

For the construction in Prop. 8.2.3 we first need suitable lifts of the one-particle time evolution operators $U(t,-\infty)$, $t \in \mathbb{R}$. This has the effect that the (second quantized) time evolution between two times $t_0 < t_1$ depends on the *entire history* of the system, i.e. on the A-field at times $t \in (-\infty, t_1]$. If we altered the electromagnetic fields in the distant past $t \ll t_0$ we might end up with a different phase for the time evolution between t_0 and t_1. This seems little desirable from a physical point of view. What we loose is not "causality" per se but rather what one might call the Cauchy - property [6] of the physical system: that the future evolution of the system is completely determined by the laws of physics, given the current state of the system at any time t_0. [7]

If in the construction of Prop. 8.2.3 we used a different "reference time" than $t = -\infty$ (which we could do), the situation would be even worse. The time evolution of the system before that particular reference time would then depend on the electromagnetic potential *in the future*. This, we would rightfully call a violation of causality.

In the geometric construction, the lift of the renormalized time evolution and its generators is determined solely by the connection i.e. by *the geometric structure* of the principle bundle. Thus, it depends only on the corresponding ("first quantized") objects in the one-particle theory and the renormalization used to maintain them inside $\mathrm{U}_{\mathrm{res}}$ or $\mathfrak{u}_{\mathrm{res}}$, respectively. The second quantized time evolution will therefore preserve the causal structure of the one-particle Dirac theory as well as the renormalization does. Therefore, we have required that the renormalization is causal in the sense that $\mathrm{T}_t(\mathsf{A})$ depends only on $\mathsf{A}(t)$ and its time-derivatives up to a finite order n. Using such a renormalization, the geometric construction assures that $\mathfrak{U}(t_1,t_0)$ depends on the external field $\mathsf{A}(t)$ (and its first n time-derivatives) only at times $t \in [t_0,t_1]$. Similarly, for fixed $t \in \mathbb{R}$, the lifted interaction Hamiltonian $\mathfrak{h}(t)$ depends only on $\mathsf{A}(t)$ and its first n+1 time-derivatives at t.

Translating these insights back into the language of time-varying Fock spaces, we derive:

[6] Or "Markov property", if we borrow the language of probability theory

[7] Or on a space-like Cauchy-surface in the general relativistic case.

Corollary 8.2.4 (Composition Property on time-varying Fock spaces)**.**
Given a unitary time-evolution $U^{\mathsf{A}}(t,t')$, there exists a family of infinite-wedge-spaces $(\mathcal{F}_t)_t$ and right-operations $R(t,t')$ such that

$$\widetilde{U}(t,t') := \mathcal{L}_{U(t,t')}\mathcal{R}_{R(t,t')} : \mathcal{F}_t \longrightarrow \mathcal{F}_{t'},\ \forall t,t' \in \mathbb{R}$$

and

$$\widetilde{U}(t_2,t_1)\,\widetilde{U}(t_1,t_0) = \widetilde{U}(t_2,t_0),\ \forall\, t_0 < t_1 < t_2 \tag{8.2.6}$$

Moreover, for all $t > t' \in \mathbb{R}$, the right-operation $R(t.t')$ depends on the A-field (and possibly its time-derivates) only in the time-interval $[t',t]$.

This is a little improvement over Prop. 6.2.1. However, the result is still not fully satisfying because the Fock spaces themselves are chosen by the renormalization. In particular, this choice is restricted by the condition that the renormalized time-evolution must be differentiable in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ which is obviously not a sensible requirement from the perspective of time-varying Fock spaces. In particular, we will leave the Fock space even if the time-evolution stays (but is not norm-differentiable) in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

The Causal Phase of the S-Matrix

If one cares about the S-matrix only, these causal properties of the second-quantized (renormalized) time-evolution might be of secondary interest. A better formulation of *causality* in terms of the second quantized S-matrix is the following:

Given an external field

$$\mathsf{A} = \mathsf{A}_1 + \mathsf{A}_2 \in \mathcal{A} \tag{8.2.7}$$

which splits in two parts with disjoint supports in time, i.e. $\exists\, r \in \mathbb{R}$ such that

$$\mathrm{supp}_t\ \mathsf{A}_1 \subset (-\infty, r)\ , \qquad \mathrm{supp}_t\ \mathsf{A}_2 \subset (r, +\infty) \tag{8.2.8}$$

That is, the field A_1 vanishes for times $t \geq r$ and the field A_2 for $t \leq r$.
Then, the phase of the S-matrix is *causal* if

$$\boxed{\mathbf{S}[\mathsf{A}] = \mathbf{S}[\mathsf{A}_1 + \mathsf{A}_2] = \mathbf{S}[\mathsf{A}_2]\,\mathbf{S}[\mathsf{A}_1]} \tag{8.2.9}$$

The importance of this causality-condition was stressed particularly by G.Scharf ([Scha]). But indeed, the discussed causality of the geometric second quantization does imply causality in the sense of Scharf: As the A-field vanishes around time r, so does the renormalization. Therefore:

$$U^{\mathsf{A}}_{ren}(r,-\infty) = U^{\mathsf{A}}(r,-\infty) = S[\mathsf{A}_1]$$
$$U^{\mathsf{A}}_{ren}(+\infty,r) = U^{\mathsf{A}}(+\infty,r) = S[\mathsf{A}_2]$$

Hence, by construction of the lift and Theorem 8.2.2:

$$\mathbf{S}[\mathsf{A}] = \mathfrak{U}(+\infty,-\infty) = \mathfrak{U}(+\infty,r)\mathfrak{U}(r,-\infty) = \mathbf{S}[\mathsf{A}_2]\,\mathbf{S}[\mathsf{A}_1] \tag{8.2.10}$$

We note that this finding seems to contradict the results in [Scha], where it is suggested that the phase of the second quantized scattering operator is completely determined by the causality condition (8.2.9). The geometric second quantization is causal, still there are plenty of freedoms left: we are free to choose a different $\widetilde{\mathrm{GL}}_{\mathrm{res}}$-invariant connection and/or different renormalizations of the time evolution.
How the latter can affect the phase of $\mathbf{S}$ is discussed in the following section.

