Univalent typoids

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Abstract

A typoid is a type equipped with an equivalence relation, such that the terms of equivalence between the terms of the type satisfy certain conditions, with respect to a given equivalence relation between them, that generalise the properties of the equality terms. The resulting weak 2-groupoid structure can be extended to every finite level. The introduced notions of typoid and typoid function generalise the notions of setoid and setoid function. A univalent typoid is a typoid satisfying a general version of the univalence axiom. We prove some fundamental facts on univalent typoids, their product and exponential. As a corollary, we get an interpretation of propositional truncation within the theory of typoids. The couple typoid and univalent typoid is a weak groupoid-analogue to the couple precategory and category in homotopy type theory.

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1 Introduction

One of the key-features of Martin-Löf's intensional type theory (ITT) (see [10], [11]) is the use of two kinds of equality for the terms a, b of a type A. The definitional, or judgmental equality $a \equiv b$, expresses that a and b are by definition equal, while the propositional equality $a =_A b$, or simpler $a = b$, is a new type, and every term $p : a =_A b$ can be understood as a proof that a and b are propositionally equal. Through the (rough) homotopic interpretation of ITT[1] and the development of homotopy type theory (HoTT), "p is a path from the point a to the point b in space A". The passage from proofs of equality to proofs of equivalence is trivial through the use of Martin-Löf's J-rule[2], the induction principle that corresponds to the inductive definition of the type family $=_A: A \to A \to \mathcal{U}$, where $\mathcal{U}$ is a fixed universe such that $A : \mathcal{U}$. The passage from proofs of equivalence to proofs of equality is non-trivial. One needs the axiom of function extensionality (FE) to generate terms of equality from terms of equivalence between functions, and Voevodsky's axiom of univalence (UA) to generate terms of equality from terms of equivalence between types in a fixed universe (see [13]).

The following question arises naturally. "Is it possible to have a common framework for all instances of getting terms of equality from terms of equivalence?"

To answer this question, we introduce the notions of typoid, a generalisation of the notion of setoid, and of univalent typoid. Another approach to this question form the notions of precategory and category in HoTT (see Chapter 9 in book-HoTT [13]). As we discuss in section 7, the couple typoid and univalent typoid is a weak groupoid-analogue to the couple precategory and category within HoTT.

A setoid is the interpretation of Bishop's notion of set, introduced in [4], in ITT (see [12], [3], [7]). It consists of a type A paired with an equivalence relation $\simeq_A: A \to A \to \mathcal{U}$

[1] This was formulated first by Voevodsky in [14], Awodey and Warren in [2], and was inspired by Hofmann and Streicher's groupoid interpretation of ITT in [9].
[2] Within the homotopic interpretation of ITT this rule is also called "path-induction".

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on A. The morphisms between setoids are the functions between their types that respect the corresponding equivalences. A typoid[3]is a type A with an equivalence relation $\simeq_{\mathcal{A}}$, such that the termss of equivalence between the terms of A satisfy certain conditions with respect to a given, one level higher, equivalence relation $\cong_{\mathcal{A}}$ between them. These conditions generalise the properties of terms of equality between the terms of A. The resulting weak 2-groupoid-structure of a typoid[4], together with its corresponding notion of morphism, can be extended to any finite level.

A univalent typoid is a typoid that satisfies a general version of Voevodsky's axiom of univalence. Roughly speaking, a univalent typoid is a typoid A such that a term $e : x \simeq_{\mathcal{A}} y$ generates a term $p : x =_A y$. In the next sections we prove some first fundamental facts on univalent types, their product and exponential. As a corollary, we get an interpretation of propositional truncation within the theory of typoids.

We work within the informal framework of univalent type theory (UTT) i.e., of ITT extended with UA, which is found in the book-HoTT [13]. All proofs not included here are omitted as straightforward.

2 Typoids and typoid functions

▶ **Definition 1.** A structure $\mathcal{A} \equiv \left(A, \simeq_{\mathcal{A}}, \mathtt{eqv}_{\mathcal{A}}, *_{\mathcal{A}}, ^{-1_{\mathcal{A}}}, \cong_{\mathcal{A}}\right)$ is called a 2-*typoid*, or simply a *typoid*, if $A : \mathcal{U}$ and $\simeq_{\mathcal{A}} : \prod_{x,y:A} \mathcal{U}$ is an equivalence relation on A such that

$$\mathtt{eqv}_{\mathcal{A}} : \prod_{x:A}(x \simeq_{\mathcal{A}} x),$$

$$*_{\mathcal{A}} : \prod_{x,y,z:A} \prod_{e:x\simeq_{\mathcal{A}} y} \prod_{d:y\simeq_{\mathcal{A}} z} x \simeq_{\mathcal{A}} z,$$

$$^{-1_{\mathcal{A}}} : \prod_{x,y:A} \prod_{e:x\simeq_{\mathcal{A}} y} y \simeq_{\mathcal{A}} x$$

and $\cong_{\mathcal{A}} : \prod_{x,y:A} \prod_{e,d:x\simeq_{\mathcal{A}} y} \mathcal{U}$ such that

$$\cong_{\mathcal{A}} (x,y) : \prod_{e,d:x\simeq_{\mathcal{A}} y} \mathcal{U}$$

is an equivalence relation on $x \simeq_{\mathcal{A}} y$, for every $x, y : A$. For simplicity we write $\mathtt{eqv}_x$ instead of $\mathtt{eqv}_{\mathcal{A}}(x)$. Moreover, If $e, e_1, d_1 : z \simeq_{\mathcal{A}} y$, $e_2, d_2 : y \simeq_{\mathcal{A}} z$, and $e_3 : z \simeq_{\mathcal{A}} w$, the following conditions are satisfied:

(Typ_1) $(\mathtt{eqv}_x *_{\mathcal{A}} e) \cong_{\mathcal{A}} e$ and $(e *_{\mathcal{A}} \mathtt{eqv}_y) \cong_{\mathcal{A}} e$.

(Typ_2) $(e *_{\mathcal{A}} e^{-1_{\mathcal{A}}}) \cong_{\mathcal{A}} \mathtt{eqv}_x$ and $(e^{-1_{\mathcal{A}}} *_{\mathcal{A}} e) \cong_{\mathcal{A}} \mathtt{eqv}_y$.

(Typ_3) $(e_1 *_{\mathcal{A}} e_2) *_{\mathcal{A}} e_3 \cong_{\mathcal{A}} e_1 *_{\mathcal{A}} (e_2 *_{\mathcal{A}} e_3)$.

(Typ_4) If $e_1 \cong_{\mathcal{A}} d_1$ and $e_2 \cong_{\mathcal{A}} d_2$, then $(e_1 *_{\mathcal{A}} e_2) \cong_{\mathcal{A}} (d_1 *_{\mathcal{A}} d_2)$.

An 1-*typoid*, or a *setoid*, is just a couple $(A, \simeq_{\mathcal{A}})$.

From now on, $\mathcal{A}$ and $\mathcal{B}$ denote typoids, i.e., $\mathcal{A} \equiv \left(A, \simeq_{\mathcal{A}}, \mathtt{eqv}_{\mathcal{A}}, *_{\mathcal{A}}, ^{-1_{\mathcal{A}}}, \cong_{\mathcal{A}}\right)$ and $\mathcal{B} \equiv \left(B, \simeq_{\mathcal{B}}, \mathtt{eqv}_{\mathcal{B}}, *_{\mathcal{B}}, ^{-1_{\mathcal{B}}}, \cong_{\mathcal{B}}\right)$. If the context is clear, we may omit the subscripts.

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[3] A similar notion is found in [5] and [6], where the additional structure of the product and coproduct of setoids is considered at level two.

[4] The more accurate term for the notion of typoid that is studied here is that of a 2-typoid, which is avoided only for simplicity.

▶ **Proposition** 2. Let $\mathcal{A}$ be a typoid, $x, y : A$, and $e, d : x \simeq y$.

(i) $(\mathtt{eqv}_x)^{-1} \cong \mathtt{eqv}_x$.

(ii) $(e^{-1})^{-1} \cong e$.

(iii) If $e \cong d$, then $e^{-1} \cong d^{-1}$.

Proof. (i) By Typ_2 we have that $\mathtt{eqv}_x * \mathtt{eqv}_x{}^{-1} \cong \mathtt{eqv}_x$ and by Typ_1 we also have $\mathtt{eqv}_x * \mathtt{eqv}_x{}^{-1} \cong \mathtt{eqv}_x{}^{-1}$, hence we get $(\mathtt{eqv}_x)^{-1} \cong \mathtt{eqv}_x$.
(ii) Since $(e^{-1})^{-1} * e^{-1} \cong \mathtt{eqv}_x$, by Typ_4 we get $\big((e^{-1})^{-1} * e^{-1}\big) * e \cong \mathtt{eqv}_x * e$, and consequently $(e^{-1})^{-1} * \mathtt{eqv}_y \cong e$, hence $(e^{-1})^{-1} \cong e$.
(iii) By Typ_4 we get $e^{-1} * e \cong e^{-1} * d$, hence $e^{-1} * d \cong \mathtt{eqv}_y$, therefore $(e^{-1} * d) * d^{-1} \cong \mathtt{eqv}_y * d^{-1}$ i.e., $e^{-1} * \mathtt{eqv}_x \cong d^{-1}$, and $e^{-1} \cong d^{-1}$. ◀

▶ **Example 3.** Using basic properties of the equality $p =_{x =_A y} q$, of the concatenation $p * q$ and of the inversion p^{-1} of equality terms, it is easy to see that the structure

$$\mathcal{A}_0 \equiv \big(A, =_{\mathcal{A}_0}, \mathtt{refl}_{\mathcal{A}_0}, *, {}^{-1_{\mathcal{A}_0}}, \cong_{\mathcal{A}_0}\big)$$

is a typoid, where the equivalence $\cong_{\mathcal{A}_0} : \prod_{x,y:A} \prod_{e,e' : x =_A y} \mathcal{U}$ is defined by

$$\cong_{\mathcal{A}_0} (x, y, e, e') \equiv (e =_{x =_A y} e'),$$

for every $x, y : A$ and $e, e' : x =_A y$. We call $\mathcal{A}_0$ the *equality* typoid, and its typoid structure the *equality* typoid structure on A.

