Constructive Topology of Bishop Spaces

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Iosif Petrakis

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Abstract

The theory of Bishop spaces (TBS) is so far the least developed approach to constructive topology with points. Bishop introduced function spaces, here called Bishop spaces, in 1967, without really exploring them, and in 2012 Bridges revived the subject. In this Thesis we develop TBS.

Instead of having a common space-structure on a set X and $\mathbb{R}$, where $\mathbb{R}$ denotes the set of constructive reals, that determines a posteriori which functions of type $X \to \mathbb{R}$ are continuous with respect to it, within TBS we start from a given class of "continuous" functions of type $X \to \mathbb{R}$ that determines a posteriori a space-structure on X. A Bishop space is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$, where $\mathbb{F}(X)$ denotes the set of all functions of type $X \to \mathbb{R}$, is a Bishop topology, or simply a topology, includes the constant maps and it is closed under addition, uniform limits and composition with the Bishop-continuous functions $\mathrm{Bic}(\mathbb{R})$ of type $\mathbb{R} \to \mathbb{R}$. The space-structure induced by a topology F on some inhabited set X is the neighborhood structure $N(F)$, where $N(F) = \{U(f) \mid f \in F\}$ and $U(f) = \{x \in X \mid f(x) > 0\}$.

The main motivation behind the introduction of Bishop spaces is that function-based concepts are more suitable to constructive study than set-based ones. Although a Bishop topology of functions F on X is a set of functions, the set-theoretic character of TBS is not that central as it seems. The reason for this is Bishop's inductive concept of the least topology $\mathcal{F}(F_0)$ generated by a given subbase $F_0 \subseteq \mathbb{F}(X)$. The definitional clauses of a Bishop space, seen as inductive rules, induce the corresponding induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$. Hence, starting with a constructively acceptable subbase F_0 the generated topology $\mathcal{F}(F_0)$ is a constructively graspable set of functions exactly because of the corresponding principle $\mathrm{Ind}_{\mathcal{F}}$. The function-theoretic character of TBS is also evident in the characterization of "continuity". If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a Bishop morphism from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \to Y$ such that $\forall_{g \in G}(g \circ h \in F)$. The Bishop morphisms are the arrows in the category of Bishop spaces **Bis**.

The development of constructive point-function topology in this Thesis takes two directions. The first is a purely topological one. We introduce and study, among other notions, the quotient, the pointwise exponential, the dual, the Hausdorff, the completely regular, the $\mathbf{2}$-compact, the pair-compact and the 2-connected Bishop spaces. We prove, among other results, the Stone-Čech isomorphism between $\mathcal{F}$ and $\rho\mathcal{F}$, the Embedding lemma, a generalized version of the Tychonoff embedding theorem for completely regular Bishop spaces, the Gelfand-Kolmogoroff theorem for fixed and completely regular Bishop spaces, a Stone-

Weierstrass theorem for pseudo-compact Bishop spaces and a Stone-Weierstrass theorem for pair-compact Bishop spaces. Of special importance is the notion of 2-compactness, a constructive function-theoretic notion of compactness for which we show that it generalizes the notion of a compact metric space. In the last chapter we initiate the basic homotopy theory of Bishop spaces.
The other direction in the development of TBS is related to the analogy between a Bishop topology F, which is a ring and a lattice, and the ring of real-valued continuous functions $C(X)$ on a topological space X. This analogy permits a direct "communication" between TBS and the theory of rings of continuous functions, although due to the classical set-theoretic character of $C(X)$ this does not mean a direct translation of the latter to the former. We study the zero sets of a Bishop space and we prove the Urysohn lemma for them. We also develop the basic theory of embeddings of Bishop spaces in parallel to the basic classical theory of embeddings of rings of continuous functions and we show constructively the Urysohn extension theorem for Bishop spaces.
The constructive development of topology in this Thesis is within Bishop's informal system of constructive mathematics BISH, inductive definitions with rules of countably many premises included.

Zusammenfassung

Die Theorie der Bishop-Räume (TBS) ist der bis heute am wenigsten weit entwickelte Ansatz zur konstruktiven Topologie mit Punkten. Bishop führte die Funktionenräume, die hier als Bishop-Räume bezeichnet werden, erstmals im Jahre 1967 ein, ohne sie näher zu untersuchen, und Bridges belebte das Thema 2012 wieder. In dieser Dissertation entwickeln wir die TBS.

Anstelle einer Räumenstruktur auf einer Menge X und den konstruktiven reellen Zahlen $\mathbb{R}$, die a posteriori bestimmen, welche Funktionen vom Typ $X \to \mathbb{R}$ stetig bezüglich dieser Struktur sind, beginnt man in TBS mit einer vorgegebenen Klasse "stetiger" Funktionen vom Typ $X \to \mathbb{R}$, die a posteriori die Struktur eines Raumes auf X definiert. Ein Bishop-Raum ist ein Paar $\mathcal{F} = (X, F)$, wobei X eine nichtleere Menge und $F \subseteq \mathbb{F}(X)$, wobei $\mathbb{F}(X)$ die Menge aller Funktionen vom Typ $X \to \mathbb{R}$ bezeichnet, eine Bishop-Topologie, oder einfach Topologie, ist, die die konstanten Abbildungen enthält und unter Addition, uniformen Grenzwerten und Komposition mit den Bishop-stetigen Funktionen $\mathrm{Bic}(\mathbb{R})$ vom Typ $\mathbb{R} \to \mathbb{R}$ abgeschlossen ist. Die Struktur eines topologischen Raums, die von einer Topologie F auf einer nichtleeren Menge X induziert wird, wird durch die Basis $N(F) = \{U(f) \mid f \in F\}$ definiert, wobei $U(f) = \{x \in X \mid f(x) > 0\}$.

Die Hauptmotivation für die Einführung von Bishop-Räumen war, dass funktionenbasierte Begriffe sich für konstruktivistische Untesuchungen besser eignen als menegenbasierte. Obwohl eine Bishop-Topologie als Menge von Funktionen definiert wird, ist dieser mengentheoretische Charakter von TBS nicht so zentral, wie es scheinen mag. Der Grund hierfür ist Bishops induktive Definition der kleinsten Topologie $\mathcal{F}(F_0)$, die von einer gegebenen Subbasis $F_0 \subseteq \mathbb{F}(X)$. Als Induktionsschema betrachtet führen die definierenden Regeln des Bishop-Raumes zu einem entsprechenden Induktionsprinzip $\mathrm{Ind}_{\mathcal{F}}$ auf $\mathcal{F}(F_0)$. Damit ist also $\mathcal{F}(F_0)$ gerade wegen $\mathrm{Ind}_{\mathcal{F}}$ auf $\mathcal{F}(F_0)$ konstruktiv greifbar, solang nur F_0 konstruktiv annehmbar ist. Der funktionentheoretische Charakter von TBS wird auch in der Definition der Stetigkeit deutlich. Sind $\mathcal{F} = (X, F)$ und $\mathcal{G} = (Y, G)$ Bishop-Räume, so ist ein Bishop-Morphismus von $\mathcal{F}$ nach $\mathcal{G}$ eine Funktion $h : X \to Y$ sodass $\forall_{g \in G}(g \circ h \in F)$. Diese Bishop-Morphismen sind die Pfeile in der Kategorie der Bishop-Räume **Bis**.

Die Entwicklung konstruktiver Topologie mit Punkten und Funktionen in dieser Dissertation geschieht in zwei Richtungen. Die erste ist rein topologisch. Wir führen u.a. den Quotienten-, den punktweisen Exponential-, den Dual-, den Hausdorff-, den vollstandig regulären, den 2-kompakten, den paar-kompakten und den 2-zusammenhängenden Bishop-Raum ein und untersuchen ihre Eigenschaften. Wir beweisen u.a. den Stone-Čech-Isomorphismus zwischen

$\mathcal{F}$ und $\rho\mathcal{F}$, das Einbettungslemma, eine verallgemeinende Form des Tychonoffschen Einbetttungssatzes für vollstandig reguläre Bishop-Räume, den Gelfand-Kolmogoroffschen Satz für Bishop-Räume, den Satz von Stone-Weierstrass für Pseudo-Kompakte Bishop-Räume und den Satz von Stone-Weierstrass für Paar-Kompakte Bishop-Räume. Von besonderer Bedeutung ist der Begriff der 2-Kompaktheit, eine konstruktive funktionentheoretische Entsprechung der Kompaktheit, die, wie wir zeigen, den Begriff des kompakten metrischen Raums verallgemeinert. Im letzten Kapitel wird die Homotopie Theorie der Bishop-Räume eingeführt.

Die andere Richtung in der Entwicklung von TBS betrifft die Analogie zwischen einer Bishop-Topologie F, die ein Ring und ein Verband ist, und dem Ring der reelwertigen stetigen Funktionen $C(X)$ auf einem topologischen Raum X. Diese Analogie erlaubt eine gewisse "Kommunikation" zwischen TBS und der Theorie der Ringe stetiger Funktionen, wobei das aufgrund des klassischen mengentheoretischen Charakters von $C(X)$ keine unmittelbare Übersetzung von letzterer in erstere gestaltet. In dieser Dissertation untersuchen wir die Nullstellenmengen eines Bishop-Raumes und beweisen für sie das Urysohnsche Lemma. Wir entwickeln auch die grundlegende Theorie von Einbettungen zwischen Bishop-Räumen in Analogie zur klassischen Theorie von Einbettungen zwischen Ringen stetiger Funktionen und geben einen konstruktiven Beweis für Urysohns Erweiterungssatz für Bishop-Räume.

Die konstruktive Entwicklung der Topologie in dieser Dissertation findet im Rahmen von Bishops informellem System für konstruktive Mathematik BISH statt, induktive Definitionen eingeschlossen.

Publications included in or related to this Thesis

1. Iosif Petrakis: Limit spaces with approximations, to appear in the *Annals of Pure and Applied Logic*.
2. Iosif Petrakis: The Urysohn Extension Theorem for Bishop Spaces, to appear in S. Artemov and A. Nerode (Eds.) *Symposium on Logical Foundations in Computer Science 2016*, LNCS.
3. Iosif Petrakis: Completely Regular Bishop Spaces, in A. Beckmann, V. Mitrana and M. Soskova (Eds.): *Evolving Computability*, CiE 2015, LNCS 9136, 2015, 302-312. DOI: 10.1007/978-3-319-20028-6 31.
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a. Publication 1 is not included in this Thesis but its content is related to possible future applications of the theory of Bishop spaces to computability at higher types. We discuss this in section 8.3.

b. Publication 2 contains sections 5.5 and 5.6 of this Thesis. Its content was first included in an earlier draft of this Thesis and only later submitted for publication.

c. Publication 3 contains Proposition 3.6.10, section 5.7, except Example 5.7.12 and Proposition 5.7.13, and section 5.8, except Proposition 5.8.7 and Corollary 5.8.8. Its content was first included in an earlier draft of this Thesis and only later submitted for publication.

d. Publication 4 is a short summary of our then current results in the theory of Bishop spaces.

e. Publication 5 is not included in this Thesis but its content is related to the formal study of subsystems of BISH, as it is mentioned in section 2.1.

Chapter 1

Introduction

> Subsequent developments of constructive topology took several forms, notable being the early work of Grayson, the formal topology, ..., the theory of frames and locales, and the theory of apartness. Bishop's idea of a function-space-based approach to topology has been largely forgotten in the midst of all this activity. Yet such an approach would seem worthy of exploration, not least in light of the burgeoning of category theory in the last 40 years. Perhaps it will lead us to a dead end; perhaps, though yielding interesting information, function spaces will not hold sufficient content to have major influence on constructive topology; but I hope that this paper may stimulate interest in function spaces as objects worthy of investigation.
>
> Douglas S. Bridges, 2012

This Thesis could have been written right after the publication of Errett Bishop's seminal book *Foundations of Constructive Analysis* in 1967. The reasons for such a late effort to develop the theory of Bishop spaces (TBS) are related on the one hand to Bishop's ambivalence on the urgency of developing abstract constructive topology, and on the other to the increasing interest on foundational aspects of Bishop's mathematics rather than purely mathematical ones. The subject of this Thesis is a constructive reconstruction of topology based on the notion of a function space, here called a Bishop space, that it was introduced by Bishop in his book of 1967, p.71, but it was not really studied until Douglas

Bridges's paper [19] of 2012. The main characteristics of TBS, as it is developed in this Thesis, are the following:

1. Points are accepted from the beginning, hence it is not a point-free approach to topology.
2. Most of its notions are function-theoretic. Set-theoretic notions are avoided or play a secondary role to its development.
3. It is constructive. We work within Bishop's informal system of constructive mathematics BISH, inductive definitions with rules of countably many premises included.
4. It has simple foundation and it follows the style of standard mathematics.

In other words, TBS is an approach to constructive point-function topology.

1.1 Why Bishop Spaces

There are various approaches to constructive topology. In point-free constructive topology we mention the theory of formal topology of Martin-Löf [63] and Sambin [81], and its latest version within the minimalist program of Sambin and Maietti [82]. As Crosilla and Schuster note in [37], pp.14-15, "formal topology is a constructive and predicative generalization of the theory of frames and locales which have been studied since the 1950s, though usually in a classical or impredicative way". In constructive topology with points belong the theory of apartness spaces developed mainly by Bridges and Vîţă in [27], Grayson's direct study of the axioms of topology using intuitionistic logic in [45] and [46], the theory of Bishop's neighborhood spaces, studied mainly by Ishihara in [55], [57]. The intuitionistic topology is the older approach to constructive topology. Brouwer's work, see for example [29], [30] and [50] was pioneering and influential to all subsequent approaches. One should also mention the work of Freudenthal [43], of Troelstra [93] and more recently of Waaldjik [98]. Recently homotopy theory became popular among logicians because of Homotopy Type Theory (HoTT) which revealed surprising connections between intuitionistic logic and higher-dimensional structures in algebraic topology and category theory. In some not so straightforward sense, HoTT is a kind of constructivisation of certain parts of the classical homotopy theory.