8.3 Holonomy of the Bundles

We have seen that the choice of the renormalization is not unique. Therefore, the renormalized time evolution is - without additional requirements - more or less arbitrary and the urging question arises how seriously it can be taken and what it can actually tell us about the physical system. However, we may hope that the parallel transport fixes the phase of the second quantized S-matrix by the method of parallel transport. This was the result stated in [LaMi96]. But here, the ambiguity in the renormalization is also problematic.

If we use two different renormalizations to implement the S-matrix to the Fock-space via parallel transport, we have two different curves in $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, both starting at the identity and ending in S, and therefore two different horizontal lifts in $\widetilde{\mathrm{U}}_{\mathrm{res}}$. Naturally, the question arises, how the result i.e. the phase of the scattering matrix depends on the choice of the renormalization. Geometrically, this new freedom is expressed by the *holonomy group* of the bundle $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

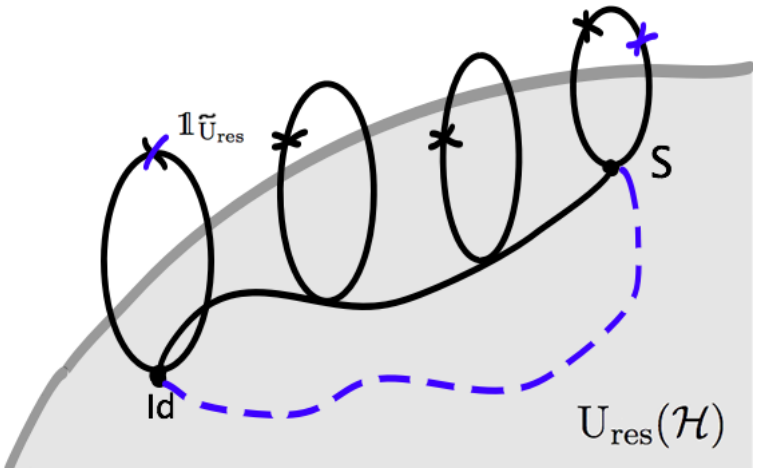


Figure 8.2: PT along different paths can end in different points in the fibre over S

Consider a principle-G-bundle $E \xrightarrow{\pi} M$ for a (finite dimensional) Lie group G, over a connected, paracompact manifold M. Let E be equipped with a connection Γ. Let u_0, u_1 two points in the base-manifold M and $c, c' : [0,1] \to M$ two piecewise differentiable curves in M from u_0 to u_1 . We take a point $p \in \pi^{-1}(u_0) \subset E$ in the fibre over the starting point u_0. We want to know if the parallel transport of p along c is the same as the parallel transport of p along c'. In other words: if the horizontal lifts of c and c', both with starting point $p \in \pi^{-1}(u_0)$, end up in the same point in the fibre $\pi^{-1}(u_1)$ above u_1. Equivalently, we can look at the closed loop $\gamma := c \circ (c')^{-1}$ in M, resulting from moving along the c first and then following c' backwards and ask, if the parallel transport of p along γ is p again or not. As the loop starts and ends in u_0, the parallel transport $P_\gamma(p)$ is certainly a point in the same fibre $\pi^{-1}(u_0)$ over u_0. Therefore, there exists a unique element $g := \mathrm{hol}(\gamma)$ in the structure group G with $P_\gamma(p) = p \cdot g$.

This motivates the following definition:

Definition 8.3.1 (Holonomy Group)**.**
Let G a finite dimensional Lie group and $E \xrightarrow{\pi} M$ a principle G bundle with connection Γ. For every point $p \in P$ we define

$$\mathrm{Hol}(\Gamma, p) := \{g \in G \mid \exists \,\text{closed loop } \gamma \text{ around } \pi(p) \text{ in } M \text{s.t. } P_\gamma(p) = p \cdot g\}$$

$\mathrm{Hol}(\Gamma, p)$ is a subgroup of G called the *holonomy group* of Γ at p.

Furthermore, we define the *restricted holonomy group*

$$\mathrm{Hol}_0(\Gamma, p) := \{g \in \mathrm{Hol}(\Gamma, p) \mid \text{the corresponding } \gamma \text{ can be chosen to be null-homotopic}\}$$

by restricting to null-homotopic curves.

In our case, the base-manifold $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ or $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, respectively, is simply-connected and therefore Hol and Hol_0 are the same.

It is easy to check that the holonomy group is indeed a group. Concatenation of two loops results in multiplication of the corresponding group elements. Parallel transport along the constant path yields the identity, and if $P_\gamma(p) = p \cdot g$, then $P_{\gamma^{-1}}(p) = p \cdot g^{-1}$, where γ^{-1} is the loop obtained by reversing the parameterization of γ, i.e. following γ backwards.
The holonomy groups at two points which can be connected by parallel transport are conjugated to each other, because, if p, q can be connected by a horizontal curve in E, we can parallel transport from q to p, then along a loop around p and back from p to q, which corresponds to parallel transport along a loop around q.

Now, we are going to show that the holonomy group of the bundle $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ at the identity equals the entire structure group U(1). We could perform the same calculation for the $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ bundle, but as we are mainly interested in lifting paths in $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ we find the unitary case more educative.

It will actually suffice to consider loops in a two-dimensional subspace. Thus, we will do the computations in $\mathrm{U}(2, \mathbb{C})$ for simplicity and embed $\mathrm{U}(2, \mathbb{C})$ into $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ in the following way: Let $(e_k)_{k \in \mathbb{Z}}$ be a basis of $\mathcal{H}$ such that $(e_k)_{k \geq 0}$ is a basis of $\mathcal{H}_+$ and $(e_k)_{k<0}$ a basis of $\mathcal{H}_-$. Now can identify U(2) with $\mathrm{U}(\mathrm{span}(e_0, e_{-1}))$ i.e.

$$\mathrm{U}(2, \mathbb{C}) \hookrightarrow \mathrm{U}_{\mathrm{res}}(\mathcal{H}); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} a & & b & \\ & \mathbb{1} & & \mathbf{0} \\ \hline c & & d & \\ & \mathbf{0} & & \mathbb{1} \end{array}\right)$$

where the identity matrices are on $(e_0)^\perp \subset \mathcal{H}_+$ and $(e_{-1})^\perp \subset \mathcal{H}_-$, respectively.