▶ **Example 4.** If $A, B : \mathcal{U}$, it is easy to see that the structure

$$\mathrm{Fun}(A,B) \equiv \big(A \to B, \simeq_{\mathrm{Fun}(A,B)}, \mathtt{eqv}_{\mathrm{Fun}(A,B)}, *_{\mathrm{Fun}(A,B)}, {}^{-1_{\mathrm{Fun}(A,B)}}, \cong_{\mathrm{Fun}(A,B)}\big)$$

is a typoid, where if $f, g : A \to B$, $H, H' : f \simeq_{\mathrm{Fun}(A,B)} g$, and $G : g \simeq_{\mathrm{Fun}(A,B)} h$, we define

$$f \simeq_{\mathrm{Fun}(A,B)} g \equiv \prod_{x:A} f(x) =_B g(x),$$

$$H *_{\mathrm{Fun}(A,B)} G \equiv \lambda(x : A).(H(x) * G(x)),$$

$$H^{-1_{\mathrm{Fun}(A,B)}} \equiv \lambda(x : A).(H(x))^{-1},$$

$$\mathtt{eqv}_f \equiv \lambda(x : A).\mathtt{refl}_{f(x)},$$

$$H \cong_{\mathrm{Fun}(A,B)} H' \equiv \prod_{x:A} H(x) =_{(f(x) =_B g(x))} H'(x).$$

We call $\mathrm{Fun}(A,B)$ the *typoid of functions* from A to B. Similarly, one can define a typoid structure on the dependent functions $\prod_{x:A} P(x)$, where $P : A \to \mathcal{U}$ is a type family over A.

▶ **Example 5.** Using Voevodsky's definition of the type $\mathtt{isequiv}(f)$ "f is an equivalence between $A, B : \mathcal{U}$" (see [13], section 2.4), it is easy to show that

$$\mathrm{Uni} \equiv \big(\mathcal{U}, \simeq_{\mathrm{Uni}}, \mathtt{eqv}_{\mathrm{Uni}}, *_{\mathrm{Uni}}, {}^{-1_{\mathrm{Uni}}}, \cong_{\mathrm{Uni}}\big)$$

is a typoid, where if $(f, u), (f', u') : A \simeq_{\mathrm{Uni}} B$ and $(g, v) : B \simeq_{\mathrm{Uni}} C$, we define

$$A \simeq_{\mathrm{Uni}} B \equiv \sum_{f : A \to B} \mathtt{isequiv}(f),$$

$$(f, u) *_{\mathrm{Uni}} (g, v) \equiv (g \circ f, w),$$

$$(f,u)^{-1_{\mathrm{Uni}}} \equiv (f^{-1}, u^{-1}),$$

$$\mathtt{eqv}_A \equiv (\mathrm{id}_A, i),$$

$$(f,u) \cong_{\mathrm{Uni}} (f',u') \equiv \prod_{x:A} f(x) =_B f'(x),$$

where $w : \mathtt{isequiv}(g \circ f), u^{-1} : \mathtt{isequiv}(f^{-1})$ and $i : \mathtt{isequiv}(\mathrm{id}_A)$. We call Uni the *universal typoid*.

▶ **Definition 6.** If $\mathcal{A}, \mathcal{B}$ are typoids, we call a function $f : A \to B$ a *typoid function*, if there are dependent functions

$$\Phi_f : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} f(x) \simeq_{\mathcal{B}} f(y),$$

$$\Phi_f^2 : \prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d),$$

which we call an 1-*associate* of f and a 2-*associate* of f with respect to Φ_f, respectively, such that for every $x,y,z : A$ and every $e_1 : x \simeq_{\mathcal{A}} y$, $e_2 : y \simeq_{\mathcal{A}} z$ the following conditions hold.

(i) $\Phi_f(x,x,\mathtt{eqv}_x) \cong_{\mathcal{B}} \mathtt{eqv}_{f(x)}$,

(ii) $\Phi_f(x,z,e_1 *_{\mathcal{A}} e_2) \cong_{\mathcal{B}} \Phi_f(x,y,e_1) *_{\mathcal{B}} \Phi_f(y,z,e_2)$.

If $\Phi_f(x,x,\mathtt{eqv}_x) \equiv \mathtt{eqv}_{f(x)}$, for every $x : A$, we call f *strict* with respect to Φ_f.

Φ_f witnesses that f preserves the equivalences between the terms of type A and B, as $\Phi_f(x,y) : x \simeq_{\mathcal{A}} y \to f(x) \simeq_{\mathcal{B}} f(y)$, while Φ_f^2 witnesses that Φ_f preserves the equivalences between equivalences, as $\Phi_f^2(x,y,e,d) : e \cong_{\mathcal{A}} d \to \Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d)$.

▶ **Proposition** 7. If $\mathcal{A}, \mathcal{B}$ are typoids and $f : A \to B$ is a typoid function, then, for every $x,y : A$ and $e : x \simeq_{\mathcal{A}} y$, we have that

$$\Phi_f(y,x,e^{-1_{\mathcal{A}}}) \cong_{\mathcal{B}} [\Phi_f(x,y,e)]^{-1_{\mathcal{B}}}.$$

▶ **Example 8.** If $\mathcal{A}_0, \mathcal{B}_0$ are equality typoids and $f : A \to B$, then f is a strict typoid function with respect to its 1-associate, the application function $\mathtt{ap}_f$, and with 2-associate with respect to $\mathtt{ap}_f$ the two-dimensional application function $\mathtt{ap}_f^2$ of f, where

$$\mathtt{ap}_f : \prod_{x,y:A} \prod_{p:x=_A y} f(x) =_B f(y),$$

$$\mathtt{ap}_f^2 : \prod_{x,y:A} \prod_{p,q:x=_A y} \prod_{r:p=_{(x=_A y)} q} \mathtt{ap}_f(x,y,p) =_{(f(x)=_B f(y))} \mathtt{ap}_f(x,y,q).$$

The properties $\mathtt{ap}_f(x,x,\mathtt{refl}_x) \equiv \mathtt{refl}_{f(x)}$ and $\mathtt{ap}_f(x,z,p*q) = \mathtt{ap}_f(x,y,p) * \mathtt{ap}_f(y,z,q)$ follow from the J-rule (see section 2.2 of [13]).

▶ **Proposition** 9. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are typoids and $f : A \to B, g : B \to C$ are typoid functions with associates Φ_f, Φ_f^2 and Φ_g, Φ_g^2, respectively, then $g \circ f : A \to C$ is a typoid function with associates

$$\Phi_{g\circ f} : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} g(f(x)) \simeq_{\mathcal{C}} g(f(y)),$$

$$\Phi_{g\circ f}^2 : \prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \Phi_{g\circ f}(x,y,e) \cong_{\mathcal{C}} \Phi_{g\circ f}(x,y,d),$$

defined for every $x, y : A, e, d : x \simeq_{\mathcal{A}} y, i : e \cong_{\mathcal{A}} d$ by

$$\Phi_{g\circ f}(x, y, e) \equiv \Phi_g\big(f(x), f(y), \Phi_f(x, y, e)\big),$$

$$\Phi^2_{g\circ f}(x, y, e, d, i) \equiv \Phi^2_g\Big(f(x), f(y), \Phi_f(x, y, e), \Phi_f(x, y, d), \Phi^2_f(x, y, e, d, i)\Big).$$

If f, g are strict with respect to Φ_f, Φ_g, then $g \circ f$ is strict with respect to $\Phi_{g\circ f}$.

If $P : A \to \mathcal{U}$ and $p : x =_A y$, the J-rule implies the existence of the transport function $p^P_* : P(x) \to P(y)$ over p (see [13], section 2.3).

▶ **Proposition 10.** If $\mathcal{A}$ is a typoid, the identity $\mathrm{id}_A : A \to A$ is a strict typoid function from $\mathcal{A}_0$ to $\mathcal{A}$ with respect to its 1-associate

$$\mathtt{idtoEqv}_{\mathcal{A}} : \prod_{x,y:A} \prod_{p:x=_A y} x \simeq_{\mathcal{A}} y,$$

$$\mathtt{idtoEqv}_{\mathcal{A}}(x, y, p) \equiv p^{P_x}_*(\mathtt{eqv}_x),$$

where $P_x : A \to \mathcal{U}$ is defined by $P_x(z) \equiv x \simeq_{\mathcal{A}} z$, for every $z : A$.