In classical topology it is standard to relate the topological properties of a given topological space $(X, \mathcal{T})$ with the properties of the set of real-valued continuous functions $C(X)$ on X and its subset $C^*(X)$ of bounded ones. As it is noted in [44], p.12, both these function rings are determined by the space X; if X is homeomorphic to Y, then $C(X)$ is isomorphic to $C(Y)$. In some cases one can also "recover" X from $C(X)$, as in the case of a compact Hausdorff space X, where X is homeomorphic to the topological spectrum of $C(X)$. Moreover, specific topological properties of X are determined by corresponding properties of $C(X)$. For example, X is connected if and only if $C(X)$ is not the direct sum of proper subrings.

Bishop somehow "reversed" the above picture. Namely, his initial object of study is an appropriate set F of functions $X \to \mathbb{R}$, the topology of functions on X, which induces a posteriori a topology of opens on X, the smallest one with respect to which all the elements of F are continuous. This act of Bishop is meaningful both constructively and classically. Note that Gillman and Jerison asked within the classical theory of rings in [44], p.271,

> "to represent a given abstract object as a family of continuous functions".

The notion of a Bishop space can be seen as an attempt to answer this fundamental question. Within our development of TBS the topology of functions, and not the induced topology of opens, is going to be the main object of study. The reason for this is that, following Bishop, we want to avoid the difficulties arising for a constructive treatment of topology if we try to "mimic" the classical development of topology based on the set-theoretic definition of a topology of opens. Hence, we try to form the topological concepts based mainly on the set of functions F and not on the induced by F topology. If someone tries to reconstruct constructively the standard topological space, like Troelstra or Grayson did, will encounter too many problems exactly because standard topology is set-theoretic, therefore too difficult to approach constructively. On the other hand the notion of function has a straightforward interpretation in constructive mathematics and it is easier to handle. Note that in BISH the notion of function is a primitive notion that it is not reduced to the notion of set.
Bishop's first main motivation for introducing the topologies of functions is the inductive character of the notion of the least topology of functions including a given set of functions F_0. This concept is in contrast to abstract subsets of the set $\mathbb{F}(X)$ of all functions of type $X \to \mathbb{R}$ such as the set $C_p(X)$ of all pointwise continuous functions, which are difficult to grasp, or, in Bishop's words, "it is not possible to get a good hold on its structure". It is not a coincidence that the topologies of functions are introduced in the Set Theory chapter of [6], where the foundations of a constructive notion of a set are laid. A bit later, in [62], Martin-Löf elaborated the notion of an inductively generated set. This motivation behind Bishop's abstract spaces of functions echoes Brouwer's much earlier critique of sets as "boxes". The notion of the least topology of functions including some F_0 is an important case-study of an inductively defined concept within BISH, the second, and the last inductive definition found in [6] after the inductive definition of a Borel set, which was later abandoned in favor to Bishop's new treatment of measure theory given in [14] and in [15]. The importance of this concept is also related to the generation of new constructive sets from old ones. In this Thesis we give a plethora of results on inductively generated Bishop spaces, hoping that the study of this specific inductive definition within TBS can be related to the study of similar inductively defined notions in other areas of constructive mathematics, like, for example, the least uniformity generated by a given family of pseudometrics.
Bishop's second main motivation for introducing function spaces is expressed in [15], p.77, as follows:

> Proximity is introduced into X classically not by giving a family of functions, but by giving a family of subsets, either open sets or neighborhoods. Classically, this is equivalent to giving a family of functions from X to $\{0, 1\}$. Constructively,

> there is a vast difference: since functions are sharply defined, whereas most sets are fuzzy around the edges, only the all-too-rare detachable subsets of X correspond to functions from X to $\{0, 1\}$. The fuzziness of sets is another reason to focus attention on function spaces instead of neighborhood spaces.

What we find important in Bishop's approach is that it can also be interpreted and developed completely classically. As the limit space, or more generally the filter spaces, form a theory of abstract convergence, the theory of Bishop spaces can form a theory of abstract continuity. A classical mathematician can read $\mathbb{R}$ as the classical continuum and the Bishop continuous functions $\mathrm{Bic}(\mathbb{R})$, which are essential to the definition of a Bishop topology, as $C(\mathbb{R})$. It is important to stress though, that the development of TBS is shaped by the use of intuitionistic logic and of positively defined concepts like the apartness relation on X induced by some topology of functions F on X. Since classical mathematics can be seen as an extension of Bishop's constructive mathematics, all propositions we prove here within BISH hold also classically. There is also a kind of symmetry between Bishop spaces and Spanier's quasi-topological spaces introduced a bit earlier in [88]. In a quasi-topological space the topological structure is generated by an abstract family of functions on compact Hausdorff spaces and with X as codomain. As a result the morphisms in the category of quasi-topological spaces and the morphisms in the category of Bishop spaces behave in a symmetric, purely function-theoretic way. As it is noted by Vogt in [96], p.545, "many topologists dislike working with things that are not topological spaces", therefore structures like the quasi-topological spaces are not as popular as some subcategories of the category of the topological spaces. We hope to show in this Thesis that the study of Bishop spaces could be of interest even to such a classical topologist.

1.2 Background

Bishop did not publish his research on the constructive reconstruction of parts of abstract topology, only his general views on the subject and summaries of his attempts. He elaborated instead the constructive theory of metric spaces, of normed and Banach spaces, of Hilbert spaces, of locally compact Abelian groups and of commutative Banach algebras. In [11], p.28, he writes:

> The constructivization of general topology is impeded by two obstacles. First, the classical notion of a topological space is not constructively viable. Second, even for metric spaces the classical notion of a continuous function is not constructively viable; the reason is that there is no constructive proof that a (pointwise) continuous function from a compact (complete and totally bounded) metric space to $\mathbb{R}$ is uniformly continuous. Since uniform continuity for functions on a compact space is the useful concept, pointwise continuity (no longer useful for proving uniform continuity) is left with no useful function to perform. Since uniform continuity cannot be formulated in the context of a general topological space, the latter concept also is left with no useful function to perform.

Bishop's syllogism "only uniform continuity is useful, uniform continuity cannot be formulated within general topological spaces, therefore general topological spaces are not useful" is sound, although recent advances in constructive analysis have revealed the utility of weaker notions of continuity like pointwise or sequential continuity. What we find more interesting as an answer to Bishop's critique on the limitations of constructive abstract topology is the following: imagine there is a constructive notion $\mathcal{F}$ of an abstract topological space which does not copy or follow the pattern of the classical topological space, and imagine that there is a constructive notion of a "continuous" function between two such spaces $\mathcal{F}$ and $\mathcal{G}$ such that although uniform continuity is not part of the definition of this notion, in many expected cases it is reduced to uniform continuity. Then one can hope that the problem in the constructivisation of topology posed by Bishop can be bypassed. There are results in TBS like Proposition 6.4.7, Corollary 5.4.5 and Theorem 3.8.11, proved by Bridges in a natural extension of BISH, which show that the notion of a Bishop space, that is Bishop's notion of a function space, and the continuity notion of a morphism between Bishop spaces realize this imagination. Of course, we do not claim to have solved in this Thesis all problems of constructive topology. We only hope to have shown that such a solution to the problem of constructive topology is possible.

It is a mystery to us though, why Bishop himself never elaborated his own concept in this direction. The mystery is even bigger, if we take into account that Bishop was an expert in this part of abstract analysis which is closer to the notion of a function space. Bishop introduced the notion of function space in [6], p.71, motivated its study, connected it to the notion of a neighborhood space, suggested to focus attention on function structures instead of neighborhood structures, he defined the fundamental to TBS inductive concept of the least function space including a given set of functions, he defined the finite product of function spaces using the least function concept, and he defined a notion of a connected function space. A characterization of connectedness and the preservation of connectedness under finite products were given as exercises. Finally, he posed the question of the "completeness" of the list of properties defining a function space. Throughout his book he refers to specific function spaces, but he did not define the morphisms between function spaces, he did not elaborate his inductive concept of the least function space, and he completely forgets about it when he discusses in [7] the problem of formalizing inductive definitions. Having already replaced with Cheng his original inductive definition of Borel sets with the theory of integration spaces and the theory of profiles that appeared in [14], he acts as if there is no second inductive definition in his system.

Nevertheless, the problem of constructivising topology still bothered Bishop. In [11], p.28, he writes:

> In [6] I was able to get along by working mostly with metric spaces and using various ad hoc definitions of continuity: one for compact (metric) spaces, another for locally compact spaces, and another for the duals of Banach spaces. The unpublished[1] manuscript [9] was an attempt to develop constructive general

[1] As Douglas Bridges suggested to me, the two manuscripts [8] and [9] are probably the same texts. In [8] one can find all these definitions mentioned in [11], but neither me nor Douglas Bridges have ever seen [9].

> topology systematically. The basic idea is that a topological space should consist of a set X, endowed with both a family of metrics and a family of boundedness notions, where a boundedness notion on X is a family S of subsets of X (called bounded subsets), whose union is X, closed under finite unions and the formation of subsets.

Bishop was not satisfied with his reconstruction of topology, and maybe this is why he never published this work. He found his theory too involved and not too broad to include properly the notion of a ball space (see [11], p.29). Thus, in discussing in [11], p.29, some of the the tasks that face constructive mathematics he refers to constructive topology, right after mentioning the primary importance of the constructive reconstruction of algebra, as follows:

> Less critical, but also of interest, is the problem of a convincing constructive foundation for general topology, to replace the ad hoc definitions in current use. It would also be good to see a constructivization of algebraic topology actually carried through, although I suspect this would not pose the critical difficulties that seem to be arising in algebra.

Bishop and Bridges kept the section of function spaces in [15] almost unchanged. The only new reference to function spaces between Bishop's book of 1967 and Bridges's paper [19] that we know of is a comment of Myhill in [67] regarding the inductive definition of the least function space and the place of the inductive definitions in the formalization of BISH within Myhill's constructive set theory. We discuss Myhill's comment in section 3.5.
Bridges talked on function spaces at the First Workshop on Formal Topology in 1997 and revived the subject of function spaces in [19]. He defined the morphisms between them, the fundamental point-point apartness and the set-set apartness relation induced by a Bishop topology of functions, for which he showed that it satisfies the axioms of an abstract set-set apartness together with the Efremovič condition. He showed that all B-continuous real-valued functions on a metric space X form a Bishop topology, that he called the metrical topology on X. Most importantly, he showed that adding the antithesis of Specker's theorem to BISH one gets that a morphism between metrical function spaces is B-continuous, producing a proof of the forward continuity theorem[2]. He also introduced the weak and the relative function spaces.
Motivated by Bridges's paper, Ishihara showed in [56] the existence of an adjunction between the category of neighbourhood spaces and the category of Φ-closed pre-function spaces, where a pre-function space is an extension of the notion of a function space.

1.3 Organisation of this Thesis

This Thesis is divided in the following chapters.

[2]In this Thesis B-continuity is discussed in section 2.4 and the forward continuity theorem in section 3.8.

Chapter 2: Bishop's fundamental notions

We discuss briefly Bishop's understanding of sets and functions and we present the basic properties of the constructive reals $\mathbb{R}$ that are used in the rest of this Thesis. We put our emphasis on Bishop's notion of a continuous function of type $\mathbb{R} \to \mathbb{R}$, and we examine the extent to which his seemingly ad hoc definition of continuity can be justified inductively.

Chapter 3: Continuity as a primitive notion

We prove the basic properties of Bishop spaces and their morphisms which are necessary to the subsequent development of TBS. We provide many abstract and concrete examples of Bishop spaces and incorporating into TBS some older results of Bridges we study the morphisms between metric spaces seen as Bishop spaces. We also study the neighborhood structure $N(F)$ on some inhabited set X that is induced by some Bishop topology F on X. Although this space-structure is used in Chapter 5 to establish some correlations between TBS and standard topology, its study is not a "real" part of TBS due to the set-theoretic character of a neighborhood space. The most important Bishop spaces are the inductively generated Bishop spaces studied in section 3.4. The induction principle that corresponds to these Bishop spaces is an important tool for proving results on them and at the same time for establishing their constructive character. The notion of a base of a Bishop space is introduced and studied.

Chapter 4: New Bishop spaces from old

We study the product Bishop topology, the weak and the relative topology, and we introduce the pointwise exponential topology which corresponds to the classical topology of the pointwise convergence, the dual of a Bishop space as a special case of the pointwise exponential topology, and the quotient Bishop spaces. We also prove a theorem of Stone-Weierstrass type for pseudo-compact Bishop spaces.

Chapter 5: Apartness in Bishop spaces

We study the point-point apartness relation and the set-set apartness relation on X induced by some topology F on X. These notions of "internal inequality" or of "internal separation" are crucial to the translation of the classical theory of the rings of continuous functions to TBS. We introduce the Hausdorff Bishop spaces with respect to a given apartness relation, we study the zero sets of a Bishop space and we prove the Urysohn lemma for them. We translate the classical theory of embeddings of rings of continuous functions into TBS and we prove the Urysohn extension theorem for Bishop spaces. The tightness of the point-point apartness induced by some topology F on X guarantees that F determines the equality of X. We introduce the completely regular Bishop spaces, and we prove the Tychonoff embedding theorem which characterises them.

Chapter 6: Compactness

We introduce the notion of a 2-compact Bishop space as a constructive function-theoretic notion of compactness suitable to TBS. The function-theoretic character of $\mathbb{2}$-compactness is based on the function-theoretic notions of a Bishop space and of a Bishop morphism. Another notion of compactness, that of pair-compactness, is also introduced. Although $\mathbb{2}$-compactness is far more superior to pair-compactness, the latter offers an immediate proof of a Stone-Weierstrass theorem for pair-compact Bishop spaces. In between we study some concrete sets, like the Cantor, the Baire space and the Hilbert cube, as Bishop spaces.