A general (piecewise differentiable) path in U(2) has the form

$$U(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

with

$$U^{-1}(t) = U^*(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} = \begin{pmatrix} a^*(t) & c^*(t) \\ b^*(t) & d^*(t) \end{pmatrix}$$

Unitarity requires (among others) $|a|^2 + |b|^2 = 1$.

The formula (7.3.1) for the parallel transport in local coordinates becomes

$$\exp\Big[-\int_{-T}^{T}\mathrm{tr}\big[\dot a(t)(a^{-1}(t)-\alpha(t))+\dot b(t)\gamma(t)\big]dt\Big]=\exp\Big[-\int_{-T}^{T}\big[\dot a(t)(a^{-1}(t)-a^{*}(t))+\dot b(t)b^{*}(t)\big]dt\Big] \tag{8.3.1}$$

We can write

$$\begin{aligned} a(t)&=r(t)\,e^{i\varphi(t)}\\ b(t)&=\sqrt{1-r^2(t)}\,e^{i\psi(t)}\end{aligned}$$

with $r(t),\varphi(t)$ and $\psi(t)$ piecewise differentiable, real functions. Our path has to stay in the neighborhood where $a(t)$ is invertible, i.e. $r(t)\neq 0$ is required. Note that $|r(t)|$ has to be ≤ 1, so $\sqrt{1-r^2(t)}$ is a real, differentiable function.
Actually, we won't even have to exploit the freedom of choosing $\psi(t)$ and can set it to zero. Then:

$$\begin{aligned}\dot a(t)&=\dot r(t)e^{i\varphi(t)}+i\dot\varphi(t)r(t)e^{i\varphi(t)}\\ \dot a(t)a^{*}(t)&=\dot r(t)r(t)+i\dot\varphi(t)r^2(t)\\ \dot a(t)a^{-1}(t)&=\dot r(t)r^{-1}(t)+i\dot\varphi(t)\\ \dot b(t)b^{*}(t)&=-r(t)\dot r(t)\end{aligned}$$

The argument of the exponential in (8.3.1) is thus

$$\begin{aligned}&\int\Big[\dot a(t)a^{*}(t)-\dot a(t)a^{-1}(t)-\dot b(t)b^{*}(t)\Big]\mathrm{d}t\\ =&\int\Big[i\,\dot\varphi(t)(r^2(t)-1)-r^{-1}(t)\dot r(t)\Big]\mathrm{d}t\end{aligned} \tag{8.3.2}$$

Now, note that the second summand is just the derivative of $\log[r(t)]$ and gives no contribution when integrated over a closed loop. We're left with

$$\exp\Big[\oint\big[i\,\dot\varphi(t)(r^2(t)-1)-r^{-1}(t)\dot r(t)\big]\mathrm{d}t\Big]=\exp\Big[i\oint\big[\dot\varphi(t)(r^2(t)-1)\big]\mathrm{d}t\Big]$$

for a closed path (around the identity) which can take any value in U(1). [8] In other words, parallel transport along a closed path (about the identity) can result in multiplication by any complex phase. The holonomy group is the whole U(1). With the same method we can show that the holonomy group of the $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$-bundle is also the entire structure group $\mathbb{C}^{\times}$.

We summarize:

Proposition 8.3.2 (Holonomy Groups)**.**
The holonomy groups of the connection Γ_Φ on the principle-bundle $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$ and of its restriction to $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ are

$$\mathrm{Hol}(\widetilde{\mathrm{GL}}_{\mathrm{res}},\Gamma_\Phi)=\mathrm{Hol}_0(\widetilde{\mathrm{GL}}_{\mathrm{res}},\Gamma_\Phi)=\mathbb{C}^{\times}$$

and

$$\mathrm{Hol}(\widetilde{\mathrm{U}}_{\mathrm{res}},\Gamma_\Phi)=\mathrm{Hol}_0(\widetilde{\mathrm{U}}_{\mathrm{res}},\Gamma_\Phi)=\mathrm{U}(1).$$

[8] E.g. parameterize over $t\in[-\pi,\pi]$. We can take $\dot\varphi(t)\equiv k$, for any $k\in\mathbb{Z}$ and for $r^2(t)$ there are smooth, positive functions with $r^2(-\pi)=r^2(\pi)=1$, such that $\int_{-\pi}^{\pi}(r^2(t)-1)\mathrm{d}t$ takes any desired value between -2π and 0.

This means that the whole freedom of the geometric phase that we have eliminated by parallel transport is being reintroduced through the ambiguity of the renormalization, just under a new name: holonomy.

We remark that on a finite-dimensional bundle we could have immediately derived this result using the *Ambrose-Singer Theorem*, which says that the Lie algebra of the holonomy group is generated by the curvature two-form. Unfortunately, generalization of this result to infinite-dimensions is usually problematic.

We can use an explicit formula to compute how the parallel transport along two curves differ. In coordinates defined by a local section σ, the holonomy-group element $\mathrm{hol}(\gamma)$ corresponding to parallel transport along a closed loop γ w.r.to the connection one-form Φ can be computed as

$$\mathrm{hol}(\gamma) = \exp\Big[\oint_{\gamma} (\sigma^*\Phi)\Big] = \exp\Big[\int_{S(\gamma)} (\sigma^*\Omega)\Big] \tag{8.3.3}$$

where $S(\gamma)$ is the surface enclosed by γ and $\Omega = \mathrm{d}\Phi$ is the curvature 2-form. The first equality follows immediately from the local expression for parallel transport, whereas the second equality is an application of Stokes theorem using $\mathrm{d}\Phi = \Omega$.
Note that application of this formula is unproblematic even on infinite-dimensional manifolds, because integration is just along 1-dimensional curves or 2-dimensional surfaces.

If we use two different renormalizations T and T$'$ to lift the S-matrix by parallel transport along a path $\gamma = \gamma[\mathrm{T}]$ as in (8.2.2) we will find that

$$\mathbf{S}[\mathsf{A}, \mathrm{T}'] = \mathrm{hol}(\gamma[\mathrm{T}'] \circ \gamma[\mathrm{T}]^{-1})\, \mathbf{S}[\mathsf{A}, \mathrm{T}] \tag{8.3.4}$$

and the phase-difference can be computed from (8.3.3).