If we consider A with the equality typoid structure as a codomain of id_A, then by Lemma 2.11.2 in [13] we have that

$$\mathtt{idtoEqv}_{\mathcal{A}_0}(x, y, p) \equiv p^{P_x}_*(\mathtt{refl}_x) \equiv p^{z\mapsto x=_A z}_*(\mathtt{refl}_x) = \mathtt{refl}_x * p = p$$

i.e., $\mathtt{idtoEqv}_{\mathcal{A}_0}(x, y, p)$ is pointwise equal to $\mathrm{id}_{x=_A y}$. In [13], section 2.10, the function $\mathtt{idtoEqv} : A =_{\mathcal{U}} B \to A \simeq_{\mathcal{U}} B$ is defined by $\mathtt{idtoEqv}(p) \equiv p^{\mathrm{id}_{\mathcal{U}}}_*$, for every $p : A =_{\mathcal{U}} B$. Since

$$\mathtt{idtoEqv}(\mathtt{refl}_A) \equiv (\mathtt{refl}_A)^{\mathrm{id}_{\mathcal{U}}}_* \equiv \mathrm{id}_{\mathrm{id}_{\mathcal{U}}(A)} \equiv \mathrm{id}_A,$$

and

$$\mathtt{idtoEqv}_{\mathrm{Uni}}(A, A, \mathtt{refl}_A) \equiv (\mathtt{refl}_A)^{P_A}_*(\mathrm{id}_A) \equiv \mathrm{id}_{P_A(A)}(\mathrm{id}_A) \equiv \mathrm{id}_A,$$

the two functions agree on $\mathtt{refl}_A$, hence by the J-rule they are pointwise equal. The definition of $\mathtt{idtoEqv}_{\mathcal{A}}$ is a common formulation of the functions $\mathtt{happly}$, related to the axiom of function extensionality, and $\mathtt{idtoEqv}$, related to the univalence axiom.

3 Product typoid

▶ **Proposition 11.** If $(A, \simeq_{\mathcal{A}}), (B, \simeq_{\mathcal{B}})$ are setoids, there are dependent functions

$$T : \prod_{z,w:A\times B} \Big((\mathtt{pr}_1(z) \simeq_{\mathcal{A}} \mathtt{pr}_1(w)) \times (\mathtt{pr}_2(z) \simeq_{\mathcal{B}} \mathtt{pr}_2(w)) \to z \simeq_{\mathcal{A}\times\mathcal{B}} w \Big),$$

$$\Upsilon : \prod_{z,w:A\times B} \Big(z \simeq_{\mathcal{A}\times\mathcal{B}} w \to (\mathtt{pr}_1(z) \simeq_{\mathcal{A}} \mathtt{pr}_1(w)) \times (\mathtt{pr}_2(z) \simeq_{\mathcal{B}} \mathtt{pr}_2(w)) \Big),$$

where

$$(x, y) \simeq_{\mathcal{A}\times\mathcal{B}} (x', y') \equiv (x \simeq_A x') \times (y \simeq_B y'),$$

such that for each $i \in \{1, 2\}$, where $C_1 \equiv A$ and $C_2 \equiv B$,

$$\prod_{z,w:A\times B} \mathtt{pr}_i\Big(\Upsilon(z, w, T(z, w, e_1, e_2))\Big) \cong_{C_i} e_i.$$

Proof. If $x : A$ and $y : B$, we find $G(x,y) : \prod_{w:A\times B} P(w)$, where

$$P(w) \equiv (x,y) \simeq_{\mathcal{A}\times\mathcal{B}} w \to (x \simeq_{\mathcal{A}} \mathtt{pr}_1(w)) \times (y \simeq_{\mathcal{B}} \mathtt{pr}_2(w)),$$

hence by the induction principle of the product type we get Υ. We define the dependent function $H : \prod_{x':A}\prod_{y':B} P((x',y'))$ by $H(x',y') \equiv \mathrm{id}_{(x,y)\simeq_{\mathcal{A}\times\mathcal{B}}(x',y')}$, and then $H(x',y') : (x,y) \simeq_{\mathcal{A}\times\mathcal{B}} (x',y') \leftrightarrow (x \simeq_{\mathcal{A}} x') \times (y \simeq_{\mathcal{A}} y')$, and $G(x,y)$ is given again by the induction principle of the product type. For T we proceed similarly. ◀

▶ **Corollary 12.** *If $(A, \simeq_{\mathcal{A}}), (B, \simeq_{\mathcal{B}})$ are setoids, then $\mathtt{pr}_1, \mathtt{pr}_2$ are setoid functions.*

Proof. We define the dependent function $\Phi_{\mathtt{pr}_1} : \prod_{z,w:A\times B}\prod_{e:z\simeq_{\mathcal{A}\times\mathcal{B}} w} \mathtt{pr}_1(z) \simeq_{\mathcal{A}} \mathtt{pr}_1(w)$ by $\Phi_{\mathtt{pr}_1}(z,w,e) \equiv \mathtt{pr}_1(\Upsilon(z,w,e))$. Similarly, we define $\Phi_{\mathtt{pr}_2}(z,w,e) \equiv \mathtt{pr}_2(\Upsilon(z,w,e))$. ◀

We use the notations $e_1 \equiv \Phi_{\mathtt{pr}_1}(z,w,e)$ and $e_2 \equiv \Phi_{\mathtt{pr}_2}(z,w,e)$.

▶ **Proposition** 13. If $\mathcal{A}, \mathcal{B}$ are typoids, then the structure

$$\mathcal{A}\times\mathcal{B} \equiv \left(A\times B, \simeq_{\mathcal{A}\times\mathcal{B}}, \mathtt{eqv}_{\mathcal{A}\times\mathcal{B}}, *_{\mathcal{A}\times\mathcal{B}}, {}^{-1_{\mathcal{A}\times\mathcal{B}}}, \cong_{\mathcal{A}\times\mathcal{B}}\right)$$

is a typoid, where for every $z, w, u : A\times B$ and $e, e' : z =_{A\times B} w, d : w =_{A\times B} u$ we define

$$\mathtt{eqv}_z \equiv T(z, z, \mathtt{eqv}_{\mathtt{pr}_1(z)}, \mathtt{eqv}_{\mathtt{pr}_2(z)}),$$
$$e *_{\mathcal{A}\times\mathcal{B}} d \equiv T(z, u, e_1 *_{\mathcal{A}} d_1, e_2 *_{\mathcal{B}} d_2),$$
$$e^{-1_{\mathcal{A}\times\mathcal{B}}} \equiv T(w, z, e_1^{-1_{\mathcal{A}}}, e_2^{-1_{\mathcal{B}}}),$$
$$e \cong_{\mathcal{A}\times\mathcal{B}} e' \equiv (e_1 \cong_{\mathcal{A}} e_1') \times (e_2 \cong_{\mathcal{B}} e_2').$$

▶ **Corollary 14.** *If $\mathcal{A}, \mathcal{B}$ are typoids, then $\mathtt{pr}_1, \mathtt{pr}_2$ are typoid functions.*

Proof. We show this only for $\mathtt{pr}_1$. The equivalence $\Phi_{\mathtt{pr}_1}(z,z,\mathtt{eqv}_z) \cong_{\mathcal{A}} \mathtt{eqv}_{\mathtt{pr}_1(z)}$ follows from the previously shown equivalence $(\mathtt{eqv}_z)_1 \cong_{\mathcal{A}} \mathtt{eqv}_{\mathtt{pr}_1(z)}$. The equivalence $\Phi_{\mathtt{pr}_1}(z,u,e *_{\mathcal{A}\times\mathcal{B}} d) \cong_{\mathcal{A}} \Phi_{\mathtt{pr}_1}(z,w,e) *_{\mathcal{A}} \Phi_{\mathtt{pr}_1}(w,u,d)$ is rewritten as $(e *_{\mathcal{A}\times\mathcal{B}} d)_1 \cong_{\mathcal{A}} (e_1 *_{\mathcal{A}} d_1)$, which follows immediately by the definition of $e *_{\mathcal{A}\times\mathcal{B}} d$ and the relation between Υ and T in Proposition 11. Since $e_1 \equiv \Phi_{\mathtt{pr}_1}(z,w,e)$ and $d_1 \equiv \Phi_{\mathtt{pr}_1}(z,w,d)$, by the definition of $e \cong_{\mathcal{A}\times\mathcal{B}} d$ we get the following 2-associate of $\mathtt{pr}_1$ with respect to $\Phi_{\mathtt{pr}_1}$

$$\Phi^2_{\mathtt{pr}_1} : \prod_{z,w:A\times B}\ \prod_{e,d:z\simeq_{\mathcal{A}\times\mathcal{B}} w}\ \prod_{i:e\cong_{\mathcal{A}\times\mathcal{B}} d} \Phi_{\mathtt{pr}_1}(z,w,e) \cong_{\mathcal{A}} \Phi_{\mathtt{pr}_1}(z,w,d),$$
$$\Phi^2_{\mathtt{pr}_1}(z,w,e,d,i) \equiv \mathtt{pr}_1(i).$$