Chapter 7: Basic homotopy theory of Bishop spaces

We provide within BISH a straightforward elementary counterpart to the basic classical homotopy theory. A similar study within formal topology was initiated by Palmgren in [71]. Since TBS is a function-theoretic approach to constructive topology, and since classical homotopy theory contains many function-theoretic concepts, it is natural to try to develop such a reconstruction within TBS. If (X, F) is a Bishop space, an F-path is a morphism from $[0, 1]$, endowed with the topology of the uniformly continuous functions, to (X, F). In contrast to the "logical" character of paths in HoTT, not every Bishop space has the path-joining property (PJP). We introduce the rich class of codense Bishop spaces, which partially generalizes the class of complete metric spaces in TBS, and we show that every codense Bishop space has the PJP. We also study Bishop spaces with the homotopy-joining or the loop homotopy-joining property. With such concepts we can start the translation of the basic facts of the classical theory of the homotopy type into TBS.

Most of the proofs in this Thesis are within BISH and most of these constructive proofs are within RICH, the subsystem of BISH named after Fred Richman, who criticized, for example in [78], the use of choice principles in constructive mathematics. In contrast to BISH, the system RICH includes no choice, like the countable choice, or the principle of dependent choices. For simplicity of presentation we do not indicate in the formulation of a proposition or a theorem if it is in RICH, although we always mention the use of choice in a proof. When a concrete principle outside BISH, like Markov's Principle, is used in a proof over BISH, we indicate it at the beginning of the proposition. When a proof uses classical logic we write CLASS at the beginning of the proposition. All results in this Thesis proved "outside" BISH have a secondary or complementary role to our study of Bishop spaces.

1.4 Contributions

We would like to think as the main contribution of this Thesis the presentation of TBS not as a collection of independent results, but as a theory with some unity and structure. More specifically, we consider the following as contributions of this Thesis.

Chapter 2: Bishop's fundamental notions

The notion of the least set of inductive continuous functions $\mathrm{IC}(\mathbb{R}, \Phi_0)$ generated by some given family Φ_0 of Bishop-continuous functions is an attempt to approach the notion of an abstract Bishop continuous function on $\mathbb{R}$ in an inductive and function-theoretic way. The development of the study of these inductively defined functions can lead to the generalization of the notion of a Bishop space, if the closure of a Bishop topology under composition with the elements of $\mathrm{Bic}(\mathbb{R})$ is replaced by the closure under composition with the elements of $\mathrm{IC}(\mathbb{R}, \Phi_0)$.

Chapter 3: Continuity as a primitive notion

The systematic study of the inductively generated Bishop spaces, the most important class of Bishop spaces. Especially, the "correction" to a comment of Myhill, who maintained in [67] that transfinite induction can be avoided in the definition of the least Bishop space. We show that this is the case only for pseudo-compact Bishop spaces.

Chapter 4: New Bishop spaces from old

a. The presentation of a base of a product of pseudo-compact spaces and a Stone-Weierstrass theorem for pseudo-compact Bishop spaces. We use only the Weierstrass approximation theorem for these results, while in standard topology one refers to compact topological spaces and uses the Stone-Weierstrass theorem. As a generalization of this result we get a Stone-Weierstrass theorem for pseudo-compact Bishop spaces.
b. The introduction of the quotient Bishop spaces. Their definition is simple and the translation of the classical theory of the quotient topological spaces into TBS is direct.

Chapter 5: Apartness in Bishop spaces

a. The designation of the canonical apartness relations induced by a Bishop topology as the main tools in the translation of some basic parts of the classical theory of $C(X)$ into TBS.
b. The study of the zero sets of a Bishop space and especially the Urysohn lemma for them.
c. The development of the various notions of embeddings of Bishop spaces and especially the proof of the Urysohn extension theorem within BISH.
d. The introduction of the completely regular Bishop spaces, the Stone-Čech theorem for Bishop spaces and the general Tychonoff embedding theorem.

Chapter 6: Compactness

a. The study of concrete spaces, like the Baire and the Cantor space, the Hilbert cube and the Cantor set, as Bishop spaces, showing that Bishop topology is a useful tool in the study of concrete sets.
b. The result that the Cantor topology is equal to the set of all uniformly continuous real-valued functions on the Cantor space.
c. The introduction of 2-compactness as a constructive function-theoretic notion of compactness suitable to TBS, and especially the result that a compact metric space endowed with the topology of the uniformly continuous functions is a 2-compact Bishop space.

Chapter 7: Basic homotopy theory of Bishop spaces

The introduction of codense Bishop spaces as an abstract, partial generalization of complete metric spaces and the translation of Palmgren's results on complete metric spaces and the path- and homotopy-joining properties from [71] into TBS.

As we explain in Chapter 8, there are still many open questions arising from our development of TBS in this Thesis. We hope to address some of them in our future work.

Chapter 2

Bishop's fundamental notions

> ... a mathematician who single-handedly showed that deep mathematics could be developed constructively, and thereby pulled the subject back from the edge of the grave.
>
> Douglas S. Bridges, 2005

Errett Bishop (1928-1983) was an outstanding mathematician with contributions in the theory of Banach spaces, like the Bishop-Phelps theorem, in the theory of complex manifolds, like the embedding theorem for an n-dimensional Stein manifold, in the theory of integral representation of points in compact convex sets, like the the Bishop-de Leeuw theorem, and in many other areas of analysis (see [13], [79]). Moreover, with his work [6] and [15] he revolutionized constructive analysis and the foundations of mathematics. He and Brouwer are the most important constructive mathematicians of the previous century.
Bishop developed the informal system of constructive mathematics BISH, a common territory between classical mathematics, intuitionism and recursive mathematics. This means that if p is a proof of a proposition Q in BISH, then p is a proof of Q interpreted in classical mathematics, and at the same time p is a proof of Q interpreted in other intuitionistic systems of mathematics like Brouwer's intuitionistic mathematics INT, or Markov's recursive mathematics RUSS. All these pairwise incompatible disciplines can be seen then as special varieties of Bishop's constructive mathematics. In Bishop's book [6], and in many publications after 1967, a large part of classical mathematics has found its constructive counterpart in BISH. In this chapter we discuss briefly Bishop's understanding of sets and functions and we present the basic properties of the constructive reals $\mathbb{R}$ that are used in the rest of this Thesis. We put our emphasis on his notion of a continuous function of type $\mathbb{R} \to \mathbb{R}$, and we examine the extent to which his ad hoc definition of continuity can be justified inductively.

2.1 Bishop sets and functions

In this section we give a very brief account of the fundamentals of the theory of sets and functions within BISH. An introduction to the several non-trivial issues concerning them can be found in the work [2] of Beeson, a formalization of BISH close to the spirit of Bishop was given by Myhill in [67], while a useful and concise presentation of Myhill's system was given by Bridges and Reeves in [21]. Roughly speaking, the fundamental properties of Bishop sets and functions are the following:

1. The concept of function is primitive, therefore it is not defined as a set.
2. There exists a primitive set $\mathbb{N}$ of natural numbers.
3. A set X is completely defined when a method to construct an abstract element of X, a method to prove that two elements of X are equal, and a proof that this equality $=_X$ on X is an equivalence relation are given.
4. There is no notion of equality between elements of sets X and Y which are not subsets of some set Z.
5. An *operation*, or a rule, or an algorithm, is a primitive notion. A *function* from a set X to a set Y is an *extensional operation* i.e., $\forall_{x \in X}(f(x) \in Y)$ and $\forall_{x,x' \in X}(x =_X x' \rightarrow f(x) =_Y f(x'))$.
6. A subset Y of X is a set for which we can show that $\forall_{y \in Y}(y \in X)$. We can define a subset of X either through a function $i : Y \rightarrow X$ such that $y_1 =_Y y_2 \leftrightarrow i(y_1) =_X i(y_2)$, or through an appropriate separation principle[1].
7. If X, Y are sets the set $\mathbb{F}(X, Y)$ of all functions from X to Y is formed, where $f =_{\mathbb{F}(X,Y)} g \leftrightarrow \forall_{x \in X}(f(x) =_Y g(x))$, for every $f, g \in \mathbb{F}(X, Y)$. The method of constructing an element of $\mathbb{F}(X, Y)$ is considered to be a proof that $\forall_{x \in X}(f(x) \in Y)$ (see [2], p.44). As it is noted in [21], p.76, the acceptance of $\mathbb{F}(X, Y)$ is a "weak substitute for the standard power set axiom".
8. The complete definitions of the intersection, union and equality of subsets of a set X are straightforward and can be found in [15], pp.68-9. Note that despite these definitions, the concept of the power set of X does not appear in [15], while Beeson mentions characteristically in [2], p.46, that "power sets seem never to be needed in mathematical practice". One could say though, that, as in the case of $\mathbb{F}(X, Y)$, the method of constructing an element Y of the power set of X is a proof of $\forall_{y \in Y}(y \in X)$.
9. Following Beeson [2], p.44, if B is a rule which associates to every element x of a set A a set $B(x)$, the *sum set*, or *disjoint union* $\sum_{x \in A} B(x)$ and the *infinite product* $\prod_{x \in A} B(x)$ are defined by

$$\sum_{x \in A} B(x) := \{(x, y) \mid x \in A \wedge y \in B(x)\},$$

[1] Bishop explicitly mentions only the first method, but he constantly uses subsets defined as the elements of a set satisfying a given property, like the continuous functions from a metric space to $\mathbb{R}$. It is clear which the method of construction and the equality of a subset are, when this is defined through separation.

$$\prod_{x \in A} B(x) := \{f \in \mathbb{F}(A, \bigcup_{x \in A} B(x)) \mid \forall_{x \in A}(f(x) \in B(x))\},$$

where the *exterior union* $\bigcup_{x \in A} B(x)$ is defined by Richman (see Ex. 2 in [15], p.78).

10. The choice principles considered in BISH are the *principle of dependent choice* (DC), which implies the *principle of countable choice* (CC), and *Myhill's axiom of nonchoice* (MNC)

$$\text{(DC)} \quad Q \subseteq X \times X \to x_0 \in X \to \forall_{x \in X} \exists_{y \in X}(Q(x, y)) \to$$
$$\to \exists_{f \in \mathbb{F}(\mathbb{N}, X)}(f(0) = x_0 \wedge \forall_{n \in \mathbb{N}}(Q(f(n), f(n+1)))).$$
$$\text{(CC)} \quad \forall_{n \in \mathbb{N}} \exists_{x \in X}(P(n, x)) \to \exists_{f \in \mathbb{F}(\mathbb{N}, X)}(\forall_{n \in \mathbb{N}}(P(n, f(n))),$$
$$\text{(MNC)} \quad \forall_{x \in X} \exists!_{y \in Y}(A(x, y)) \to \exists_{f : X \to Y} \forall_{x \in X}(A(x, f(x))).$$

11. Inductive definitions of sets, especially of sets of functions. Although Bishop replaced his initial inductive definition of a Borel set with the non-inductive theory of integration spaces and the theory of profiles and he neglected his inductively defined notion of a least function space, inductively defined sets are central to the development of TBS. Myhill included them in an extension of his formal system for BISH and despite the older standard view that "inductive definitions seem to be irrelevant for constructive analysis" (see [2], p.45), modern developments in constructive mathematics have revealed their importance (see, for example, the work of Coquand on constructive combinatorics and the inductive character of formal topology). The importance of inductive definitions is closely connected to Beeson's important question on the legitimacy of quantifying over $\mathbb{F}(X, Y)$ (see [2], p.46). If a set of functions F is defined via some appropriate inductive rules, the corresponding induction principle guarantees the legitimacy of the quantification over F.

There are many issues regarding the above fundamental properties of Bishop sets and functions, that cannot be studied in this Thesis (see also section 8.3). Even the exact formulation of them is a non-trivial enterprise. For example, if we define an equality $=$ on X as an equivalence relation on X, then we use the notion of a relation i.e., of a certain subset of $X \times X$ (see [15], p.23). In this case we need to define the notion of subset (and product) first which clearly rests on the notion of equality. More importantly, in a formal approach, like Myhill's, an axiom guarantees the existence of $\mathbb{F}(X, Y)$, while Bishop himself expressed reasonable doubts on the constructive character of the abstract $\mathbb{F}(X, Y)$. In [15], p.67, we read

> When X is not countable, the set $\mathbb{F}(X, Y)$ seems to have little practical interest, because to get a hold on its structure is too hard. For instance, it has been asserted by Brouwer that all functions in $\mathbb{F}(\mathbb{R}, \mathbb{R})$ are continuous, but no acceptable proof of this assertion is known.

That's also why formal theories of numbers and number-theoretic sequences only, like Howard's and Kreisel's system H in [53], Kleene's system M (see [77]), the system of

elementary analysis EL (see [92]) and Veldman's system BIM (see [95] and [72]), were introduced to formalize proper parts of BISH. But still one has to answer Beeson's question in [2], p.46, "why are we able to quantify over $\mathbb{N}^{\mathbb{N}}$?". Although we do not address these questions in this Thesis, we think that the study of inductively defined subsets of Bishop spaces, which is in the heart of TBS, may lead to some proper distinctions between completely presented and just defined subsets of Bishop spaces, and consequently between "proper" quantification over completely presented subsets and just "ideal" quantification over defined subsets of Bishop spaces.
Almost of equal importance to a given equality on a set X is the presence of a positively defined inequality, or (point-point) apartness relation, on X. Since the logical inequality $\neq$ on X, where $x \neq y := (x = y \to \bot)$, does not behave constructively as smoothly as classically, we need other relations to play the role of inequality. Such inequalities for the case of the intuitionistic $\mathbb{R}$ were defined first by Brouwer. One can go further and define also a contradiction $\bot_X$ specific to X in order to limit the logic outside X as much as possible. In this case a set can be seen as a structure $(X, =_X, \Join_X, \bot_X)$ together with a method describing the construction of its elements.