8.4 Outlook: Gauge Invariance

We have already seen that gauge-transformations are not implementable on the Fock space. We know that this must be true, because a gauge-transformation

$$\mathcal{G} \ni g : \Psi(x) \to e^{i\,\Lambda_g(x)}\Psi(x),\ \Lambda_g \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$$

changes the spatial component of the A-field and therefore the polarization class.
This fact is troubling, but it's not necessarily a disaster. It just tells us that we might have to sacrifice the naive idea about gauge invariance that the symmetry translates directly from the one-particle theory to the multi-particle theory. At this level of description, the significance of a gauge transformation in the second quantized theory would be unclear anyways. Thus, one should try to figure out what objects in the theory can be taken seriously and are required to be gauge-invariant. If we want to take the S-matrix seriously, for example, we should require that its gauge-invariance carries over from the one-particle theory to the second quantized theory. This is in fact of particular importance, because then:

$$\mathbf{S}[\mathsf{A}_\mu] = \mathbf{S}[\mathsf{A}_\mu - \epsilon\partial_\mu\Lambda],\ \forall\Lambda \in C_c^\infty(\mathbb{R}^4, \mathbb{R}) \tag{8.4.1}$$

implies by differentiation w.r.to ϵ at $\epsilon = 0$

$$0 = -\int \mathrm{d}x\, \frac{\delta}{\delta\mathsf{A}_\mu(x)}\mathbf{S}[\mathsf{A}]\,\partial_\mu\Lambda = \int \mathrm{d}x\,\Lambda(x)\partial_\mu \frac{\delta}{\delta\mathsf{A}_\mu(x)}\mathbf{S}[\mathsf{A}] \tag{8.4.2}$$

and thus, with the definition (8.0.2) of the current density:

$$\boxed{\partial_\mu \frac{\delta\mathbf{S}}{\delta\mathsf{A}_\mu(x)} = \partial_\mu\, j^\mu(x) \equiv 0} \tag{8.4.3}$$

So, the gauge-invariance of the second-quantized S-matrix *is* physically significant because it implies the continuity equation (8.4.3) for the current density.
(Actually, equation (8.0.4) shows that a much weaker condition would suffice.
The current density will be gauge-invariant if the first distributional derivative $\frac{\delta\varphi}{\delta A(x)}$ of the phase of $\mathbf{S}$ is gauge-invariant. Then, the continuity equation follows analogously from $j^\mu[\mathsf{A}_\mu] = j^\mu[\mathsf{A}_\mu - \epsilon\partial_\mu\Lambda]$.)

The geometric second quantization by parallel transport is -a priori- not gauge-invariant. Although the one-particle S-operator is invariant under compactly supported gauge transformations, the unitary time evolution and the renormalization are *not*. Therefore, if we use parallel transport to lift the S-matrix to $\widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ once for the external field $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and once for $\mathsf{A}' = \mathsf{A}_\mu - \partial_\mu\Lambda,\ \Lambda \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$ we will perform the parallel transport along different paths in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and again, the lifted S-matrix can differ by any complex phase.[9] However, we suggest that it might be possible to define the renormalization precisely in such a way that the gauge-transformation of the renormalization and of the time evolution cancel out. We will call such a renormalization *gauge-covariant*.

Suppose that a (Schrödinger picture) renormalization $\mathrm{T} : \mathbb{R} \times \mathcal{A} \to \mathrm{U}(\mathcal{H})$ satisfies

$$\mathrm{T}\big(\mathsf{A}(t) - \partial_\mu\Lambda(t,\underline{x})\big) = e^{i\Lambda(t,\underline{x})}\ \mathrm{T}\big(\mathsf{A}(t)\big),\ \forall\Lambda \in C_c^\infty(\mathbb{R}^4, \mathbb{R}) \tag{8.4.4}$$

By Theorem 6.2.3 this is compatible, with the requirements on T [10].

[9] An explicit computation of this gauge-anomaly is carried out [LaMi96].

[10] By [Thm. 6.2.3] : $e^{i\Lambda(t,x)} \in \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, C(\underline{\mathsf{A}}), \mathcal{H}, C(\underline{\mathsf{A}} + \underline{\nabla}\Lambda)\big)$ and so $e^{i\Lambda(t,x)}\ \mathrm{T}(\underline{\mathsf{A}})$ is in $\mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, C(\underline{\mathsf{A}}), \mathcal{H}, C(\underline{\mathsf{A}} + \underline{\nabla}\Lambda)\big) \cdot \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, [\mathcal{H}_+], \mathcal{H}, C(\underline{\mathsf{A}})\big) = \mathrm{U}^0_{\mathrm{res}}\big(\mathcal{H}, [\mathcal{H}_+], \mathcal{H}, C(\underline{\mathsf{A}} + \underline{\nabla}\Lambda)\big)$.

Then, the renormalized (Schrödinger picture) time evolution was invariant under gauge-transformation because

$$\begin{aligned} U^{\mathsf{A}}_{\mathrm{ren}}(t,t') &= \mathrm{T}^*(\mathsf{A}(t))\, U^{\mathsf{A}}(t,t')\ \mathrm{T}(\mathsf{A}(t')) \\ &\xrightarrow{g} U^{\mathsf{A}-\partial\Lambda}_{ren}(t,t') = \mathrm{T}^*\big(\mathsf{A}(t)-\partial_\mu\Lambda(t)\big)\, U^{\mathsf{A}-\partial\Lambda}(t,t')\, T\big(\mathsf{A}(t')-\partial_\mu\Lambda(t)\big) \\ &= \mathrm{T}^*(\mathsf{A}(t))\, e^{-i\Lambda(t,\underline{x})}\, U^{\mathsf{A}-\partial\Lambda}(t,t')\, e^{i\Lambda(t,\underline{x})}\ \mathrm{T}(\mathsf{A}(t')) \\ &= \mathrm{T}^*(\mathsf{A}(t))\, U^{\mathsf{A}}(t,t')\ \mathrm{T}(\mathsf{A}(t')) = U^{\mathsf{A}}_{ren}(t,t') \end{aligned}$$

So with a gauge-covariant renormalization -if one exists- the lift of the S-matrix would not change under gauge-transformation, but neither would the lift of the time evolution for intermediate times. I would find this to be a nice turn on gauge-symmetry. The renormalization would de facto pick out one representative of the gauge-class (deRham cohomology class) of the vector potential and we wouldn't see gauge-transformations in the second quantized theory at all.