◀

▶ **Corollary 15.** *If $\mathcal{A}, \mathcal{B}$ are typoids, $z, w : A\times B$, and $e : z \simeq_{\mathcal{A}\times\mathcal{B}} w$, then*

$$T(z,w,e_1,e_2) \cong_{\mathcal{A}\times\mathcal{B}} e.$$

Proof. If $\tau \equiv T(z,w,e_1,e_2)$, then $\tau \cong_{\mathcal{A}\times\mathcal{B}} e \equiv (\tau_1 \cong_{\mathcal{A}} e_1) \times (\tau_2 \cong_{\mathcal{A}} e_2)$. By Proposition 11 $\tau_1 \equiv \mathtt{pr}_1(\Upsilon(z,w,T(z,w,e_1,e_2))) \cong_{\mathcal{A}} e_1$ and similarly $\tau_2 \cong_{\mathcal{B}} e_2$. ◀

▶ **Corollary 16.** *If $\mathcal{A}, \mathcal{B}$ are typoids, $z, w : A\times B$, and $e_1, d_1 : \mathtt{pr}_1(z) \simeq_{\mathcal{A}} \mathtt{pr}_1(w)$, $e_2, d_2 : \mathtt{pr}_2(z) \simeq_{\mathcal{A}} \mathtt{pr}_2(w)$ such that $e_1 \cong_{\mathcal{A}} d_1$ and $e_2 \cong_{\mathcal{A}} d_2$, then*

$$T(z,w,e_1,e_2) \cong_{\mathcal{A}\times\mathcal{B}} T(z,w,d_1,d_2).$$

Proof. If $\tau \equiv T(z,w,e_1,e_2)$ and $\tau' \equiv T(z,w,d_1,d_2)$, then by Proposition 11 we get $\tau_1 \equiv \mathtt{pr}_1(\Upsilon(z,w,T(z,w,e_1,e_2)) \cong_{\mathcal{A}} e_1$ and $\tau_2 \equiv \mathtt{pr}_2(\Upsilon(z,w,T(z,w,e_1,e_2)) \cong_{\mathcal{B}} e_2$. Similarly we have that $\tau_1' \equiv \mathtt{pr}_1(\Upsilon(z,w,T(z,w,d_1,d_2)) \cong_{\mathcal{A}} d_1$ and $\tau_2' \equiv \mathtt{pr}_2(\Upsilon(z,w,T(z,w,d_1,d_2)) \cong_{\mathcal{B}} d_2$. By our hypothesis and the definition of $\cong_{\mathcal{A}\times\mathcal{B}}$ we get $\tau \cong_{\mathcal{A}\times\mathcal{B}} \tau'$. ◀

4 Univalent typoids

▶ **Definition 17.** A typoid $\mathcal{A}$ is called *univalent*, if there are dependent functions

$$\mathtt{Ua}_{\mathcal{A}} : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} x =_A y,$$

$$\mathtt{Ua}^2_{\mathcal{A}} : \prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \mathtt{Ua}_{\mathcal{A}}(x,y,e) = \mathtt{Ua}_{\mathcal{A}}(x,y,d)$$

such that for every $x, y : A, p : x =_A y$ and $e : x \simeq_{\mathcal{A}} y$ we have that

$$\mathtt{Ua}_{\mathcal{A}}(x,y,\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,p)) = p,$$

$$\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)) \cong_{\mathcal{A}} e,$$

where $\mathtt{IdtoEqv}_{\mathcal{A}}$ is an 1-associate of id_A (from $\mathcal{A}_0$ to $\mathcal{A}$) with respect to which id_A is strict[5]. We call a univalent typoid *strictly* univalent, if $\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x) \equiv \mathtt{refl}_x$.

The equality typoid $\mathcal{A}_0$ is strictly univalent, if we consider

$$\mathtt{IdtoEqv}_{\mathcal{A}_0}(x,y,p) \equiv p \equiv \mathtt{Ua}_{\mathcal{A}_0}(x,y,p),$$

for every $x, y : A$ and $p : x \simeq_{\mathcal{A}_0} y$. The function extensionality axiom guarantees that the typoid of functions $\mathrm{Fun}(A,B)$ is univalent, and Voevodsky's univalence axiom that the universal typoid Uni is univalent. We need only to explain why the functions $\mathtt{funext}$ and $\mathtt{ua}$ satisfy the above conditions: if $H, H' : f \simeq_{\mathrm{Fun}(A,B)} g$ such that $H \cong_{\mathrm{Fun}(A,B)} H'$, then $\mathtt{funext}(H) = \mathtt{funext}(H')$, and if $(f,u),(g,w) : A \simeq_{\mathrm{Uni}} B$ such that $(f,u) \cong_{\mathrm{Uni}} (g,w)$, then $\mathtt{ua}((f,u)) = \mathtt{ua}((g,w))$, respectively. By the function extensionality axiom if $H, H' : f \simeq_{\mathrm{Fun}(A,B)} g$, then there is $p : H = H'$, hence the application function of $\mathtt{funext}$ satisfies $\mathtt{ap}_{\mathtt{funext}}(p) : \mathtt{funext}(H) = \mathtt{funext}(H')$. By Theorem 2.7.2 of [13] we have that

$$((f,u) =_{A \simeq_{\mathrm{Uni}} B} (g,w)) \simeq_{\mathrm{Uni}} \sum_{p:f=g} \left(p_*^{f \mapsto \mathtt{isequiv}(f)}(u) = w \right).$$

By the function extensionality axiom the hypothesis $(f,u) \cong_{\mathrm{Uni}} (g,w)$ implies that $f = g$, while a term of type $p_*^{f \mapsto \mathtt{isequiv}(f)}(u) = w$ is found by the equality of all terms of type $\mathtt{isequiv}(g)$. Hence the hypothesis $(f,u) \cong_{\mathrm{Uni}} (g,w)$ implies $(f,u) =_{A \simeq_U B} (g,w)$ and we use the application function of $\mathtt{ua}$ to get a term of type $\mathtt{ua}((f,u)) = \mathtt{ua}((g,w))$.

The next proposition is a common reformulation of properties of the functions $\mathtt{funext}$ and $\mathtt{ua}$ found in sections 2.9 and 2.10 of [13], respectively.

▶ **Proposition** 18. If $\mathcal{A}$ is a univalent typoid, the identity function $\mathrm{id}_A : A \to A$ is a typoid function from $\mathcal{A}$ to $\mathcal{A}_0$, with $\mathtt{Ua}^2_{\mathcal{A}}$ as a 2-associate of id_A with respect to its 1-associate $\mathtt{Ua}_{\mathcal{A}}$.

Proof. We show that $\Phi_{\mathrm{id}_A} \equiv \mathtt{Ua}_{\mathcal{A}}$ and $\Phi^2_{\mathrm{id}_A} \equiv \mathtt{Ua}^2_{\mathcal{A}}$ are associates of id_A. By definition we have that $\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x) \equiv \mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{IdtoEqv}_{\mathcal{A}}(x,x,\mathtt{refl}_x)) = \mathtt{refl}_x$. Since

[5] In Proposition 10 we showed the existence of such an associate of id_A but in general we don't need to use the specific definition of $\mathtt{idtoEqv}_{\mathcal{A}}$. This is crucial in proving that the product of univalent typoids is a univalent typoid. For this reason we use a different notation for this abstract 1-associate of id_A. Using the J-rule one shows that $\mathtt{idtoEqv}_{\mathcal{A}}$ and $\mathtt{IdtoEqv}_{\mathcal{A}}$ are pointwise equal, hence by function extensionality they are equal.

$\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e_1)) \cong_{\mathcal{A}} e_1$ and $\mathtt{IdtoEqv}_{\mathcal{A}}(y,z,\mathtt{Ua}_{\mathcal{A}}(y,z,e_2)) \cong_{\mathcal{A}} e_2$, by Typ_4 we get

$$\begin{aligned} e_1 *_{\mathcal{A}} e_2 &\equiv_{\mathcal{A}} \mathtt{IdtoEqv}_{\mathcal{A}}(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e_1)) *_{\mathcal{A}} \mathtt{IdtoEqv}_{\mathcal{A}}(y,z,\mathtt{Ua}_{\mathcal{A}}(y,z,e_2)) \\ &\equiv_{\mathcal{A}} \mathtt{IdtoEqv}_{\mathcal{A}}(x,z,[\mathtt{Ua}_{\mathcal{A}}(x,y,e_1) *_{\mathcal{A}} \mathtt{Ua}_{\mathcal{A}}(y,z,e_2)])) \end{aligned}$$

If $B \equiv \mathtt{Ua}_{\mathcal{A}}\big(x,z,\mathtt{IdtoEqv}_{\mathcal{A}}(x,z,[\mathtt{Ua}_{\mathcal{A}}(x,y,e_1) *_{\mathcal{A}} \mathtt{Ua}_{\mathcal{A}}(y,z,e_2)])\big)$, by the existence of $\mathtt{Ua}^2_{\mathcal{A}}$ we get a term of type $\mathtt{Ua}_{\mathcal{A}}(x,z,e_1 *_{\mathcal{A}} e_2) = B$, hence a term of type $\mathtt{Ua}_{\mathcal{A}}(x,z,e_1 *_{\mathcal{A}} e_2) = \mathtt{Ua}_{\mathcal{A}}(x,y,e_1) *_{\mathcal{A}} \mathtt{Ua}_{\mathcal{A}}(y,z,e_2)$. ◀

▶ **Theorem 19.** *Let $\mathcal{A},\mathcal{B}$ be typoids and $f : A \to B$.*
(i) If $\mathcal{A}$ is univalent, then f is a typoid function.
(ii) If $\mathcal{A}$ is strictly univalent, then f is a strict typoid function with respect to its 1-associate given in the proof of (i).