Definition 2.1.1. *If X is an inhabited set and $=_X$ is an equality on X, a point-point apartness relation on $(X, =)$ is a binary relation $\Join_X$ on X satisfying the following conditions:*
(Ap1) $\forall_{x,y \in X}(x =_X y \to x \Join y \to \bot)$.
(Ap2) $\forall_{x,y \in X}(x \Join y \to y \Join x)$.
(Ap3) $\forall_{x,y \in X}(x \Join y \to \forall_{z \in X}(x \Join z \vee y \Join z))$.
An apartness relation $\Join$ on X is called tight, if $\neg(x \Join y) \to x =_X y$, for every $x, y \in X$.

It is immediate to see that classically the negation of an equivalence relation on X is an apartness relation on X, an apartness relation on X is the negation of an equivalence relation on X, and the only tight apartness relation on $(X, =)$ is $\neq$.

Definition 2.1.2. *A mapping $e : (X, \Join_X) \to (Y, \Join_Y)$ preserves apartness, if*

$$\forall_{x_1, x_2 \in X}(x_1 \Join_X x_2 \to e(x_1) \Join_Y e(x_2)),$$

and it is called strongly continuous, or apartness-continuous, if

$$\forall_{x_1, x_2 \in X}(e(x_1) \Join_Y e(x_2) \to x_1 \Join_X x_2).$$

The next proof requires the use of Markov's principle.

Proposition 2.1.3 (MP)**.** *If $e : (X, \Join_X) \to (Y, \Join_Y)$, and $\Join_X$ is tight, then e is strongly continuous.*

Proof. We fix $x_1, x_2 \in X$ such that $e(x_1) \Join_Y e(x_2)$ and suppose that $\neg(x_1 \Join_X x_2)$. Since $\Join_X$ on X is tight, we get that $x_1 = x_2$, which implies $e(x_1) = e(x_2)$. This together with $e(x_1) \Join e(x_2)$ implies by Ap1 the absurdity $\bot$. Hence, we showed $\neg\neg(x_1 \Join x_2)$, and by MP we get $(x_1 \Join x_2)$. $\square$

As we explain later (see Remark 2.3.12), the Bishop continuous functions and the continuous functions between metric spaces are shown to be $\bowtie$-continuous without the use of MP.

Definition 2.1.4. *If $A \subseteq X$ and $\bowtie_X$ is a given apartness relation on X we denote by $\bowtie_A$ its restriction to A^2. If $\bowtie_X, \bowtie_Y$ are apartness relations on X and Y, respectively, we denote by $\bowtie_{X\times Y}$ the product apartness relation on $X \times Y$, defined by*

$$(x_1, y_1) \bowtie_{X\times Y} (x_2, y_2) :\leftrightarrow x_1 \bowtie_X x_2 \vee y_1 \bowtie_Y y_2,$$

and if $\Phi \subseteq \mathbb{F}(X, Y)$ we denote by $\underset{\rightarrow}{\bowtie}$ the apartness relation defined, for every $h_1, h_2 \in \Phi$, by

$$h_1 \underset{\rightarrow}{\bowtie} h_2 :\leftrightarrow \exists_{x\in X}(h_1(x) \bowtie_Y h_2(x)).$$

If $\bowtie_Y$ is tight, then $\underset{\rightarrow}{\bowtie}$ is tight, since

$$\begin{aligned}\neg(h_1 \bowtie h_2) &\leftrightarrow \neg\exists_{x\in X}(h_1(x) \bowtie_Y h_2(x))\\ &\rightarrow \forall_{x\in X}(\neg(h_1(x) \bowtie_Y h_2(x)))\\ &\rightarrow \forall_{x\in X}(h_1(x) =_Y h_2(x))\\ &\rightarrow h_1 = h_2.\end{aligned}$$

Following [15], p.17, a function $f : X \rightarrow Y$ is 1−1, or an *injection*, if $\forall_{x,y\in X}(f(x) = f(y) \rightarrow x = y)$. Note that we use the positive formulation of injectivity of a function and not its contrapositive, which is negatively formulated and only classically equivalent to it. It is obvious then that $\forall_{y\in f(X)}\exists!_{x\in X}(f(x) = y)$ and the inverse function $f^{-1} : f(X) \rightarrow X$ is well-defined. If $f : X \rightarrow Y$ preserves apartness and $\bowtie_X$ is tight, then f is an injection; suppose that $f(x) = f(y)$, which by Ap1 implies $\neg(f(x) \bowtie_Y f(y))$. By contraposing the implication in the definition of the preservation of apartness we get that $\neg(x \bowtie_X y)$, and by the tightness of $\bowtie_X$ we get that $x = y$. The next definition follows [19], p.104.

Definition 2.1.5. *If X is an inhabited set, $=_X$ is an equality on X, and $\bowtie_X$ is a point-point apartness relation on X, a set-set apartness relation on $(X, =, \bowtie_X)$ is a binary relation $\bowtie$ on the subsets of X satisfying the following conditions:*

(AP1) $X \bowtie \emptyset$.
(AP2) $-A \subseteq \sim A$.
(AP3) $(A_1 \cup A_2) \bowtie (B_1 \bowtie B_2) \leftrightarrow \forall_{i,j\in\{1,2\}}(A_i \bowtie B_j)$.
(AP4) $-A \subseteq \sim B \rightarrow -A \subseteq -B$.
(AP5) $x \in -A \rightarrow \exists_{B\subseteq X}(x \in -B \wedge X = (-A) \cup B)$,

where the complement $\sim A$, and the apartness complement $-A$ of A are defined by

$$\sim A := \{x \in X \mid \forall_{a\in A}(x \bowtie_X a)\},$$

$$-A := \{x \in X \mid x \bowtie A\}.$$

If $(X, \bowtie_X)$ and $(Y, \bowtie_Y)$ are set-set apartness spaces, a function $f : X \rightarrow Y$ is called *strongly continuous*, if $f(A) \bowtie_Y f(B) \rightarrow A \bowtie_X B$, for every $A, B \subseteq X$. This is a natural notion of morphism in the category of apartness spaces (see [27]). We denote by $\mathrm{Const}(X, Y)$ the set of constant functions from X to Y and by $A \between B$ the fact that the intersection $A \cap B$ is inhabited.

2.2 Bishop reals

In this section we present the definition of Bishop reals $\mathbb{R}$ and some basic results on $\mathbb{R}$ which we use in the rest of this Thesis. In order to avoid being too lengthy we refer, when necessary, to properties of $\mathbb{R}$ that can be found in the corresponding literature. In the next fundamental definition we follow [15], pp.18-22 and p.24. For an excellent introduction to constructive $\mathbb{R}$ and its basic properties see [10].

Definition 2.2.1. *A Bishop real number, or a constructive real number, x is a sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in \mathbb{Q}$, for every $n \in \mathbb{N}$, such that*

$$\forall_{n,m\in\mathbb{N}^+}(|x_m - x_n| \leq \frac{1}{m} + \frac{1}{n}).$$

If $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are Bishop reals, the $=_{\mathbb{R}}, >_{\mathbb{R}}, \geq_{\mathbb{R}}, \bowtie_{\mathbb{R}}, +, \cdot, ^{-1}$ and absolute value $|.|$ are[2] *defined, respectively, by*

$$x = y :\leftrightarrow \forall_{n\in\mathbb{N}}(|x_n - y_n| \leq \frac{2}{n}),$$

$$x > 0 :\leftrightarrow \exists_{n\in\mathbb{N}}(x_n > \frac{1}{n}),$$

$$x \geq 0 :\leftrightarrow \forall_{n\in\mathbb{N}}(x_n \geq -\frac{1}{n}),$$

$$a \bowtie_{\mathbb{R}} b :\leftrightarrow a > b \vee a < b,$$

$$x + y := (x_{2n} + y_{2n})_{n\in\mathbb{N}},$$

$$\max\{x, y\} := (\max\{x_n, y_n\})_{n=1}^{\infty},$$

$$K_x := \min\{k \in \mathbb{N} \mid n > |x_1| + 2\},$$

$$x \cdot y := (x_{2kn} \cdot y_{2kn})_{n\in\mathbb{N}}, \quad k := \max\{k_x, k_y\},$$

$$x \bowtie_{\mathbb{R}} 0 \to \exists_{N>0}\forall_{m\geq N}(|x_m| \geq \frac{1}{N}),$$

$$x^{-1} := (y_n)_{n\in\mathbb{N}},$$

$$y_n := \begin{cases} \frac{1}{x_{N^3}} & , \textit{ if } n < N \\ \frac{1}{x_{nN^2}} & , \textit{ if } n \geq N, \end{cases}$$

$$|x| := \max\{x, -x\}.$$

If $q \in \mathbb{Q}$, then $q^* = (q_n)_{n\in\mathbb{N}} \in \mathbb{R}$, where $q_n = q$, for every $n \in \mathbb{N}$.

[2] Usually we denote these relations omitting the subscript.

Proposition 2.2.2 (Properties of $<$)**.** *If $x, y \in \mathbb{R}$, the following hold:*
(i) $x \geq y \leftrightarrow \neg(x < y)$.
(ii) $x + y > 0 \to x > 0 \vee y > 0$.
(iii) (Tri) $x < y \to \forall_{z \in \mathbb{R}}(x < z \vee z < y)$.
(iv) $x \cdot y < 0 \to x < 0 \vee y < 0$.

Proof. (i)-(iii) See [15], p.26
(iv) See [15], p.62, Exercise 5. □

Proposition 2.2.3. *Suppose that $x, y, z \in \mathbb{R}$. Then the following hold:*
(i) If $x \leq z$ and $y \leq z$, then $\max\{x, y\} \leq z$.
(ii) If $x \geq z$ and $y \geq z$, then $\min\{x, y\} \geq z$.
(iii) $x \leq y \leftrightarrow x \wedge y = x \leftrightarrow x \vee y = y$.
(iv) $x > 0 \to y > 0 \to \min\{x, y\} > 0$

Proof. (i) By definition we have that $(z-x)_n = z_{2n} - x_{2n} \geq -\frac{1}{n}$ and $(z-y)_n = z_{2n} - y_{2n} \geq -\frac{1}{n}$, for every n. Since $x_{2n}, y_{2n} \in \mathbb{Q}$ and $\max\{x_{2n}, y_{2n}\} = x_{2n} \vee y_{2n}$, we get that $(z - \max\{x, y\})_n = z_{2n} - \max\{x, y\}_{2n} = z_{2n} - \max\{x_{2n}, y_{2n}\} \geq -\frac{1}{n}$.
(ii) Since $\min\{x, y\} = -\max\{-x, -y\}$ and $x \geq z \to -x \leq -z$, $y \geq z \to -y \leq -z$, then by (i) $\max\{-x, -y\} \leq -z$, therefore $z \leq -\max\{-x, -y\} = \min\{x, y\}$.
(iii) We show only the first equivalence. The equivalence $x \leq y \leftrightarrow x \vee y = y$ is shown similarly. If $x \wedge y = x$, then $y \geq x \wedge y = x$. If $x \leq y$, and since $x \leq x$, then by (ii) we get that $x \leq x \wedge y$. Since $x \wedge y \leq x$, we get the required equality.
(iv) If $x, y > 0$, then $x > \frac{1}{n}, y > \frac{1}{m}$, hence $x, y > \frac{1}{n+m}$, and by the (ii) we get that $\min\{x, y\} \geq \frac{1}{n+m} > 0$. For the converse, we have that $x, y \geq \min\{x, y\} > 0$. □

Proposition 2.2.4 (Properties of $|.|$)**.** *If $x, y \in \mathbb{R}$, the following hold:*
(i) $x > 0 \to |x| = x$.
(ii) $|x| > 0 \to x > 0 \vee x < 0$.

Proof. (i) On the one hand $x, -x \leq \max\{x, -x\}$ and on the other $-x < x, x \leq x \to \max\{x, -x\} \leq x$ i.e., $\max\{x, -x\} = x$.
(ii) Suppose that $|x| > 0$. By Tri we have that $x > 0 \vee x < |x|$. If $x > 0$ we are done. If $x < |x|$, then we conclude that $|x| = -x > 0 \leftrightarrow x < 0$; if $x < \max\{x, -x\}$, then $-x \geq x$, since if we suppose $-x < x$, then $\max\{x, -x\} \leq x < \max\{x, -x\}$, which is absurd. Hence $\max\{x, -x\} \geq -x$, and since $-x \leq \max\{x, -x\}$, we get $\max\{x, -x\} = x$. □

Next we show some necessary properties of $\mathbb{R}$ to the proof of Proposition 5.3.3, which are not explicitly formulated in [15], using properties of $\mathbb{R}$ proved in [15], p.23 and p.30.

Proposition 2.2.5. *If $x, y, a_i, a_n \in \mathbb{R}$, the following hold:*
(i) $|x| = 0 \to x = 0$.
(ii) $|x| + |y| = 0 \to x = y = 0$.
(iii) $\forall_{1 \leq i \leq N}(a_i \geq 0) \to \sum_{i=1}^{N} a_i = 0 \to a_i = 0$, *for every i.*
(iv) $\forall_n(a_n \geq 0) \to \sum_{i=1}^{\infty} a_n = 0 \to a_n = 0$, *for every n.*

(v) $x^2 = 0 \to x = 0$.
(vi) $x^n = 0 \to x = 0$, *for every* $n > 2$.