Note that such a gauge-covariant renormalization would in particular act by "gauging the field away" whenever this is possible, which seems like a sensible approach. However, such a renormalization hasn't been explicitely constructed yet.

Conjecture (Gauge Invariant Renormalization)**.**
There exist gauge-invariant renormalizations satisfying (8.4.4).

A quasi-renormalization can in principle depend on the spatial component only and might be required to act in a similar way by gauging away the vector-part of the A-field.

Remark 8.4.1 (Hodge Decomposition)**.**
For future discussions it might be interesting to know how restrictive the condition of "gauge-covariance" is on the renormalization. This is easiest to answer in a setting where $\mathcal{A}$ is realized as the space of smooth one-forms $\Omega^1(M,\mathbb{R})$ on a closed manifold M.
Then, we have the *Hodge decomposition*:

$$\Omega^1(M) = \mathrm{im}(\mathrm{d}) \oplus \mathrm{im}(\delta) \oplus \mathrm{Harm}^1(M)$$

with

· $\mathrm{im}(\mathrm{d})$ the space of exact one-forms $\{df \mid f \in C^\infty(M)\}$

· $\mathrm{im}(\delta)$ the space of co-exact one-forms $\{\delta\beta \mid \beta \in \Omega^2(M)\}$

· $\mathrm{Harm}^1(M)$ the space of harmonic one-forms $\{\alpha \in \Omega^1(M) \mid \Delta\alpha = 0\}$

The condition of gauge-covariance would then determine the renormalization only on exact one-forms, i.e. on $\mathrm{im}(d) \subset \Omega^1(M)$ by $\mathrm{T}(\mathrm{d}\Lambda) = e^{i\Lambda}$.

Chapter 9

Résumé

We have presented two methods for second quantization of the unitary time evolution in the external field setting of QED. Given the inevitable problems that we have encountered, the two alternatives seem to be the best that can be done in the existing framework. In chapter 6 we followed [DeDuMeScho] and realized the time evolution as unitary transformations between time-varying Fock spaces. In chapter 8 we introduced the concept of a quasi-renormalization used to map the time-evolution back to $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ which makes it implementable on the standard Fock space. In both cases, the lifts of the time evolution operators to the Fock space are unique up to a "geometric phase". We have proved that the two descriptions are dual to each other and that we can use the (quasi-)renormalization to translate between them. We have also shown how the theorems about the implementability of unitary transformations on those Fock spaces can be related to well-known results from representation theory (chapter 5). In chapter 7 we have introduced the "Langmann-Mickelsson" connection on the principle bundle $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. Parallel transport w.r.to this connection defines a unique second quantization of a family of unitary operators including a differentiable prescription for the phase. However, if we want to apply the method of parallel transport to the unitary time-evolution we will always (except for the most trivial cases) need a renormalization in the sense of Def. 8.1.4 which makes the renormalized time-evolution differentiable w.r.to to the differentiable structure in $\widetilde{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H})$. Still, in a certain sense, compared to the construction [DeDuMeScho] we have eliminated the U(1)-freedom of the geometric phase through the additional structure of a bundle-connection. But the fact that we have to renormalize the time evolution to make it differentiable even for pure electric potentials, when it *is* already implementable, is a serious drawback. For once, because it means that we cannot use the method of "geometric second quantization" to lift those time-evolutions (without spoiling them by a renormalization). Secondly, because translating back into the formalism of time-varying Fock spaces, the requirement of differentiability has no meaningful analogue and the renormalization will correspond to a rather peculiar choice of Fock spaces (infinite-wedge-spaces). In particular, the Fock spaces will change, even if the polarization class doesn't.

One advantage of the renormalized theory might be that it allows second quantization of (renormalized) Hamiltonians, which are uniquely determined by the bundle-connection. However, the renormalization introduces a bunch of artificial terms in the renormalized Hamiltonian (8.1.9) and so the physical relevance of those objects is unclear.

It is important to realize that at this point (which is more or less the status after [LaMi96] and [DeDuMeScho]) the ambiguities in the construction of the time evolution are too vast to provide meaningful physical content. In the second-quantization procedure on time-varying Fock spaces this fact could be somewhat concealed because the intuition that comes with it might suggest that the physical state is what it is (modulo phase) and only the mathematical space that inhibits it requires additional specification. But in the (quasi-)renormalized theory, this ambiguity translates into the freedom to lift *any* unitary operator to the (fixed) Fock space as long as it stays in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and therefore -without additional structure- the time evolution doesn't really tell us anything. In other words: we do not even know what physical quantities should characterize the states represented on different Fock spaces. Without additional ingredients in the theory specifying the (instantaneous) vacuum states, we cannot even say how many particles and anti-particles exist at a given time.[1]

The situation is better in the asymptotic case, if we study the S-matrix only. Before the interaction is turned on and after it is turned off we are in the standard polarization class of $\mathcal{H}_+$ with a more or less canonical construction of the Fock space and so we can compare "in states" (at $t = -\infty$) and "out states" (at $t = +\infty$) with respect to the same vacuum. Second quantization of the S-matrix and not of the entire time evolution is the problem that was actually treated in [LaMi96]. Yet, as we have argued in §8.3. the result stated in [LaMi96] that *"the phase is uniquely determined ... by the geometric structure of the central extension of the group of one-particle (renormalized) time evolution operators"* is too optimistic. Different renormalizations will lead to different phases of the second quantized S-matrix. The ambiguity is expressed by the holonomy group of the $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ bundle which we have computed to equal the entire structure group U(1). Therefore, if we don't take the time evolution for intermediate times (i.e. the path from $\mathbb{1}$ to S) seriously at all, the entire freedom of the geometric phase supposedly eliminated by the parallel transport reappears in different disguise. We emphasize that the authors themselves have revised their initial statement and addressed this fact in a later publication [Mi98].