Proof. (i) Through the correspondences

$$x \simeq_{\mathcal{A}} y \overset{\mathtt{Ua}_{\mathcal{A}}(x,y)}{\longrightarrow} x =_A y \overset{\mathtt{ap}_f(x,y)}{\longrightarrow} f(x) =_B f(y) \overset{\mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(y))}{\longrightarrow} f(x) \simeq_{\mathcal{B}} f(y)$$

we define the dependent function $\Phi_f : \prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}} y} f(x) \simeq_{\mathcal{B}} f(y)$ by

$$\Phi_f(x,y,e) \equiv \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(y),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e))\Big).$$

By the proof of Proposition 18 there is $r : \mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x) = \mathtt{refl}_x$, and by $\mathtt{ap}^2_f$ we get a term $r' : \mathtt{ap}_f(x,x,\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x)) = \mathtt{ap}_f(x,x,\mathtt{refl}_x) \equiv \mathtt{refl}_{f(x)}$. Since $\mathtt{IdtoEqv}^2_{\mathcal{B}}$ is of type

$$\prod_{x',y':B}\ \prod_{p',q':x'=_B y'}\ \prod_{r':p'=q'} \mathtt{IdtoEqv}_{\mathcal{B}}(x',y',p') \cong_{\mathcal{B}} \mathtt{IdtoEqv}_{\mathcal{B}}(x',y',q'),$$

$\mathtt{IdtoEqv}^2_{\mathcal{B}}(f(x),f(x),\mathtt{ap}_f(x,x,\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x)),\mathtt{refl}_{f(x)},r')$ is of type

$$\mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(x),\mathtt{ap}_f(x,x,\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x))) \cong_{\mathcal{B}} \mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(x),\mathtt{refl}_{f(x)}).$$

By the definition of Φ_f and the fact that $\mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(x),\mathtt{refl}_{f(x)}) \equiv \mathtt{eqv}_{f(x)}$ we get $\Phi_f(x,x,\mathtt{eqv}_x) \cong_{\mathcal{B}} \mathtt{eqv}_{f(x)}$. By the existence of $\mathtt{ap}^2_f$ and $\mathtt{IdtoEqv}^2_{\mathcal{B}}$ we get

$$\begin{aligned} \Phi_f(x,z,e *_{\mathcal{A}} d) &\equiv \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(z),\mathtt{ap}_f(x,z,\mathtt{Ua}_{\mathcal{A}}(x,z,e *_{\mathcal{A}} d))\Big) \\ &\cong_{\mathcal{B}} \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(z),\mathtt{ap}_f(x,z,\mathtt{Ua}_{\mathcal{A}}(x,y,e) *_{\mathcal{B}} \mathtt{Ua}_{\mathcal{A}}(y,z,d))\Big) \\ &\cong_{\mathcal{B}} \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(z),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)) *_{\mathcal{B}} \mathtt{ap}_f(y,z,\mathtt{Ua}_{\mathcal{A}}(y,z,d))\Big) \\ &\cong_{\mathcal{B}} \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(y),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e))\Big) *_{\mathcal{B}} \\ &\qquad \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(y),f(z),\mathtt{ap}_f(y,z,\mathtt{Ua}_{\mathcal{A}}(y,z,d))\Big) \\ &\equiv \Phi_f(x,y,e) *_{\mathcal{B}} \Phi_f(y,z,d). \end{aligned}$$

We define $\Phi^2_f : \prod_{x,y:A}\prod_{e,d:x\simeq_{\mathcal{A}} y}\prod_{i:e\cong_{\mathcal{A}} d} \Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d)$ by

$$\Phi^2_f(x,y,e,d,i) \equiv \mathtt{IdtoEqv}^2_{\mathcal{B}}\Bigg(f(x),f(y),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,d)),$$
$$\mathtt{ap}^2_f\Big(x,y,\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,d)),\mathtt{Ua}^2_{\mathcal{A}}(x,y,e,d,i)\Big)\Bigg).$$

Since the term $\mathtt{Ua}^2_{\mathcal{A}}(x,y,e,d,i)$ is of type $\mathtt{Ua}_{\mathcal{A}}(x,y,e) = \mathtt{Ua}_{\mathcal{A}}(x,y,d)$ and the terms $\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e))$ and $\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,d))$ are of type $f(x) =_B f(y)$, the term

$$\mathtt{ap}^2_f\Big(x,y,\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)),\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,d)),\mathtt{Ua}^2_{\mathcal{A}}(x,y,e,d,i)\Big)$$

is of type $\mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)) = \mathtt{ap}_f(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,d))$. Hence by the type of $\mathtt{IdtoEqv}^2_{\mathcal{B}}$ and the definition of Φ_f we get that $\Phi^2_f(x,y,e,d,i)$ is of type $\Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d)$.
(ii) By the proof of Proposition 10 we have that

$$\begin{aligned}
\Phi_f(x,x,\mathtt{eqv}_x) &\equiv \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(x),\mathtt{ap}_f(x,x,\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x))\Big)\\
&\equiv \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x),f(x),\mathtt{ap}_f(x,x,\mathtt{refl}_x))\Big)\\
&\equiv \mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(x),\mathtt{refl}_{f(x)}))\\
&\equiv \mathtt{eqv}_{f(x)}.
\end{aligned}$$

◀

▶ **Proposition** 20. Let $\mathcal{A},\mathcal{B}$ be typoids, $x,y : A$, and $f : A \to B$ a typoid function with 1-associate Φ_f. If $\mathcal{B}$ is univalent, the following diagram commutes

$$\begin{tikzcd}
x =_A y \arrow[r, "\mathtt{ap}_f(x,y)"] \arrow[d, "\mathtt{IdtoEqv}_{\mathcal{A}}(x,y)"'] & f(x) =_B f(y) \\
x \simeq_{\mathcal{A}} y \arrow[r, "\Phi_f(x,y)"'] & f(x) \simeq_{\mathcal{B}} f(y). \arrow[u, "\mathtt{Ua}_{\mathcal{B}}(f(x),f(y))"']
\end{tikzcd}$$

Proof. In order to use the J-rule, we define $C : \prod_{x,y:A}\prod_{p:x=_A y}\mathcal{U}$ by

$$C(x,y,p) \equiv \mathtt{Ua}_{\mathcal{B}}\bigg(f(x),f(y),\Phi_f\big(x,y,\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,p)\big)\bigg) = \mathtt{ap}_f(x,y,p).$$

If $i : \Phi_f(x,x,\mathtt{eqv}_x) \cong_{\mathcal{B}} \mathtt{eqv}_{f(x)}$, then $\mathtt{Ua}^2_{\mathcal{B}}(f(x),f(x),\Phi_f(x,x,\mathtt{eqv}_x),\mathtt{eqv}_{f(x)},i)$ is a term of type $\mathtt{Ua}_{\mathcal{B}}(f(x),f(x),\Phi_f(x,x,\mathtt{eqv}_x)) = \mathtt{Ua}_{\mathcal{B}}(f(x),f(x),\mathtt{eqv}_{f(x)})$. Since we have that $\mathtt{Ua}_{\mathcal{B}}(f(x),f(x),\mathtt{eqv}_{f(x)}) = \mathtt{refl}_{f(x)}$, the type

$$\begin{aligned}
C(x,x,\mathtt{refl}_x) &\equiv \mathtt{Ua}_{\mathcal{B}}(f(x),f(x),\Phi_f(x,x,\mathtt{IdtoEqv}_{\mathcal{B}}(x,x,\mathtt{refl}_x))) = \mathtt{ap}_f(x,x,\mathtt{refl}_x)\\
&\equiv \mathtt{Ua}_{\mathcal{B}}(f(x),f(x),\Phi_f(x,x,\mathtt{eqv}_x)) = \mathtt{ap}_f(x,x,\mathtt{refl}_x)
\end{aligned}$$

is inhabited, and the J-rule can be used. ◀

▶ **Corollary 21.** *If $\mathcal{A},\mathcal{B}$ are univalent typoids, $x,y : A$, and $f : A \to B$, the following diagram commutes*

$$\begin{tikzcd}
x \simeq_{\mathcal{A}} y \arrow[r, "\Phi_f(x,y)"] \arrow[d, "\mathtt{Ua}_(x,y)"'] & f(x) \simeq_{\mathcal{B}} f(y) \arrow[d, "\mathtt{Ua}_{\mathcal{B}}(f(x),f(y))"] \\
x =_A y \arrow[r, "\mathtt{ap}_f(x,y)"'] & f(x) =_B f(y).
\end{tikzcd}$$

where Φ_f is the 1-associate of f determined by Theorem 19.