Proof. (i) If $x = (x_n)$, then $|x| = \max(x, -x) = (\max(x_n, -x_n)) = (|x_n|)$. By the equality $|x| = 0$ and the transitivity of equality we get $|x| = 0^*$. By the definition of equality between reals we have that $|x| = 0^* \leftrightarrow |x_n| \leq \frac{2}{n}$, for each n, something which is equivalent to $x = 0$.
(ii) We have again that $0^* \leq |x| \leq |x| + |y| \leq 0^*$ which implies that $|x| = 0^*$. Similarly we get that $|y| = 0$.
(iii) This requires a trivial induction. The case $n = 2$ is just the case (ii).
(iv) We fix some a_{n_0} and we use the inequalities

$$0 \leq \sum_{k=1}^{n_0} a_k \leq \sum_{n=1}^{\infty} a_n \leq 0,$$

where the middle inequality is justified as follows: consider the constant sequence $\sum_{k=1}^{n_0} a_k$ and the sequence $(\sum_{k=1}^{n_0+n} a_k)_n$; then each term of the first is less or equal than the corresponding term of the second, therefore the limit of the first is less or equal than the term of the limit of the second.
(v) and (vi) Using the fact that multiplication with a positive number preserves the ordering of reals (Proposition 2.11(i) in [15], p.23), we get that $x > 0 \to x^n > 0$, and $x < 0 \to x^{2n} > 0$, while $x < 0 \to x^{2n+1} < 0$, for every $n \geq 1$. Suppose that $x^2 = 0$. If $x > 0$, then $x^2 > 0$, hence $x \leq 0$, while if $x < 0$, then $x^2 > 0$, hence $x \geq 0$. If $n > 2$, we work similarly. If $x^{2n} = 0$, then if $x < 0$, then $x^{2n} > 0$, hence $x \geq 0$, while if $x > 0$, then $x^{2n} > 0$, hence $x \leq 0$. If $x^{2n+1} = 0$, then if $x < 0$, then $x^{2n+1} < 0$, hence $x \geq 0$, while if $x > 0$, then $x^{2n+1} > 0$, hence $x \leq 0$. ☐

The apartness $\bowtie_{\mathbb{R}}$ is tight (see also the remark following Proposition 5.1.5); since $\neg(p \vee q) \to \neg p \wedge \neg q$, we have that $\neg(x < y \vee y < x) \to \neg(x < y) \wedge \neg(y < x)$, which implies that $x \geq y$ and $y \geq x$ i.e., $x = y$. The tightness of $\bowtie_{\mathbb{R}}$ is clearly very useful in proving the equality between two reals. For example, if $(a_n)_n \subseteq \mathbb{R}$ and $a_n \to a$ is defined by

$$\forall_{\epsilon > 0} \exists_{n_0(\epsilon) \in \mathbb{N}} \forall_{n \geq n_0(\epsilon)} (|a_n - a| < \epsilon),$$

we show the uniqueness of sequential convergence in $\mathbb{R}$: $(a_n \to a) \to (a_n \to b) \to a = b$ by supposing that $|a - b| = \epsilon > 0$ and for every $n \geq \max\{n_{0,a}(\frac{\epsilon}{2}), n_{0,b}(\frac{\epsilon}{2})\}$ we have that $\epsilon = |a - b| \leq |a - a_n| + |a_n - b| < \epsilon$. Next we show that this definition of the apartness relation is equivalent to the one given in [15], p.72. Note that the *canonical* apartness relation induced by a metric d on X defined by is defined by $x \bowtie_d y \leftrightarrow d(x, y) > 0$, for every $x, y \in X$, and the standard metric on $\mathbb{R}$ is defined by $d(a, b) := |a - b|$, for every $a, b \in \mathbb{R}$.

Remark 2.2.6. *If* $x, y \in \mathbb{R}$, *then* $x \bowtie_{\mathbb{R}} y \leftrightarrow |x - y| > 0$.

Proof. If $x > y$, then $x - y > 0$, and by Proposition 2.2.4 we have that $|x - y| = x - y > 0$. If $x < y$, we work similarly. For the converse we apply Proposition 2.2.4(ii) on $x - y$. ☐

The next property of the $\bowtie_{\mathbb{R}}$ is used in the proof of Proposition 3.4.11.

Proposition 2.2.7. *If $x, y, z, w \in \mathbb{R}$, then $x + y \bowtie_{\mathbb{R}} z + w \to x \bowtie_{\mathbb{R}} z \vee y \bowtie_{\mathbb{R}} w$.*

Proof. By the previous equivalence we have that $0 < |(x+y)-(z+w)| = |(x-z)+(y-w)| \leq |x - z| + |y - w|$, while by Proposition 2.2.2(ii) we have that $|x - z| > 0$ or $|y - w| > 0$ i.e., $x \bowtie z$ or $y \bowtie w$. $\square$

The next lemma is used in Chapter 7 and it is mentioned without proof in [71].

Lemma 2.2.8. *If $a, b, c \in \mathbb{R}$ such that $a < b < c$, D_{ab} is dense in $[a, b]$ and D_{bc} is dense in $[b, c]$, then $D = D_{ab} \cup D_{bc}$ is dense in $[a, c]$.*

Proof. We fix some $x \in [a, c]$ and we find some $d \in D$ such that d is less than ϵ-close to x, where without loss of generality $0 < \frac{\epsilon}{2} < \frac{c-b}{10}$. We use repeatedly the constructive trichotomy Tri. Since $x < x + \frac{\epsilon}{2}$ we have that

$$b > x \ \vee \ b < x + \frac{\epsilon}{2}.$$

If $b > x$, then $x \in [a, b]$ and we use the fact that $D_{ab} \subseteq D$ is dense in $[a, b]$. Suppose next that $b < x + \frac{\epsilon}{2}$. By Tri again we split in the following two cases:

$$x + \frac{\epsilon}{2} > \frac{b+c}{2} \ \vee \ x + \frac{\epsilon}{2} < c.$$

If $x + \frac{\epsilon}{2} > \frac{b+c}{2}$, then $x > b$, therefore $x \in [b, c]$ and we work as above. To show this we suppose that $\frac{\epsilon}{2} = \frac{c-b}{10} - \tau$, for some $\tau > 0$. Hence,

$$\begin{aligned}
x + \frac{\epsilon}{2} > \frac{b+c}{2} &\leftrightarrow x + \frac{\epsilon}{2} - \frac{b}{2} - \frac{c}{2} > 0 \\
&\leftrightarrow x + \frac{c-b}{10} - \tau - \frac{b}{2} - \frac{c}{2} > 0 \\
&\leftrightarrow x - (\frac{b}{2} + \frac{b}{2}) + \frac{b}{2} + \frac{c-b}{10} - \tau - \frac{c}{2} = \rho, \text{ for some } \rho > 0, \\
&\leftrightarrow x - b = \tau + \rho + (\frac{c-b}{2} - \frac{c-b}{10}) > 0 \\
&\leftrightarrow x > b.
\end{aligned}$$

It remains to consider the case $b < x + \frac{\epsilon}{2} < c$. Since $x + \frac{\epsilon}{2} \in [b, c]$, there exists some $d \in D_{bc}$ such that $|d - (x + \frac{\epsilon}{2})| < \frac{\epsilon}{2}$ i.e., $-\frac{\epsilon}{2} < d - x - \frac{\epsilon}{2} < \frac{\epsilon}{2}$, or $0 < d - x < \epsilon$, which implies that $-\epsilon < d - x < \epsilon$, or equivalently $|d - x| < \epsilon$. $\square$

A useful example of a dense subset of $[a, b]$ used by Palmgren in [71] is the set

$$D_{ab} = \{a + (b - a)\frac{k}{2^n} \mid n \in \mathbb{N} \wedge k \in \{0, 1, \dots, 2^n\}\}.$$

If $n = 0$, then $k = 0, 1$ and we get the elements a and b. If $n = 1$, then $k = 0, 1, 2$ and we get the elements a, $\frac{a+b}{2}$ and b. If $n = 2$, then $k = 0, 1, 2, 3, 4$ and we get the elements a, $\frac{3a+b}{4}\frac{a+b}{2}$, $\frac{a+3b}{4}$ and b, and so on. It is useful to work with these countable dense subsets because if $a < b < c$, then $D_{ab} \cap D_{bc} = \{b\}$, a necessary fact in the proof of Theorem 7.3.6. Clearly, one could work with an arbitrary dense set D_{ab} and add to it both a and b. For reasons that have to do with the possible translation of constructive analysis to Type Theory it is useful to understand a dense subset D of a metric space (X, d) as a pair (D, D^*), where $D^* : \mathbb{N} \to X$ and $D = D^*(\mathbb{N})$ satisfies the density property.
If $f, g, h \in \mathbb{F}(X)$, we use the following notations

$$[f \geq g] = \{x \in X \mid f(x) \geq g(x)\},$$

$$[f \leq g] = \{x \in X \mid f(x) \leq g(x)\},$$

$$[f = g] = \{x \in X \mid f(x) = g(x)\},$$

$$[h \leq f \leq g] = [h \leq f] \cap [f \leq g].$$

According to Bishop and Bridges [15], p.85, if (X, d) is an inhabited metric space, $B \subseteq X$ is a *bounded subset* of X, if there is some $x_0 \in X$ such that $B \cup \{x_0\}$ with the induced metric is a bounded metric space. Simplifying our exposition, we consider that the inclusion map of a subset is the identity (see [15], p.68), therefore the induced metric on $B \cup \{x_0\}$ is reduced to the relative metric on $B \cup \{x_0\}$. Hence we use the following definition.

Definition 2.2.9. *A bounded subset B of an inhabited metric space X is a triplet (B, x_0, M), where $x_0 \in X, B \subseteq X$, and $M > 0$ is a bound for $B \cup \{x_0\}$.*

If (B, x_0, M) is a bounded subset of X then $B \subseteq \mathcal{B}(x_0, M)$, where $\mathcal{B}(x_0, M)$ is the open sphere of radius M about x_0, and $(\mathcal{B}(x_0, M), x_0, 2M)$ is also a bounded subset of X. In other words, a bounded subset of X is included in an inhabited bounded subset of X which is also metric-open i.e., it includes an open ball of every element of it, a fact used in the proof of Lemma 4.7.13.

2.3 Bishop continuity

The uniform continuity theorem (UCT), according to which a real-valued pointwise continuous function on $[a, b]$ is uniformly continuous, is true in CLASS and in INT (as a consequence of the fan theorem), while it is false in RUSS (see [20], p.59). Bishop was very suspicious, to say the least, towards Brouwer's proof of the fan theorem and his use of the choice sequences (e.g., see [6], p.6). He was equally suspicious to the use of formal methods in solving the foundational issues of mathematics (see again [6], p.6, and mainly [12]). Thus he found a way to get around the problematic character of UCT by incorporating it to his concept of continuity, considering that the concept of pointwise continuity was far less important than that of uniform continuity[3]. This choice of continuity of Bishop is essential to the neutral character of BISH with respect to CLASS, INT and RUSS.

Definition 2.3.1. *We denote the set of all functions of type $X \to \mathbb{R}$ by $\mathbb{F}(X)$, and the constant function on X with value $a \in \mathbb{R}$ by $\overline{a}$. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called Bishop continuous, or simply continuous, if it is uniformly continuous on every bounded subset B of $\mathbb{R}$ i.e., for every bounded subset B of $\mathbb{R}$ and for every $\epsilon > 0$ there exists $\omega_{\phi,B}(\epsilon) > 0$ such that*

$$\forall_{x,y \in B}(|x - y| \leq \omega_{\phi,B}(\epsilon) \to |\phi(x) - \phi(y)| \leq \epsilon),$$

where the function

$$\omega_{\phi,B} : \mathbb{R}^+ \to \mathbb{R}^+,$$

$$\epsilon \mapsto \omega_{\phi,B}(\epsilon)$$

is called a modulus of continuity for ϕ on B. A continuous function is also denoted as a pair $(\phi, (\omega_{\phi,B})_{B \subseteq^b \mathbb{R}})$, and two continuous functions $(\phi_1, (\omega_{\phi_1,B})_B)$, $(\phi_2, (\omega_{\phi_2,B})_B)$ are equal, if $\phi_1(x) = \phi_2(x)$, for every $x \in \mathbb{R}$. We denote the set of Bishop continuous functions by $\mathrm{Bic}(\mathbb{R})$. *Similarly,* $\mathrm{Bic}(Y)$ *denotes the set of real-valued continuous functions defined on some $Y \subseteq \mathbb{R}$ such that they are uniformly continuous on every bounded subset of Y.*

Bishop defined continuous functions defined on intervals of reals only, but it is useful to extend his definition to functions defined on an arbitrary subset of $\mathbb{R}$. For example, we will refer later to the set $\mathrm{Bic}(\mathbb{Z})$. At first sight it seems that the definition of Bishop continuity rests on quantification over the power set of $\mathbb{R}$:

$$\mathrm{Bic}(\mathbb{R})(\phi) :\leftrightarrow \forall_{B \in \mathcal{P}(\mathbb{R})}(\text{bounded}(B) \to \phi_{|B} \text{ is uniformly continuous}).$$

That would be problematic - in the literature this is considered an impredicative definition - since the notion of the power set of $\mathbb{R}$ is constructively suspicious, as we discussed in section 2.1. It is clear though, that it suffices to quantify over $\mathbb{N}$ i.e.,

$$\mathrm{Bic}(\mathbb{R})(\phi) \leftrightarrow \forall_{n \in \mathbb{N}}(\phi_{|[-n,n]} \text{ is uniformly continuous}),$$

since a bounded subset of $\mathbb{R}$ is by definition given as a triplet (B, x_0, M) and since $B \subseteq (x_0 - M, x_0 + M)$, for some $M > 0$, we get that $(x_0 - M, x_0 + M) \subseteq [-n, n]$, where $n = \max\{N_1, N_2\}$ and $N_1, N_2 \in \mathbb{N}$ such that $N_1 > x_0 + M$ and $-N_2 < x_0 - M$ by the Archimedean property of reals[4]. Hence, the uniform continuity of ϕ on $[-n, n]$ implies its uniform continuity on B.

Next we show that Bishop continuity of functions defined on $\mathbb{R}$ is preserved under composition. In order to do this we prove some simple lemmas. First we give for the sake of

[3] In [15], p.66, it is mentioned that pointwise continuity is not used anywhere in the book. As Bridges remarks in [19], p.101, though, later research showed other kinds of continuity like sequential continuity have a "significant role to play in constructive analysis".