We also want to point out that the connection itself constitutes a choice and an additional structure that we impose on the theory. We have presented the Langmann-Mickelsson connection (or its right-invariant analogue) as arguably the nicest and most natural alternative but we should keep in mind that the connection is not god-given. We have specified this freedom in §7.2.1.

In conclusion, we must note that the connection alone does not fix the geometric phase in a unique way. However, the construction has other important benefits. The most positive result of this work is probably that second quantization by parallel transport preserves the semi-group structure of the time-evolution and the causal structure of the one-particle theory. In particular, the phase of the second quantized S-matrix is causal in the sense of Scharf ([Scha]). This fact was also mentioned before in [Mi98], however without proof.

[1] This is one of the big problems of Quantum field theory in general. QFT on curved space-time is a good conscious raiser for this issue. Also spectacular phenomena like the Unruh effect or Hawking radiation can be traced back to it.

So, is there a time evolution in QED?

At the moment the answer is clearly **no**. The construction on time-varying Fock spaces as well as the "renormalized theory" (with or without parallel transport) are too arbitrary to provide a physically meaningful time evolution.

In order to make physical sense of the existing framework we either need additional ingredients to reduce the ambiguities in the presented constructions or we need to identify physical quantities that are meaningful and well-defined *despite* all those ambiguities.
For the first alternative, we suggest that further study of the renormalizations might be a good way to proceed along the paths developed in this work. Concretely, we suggest that it might be possible to reduce the ambiguities by posing adequate requirements on the choice of the (quasi-)renormalization. Those requirements should be motivated by physical principles. *Gauge invariance* and *Lorentz invariance* are the most evident features that should be incorporated in the construction (c.f. §8.4). However in all honesty I must say that even if future results along those lines would turn out as positive as one can hope for, I would find the solutions very constructed and ultimately unsatisfying from a physical and an aesthetical point of view.

The results of this work might very well be undestood as sustaining the widespread believe that *not the time evolution but only the S-matrix should be taken seriously in relativistic Quantum field theory.* Within the boundaries of the existing theory, there is certainly truth to that. However, as a matter of principle I would insist that a fundamental theory has to be able to describe the universe as it is *right now* and not just the change between $t = -\infty$ and $t = +\infty$. Therefore, I find this position unacceptable as a fundamental standpoint.
A good argument for that philosophy that could make me change my mind would be for example if we found that the reason for the failure of the time evolution was relativity. I.e. if we found that the relationship between different states of a Quantum mechanical system must be purely operational because Lorentz invariance doesn't allow an evolution w.r.to a fixed time-scale. But our situation is nothing of this kind.

It is very important to note the following: the one and only reason why things work better in the asymptotic case is because we have assumed that then *the interaction is turned off*.[2] In my opinion, this points only to the fact that the theory itself is deeply and fundamentally flawed. [3]

[2] Or at least falling off quickly, if the regularity conditions on the fields are weakened.

[3] To my knowledge, there is no rigorous formulation of the fully quantized theory, but I am not aware of any reason why things should get better better if the electromagnetic field is quantized and treated as a further dynamic quantity. Rather the opposite seems to be the case.

Appendix A

A.1 Commensurable Polarizations and Polarization Classes

Definition (Commensurability)**.**
Two polarizations $V, W \in \mathtt{Pol}(\mathcal{H})$ are called *commensurable* if $V \cap W$ has finite codimension in both V and W.

Proposition (Commensurable Polarizations and Polarization Classes)**.**
Let $V \in \mathtt{Pol}(\mathcal{H})$. We denote by $\mathrm{Gr}(\mathcal{H}, V)$ the restricted Grassmannian of V i.e. its polarization class endowed with the structure of a complex Hilbert-manifold modeled on $I_2(V, V^\perp)$. Then the set of all polarizations $W \in \mathtt{Pol}(\mathcal{H})$ commensurable with V is dense in $\mathrm{Gr}(\mathcal{H}, V)$ and for any such W, $\mathrm{charge}(V, W)$ coincides with (1.2.7), i.e.

$$\mathrm{charge}(V, W) = \mathrm{ind}(P_V|_{W \to V}) = \dim(V/(V \cap W)) - \dim(W/(V \cap W))$$

Proof. Let $W \in \mathtt{Pol}(\mathcal{H})$ commensurable with V.
i) Claim: $P_W - P_V$ is of Hilbert-Schmidt type:
Obviously, $P_W - P_V$ is zero on $V \cap W$ as well as on $(V + W)^\perp$, thus, if V and W are commensurable, $P_W - P_V$ is non-zero on a finite dimensional subspace only and therefore of Hilbert-Schmidt type.
ii) Claim: $\mathrm{charge}(V, W) = \dim(V/(V \cap W)) - \dim(W/(V \cap W))$:
Write $W = (W \cap V) \oplus \ker(P_V|_W) \oplus R$. Then, $\dim(\ker(P_V|_W) \oplus R) < \infty$ and $\mathrm{ind}(P_V|_W) = \dim\ker(P_V|_W) - \dim\mathrm{coker}(P_V|_W) = \dim(W/((V \cap W) \oplus R)) - \dim(V/((V \cap W) \oplus P_V(R))) = \dim(W/(V \cap W)) - \dim(V/(V \cap W))$
since $P_V|_R$ is a finite dimensional isomorphism.
iii)Claim: $\{W \in \mathtt{Pol}(\mathcal{H}) | W$ commensurable with $V\} \subset \mathrm{Gr}(V)$ is dense.
We have to make use of the manifold structure of $\mathrm{Gr}(\mathcal{H})$ introduced in §2.3.
Define $W' := \mathrm{im}(P_V|_W) \oplus \ker(P_V|_W)$. W' is commensurable with V, since $P_V|_W$ is a Fredholm operator and this implies $\dim(W/(V \cap W')) = \dim\ker(P_V|_W) < \infty$ as well as $\dim(V/(V \cap W')) = \dim\mathrm{coker}(P_V|_W) < \infty$. Now, there is an operator $T : W' \to (W')^\perp$ with $W = \mathrm{Graph}(T) \subset \mathcal{H}$. This operator is given by:

$$T|_{\ker(P_V|_W)} = 0 \text{ and } T|^\perp_{\ker(P_V|_W)} = (P_W - P_V)|^\perp_{\ker(P_V|_W)}$$

Thus, T is a Hilbert-Schmidt operator (i.e. W lies in the domain of the chart around W' as defined in (2.3.1)). Now, if we note that the graph of any finite-rank operator $G : W' \to (W')^\perp$ is also commensurable with V, the claim follows directly from the fact, that any Hilbert-Schmidt operator can be approximated by finite-rank operators (in the Hilbert-Schmidt norm, which is the relevant norm here). □

A.2 Derivation of the Charge Conjugation

We present a general derivation of the charge conjugation operator $\mathcal{C}$ mapping negative energy solutions of the Dirac equation to positiv energy solutions with opposite charge.