For the next two results we use the following dependent functions (see [13], section 2.6)

$$^{=}\mathtt{pair} : \prod_{z,w:A\times B} \Big(z =_{A\times B} w \to (\mathtt{pr}_1(z) =_A \mathtt{pr}_1(w)) \times (\mathtt{pr}_2(z) =_B \mathtt{pr}_2(w)) \Big),$$

$$\mathtt{pair}^{=} : \prod_{z,w:A\times B} \Big((\mathtt{pr}_1(z) =_A \mathtt{pr}_1(w)) \times (\mathtt{pr}_2(z) =_B \mathtt{pr}_2(w)) \to z =_{A\times B} w \Big),$$

which are the $=$-version of the functions of Υ and T, respectively, and they satisfy

$$^{=}\mathtt{pair}(z, z, \mathtt{refl}_z) \equiv (\mathtt{refl}_{\mathtt{pr}_1(z)}, \mathtt{refl}_{\mathtt{pr}_2(z)}),$$

$$\mathtt{pair}^{=}(z, z, (\mathtt{refl}_{\mathtt{pr}_1(z)}, \mathtt{refl}_{\mathtt{pr}_2(z)})) \equiv \mathtt{refl}_z.$$

▶ **Lemma 22.** *If $\mathcal{A}, \mathcal{B}$ are typoids such that $\mathrm{id}_A, \mathrm{id}_B$ are strict with respect to the their 1-associates $\mathtt{IdtoEqv}_{\mathcal{A}}, \mathtt{IdtoEqv}_{\mathcal{B}}$, respectively, then $\mathrm{id}_{A\times B}$ is strict with respect to its 1-associate*

$$\mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}} : \prod_{z,w:A\times B} \prod_{p:z=_{A\times B}w} z \simeq_{\mathcal{A}\times\mathcal{B}} w,$$

$$\mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}(z, w, p) \equiv T(z, w, e_1, e_2),$$

$$e_1 \equiv \mathtt{IdtoEqv}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), p_1), \quad e_2 \equiv \mathtt{IdtoEqv}_{\mathcal{B}}(\mathtt{pr}_2(z), \mathtt{pr}_2(w), p_2),$$

$$p_1 \equiv \mathtt{pr}_1(^{=}\mathtt{pair}(z, w, p)), \quad p_2 \equiv \mathtt{pr}_2(^{=}\mathtt{pair}(z, w, p)).$$

▶ **Theorem 23.** *If $\mathcal{A}, \mathcal{B}$ are univalent typoids, then $\mathcal{A} \times \mathcal{B}$ is a univalent typoid.*

Proof. We need to define dependent functions

$$\mathtt{Ua}_{\mathcal{A}\times\mathcal{B}} : \prod_{z,w:A\times B} \prod_{e:z\simeq_{\mathcal{A}\times\mathcal{B}}w} z =_{A\times B} w,$$

$$\mathtt{Ua}^2_{\mathcal{A}\times\mathcal{B}} : \prod_{z,w:A\times B} \prod_{e,d:z\simeq_{\mathcal{A}\times\mathcal{B}}e} \prod_{i:e\cong_{\mathcal{A}\times\mathcal{B}}d} \mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z, w, e) = \mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z, w, d)$$

such that for every $z, w : A \times B, p : z =_{A\times B} w$ and $e : z \simeq_{\mathcal{A}\times\mathcal{B}} w$

$$\mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z, w, \mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}(z, w, p)) = p \;\;\&\;\; \mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}(z, w, \mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z, w, e)) \cong_{\mathcal{A}\times\mathcal{B}} e,$$

where $\mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}$ is defined in Lemma 22. If $e' : z \simeq_{\mathcal{A}\times\mathcal{B}} w$, we define

$$\mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z, w, e') \equiv \mathtt{pair}^{=}(z, w, {p_1}', {p_2}'),$$

$${p_1}' \equiv \mathtt{Ua}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), {e_1}') : \mathtt{pr}_1(z) =_A \mathtt{pr}_1(w),$$

$${p_2}' \equiv \mathtt{Ua}_{\mathcal{B}}(\mathtt{pr}_2(z), \mathtt{pr}_2(w), {e_2}') : \mathtt{pr}_2(z) =_B \mathtt{pr}_2(w),$$

$${e_1}' \equiv \mathtt{pr}_1(\Upsilon(z, w, e)) : \mathtt{pr}_1(z) \simeq_{\mathcal{A}} \mathtt{pr}_1(w),$$

$${e_2}' \equiv \mathtt{pr}_2(\Upsilon(z, w, e)) : \mathtt{pr}_2(z) \simeq_{\mathcal{A}} \mathtt{pr}_2(w).$$

If $e' \equiv \mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}(z, w, p) \equiv T(z, w, e_1, e_2)$, where $p : z =_{\mathcal{A}\times\mathcal{B}} w$ and e_1, e_2 are defined in Lemma 22, then by Proposition 11 ${e_1}' \equiv \mathtt{pr}_1(\Upsilon(z, w, T(z, w, e_1, e_2))) \cong_{\mathcal{A}} e_1$, and proceeding similarly we get ${e_2}' \cong_{\mathcal{B}} e_2$. With the use of $\mathtt{Ua}^2_{\mathcal{A}}$ we have that

$$\begin{aligned}
{p_1}' &\equiv \mathtt{Ua}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), {e_1}') \\
&= \mathtt{Ua}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), e_1) \\
&\equiv \mathtt{Ua}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), \mathtt{IdtoEqv}_{\mathcal{A}}(\mathtt{pr}_1(z), \mathtt{pr}_1(w), p_1)) \\
&= p_1,
\end{aligned}$$

and proceeding similarly we get $p_2{}' = p_2$. With the use of $\mathtt{ap}_{\mathtt{pair}^{=}}$ and the property of inversion between $\mathtt{pair}^{=}$ and ${}^{=}\mathtt{pair}$, shown in section 2.6 of [13], we get

$$\begin{aligned}\mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z,w,e') &= \mathtt{pair}^{=}(z,w,p_1,p_2)\\ &\equiv \mathtt{pair}^{=}(z,w,\mathtt{pr}_1({}^{=}\mathtt{pair}(z,w,p)),\mathtt{pr}_2({}^{=}\mathtt{pair}(z,w,p)))\\ &= p.\end{aligned}$$

Using the definitions in Lemma 22 and the functions $\mathtt{ap}_{\mathtt{pr}_1}$ and $\mathtt{IdtoEqv}^2_{\mathcal{A}}$, if $p \equiv \mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z,w,e') \equiv \mathtt{pair}^{=}(z,w,\mathtt{pr}_1{}',\mathtt{pr}_2{}')$, then

$$p_1 \equiv \mathtt{pr}_1({}^{=}\mathtt{pair}(\mathtt{pair}^{=}(z,w,p_1{}',p_2{}'))) = \mathtt{pr}_1((p_1{}',p_2{}')) = p_1{}',$$

$$e_1 \cong_{\mathcal{A}} \mathtt{IdtoEqv}_{\mathcal{A}}(\mathtt{pr}_1(z),\mathtt{pr}_1(w),p_1{}') \equiv \mathtt{IdtoEqv}_{\mathcal{A}}(\mathtt{pr}_1(z),\mathtt{pr}_1(w),\mathtt{Ua}_{\mathcal{A}}(\mathtt{pr}_1(z),\mathtt{pr}_1(w),e_1{}')) \cong_{\mathcal{A}} e_1{}',$$

and proceeding similarly we get $p_2 = p_2{}'$ and $e_2 = e_1{}'$. By Corollaries 16 and 15 we get

$$\mathtt{IdtoEqv}_{\mathcal{A}\times\mathcal{B}}(z,w,\mathtt{Ua}_{\mathcal{A}\times\mathcal{B}}(z,w,e')) \equiv T(z,w,e_1,e_2) \cong_{\mathcal{A}\times\mathcal{B}} T(z,w,e_1{}',e_2{}') \cong_{\mathcal{A}\times\mathcal{B}} e'.$$

In order to define $\mathtt{Ua}^2_{\mathcal{A}\times\mathcal{B}}$ let $e',d' : z \simeq_{\mathcal{A}\times\mathcal{B}} w$, $i' : e' \cong_{\mathcal{A}\times\mathcal{B}} d' \equiv (e_1{}' \cong_{\mathcal{A}} d_1{}') \times (e_2{}' \cong_{\mathcal{B}} d_2{}')$, and $i_1{}' \equiv \mathtt{pr}_1(i'), i_2{}' \equiv \mathtt{pr}_2(i')$. Since $\mathtt{Ua}^2_{\mathcal{A}}(\mathtt{pr}_1(z),\mathtt{pr}_1(w),e_1{}',d_1{}',i_1{}') : p_1{}' = q_1{}'$ and $\mathtt{Ua}^2_{\mathcal{B}}(\mathtt{pr}_2(z),\mathtt{pr}_2(w),e_2{}',d_2{}',i_2{}') : p_2{}' = q_2{}'$, we define

$$\begin{aligned}\mathtt{Ua}^2_{\mathcal{A}\times\mathcal{B}}(z,w,e',d',i') \equiv \mathtt{ap}_{\mathtt{pair}^{=}(z,w)}\Big((p_1{}',p_2{}'),(q_1{}',q_2{}'),\mathtt{Ua}^2_{\mathcal{A}}(\mathtt{pr}_1(z),\mathtt{pr}_1(w),e_1{}',d_1{}',i_1{}'),\\ \mathtt{Ua}^2_{\mathcal{B}}(\mathtt{pr}_2(z),\mathtt{pr}_2(w),e_2{}',d_2{}',i_2{}')\Big),\end{aligned}$$

where $\mathtt{ap}_{\mathtt{pair}^{=}(z,w)}((p_1{}',p_2{}'),(q_1{}',q_2{}')) : (p_1{}' = q_1{}') \times (p_2{}' = q_2{}') \to \mathtt{pair}^{=}(z,w,p_1{}',p_2{}') = \mathtt{pair}^{=}(z,w,q_1{}',q_2{}')$. ◀

▶ **Proposition** 24. Let $\mathcal{A}\times\mathcal{B}$ be univalent. If (B,b) is a pointed type i.e., $b : B$, then $\mathcal{A}$ is univalent. If (A,a) is a pointed type, then $\mathcal{B}$ is univalent.