[4] To show that for every $x \in \mathbb{R}$ there exists some $n \in \mathbb{N}$ such that $x < n$ we can use the Lemma 2.15 in [15], p.25, which implies the existence of some $q \in \mathbb{Q}$ such that $x < q < x + 1$. Note that if we consider $n \in \mathbb{N}$ such that $|x| < n$, we get $-n < -|x| \leq x$.

completeness a proof of a fact, which is used without proof in the proof of Proposition 4.6 in [15], p.38. If we fix some $\epsilon > 0$ and choose reals $a = a_0 < a_1 < \ldots < a_n = b$, for some n, such that $a_{i+1} - a_i \leq \epsilon$, for every $i \in \{0, \ldots, n-1\}$, then we cannot automatically conclude that for every $x \in [a,b]$ there exists some i such that $|x - a_i| \leq \epsilon$, since we cannot accept constructively that any $x \in [a,b]$ is in some $[a_i, a_{i+1}]$, for some $i \in \{0, \ldots, n-1\}$. We need the constructive trichotomy to get this simple fact.

Lemma 2.3.2. *If $\epsilon > 0$ and $a, a_0, a_1, \ldots, a_n, b \in \mathbb{R}$, for some n, such that $a = a_0 < a_1 < \ldots < a_n = b$ and $a_{i+1} - a_i \leq \epsilon$, for each $i \in \{0, \ldots, n-1\}$, then for each $x \in [a,b]$ there exists some i such that $|x - a_i| \leq \epsilon$.*

Proof. We fix some $x \in [a,b]$ and by Tri we have that $x < a_1$ or $x > a_0$. In the first case we get that $a \leq x < a_1 \rightarrow x - a < a_1 - a \leq \epsilon$. Suppose next that $x > a$. By Tri again we have that $x < a_2$ or $x > a_1$. In the first case we get that $a_1 - \epsilon \leq a_0 < x < a_2 \leq a_1 + \epsilon \rightarrow |x - a_1| < \epsilon$. If $x > a_1$, then we have that $x < a_3$ or $x > a_2$ and we work as in the previous case. Going on like that we reach the case $x > a_{n-2}$ split into $x < b$, which we handle as previously, and the case $x > a_{n-1}$, which gives $x - a_{n-1} \leq b - a_{n-1} \leq \epsilon$. $\square$

It is clear that with the previous lemma we can show that $[a,b]$ is totally bounded (in [15], p.96, and in [20], p.28, it is mentioned without proof that $[a,b]$ is compact). Next follows a standard useful corollary (see also [20], p.115). We include its proof to stress the necessity of the previous lemma.

Proposition 2.3.3. *If $f : [a,b] \rightarrow \mathbb{Z}$ is uniformly continuous, then f is constant.*

Proof. If ω_f is the modulus of continuity of f, we have that $\forall_{x,y \in [a,b]}(|x - y| \leq \omega_f(\frac{1}{2}) \rightarrow |f(x) - f(y)| \leq \frac{1}{2})$, which implies, clearly, that in this case $f(x) = f(y)$. We choose reals $a = a_0 < a_1 < \ldots < a_n = b$, for some n, such that $a_{i+1} - a_i \leq \omega_f(\frac{1}{2})$, for each $i \in \{0, \ldots, n-1\}$, therefore $f(a) = f(a_1) = \ldots = f(a_{n-1}) = f(b) = c$. By Lemma 2.3.2 we have that for each $x \in [a,b]$ there exists some i such that $|x - a_i| \leq \omega_f(\frac{1}{2})$ which implies that $f(x) = f(a_i) = c$. $\square$

Proposition 2.3.4. *If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $B \subseteq^b \mathbb{R}$, then $\phi(B) \subseteq^b \mathbb{R}$.*

Proof. If $B \subseteq^b \mathbb{R}$, then there exists $M > 0$ such that $B \subseteq [-M, M]$. Since ϕ is uniformly continuous on $[-M, M]$, we get by Proposition 4.6 in [15], p.38, that $\phi(B) \subseteq f([-M, M]) \subseteq [\inf \phi, \sup \phi]$, where $\inf \phi = \inf\{\phi(x) \mid x \in [a,b]\}$ and $\sup \phi = \sup\{\phi(x) \mid x \in [a,b]\}$. $\square$

Proposition 2.3.5. *Suppose that $a \in \mathbb{R}$, $f \in \mathbb{F}(\mathbb{R})$ and $\phi, \theta \in \mathrm{Bic}(\mathbb{R})$ with $\omega_{\phi,B}, \omega_{\theta,B}$, for every $B \subseteq^b \mathbb{R}$, respectively.*
(i) $\overline{a} \in \mathrm{Bic}(\mathbb{R})$ with $\omega_{\overline{a},B} = \overline{c}$, for every $B \subseteq^b \mathbb{R}$, where c is any positive real.
(ii) $\phi + \theta \in \mathrm{Bic}(\mathbb{R})$ with $\omega_{\phi+\theta,B}(\epsilon) = \min\{\omega_{\phi,B}(\frac{\epsilon}{2}), \omega_{\theta,B}(\frac{\epsilon}{2})\}$, for every $B \subseteq^b \mathbb{R}$ and $\epsilon > 0$.
(iii) $\theta \circ \phi \in \mathrm{Bic}(\mathbb{R})$ with $\omega_{\theta \circ \phi,B} = \omega_{\phi,B} \circ \omega_{\theta,\phi(B)}$, for every $B \subseteq^b \mathbb{R}$.
(iv) If for every $\epsilon > 0$ there exists some $\phi_\epsilon \in \mathrm{Bic}(\mathbb{R})$ such that $\forall_{x \in \mathbb{R}}(|\phi_\epsilon(x) - f(x)| \leq \epsilon)$, then $f \in \mathrm{Bic}(\mathbb{R})$ with $\omega_{f,B}(\epsilon) = \omega_{\phi_{\frac{\epsilon}{3}},B}(\frac{\epsilon}{3})$, for every $\epsilon > 0$ and $B \subseteq^b \mathbb{R}$.
(v) $|\phi| \in \mathrm{Bic}(\mathbb{R})$ and $\omega_{|\phi|,B}(\epsilon) = \omega_{\phi,B}(\epsilon)$, for every $\epsilon > 0$ and every $B \subseteq^b \mathbb{R}$.

Proof. (i) and (ii) are trivial. For (iii), if $(\phi, (\omega_{\phi,B})_B)$ and $(\theta, (\omega_{\theta,B})_B)$ are given, then by Proposition 2.3.4 we have that $\phi(B) \subseteq^b \mathbb{R}$, for every $B \subseteq^b \mathbb{R}$. If $\epsilon > 0$, then for every $x, y \in B$ we get that

$$\begin{aligned}|x - y| \leq \omega_{\phi,B}(\omega_{\theta,\phi(B)}(\epsilon)) &\to |\phi(x) - \phi(y)| \leq \omega_{\theta,\phi(B)}(\epsilon) \\ &\to |\theta(\phi(x)) - \theta(\phi(y))| \leq \epsilon.\end{aligned}$$

(iv) If $\epsilon > 0$, $B \subseteq^b \mathbb{R}$ and $|x - y| \leq \omega_{\phi_{\frac{\epsilon}{3}},B}(\frac{\epsilon}{3})$, for every $x, y \in B$, then

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - \phi_{\frac{\epsilon}{3}}(x) + \phi_{\frac{\epsilon}{3}}(x) - \phi_{\frac{\epsilon}{3}}(y) + \phi_{\frac{\epsilon}{3}}(y) - f(y)| \\ &\leq |f(x) - \phi_{\frac{\epsilon}{3}}(x)| + |\phi_{\frac{\epsilon}{3}}(x) - \phi_{\frac{\epsilon}{3}}(y)| + |\phi_{\frac{\epsilon}{3}}(y) - f(y)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

(v) It follows by the property of constructive reals $||x| - |y|| \leq |x - y|$ (see [92], p.264). $\square$

Clearly, the identity function $\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$ with modulus of continuity $\omega_{\mathrm{id}_{\mathbb{R}},B} = \mathrm{id}_{\mathbb{R}^+}$, for every $B \subseteq^b \mathbb{R}$. If $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{F}(X)$, we denote by $f_n \stackrel{p}{\to} f$ that f_n *converges pointwise* to f and by $f_n \stackrel{u}{\to} f$ that f_n *converges uniformly* to f i.e.,

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_{x\in\mathbb{R}}(|f_n(x) - f(x)| \leq \epsilon).$$

For simplicity we may denote the natural n_0 corresponding to $\epsilon > 0$ by $n_0(\epsilon)$. Note that in [15], p.41, the convergence $f_n \stackrel{u}{\to} f$ is defined as uniform convergence of f_n to f on every compact subinterval of $\mathbb{R}$ (II), while in the same book, p.86, the uniform convergence $f_n \stackrel{u}{\to} f$, where $f : X \to Y$ and Y is a metric space, is defined as above (I). It is clear that (I) $\to$ (II). Classically there exist functions $f_n, f : [0, 1)$ such that $f_n \stackrel{u}{\to} f$ on every compact interval $[0, a]$, $0 < a < 1$, while $f_n \stackrel{u}{\nrightarrow} f$ on $[0, 1]$ Hence (II) doesn't imply (I). Actually the two definitions clash each other. For reasons related with the definition of a Bishop space we use the definition (I). The next simple fact, although it can be formulated with respect to uniform convergence, is formulated in a way that conforms to the definition of a Bishop space (see our comment after the proof of Proposition 3.1.2) and it is used in the proof of Proposition 3.4.11.

Proposition 2.3.6. *Suppose that X is an inhabited set and $f \in \mathbb{F}(X)$ satisfying the condition $\forall_\epsilon\exists_{g\in\mathbb{F}(X)}\forall_{x\in X}(|g(x) - f(x)| \leq \epsilon)$. If*

$$\Phi = \{g \in \mathbb{F}(X) \mid \exists_{\epsilon>0}\forall_{x\in X}(|g(x) - f(x)| \leq \epsilon)\}$$

and $x, y \in X$ such that $f(x) \bowtie_{\mathbb{R}} f(y)$, then there is some $g \in \Phi$ such that $g(x) \bowtie_{\mathbb{R}} g(y)$.

Proof. If $0 < \epsilon = |f(x) - f(y)|$, let $g \in \mathbb{F}(X)$ such that $|g(x) - f(x)| \leq \epsilon$, for every $x \in X$. Since $0 < \epsilon \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \leq \frac{\epsilon}{4} + |g(x) - g(y)| + \frac{\epsilon}{4} = \frac{\epsilon}{2} + |g(x) - g(y)|$, we get that $0 < \frac{\epsilon}{2} \leq |g(x) - g(y)|$ i.e., $g(x) \bowtie_{\mathbb{R}} g(y)$. $\square$

Definition 2.3.7. *A locally compact metric space is an inhabited metric space (X, d) each bounded subset of which is included in a compact subset of X. A compact metric space is one which is complete and totally bounded.*

The definition of local compactness is given in [6], p.102, while, as it is noted in [15], p.125, the definition of metric compactness is Brouwer's. Clearly, $(\mathbb{R}, d_{\mathbb{R}})$ is locally compact and a compact space is also locally compact. Following [15], p.126, a *concrete*[5] open interval of reals (a, b) is not locally compact, since a locally compact metric space is complete (see [15], p.110)[6]. Clearly, the concrete compact intervals $[a, b]$ of reals are compact (see [15], p.96).

Definition 2.3.8. *If X is a locally compact metric space, a function $f : X \to \mathbb{R}$ is called Bishop continuous, or simply continuous, if f is uniformly continuous on every bounded subset of X i.e., there is a function*

$$\omega_{f,B} : \mathbb{R}^+ \to \mathbb{R}^+,$$

$$\epsilon \mapsto (\omega_{f,B})(\epsilon),$$

for every bounded subset B of X, the modulus of continuity of f on B. We denote by $\mathrm{Bic}(X)$ *the set of all Bishop continuous functions from X to $\mathbb{R}$. Equality on* $\mathrm{Bic}(X)$ *is defined as in the definition of* $\mathrm{Bic}(\mathbb{R})$.

As in the case of $\mathrm{Bic}(\mathbb{R})$ one has at first the impression that the above definition requires quantification over the power set of X i.e.,

$$\mathrm{Bic}(X)(f) :\leftrightarrow \forall_{B \in \mathcal{P}(X)}(\text{bounded}(B) \to f_{|B} \text{ is uniformly continuous}).$$

One easily avoids such a quantification since, if x_0 inhabits X, then for every bounded subset $(B, {x_0}', M)$ of X we have that there is some $n \in \mathbb{N}$ such that $n > 0$ and

$$B \subseteq [d_{x_0} \leq \overline{n}] = \{x \in X \mid d(x_0, x) \leq n\};$$

if $x \in B$, then $d(x, x_0) \leq d(x, {x_0}') + d({x_0}', x_0) \leq M + d({x_0}', x_0)$, therefore $x \in [d_{x_0} \leq \overline{n}]$, for some $n > M + d({x_0}', x_0)$. Hence, we can write

$$\mathrm{Bic}(X)(f) \leftrightarrow \forall_{n \in \mathbb{N}}(f_{|[d_{x_0} \leq \overline{n}]} \text{ is uniformly continuous}),$$

since $[d_{x_0} \leq \overline{n}]$ is trivially a bounded subset of X. In case one used the definition of Bishop continuity demanding the uniform continuity of f on every compact subset of X, then one can also reduce this definition to a formula including only quantification over $\mathbb{N}$ and reference only to compact subsets of X, but then he needs to use the less trivial result that every subset $[d_{x_0} \leq \overline{a}]$ of X, where $a \in \mathbb{R}$, is either void or compact for all except countably many reals a (see [15], p.110).

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[5] An *abstract* interval I of $\mathbb{R}$ is defined by the property $\forall_{x,y \in I} \forall_{z \in \mathbb{R}}(x < z < y \to z \in I)$.

[6] This is in complete contrast to the classical notion of local compactness i.e., for every $x \in X$ there exists a compact neighborhood of x, with respect to which $(0, 1)$ is locally compact.