More precisely, we want a transformation $\mathcal{C}$ satisfying

i) $\mathcal{C}H(e)\mathcal{C}^{-1} = -H(-e)$

ii) $i\hbar\frac{\partial}{\partial t}\Psi = H(e)\Psi \iff i\hbar\frac{\partial}{\partial t}\mathcal{C}\Psi = H(-e)\mathcal{C}\Psi$

for H(e) the Dirac-Hamiltonian with charge e and any eigenstate Ψ.
i) and ii) imply

$$i\hbar\frac{\partial}{\partial t}\mathcal{C}\Psi = -\mathcal{C}H(e)\Psi$$

For an eigenstate $\Psi \neq 0$ this is only possible, if $\mathcal{C}$ is *anti-linear*. We can thus write

$$\mathcal{C}\Psi = \mathcal{C}\Psi^{cc.}$$

(cc. denotes complex conjugation).

Now we take a look at the Hamiltonian:

$$H(e) = -i\underline{\alpha}\cdot\underline{\nabla} - e\underline{\alpha}\cdot\underline{A} + m\beta + e\Phi$$

and observe that in order to satisfy i), we need

$$\beta C = -C\beta^{cc.}\ ;\ \alpha_k\, C = C\,\alpha_k^{cc.} \tag{A.2.1}$$

This, together with the (anti-) commutation relations for the α matrices (or γ matrices, respectively) are enough to determine the form of the charge conjugation operator in any given representation. In particular, note that

$$\{\gamma^i, \alpha^j\} = 0, \text{ for } i = 0 \text{ or } i = j$$
$$\left[\gamma^i, \alpha^j\right] = 0, \ else$$

We look at the two most common examples:

Standard Representation: Only α^2 (and thus γ^2) is imaginary, all the other matrices are real. Thus $C = const.\,\gamma^2$. Conventionally: $C = i\gamma^2 = i\beta\alpha^2$

Standard Representation: All the γ's are imaginary. In particular, β is purely imaginary and all the $\alpha-$matrices are real. Hence, looking at (A.2.1), we see that we can take $C = \mathbb{1}$, i.e. charge-conjugation is just complex conjugation.

A.3 Miscellaneous

Lemma A.3.1 (For use in (5.1.21))**.**
For all $U \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ it is true that

$$\dim\ker(U_{++}) = \dim\ker(U_{--}^*)$$
$$\dim\ker(U_{--}) = \dim\ker(U_{++}^*)$$

Proof. With respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we write $U = U_{\mathrm{even}} + U_{\mathrm{odd}}$, with U_{even} the diagonal parts and U_{odd} the off-diagonal parts. $UU^* = U^*U = \mathbb{1}$ then implies

i) $(U^*U)_{\mathrm{even}} = U_{\mathrm{even}}^* U_{\mathrm{even}} + U_{\mathrm{odd}}^* U_{\mathrm{odd}} = \mathbb{1}$

ii) $(U^*U)_{\mathrm{odd}} = U_{\mathrm{even}}^* U_{\mathrm{odd}} + U_{\mathrm{odd}}^* U_{\mathrm{even}} = 0$

iii) $(UU^*)_{\mathrm{even}} = U_{\mathrm{even}} U_{\mathrm{even}}^* + U_{\mathrm{odd}} U_{\mathrm{odd}}^* = \mathbb{1}$

iv) $(UU^*)_{\mathrm{odd}} = U_{\mathrm{even}} U_{\mathrm{odd}}^* + U_{\mathrm{odd}} U_{\mathrm{even}}^* = 0$

ii) implies $\qquad U_{\mathrm{odd}}\ker(U_{\mathrm{even}}) \subset \ker(U_{\mathrm{even}}^*)$

since $\qquad U_{\mathrm{even}}^* U_{\mathrm{odd}}\, x = -U_{\mathrm{odd}}^* U_{\mathrm{even}}\, x = 0,\ \forall x \in \ker(U_{\mathrm{even}}).$

Similarly, iv) implies $\qquad U_{\mathrm{odd}}^*\ker(U_{\mathrm{even}}^*) \subset \ker(U_{\mathrm{even}}).$

We conclude:

$$U_{\mathrm{odd}}\ \ker(U_{\mathrm{even}}) \subset \ker(U_{\mathrm{even}}^*) \overset{iii)}{=} U_{\mathrm{odd}} U_{\mathrm{odd}}^* \ker(U_{\mathrm{even}}^*) \subset U_{\mathrm{odd}}\ker(U_{\mathrm{even}})$$

And thus

$$U\ker(U_{\mathrm{even}}) = U_{\mathrm{odd}}\ker(U_{\mathrm{even}}) = \ker(U_{\mathrm{even}}^*)$$

Separating w.r.to the $\pm-$splitting yields

$$U\ker(U_{++}) = \ker(U_{--}^*) \text{ and } U\ker(U_{--}) = \ker(U_{++}^*) \tag{A.3.1}$$

which implies the claimed identities.