Proof. We only sketch the main idea of the proof for the first case. If $e : x \simeq_{\mathcal{A}} x'$, then $(e,\mathtt{eqv}_b) : (x,b) \simeq_{\mathcal{A}\times\mathcal{B}} (x',b)$. The resulting term of type $(x,b) =_{A\times B} (x',b)$ generates a term of type $x =_A x'$. All details are handled as in the proof of Theorem 23. ◀

5 Exponential typoid

If $\mathcal{A},\mathcal{B}$ are typoids and $f : A \to B$, the type "f is a typoid function from $\mathcal{A}$ to $\mathcal{B}$" is

$$\begin{aligned}\mathtt{Typfun}(f) \equiv \sum_{\Phi_f : \prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}} y} f(x)\simeq_{\mathcal{B}} f(y)} \Bigg[\Bigg(\prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}} y}\prod_{d:y\simeq_{\mathcal{A}} z}\\ \Big(\Phi_f(x,x,\mathtt{eqv}_x) \cong_{\mathcal{B}} \mathtt{eqv}_{f(x)}\Big) \times \Big(\Phi_f(x,z,e *_{\mathcal{A}} d) \cong_{\mathcal{B}} \Phi_f(x,y,e) *_{\mathcal{B}} \Phi_f(y,z,d)\Big)\Bigg)\times\\ \times\Bigg(\prod_{x,y:A}\prod_{e,d:x\simeq_{\mathcal{A}} y}\prod_{i:e\cong_{\mathcal{A}} d} \Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d)\Bigg)\Bigg].\end{aligned}$$

A canonical term of type $\texttt{Typfun}(f)$ is a pair $(\Phi_f, (U, \Phi^2_f))$, or simply a triplet (Φ_f, U, Φ^2_f), where U is a term of the first type of the outer product and Φ^2_f is a term of the second. The *exponential* of $\mathcal{A}, \mathcal{B}$ is the typoid $\mathcal{B}^{\mathcal{A}} := \big(B^A, \simeq_{\mathcal{B}^{\mathcal{A}}}, \texttt{eqv}_{\mathcal{B}^{\mathcal{A}}}, *_{\mathcal{B}^{\mathcal{A}}}, {}^{-1_{\mathcal{B}^{\mathcal{A}}}}, \cong_{\mathcal{B}^{\mathcal{A}}}\big)$, where

$$B^A \equiv \sum_{f:A\to B} \texttt{Typfun}(f).$$

If $\phi \equiv (f, \Phi_f, U, \Phi^2_f)$ and $\theta \equiv (g, \Phi_g, W, \Phi^2_g)$ are two canonical terms of type B^A, we define

$$\phi \simeq_{\mathcal{B}^{\mathcal{A}}} \theta \equiv \sum_{\Theta_{f,g}:\prod_{x:A} f(x)\simeq_{\mathcal{B}} g(x)} \left(\prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}} y} \Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,g}(y) \cong_{\mathcal{B}} \Theta_{f,g}(x) *_{\mathcal{B}} \Phi_g(x,y,e)\right).$$

A canonical term e of type $\phi \simeq_{\mathcal{B}^{\mathcal{A}}} \theta$ is a pair $(\Theta_{f,g}, \Theta^2_{f,g})$, where

$$\Theta^2_{f,g} : \prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}} y} \Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,g}(y) \cong_{\mathcal{B}} \Theta_{f,g}(x) *_{\mathcal{B}} \Phi_g(x,y,e)$$

i.e., $\Theta^2_{f,g}(x,y,e)$ is a proof that the following diagram commutes

$$\begin{tikzcd}
f(x) \arrow[r, equal, "\Phi_f(x,y,e)"] \arrow[d, equal, "\Theta_{f,g}(x)"'] & f(y) \arrow[d, equal, "\Theta_{f,g}(y)"] \\
g(x) \arrow[r, equal, "\Phi_g(x,y,e)"'] & g(y).
\end{tikzcd}$$

If ϕ is a canonical term of type B^A we define $\texttt{eqv}_\phi : \phi \simeq_{\mathcal{B}^{\mathcal{A}}} \phi$ as the pair $(\Theta_{f,f}, \Theta^2_{f,f})$, where $\Theta_{f,f} \equiv \lambda(x:A).\texttt{eqv}_{f(x)} : \prod_{x:A} f(x) \simeq_{\mathcal{B}} f(x)$ and $\Theta^2_{f,f}(x,y,e)$ is the obvious proof that the following diagram commutes

$$\begin{tikzcd}
f(x) \arrow[r, equal, "\Phi_f(x,y,e)"] \arrow[d, equal, "\texttt{eqv}_{f(x)}"'] & f(y) \arrow[d, equal, "\texttt{eqv}_{f(y)}"] \\
g(x) \arrow[r, equal, "\Phi_g(x,y,e)"'] & g(y).
\end{tikzcd}$$

If $\phi \equiv (f, \Phi_f, U, \Phi^2_f), \theta \equiv (g, \Phi_g, W, \Phi^2_g), \eta \equiv (h, \Phi_h, V, \Phi^2_h)$ are canonical terms of type B^A and $e \equiv (\Theta_{f,g}, \Theta^2_{f,g}) : \phi \simeq_{\mathcal{B}^{\mathcal{A}}} \theta$ and $d \equiv (\Theta_{g,h}, \Theta^2_{g,h}) : \theta \simeq_{\mathcal{B}^{\mathcal{A}}} \eta$, we define

$$e *_{\mathcal{B}^{\mathcal{A}}} d \equiv (\Theta_{f,h}, \Theta^2_{f,h}) : \phi \simeq_{\mathcal{B}^{\mathcal{A}}} \eta$$

$$\Theta_{f,h} \equiv \lambda(x:A).\Theta_{f,g}(x) *_{\mathcal{B}} \Theta_{g,h}(x).$$

A term $\Theta^2_{f,h}(x,y,e)$ of type

$$\Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,h}(y) \cong_{\mathcal{B}} \Theta_{f,h}(x) *_{\mathcal{B}} \Phi_h(x,y,e) \equiv$$

$$\Phi_{f(x,y,e)} *_{\mathcal{B}} (\Theta_{f,g}(y) *_{\mathcal{B}} \Theta_{g,h}(y)) \cong_{\mathcal{B}} (\Theta_{f,h}(x) *_{\mathcal{B}} \Theta_{g,h}(x)) *_{\mathcal{B}} \Phi_h(x,y,e)$$

is found through the commutativity of the following outer diagram

$$\begin{tikzcd}
f(x) \arrow[r, equal, "\Phi_f(x,y,e)"] \arrow[d, equal, "\Theta_{f,g}(x)"'] & f(y) \arrow[d, equal, "\Theta_{f,g}(y)"] \\
g(x) \arrow[r, equal, "\Phi_g(x,y,e)"'] \arrow[d, equal, "\Theta_{g,h}(x)"'] & g(y) \arrow[d, equal, "\Theta_{g,h}(y)"] \\
h(x) \arrow[r, equal, "\Phi_h(x,y,e)"'] & h(y),
\end{tikzcd}$$

and rests on the commutativity of the inner diagrams. If $e \equiv (\Theta_{f,g}, \Theta^2_{f,g}) : \phi \simeq_{\mathcal{B}^{\mathcal{A}}} \theta$, let

$$e^{-1_{\mathcal{B}^{\mathcal{A}}}} \equiv (\Theta^{-1}_{f,g}, [\Theta^2_{f,g}]^{-1}) : \theta \simeq_{\mathcal{B}^{\mathcal{A}}} \phi,$$

where $\Theta^{-1}_{f,g} : \prod_{x:A} g(x) \simeq_{\mathcal{B}} f(x)$ is defined by

$$\Theta^{-1}_{f,g}(x) \equiv [\Theta_{f,g}(x)]^{-1_{\mathcal{B}}},$$

for every $x : A$, and $[\Theta^2_{f,g}]^{-1}(y, x, e)$ is a term of type $\Phi_g(y, x, e) *_{\mathcal{B}} \Theta_{f,g}(x)^{-1} \cong_{\mathcal{B}} \Theta_{f,g}(y)^{-1} *_{\mathcal{B}} \Phi_f(y, x, e)$ i.e., a proof of the commutativity of the following diagram

$$\begin{tikzcd}
g(x) \arrow[r, equal, "\Phi_g(y,x,e)"] \arrow[d, equal, "\Theta_{f,g}(x)^{-1}"'] & g(y) \arrow[d, equal, "\Theta_{f,g}(y)^{-1}"] \\
f(x) \arrow[r, equal, "\Phi_f(y,x,e)"'] & f(y),
\end{tikzcd}$$

which rests on the commutativity of the diagram that corresponds to the term $\Theta^2_{f,g}(x, y, e^{-1})$.