Next we show a trivial, but useful fact. Note that we use the obvious fact that a totally bounded space is bounded. Actually, if a metric space X admits only one finite ϵ-approximation, then it is bounded: if for some $\epsilon > 0$ there exist $y_1, \ldots, y_n$, for some n, such that for each $x \in X$ there exists i such that $d(x, y_i) \leq \epsilon$, then, if we fix $x, y \in X$, we have that

$$d(x,y) \leq d(x,y_i) + d(y_i,y_j) + d(y_j,y) \leq \epsilon + M + \epsilon,$$
$$M = \max\{d(y_i,y_j) \mid i,j \in \{1,\ldots,n\}\}.$$

Proposition 2.3.9. *Suppose that (X, d) is a locally compact metric space and $f : X \to \mathbb{R}$. Then the following are equivalent:*
(i) f is uniformly continuous on every bounded subset of X.
(ii) f is uniformly continuous on every totally bounded subset of X.
(iii) f is uniformly continuous on every compact subset of X.

Proof. (i) $\to$ (ii) A totally bounded subset of X is bounded.
(ii) $\to$ (iii) A compact subset of X is totally bounded.
(iii) $\to$ (i) A bounded subset B of X is included in a compact subset K of X, hence if f is uniformly continuous on K, it is uniformly continuous on B. $\square$

Note that if $X = \mathbb{R}$, then (i) - (iii) are equivalent to (iv) and (v), where
(iv) f is uniformly continuous on every compact interval of $\mathbb{R}$,
(v) f is uniformly continuous on every compact interval of $\mathbb{R}$ with rational end points,
since a compact interval is a compact subset of $\mathbb{R}$ and a compact subset of $\mathbb{R}$ is included in a compact interval. The equivalence between (iv) and (v) is trivial.
As expected, $\mathrm{Bic}(X)$ shares all properties of $\mathrm{Bic}(\mathbb{R})$ found in Proposition 2.3.5.

Proposition 2.3.10. *Suppose that (X, d) is a locally compact metric space, $h \in \mathbb{F}(X)$, $f, g \in \mathrm{Bic}(X)$, $\phi \in \mathrm{Bic}(\mathbb{R})$, and $a \in \mathbb{R}$. Then $\overline{a}, f+g, \phi \circ f \in \mathrm{Bic}(X)$. Moreover, if for every $\epsilon > 0$ there exists some $f_\epsilon \in \mathrm{Bic}(X)$ such that $\forall_{x\in\mathbb{R}}(|f_\epsilon(x) - f(x)| \leq \epsilon)$, then $f \in \mathrm{Bic}(X)$. All corresponding moduli of continuity are as in Proposition 2.3.5.*

Proof. All parts of the proof follow the proof of Propositions 2.3.5. We only explain why $f \in \mathrm{Bic}(X) \to B \subseteq^{b(d)} X \to f(B) \subseteq^{b} \mathbb{R}$. Since there exists some compact subset K of X such that $B \subseteq K$, hence $f(B) \subseteq f(K)$, we conclude[7] the boundedness of $f(B)$ by the boundedness of $f(K)$ (K is totally bounded and by Proposition 4.2 in [15], p.94 we have that $f(K)$ is also totally bounded). $\square$

Remark 2.3.11. *If X is a metric space and $h \in \mathrm{Bic}(X)$, then h is pointwise continuous.*

Proof. We fix some $x_0 \in X$ and we show that for each $x \in X$ we get that $d(x, x_0) \leq \delta_{h,x_0}(\epsilon) \to d(h(x), h(x_0)) \leq \epsilon$. If $M > 0$, then the neighborhood $B(x_0, M) \subseteq^{b} X$, since $d(x, x') \leq d(x, x_0) + d(x_0, x') \leq 2M$, for each $x, x' \in B(x_0, M)$. By definition $h^* = h_{|B(x_0,M)}$ is uniformly continuous, hence $d(x, x') \leq \omega_{h^*}(\epsilon) \to d(h(x), h(x')) \leq \epsilon$, for every $x, x' \in B(x_0, M)$. It suffices then to define $\delta_{h,x_0}(\epsilon) = \min\{M, \omega_{h^*}(\epsilon)\}$. $\square$

[7]Clearly, this argument is a direct generalization of the proof of Proposition 2.3.4.

The next simple fact implies that every $\phi \in \mathrm{Bic}(\mathbb{R})$ is strongly continuous, an elementary fact which is stated without proof in [15], p.40. For the sake of completeness we include here its proof, mentioned to us by D. S. Bridges. A more complex proof uses the fact that the function $x \mapsto x^n$ and every polynomial with rational coefficients are strongly continuous, the Weierstrass approximation theorem and Proposition 2.3.6. Similarly, we conclude that every (uniformly) continuous function between two metric spaces is strongly continuous:

$$d(f(x), f(y)) > 0 \to d(x, y) > 0.$$

Remark 2.3.12. *Suppose that $A \subseteq \mathbb{R}$, $\phi \in \mathbb{F}(A)$, X, Y are metric spaces and $f : X \to Y$.*
(i) If ϕ is pointwise continuous, then ϕ is strongly continuous.
(ii) If f is pointwise continuous, then f is strongly continuous with respect to the canonical apartness relations on X and Y induced by the corresponding metrics.

Proof. (i) Suppose that $0 < \epsilon = |\phi(b) - \phi(a)|$, for some $a, b \in A$. By the continuity of ϕ at a there exists $\delta(\frac{\epsilon}{2}) > 0$ such that if $x \in A$ and $|x-a| < \delta(\frac{\epsilon}{2})$, we have $|\phi(x)-\phi(a)| < \frac{\epsilon}{2}$. Clearly, $\neg(|a - b| < \delta(\frac{\epsilon}{2}))$ i.e., $|a - b| \geq \delta(\frac{\epsilon}{2}) > 0$. The proof of (ii) is the obvious generalization of the proof of (i). □

It is easy to see that pointwise continuity implies sequential continuity; if $x_n \to x$, the continuity of h at x gives that $d(y, x) \leq o(\epsilon) \to d(h(y), h(x)) \leq \epsilon$, for each $y \in X$. Hence,

$$n \geq n_0(o(\epsilon)) \to d(x_n, x) \leq o(\epsilon) \to d(h(x_n, h(x)) \leq \epsilon.$$

Consequently, a function in $\mathrm{Bic}(\mathbb{R})$ is sequentially continuous. It is not true in BISH that every strongly continuous function of type $\mathbb{R} \to \mathbb{R}$ is in $\mathrm{Bic}(\mathbb{R})$, since it is easy to find in CLASS strongly continuous functions of this type which are not continuous; take, for example, $f(x) = -x$, if $x \in [-1, 1]$, and $f(x) = x$, elsewhere. In [98] Waaldijk defined a Bishop continuous function on a metric space X to be continuous on every compact subset of X and pointwise continuous. If X is locally compact though, Remark 2.3.11 shows that pointwise continuity is redundant. Clearly, it is not redundant, if X is an abstract space, but we don't see the necessity to include pointwise continuity in the definition of Bishop continuity. As it is noted by Bridges and Vîţă in [27], p.154,

> ... to produce an example of a continuous but not uniformly continuous mapping $f : [0, 1] \to \mathbb{R}$ we need to add a hypothesis such as Church-Markov-Turing Thesis.

It is clear that Bishop's definition of a locally compact metric space is such that guarantees almost automatically the closure of $\mathrm{Bic}(X)$ under composition with elements of $\mathrm{Bic}(\mathbb{R})$. If X, Y are metric spaces, the initial definition of Bishop of the continuity of some $f : X \to Y$ f is that f is uniformly continuous on every *compact* subset of X. By Proposition 2.3.9, which holds for arbitrary codomain Y, it is equivalent to the one we use, if X is locally compact. But when X is not locally compact, if f is uniformly continuous on every compact subset of X, then it is not always the case that f is uniformly continuous on every bounded subset of X. For example, the inverse function

$$^{-1} : (0, 1) \to (1, +\infty),$$

$$x \mapsto \frac{1}{x},$$

is uniformly continuous on every compact subset of $(0, 1)$, but not on every bounded subset of it; since

$$|\frac{1}{n} - \frac{1}{n+1}| = \frac{1}{n(n+1)} \stackrel{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad |f(\frac{1}{n}) - f(\frac{1}{n+1})| = 1$$

we get that $^{-1}$ is not uniformly continuous on the $d_{\mathbb{R}}$-bounded subset $\{\frac{1}{n} \mid n \geq 2\}$, since it is a Cauchy sequence, of $(0, 1)$ (take $\epsilon = \frac{1}{2}$ in the definition of uniform continuity). Since $(0, 1)$ is not locally compact, the inverse function cannot be called continuous with respect to the aforementioned definition. But for intervals like $(0, 1)$ applies another definition of Bishop, see [15], p.38, or in [6], p.34, according to which a real-valued function f on an *arbitrary* interval J of $\mathbb{R}$ is continuous, if it is continuous on every compact subinterval I of J. Hence, the inverse function *is* continuous in this sense, and clearly the two definitions do not contradict each other[8]. Thus, Bishop's standard definition using compactness is, from this point of view, "better" than the one considered here (and by Bridges in [19]) using boundedness, when the domain is not locally compact. To that we can only say now that boundedness is simpler to handle than compactness and that in our work the "continuity" of the inverse function is not necessary, only its continuity in an interval of the form $[c, \infty)$, where $c > 0$, which is locally compact, is used (see Theorem 5.4.8 and its important consequences).

2.4 B-continuity

As a result of the discussion in the previous section, an abstract notion of Bishop continuity of a function $\phi : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, can be defined as follows.

Definition 2.4.1. *If $P(B)$ is a property on subsets of $A \subseteq \mathbb{R}$, then $\phi : A \to \mathbb{R}$ is called Bishop P-continuous, or simply P-continuous, if it is uniformly continuous on every P-subset of A i.e., on every $B \subseteq A$ such that $P(B)$ i.e., a P-continuous function is a pair $(\phi, (\omega_{\phi,B})_{P(B)})$, where $(\omega_{\phi,B})_{P(B)}$ is the modulus of P-continuity of ϕ on A. The set of all P-continuous functions on A is denoted by* $\mathrm{Bic}(P, A)$ *with the obvious definition of equality between such pairs.*

[8]In [86], p.2078, it is mentioned that "a drawback of Bishop's continuity is that it does not include the reciprocal function, for the lack of a locally compact domain". It is the complementary definition of Bishop though, which guarantees its continuity.

For simplicity we write *b-continuity* instead of *bounded*-continuity. Clearly, we have that $\mathrm{Bic}(\mathbb{R}) = \mathrm{Bic}(b, \mathbb{R})$. As we have already said, continuity was introduced by Bishop as *compactness*-continuity, which we write *c-continuity*. If we denote *totally-bounded-continuity* or *tb-continuity* on A by $\mathrm{Bic}(tb, A)$, we have that $c \subseteq tb \subseteq b$ and $\mathrm{Bic}(b, A) \subseteq \mathrm{Bic}(tb, A) \subseteq \mathrm{Bic}(c, A)$, while Proposition 2.3.9 says that $\mathrm{Bic}(b, \mathbb{R}) = \mathrm{Bic}(tb, \mathbb{R}) = \mathrm{Bic}(c, \mathbb{R})$. We call a P-continuity $\mathbb{R}$-*invariant*, if

$$f \in \mathrm{Bic}(P, \mathbb{R}) \to P(A) \to P(f(A)).$$

Proposition 2.3.4 says that b-continuity is $\mathbb{R}$-invariant, while Proposition 2.3.9 together with the fact that uniform continuity preserves total boundedness (Proposition 4.2 in [15], p.94) imply that tb-continuity is also $\mathbb{R}$-invariant. On the other hand, c-continuity is not $\mathbb{R}$-invariant; in RUSS there is a uniformly continuous function e defined on $[0, 1]$ which is onto $(0, 1]$. This function e is established e.g., in [20], Section 6.2 (Corollary 2.9, p.129).

Proposition 2.4.2. *Suppose that P-continuity is $\mathbb{R}$-invariant. If $A \subseteq \mathbb{R}$, $f \in \mathrm{Bic}(P, A)$ and $g \in \mathrm{Bic}(P, f(A))$, then $g \circ f \in \mathrm{Bic}(P, A)$.*

Proof. If we fix some $B \subseteq A$ such that $P(B)$, then we have by the invariance of P that $P(f(B))$, and then we work exactly as in the proof of Proposition 2.3.5(iii). $\square$

Since c-continuity is not $\mathbb{R}$-invariant, Bishop could not prove a similar closure of c-continuity under composition in an obvious way. A subclass of his continuous functions $\mathrm{Bic}(c, A)$ though, satisfies a kind of composition closure, namely all *c-invariant* functions i.e., all functions $f : A \to \mathbb{R}$ which map compact subintervals of A to compact subintervals of $f(A)$. As it is explained in [86], p.2078, it is simple to see that if $f(A)$ is a locally compact subset of $\mathbb{R}$, then the composition $g \circ f$ is in $\mathrm{Bic}(c, A)$. The fact that c-continuity is not preserved generally by the operation of composition is considered a drawback of Bishop's notion of continuity. Clearly, if $f \in \mathrm{Bic}(b, A)$ and $f(A)$ is not locally compact, then b-continuity is also not obviously closed under composition within BISH. In TBS we compose mainly with functions of type $\mathbb{R} \to \mathbb{R}$ and we work with b-continuity.

In order to remedy the problem of the closure of his continuity under composition within *abstract* metric spaces, Bishop introduced in the unpublished manuscript [8] another notion of continuity which is identical to his previous one when the metric space is locally compact, and it is closed under composition.

Definition 2.4.3. *If X is a metric space and $A \subseteq X$, A is called a compact image, $A \subseteq^{ci} X$, if there exists some compact metric space K and a uniformly continuous function $h : K \to A$ such that $h(K) = A$.*

Classically, a compact image is also compact, something which is not the case within BISH. For example, $(0, 1]$ is a compact image in RUSS, since the aforementioned function e maps $[0, 1]$ onto $(0, 1]$, but it is not compact. Clearly, a compact image is a totally bounded set. Next we follow the terminology in [19].