□

Norm Estimates for the Dyson Series, Lemma 8.1.8

For the terms in the Dyson series (8.1.8) we proove the norm-estimates [Lem. 8.1.8]:

$$\|U_n(t,t')\| \leq \frac{1}{n!}\left(\int_{t'}^{t} \|V(s)\| \mathrm{d}s\right)^n,\ \forall n \geq 0$$

and

$$\|[\epsilon, U_1(t,t')]\|_2 \leq \int_{t'}^{t} \|[\epsilon, V(s)]\|_2\ \mathrm{d}s$$

$$\|[\epsilon, U_n(t,t')]\|_2 \leq \frac{1}{(n-2)!}\int_{t'}^{t} \|[\epsilon, V(s)]\|_2\ \mathrm{d}s \left(\int_{t'}^{t} \|V(r)\| \mathrm{d}r\right)^{n-1},\ \forall n \geq 1$$

Proof. Obviously, $\|U_0(t,t')\| = \|\mathbb{1}\| = 1,\ \forall t,t'$. And for $n \geq 1$:

$$\begin{aligned}
U_n(t,t') &= (-i)^n \int_{t'}^{t} V_I(s_1) \int_{t'}^{s_1} V_I(s_2) \dots \int_{t'}^{s_{n-1}} V_I(s_n)\, \mathrm{d}s_1 \dots \mathrm{d}s_n \\
\Rightarrow\ \|U_n(t,t')\| &\leq \int_{t'}^{t} \|V_I(s_1)\| \int_{t'}^{s_1} \|V_I(s_2)\| \dots \int_{t'}^{s_{n-1}} \|V_I(s_n)\|\, \mathrm{d}s_1 \dots \mathrm{d}s_n \\
&= \int_{t'}^{t} \|V(s_1)\| \int_{t'}^{s_1} \|V(s_2)\| \dots \int_{t'}^{s_{n-1}} \|V(s_n)\|\, \mathrm{d}s_1 \dots \mathrm{d}s_n \\
&= \frac{1}{n!} \Big(\int_{t'}^{t} \|V(s)\| \mathrm{d}s \Big)^n
\end{aligned}$$

For the I_2-estimates we first note that conjugation with $e^{iD_0 t}$ doesn't change the Hilbert-Schmidt norm of the odd parts. Thus:

$$\|[\epsilon, U_1(t,t')]\|_2 = \| \int_{t'}^{t} [\epsilon, V_I(s)] \mathrm{d}s \|_2 \leq \int_{t'}^{t} \|[\epsilon, V_I(s)]\|_2\, \mathrm{d}s = \int_{t'}^{t} \|[\epsilon, V(s)]\|_2\, \mathrm{d}s$$

Furthermore,

$$\begin{aligned}
\|[\epsilon, U_{n+1}(t,t')]\|_2 &\leq \int_{t'}^{t} \|[\epsilon, V_I(s) U_n(s,t')]\|_2\, \mathrm{d}s \\
&\leq \int_{t'}^{t} \|[\epsilon, V_I(s)]\, U_n(s,t')\|_2\, \mathrm{d}s + \int_{t'}^{t} \|V_I(s) [\epsilon, U_n(s,t')]\|_2\, \mathrm{d}s \\
&\leq \int_{t'}^{t} \|[\epsilon, V_I(s)]\|_2\, \|U_n(s,t')\|\, \mathrm{d}s + \int_{t'}^{t} \|V_I(s)\|\, \|[\epsilon, U_n(s,t')]\|_2\, \mathrm{d}s \\
&= \int_{t'}^{t} \|[\epsilon, V(s)]\|_2\, \|U_n(s,t')\|\, \mathrm{d}s + \int_{t'}^{t} \|V(s)\|\, \|[\epsilon, U_n(s,t')]\|_2\, \mathrm{d}s
\end{aligned}$$

For n=1 this yields

$$\begin{aligned}
\|[\epsilon, U_2(t,t')]\|_2 &\leq \int_{t'}^{t} \int_{t'}^{s} \big(\|[\epsilon, V(s)]\|_2\, \|V(r)\| + \|V(s)\|\, \|[\epsilon, V(r)]\|_2 \big)\, \mathrm{d}r\, \mathrm{d}s \\
&= \frac{1}{2} \int_{t'}^{t} \int_{t'}^{t} \big(\|[\epsilon, V(s)]\|_2\, \|V(r)\| + \|V(s)\|\, \|[\epsilon, V(r)]\|_2 \big)\, \mathrm{d}r\, \mathrm{d}s \\
&= \int_{t'}^{t} \|[\epsilon, V(s)]\|_2\, \mathrm{d}s \int_{t'}^{t} \|V(s)\|\, \mathrm{d}s
\end{aligned}$$

And thus, inductively, for $n \geq 2$

$$
\begin{aligned}
&\|[\epsilon, U_{n+1}(t,t')]\|_2 \leq \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \|U_n(s,t')\| \, \mathrm{d}s + \int_{t'}^{t} \|V(s)\| \, \|[\epsilon, U_n(s,t')]\|_2 \, \mathrm{d}s \\
\leq &\frac{1}{n!} \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \Big(\int_{t'}^{s} \|V(r)\| \mathrm{d}r \Big)^n \mathrm{d}s + \frac{1}{(n-2)!} \int_{t'}^{t} \|V(s)\| \int_{t'}^{s} \|[\epsilon, V(s')]\|_2 \, \mathrm{d}s' \Big(\int_{t'}^{s} \|V(r)\| \mathrm{d}r \Big)^{n-1} \mathrm{d}s \\
\leq &\frac{1}{n!} \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \mathrm{d}s \, \Big(\int_{t'}^{t} \|V(r)\| \mathrm{d}r \Big)^n + \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \mathrm{d}s \, \frac{(n-1)}{(n-1)!} \int_{t'}^{t} \|V(s)\| \Big(\int_{t'}^{s} \|V(r)\| \mathrm{d}r \Big)^{n-1} \mathrm{d}s \\
= &\frac{1}{n!} \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \mathrm{d}s \, \Big(\int_{t'}^{t} \|V(r)\| \mathrm{d}r \Big)^n + \frac{(n-1)}{n!} \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \mathrm{d}s \, \Big(\int_{t'}^{t} \|V(r)\| \mathrm{d}r \Big)^n \\
= &\frac{1}{(n-1)!} \int_{t'}^{t} \|[\epsilon, V(s)]\|_2 \, \mathrm{d}s \, \Big(\int_{t'}^{t} \|V(r)\| \mathrm{d}r \Big)^n
\end{aligned}
$$

□

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und ohne Benutzung anderer als der von mir angegebenen Quellen und Hilfsmittel verfasst habe.

Academic Honor Principle

I hereby certify that I have completed the present thesis independently and without the use of sources other than those stated in the thesis.

Dustin Lazarovici
München, 4. Januar 2011