▶ **Proposition** 25. Let $\mathcal{A}, \mathcal{B}$ be typoids. If $\mathcal{B}$ is univalent, then $\mathcal{B}^{\mathcal{A}}$ is univalent.

Proof. We only sketch the main idea of the proof. If $\phi \equiv (f, \Phi_f, U, \Phi^2_f)$ and $\theta \equiv (g, \Phi_g, W, \Phi^2_g)$ are two canonical terms of type B^A, and if $(\Theta_{f,g}, \Theta^2_{f,g})$ is a canonical term of type $\phi \simeq_{\mathcal{B}^{\mathcal{A}}} \theta$, then $\Theta_{f,g} : \prod_{x:A} f(x) \simeq_{\mathcal{B}} g(x)$. Since $\mathcal{B}$ is univalent, every term $\Theta_{f,g}(x) : f(x) \simeq_{\mathcal{B}} g(x)$ generates a term of type $f(x) =_B g(x)$. Consequently, we get a term of type $f =_{A \to B} g$. Using Theorem 2.7.2 in [13], we can construct a term of type $\phi = \theta$. ◀

6 Truncated typoids

In [13], section 3.7, the notion of propositional truncation of a type A is implemented through the higher inductive type $||A||$. Here we use the "truncated typoid" $\mathcal{A}^t$ to interpret this notion. Starting from a typoid structure on a type A we define a new typoid stucture on A, which behaves accordingly. Hence, we keep the same type and we change the typoid structure, while in the theory of HITs the type is changed. We denote the unit type by $\mathbf{1}$.

▶ **Definition 26.** If $A : \mathcal{U}$, we call the typoid $\mathcal{A}^t \equiv \big(A, \simeq_{\mathcal{A}^t}, \mathtt{eqv}_{\mathcal{A}^t}, *_{\mathcal{A}^t}, {}^{-1_{\mathcal{A}^t}}, \cong_{\mathcal{A}^t}\big)$ *truncated*, where for every $x, y, z : A$, $e, e' : x \simeq_{\mathcal{A}^t} y$, and $d : y \simeq_{\mathcal{A}^t} z$, let

$$x \simeq_{\mathcal{A}^t} y \equiv \mathbf{1}, \quad \mathtt{eqv}_{\mathcal{A}^t}(x) \equiv 0_{\mathbf{1}}, \quad *_{\mathcal{A}^t}(x, y, z, e, d) \equiv 0_{\mathbf{1}},$$

$${}^{-1_{\mathcal{A}^t}}(x, y, e) \equiv 0_{\mathbf{1}}, \quad \cong_{\mathcal{A}^t}(x, y, e, e') \equiv (e = e').$$

The proof that $\mathcal{A}^t$ is a typoid is immediate. One needs only to take into account that $\mathtt{isProp}(\mathbf{1})$, where $\mathtt{isProp}(A) \equiv \prod_{x,y:A}(x =_A y)$.

▶ **Proposition** 27. If $A : \mathcal{U}$, $\mathcal{B}$ is a typoid and $f : B \to A$, then f is a typoid function from $\mathcal{B}$ to $\mathcal{A}^t$.

Proof. Let $x, y, z : B$, $e, e' : x \simeq_{\mathcal{B}} y$ $i : e \cong_{\mathcal{B}} e'$, and $d : y \simeq_{\mathcal{B}} z$ We define $\Phi_f(x, y, e) \equiv 0_{\mathbf{1}}$, hence $\Phi_f(x, y, e) : f(x) \simeq_{\mathcal{A}^t} f(y)$, and we also define $\Phi^2_f(x, y, e, e', i) \equiv \mathtt{refl}_{0_{\mathbf{1}}}$, hence $\Phi^2_f(x, y, e, e', i) : 0_{\mathbf{1}} =_{\mathbf{1}} 0_{\mathbf{1}}$. Clearly, $\Phi_f(x, x, \mathtt{eqv}_x) \equiv 0_{\mathbf{1}} \equiv e_{f(x)}$, and $\Phi_f(x, z, e_1 *_{\mathcal{B}} e_2) \equiv 0_{\mathbf{1}} = (0_{\mathbf{1}} *_{\mathcal{A}^t} 0_{\mathbf{1}}) \equiv \Phi_f(x, y, e_1) *_{\mathcal{A}^t} \Phi_f(y, z, e_2)$. ◀

▶ **Corollary 28.** *If $A, B : \mathcal{U}$ and $f : B \to A$, then f is a typoid function from $\mathcal{B}^t$ to $\mathcal{A}^t$.*

If $f : B \to A$, we use the notation f^t for f to indicate that we view f as a typoid function from B^t to A^t. In the next proof we use the type $\mathtt{isSet}(A) \equiv \prod_{x,y:A} \prod_{p,q:x=_A y}(p = q)$.

▶ **Proposition** 29. If $A : \mathcal{U}$ such that $\mathtt{isProp}(A)$, then $\mathcal{A}^t$ is univalent.

Proof. If $\Omega : \mathtt{isProp}(A)$, $d, e : \mathbf{1}$ and $i : d \cong_{\mathcal{A}^t} e \equiv (d = e)$, we define

$$\mathtt{Ua}_{\mathcal{A}^t}(x, y, d) \equiv \Omega(x, y), \qquad \mathtt{Ua}^2_{\mathcal{A}^t}(x, y, d, e, i) \equiv \mathtt{refl}_{\Omega(x,y)},$$

hence $\mathtt{Ua}_{\mathcal{A}^t}(x, y, d) : x =_A y$ and $\mathtt{Ua}^2_{\mathcal{A}^t}(x, y, d, e, i)$ is a term of type $\Omega(x, y) = \Omega(x, y) \equiv \mathtt{Ua}_{\mathcal{A}^t}(x, y, d) = \mathtt{Ua}_{\mathcal{A}^t}(x, y, e)$. Let $\mathtt{IdtoEqv}_{\mathcal{A}^t}$ be an 1-associate of id_A seen as function from $\mathcal{A}_0$ to $\mathcal{A}^t$. Since $\mathtt{IdtoEqv}_{\mathcal{A}^t}(x, y, \mathtt{Ua}_{\mathcal{A}^t}(x, y, d)) : \mathbf{1}$, and $d : \mathbf{1}$, we get $\mathtt{IdtoEqv}_{\mathcal{A}^t}(x, y, \mathtt{Ua}_{\mathcal{A}^t}(x, y, d)) = d$, i.e., $\mathtt{IdtoEqv}_{\mathcal{A}^t}(x, y, \mathtt{Ua}_{\mathcal{A}^t}(x, y, d)) \cong_{\mathcal{A}^t} d$. Since $\mathtt{Ua}_{\mathcal{A}^t}(x, y, \mathtt{IdtoEqv}_{\mathcal{A}^t}(x, y, p)) : x =_A y$, $p : x =_A y$ and $\mathtt{isProp}(A) \to \mathtt{isSet}(A)$ (see Lemma 3.3.4 of [13]), we conclude that $\mathtt{Ua}_{\mathcal{A}^t}(x, y, \mathtt{IdtoEqv}_{\mathcal{A}^t}(x, y, p)) = p$. ◀

Using Theorem 19(i) we get the following corollary.

▶ **Corollary 30.** *If $A : \mathcal{U}$ such that $\mathtt{isProp}(A)$, $\mathcal{B}$ is a typoid and $f : A \to B$, then f is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}$.*

▶ **Proposition** 31. If $A : \mathcal{U}$, $\mathcal{B}$ is a typoid such that $\mathtt{isProp}(B)$, and $f : A \to B$, then f is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}$.

Proof. By Corollary 28 we have that f is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}^t$, while by Corollary 30 id_B is a typoid function from $\mathcal{B}^t$ to $\mathcal{B}$. By Proposition 9 we conclude that $f \equiv \mathrm{id}_B \circ f$ is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}$. ◀

7 Concluding remarks and future work

We presented here some first results on the fundamental properties of general typoids and univalent typoids. Our main goal was to establish a common framework for all instances of types that behave in a "univalent way". The structure of typoid is a weak groupoid analogue to the notion of precategory in HoTT, and the notion of a univalent typoid is a weak groupoid analogue to the notion of category in HoTT[6].

A precategory in HoTT is a type A such that for every $a, b : A$ there is a "set" $\hom(a, b)$ satisfying the expected conditions of a category. If $a \cong b$ is the type of the corresponding

[6] See Chapter 9 in book-HoTT [13]. The notion of category in HoTT was first considered in [9], and it was formalised in [1].

notion of isomorphism, a precategory is a category, if the obvious function that sends a term of $a =_A b$ to a term of type $a \cong b$ is an equivalence. The analogy between typoids/univalent typoids and precategories/categories suggests naturally the study of topics, like the Rezk completion i.e., the universal way to replace a precategory by a category, within the context of typoids[7].

We plan to study the category of typoids and other categorical properties of univalent typoids in a subsequent work. More specific examples of univalent typoids are expected to be found in [8], where with the use of UA a general theorem that "isomorphism implies equality" is shown for many algebraic structures. The implementation of more higher inductive types as appropriate typoids is another topic of future work.

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