Definition 2.4.4. *If X, Y are metric spaces, a function $h : X \to Y$ is called B-continuous, if for every $A \subseteq^{ci} X$, and for every $\epsilon > 0$, we have that*

$$\forall_{a \in A} \forall_{x \in X} (d(a, x) \leq \omega_{h,A}(\epsilon) \to d(h(a), h(x)) \leq \epsilon),$$

where the function $\omega_{h,A} : \mathbb{R}^+ \to \mathbb{R}^+$ is called a modulus of B-continuity of h on A. We denote by $B(X, Y)$ the set of B-continuous functions from X to Y, and by $\mathrm{Bic}(X, Y)$ the functions from X to Y which are uniformly continuous on every bounded subset of X.

I.e., in B-continuity we have the following change of quantifiers with respect to the usual uniform continuity on some set A:

$$\forall_{a \in A} \forall_{b \in A} \rightsquigarrow \forall_{a \in A} \forall_{x \in X}.$$

Since within BISH every compact metric space is the compact image of the Cantor space $2^{\mathbb{N}}$ (see e.g., [20], p.106), and the uniform continuity is preserved under the composition $2^{\mathbb{N}} \xrightarrow{\pi} K \xrightarrow{h} A$, we get that A is a compact image if and only if there is some uniformly continuous $h : 2^{\mathbb{N}} \to A$ which is onto A. Clearly, a compact set is a compact image; either by considering id_A, or the aforementioned uniformly (quotient map) $\pi : 2^{\mathbb{N}} \to A$. It is with this equivalent description of a compact image that one can avoid the seemingly necessary quantification over the power set of X in the formulation of B-continuity. Namely, we can write

$$B(h) \leftrightarrow \forall_{e:2^{\mathbb{N}} \xrightarrow{uc} X} \forall_{\alpha \in 2^{\mathbb{N}}} \forall_{x \in X} (d(e(\alpha), x) \leq \omega_{h,e}(\epsilon) \to d(h(e(\alpha)), h(x)) \leq \epsilon),$$

where the function $\omega_{h,e} : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of B-continuity of h on $e(2^{\mathbb{N}})$ and only quantification over the function spaces $\mathbb{F}(2^{\mathbb{N}}, X), 2^{\mathbb{N}}$ and over X is used.

Proposition 2.4.5. *Suppose that X, Y and Z are metric spaces.*
(i) If $h \in B(X, Y)$, then h is uniformly continuous on every compact subset of X.
(ii) If X is locally compact, then $B(X, Y) = \mathrm{Bic}(X, Y)$.
(iii) If $h \in B(X, Y)$ and $g \in B(Y, Z)$, then $g \circ h \in B(X, Z)$.

Proof. (i) Since a compact subset of X is a compact image, it follows trivially by the (symbolic) implication $\forall_{a \in A} \forall_{x \in X} \to \forall_{a \in A} \forall_{b \in A}$.
(ii) By (i) and Proposition 2.3.9 we have that $B(X, Y) \subseteq \mathrm{Bic}(X, Y)$. We show that $\mathrm{Bic}(X, Y) \subseteq B(X, Y)$. We fix some $A \subseteq^{ci} X$ and let M be a bound of it. If $l > 0$, then the set $A_l := \{x \in X \mid \exists_{a \in A}(d(x, a) \leq l)\}$ is bounded, since $d(x, x') \leq d(x, a) + d(a, a') + d(a', x') \leq l + M + l$, for each $x, x' \in A_l$, and clearly $A \subseteq A_l$. By hypothesis h is uniformly continuous on A_l i.e., $\forall_{x,x' \in A_l}(d(x, x') \leq \omega_{h,A_l}(\epsilon) \to d(h(x), h(x')) \leq \epsilon)$. We define $\omega_{h,A} = \min\{l, \omega_{h,A_l}(\epsilon)\}$. If $a \in A$, $x \in X$ and $d(a, x) \leq \omega_{h,A}$ we get that $d(a, x) \leq l$, therefore $x \in A_l$ and then by $d(a, x) \leq \omega_{h,A_l}$ we get that $d(h(a), h(x)) \leq \epsilon$.
(iii) If $A \subseteq^{ci} X$, then h is uniformly continuous on A, therefore $h(A) \subseteq^{ci} Y$. As expected, we get that $\omega_{g \circ h,A}(\epsilon) = \omega_{h,A}(\omega_{g,h(A)}(\epsilon))$, for every $\epsilon > 0$. $\square$

2.5 Inductive continuous functions

As we have already seen in section 1.2, Bishop, in [11], p.28, admitted the ad hoc character of his various definitions of continuity. He tried to overcome this problem by developing in [8] the theory of stratified spaces, a theory that he didn't find convincing for it was too involved, as he admitted in [11], p.29.
If we concentrate on the definition of $\phi \in \mathrm{Bic}(\mathbb{R})$, we see that on the one hand this definition serves the extremely useful from the philosophical point of view, avoidance of the uniform continuity theorem within BISH, but on the other it gives no explanation why continuity of some $\phi : \mathbb{R} \to \mathbb{R}$ should mean uniform continuity of ϕ on every bounded (or compact) subset of $\mathbb{R}$. Although this definition does not require quantification over the power set of $\mathbb{R}$, it is important to address its ad hoc character. For that we may study possible inductive notions of continuity following the example of Tait's inductive notion of a continuous function of type $2^{\mathbb{N}} \to \mathbb{N}$ and its generalization to an inductive notion of a continuous function of type $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ found in [47]. This section can be seen as a first attempt to this direction.

Definition 2.5.1. *If $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, we define*

$$U(g, f, \epsilon) :\leftrightarrow \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) :\leftrightarrow \forall_{\epsilon>0}\exists_{g\in\Phi}(U(g, f, \epsilon)).$$

Note that in general we cannot conclude from $U(g, f, \epsilon)$, that $||f-g||_\infty = \sup\{|f(x)-g(x)| \mid x \in X\} \leq \epsilon$, since it is not always the case constructively that this supremum exists.

Definition 2.5.2. *If $A \subseteq \mathbb{R}$, the set $\mathrm{IC}(A)$ of inductive continuous functions on A is the subset of $\mathbb{F}(A)$ defined by the following inductive rules*

$$\frac{}{\mathrm{id}_A \in \mathrm{IC}(A)}\ , \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathrm{IC}(A)}\ , \quad \frac{f, g \in \mathrm{IC}(A)}{f + g \in \mathrm{IC}(A)}\ ,$$

$$\frac{f, g \in \mathrm{IC}(A)}{f \cdot g \in \mathrm{IC}(A)}\ , \quad \frac{(g \in \mathrm{IC}(A),\ U(g, f, \epsilon))_{\epsilon>0}}{f \in \mathrm{IC}(A)}\ .$$

Note that the most complex inductive rule above can be replaced by the rule

$$\frac{g_1 \in \mathrm{IC}(A)\ \wedge\ U(g_1, f, \frac{1}{2}),\ g_2 \in \mathrm{IC}(A)\ \wedge\ U(g_2, f, \frac{1}{2^2}),\ g_3 \in \mathrm{IC}(A)\ \wedge\ U(g_3, f, \frac{1}{2^3}), \ldots}{f \in \mathrm{IC}(A))}\ .$$

If P is any property on $\mathbb{F}(A)$, the above rules induce the following induction principle $\mathrm{Ind}_{\mathrm{IC}(A)}$ on $\mathrm{IC}(A)$:

$$\begin{aligned}
&P(\mathrm{id}_A) \to \\
&\forall_{a\in\mathbb{R}}(P(\overline{a})) \to \\
&\forall_{f,g\in\mathrm{IC}(A)}(P(f) \to P(g) \to P(f + g) \wedge P(f \cdot g)) \to \\
&\forall_{f\in\mathrm{IC}(A)}(\forall_{\epsilon>0}\exists_{g\in\mathrm{IC}(A)}(P(g)\ \wedge\ U(g, f, \epsilon)) \to P(f)) \to \\
&\forall_{f\in\mathrm{IC}(A)}(P(f)).
\end{aligned}$$

Proposition 2.5.3. $\forall_{f \in \mathrm{IC}(A)} \forall_{g \in \mathrm{IC}(f(A))} (g \circ f \in \mathrm{IC}(A))$.

Proof. We fix some $f \in \mathrm{IC}(A)$ and we show inductively that $\forall_{g \in \mathrm{IC}(f(A))}(g \circ f \in \mathrm{IC}(A))$. Clearly, $\mathrm{id}_{f(A)} \circ f = f \in \mathrm{IC}(A)$, and if $a \in \mathbb{R}$, then $\overline{a} \circ f$ is the constant function $\overline{a}$ on A. Suppose next that $g_1, g_2 \in \mathrm{IC}(f(A))$ such that $g_1 \circ f, g_2 \circ f \in \mathrm{IC}(A)$. Then $(g_1 + g_2) \circ f = (g_1 \circ f) + (g_2 \circ f) \in \mathrm{IC}(A)$, and $(g_1 \cdot g_2) \circ f = (g_1 \circ f) \cdot (g_2 \circ f) \in \mathrm{IC}(A)$. If $h \in \mathrm{IC}(f(A))$ and $\forall_{\epsilon > 0} \exists_{g \in \mathrm{IC}(f(A))}(g \circ f \in \mathrm{IC}(A) \ \wedge \ U(g, h, \epsilon))$, then, since $U(g, h, \epsilon) \to U(g \circ f, h \circ f, \epsilon)$, for every $\epsilon > 0$, we get that the last clause of $\mathrm{IC}(A)$ is satisfied for $h \circ f$. $\square$

Proposition 2.5.4. *If $B \subseteq A \subseteq \mathbb{R}$, then $\forall_{f \in \mathrm{IC}(A)}(f_{|B} \in \mathrm{IC}(B))$.*

Proof. Clearly, $(\mathrm{id}_A)_{|B} = \mathrm{id}_{|B}$, and if $a \in \mathbb{R}$, then $\overline{a}_{|B}$ is the constant function $\overline{a}$ on B. Suppose next that $g_1, g_2 \in \mathrm{IC}(A)$ such that $g_{1|B}, g_{2|B} \in \mathrm{IC}(B)$. Then $(g_1 + g_2)_{|B} = g_{1|B} + g_{2|B} \in \mathrm{IC}(B)$, and $(g_1 \cdot g_2)_{|B} = g_{1|B} \cdot g_{2|B} \in \mathrm{IC}(B)$. If $f \in \mathrm{IC}(A)$ and $\forall_{\epsilon > 0} \exists_{g \in \mathrm{IC}(A)}(g_{|B} \in \mathrm{IC}(B) \ \wedge \ U(g, f, \epsilon))$, then, since $U(g, f, \epsilon) \to U(g_{|B}, h_{|B}, \epsilon)$, for every $\epsilon > 0$, we get that the last clause of $\mathrm{IC}(B)$ is satisfied for $f_{|B}$. $\square$

Corollary 2.5.5. *If $f \in \mathrm{IC}(A)$ and $g \in \mathrm{IC}(B)$, where $f(A) \subseteq B$, then $g \circ f \in \mathrm{IC}(A)$.*

Proof. By Proposition 2.5.4 we have that $g_{|f(A)} \in \mathrm{IC}(f(A))$, and by Proposition 2.5.3 we get that $g \circ f = g_{|f(A)} \circ f \in \mathrm{IC}(A)$. $\square$

If $A = \mathbb{R}$, then not every element of $\mathrm{IC}(\mathbb{R})$ is uniformly continuous, since $f(x) = x^2 \in \mathrm{IC}(\mathbb{R})$, but f is not uniformly continuous on $\mathbb{R}$. Things change when A is a proper compact interval.

Proposition 2.5.6. *If $a, b \in \mathbb{R}$ such that $a < b$, then $\mathrm{IC}([a, b])$ is equal to the set $C_u([a, b])$ of all uniformly continuous real-valued functions on $[a, b]$.*

Proof. A polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ on $[a, b]$ is in $\mathrm{IC}([a, b])$, since $\mathrm{id}_{[a,b]} \in \mathrm{IC}([a, b])$, and $\mathrm{IC}([a, b])$ is closed under addition and the multiplication with some $\lambda \in \mathbb{R}$, since $\lambda f = \overline{\lambda} \cdot f$. By the Weierstrass approximation theorem, proved in [15], p.109, and the last rule of $\mathrm{IC}([a, b])$ we get that every uniformly continuous function on $[a, b]$ is in $\mathrm{IC}([a, b])$. For the converse inclusion we show inductively that $\mathrm{IC}([a, b]) \subseteq C_u([a, b])$. Except of the rule for the product all the other steps follow the line of proof of Proposition 2.3.5. Since $xy = \frac{1}{2}(x^2 + y^2 - (x - y)^2)$, for every $x, y \in \mathbb{R}$, it suffices to show that if $f \in \mathrm{IC}([a, b])$ such that $f \in C_u([a, b])$, then $f^2 \in C_u([a, b])$. It is easy to see that $\forall_{f \in \mathrm{IC}([a,b])}(f \text{ is bounded})$; clearly, $\mathrm{id}_{[a,b]}$ and $\overline{a}$ are bounded, for every $a \in \mathbb{R}$. If $M_f, M_g > 0$ are bounds for $f, g \in \mathrm{IC}([a, b])$, then $M_f + M_g$ and $M_f \cdot M_g$ are bounds for $f + g$ and $f \cdot g$, respectively. If $f \in \mathrm{IC}([a, b])$ and for every $\epsilon > 0$ there is some $g \in \mathrm{IC}([a, b])$ such that $U(g, f, \epsilon)$ and M_g is bound of g, then if we fix some $\epsilon > 0$, then, since $|f(x)| \leq |f(x) - g(x)| + |g(x)|$, $\epsilon + M_g$ is a bound of f. If $f \in \mathrm{IC}([a, b])$ and $M_f > 0$ is a bound for f, then

$$\begin{aligned} |f(x)^2 - f(y)^2| &= |f(x) - f(y)||f(x) + f(y)| \\ &\leq |f(x) - f(y)| 2M_f, \end{aligned}$$

hence f^2 is uniformly continuous on $[a, b]$ with $\omega_{f^2}(\epsilon) = \omega_f(\frac{\epsilon}{2M_f})$. $\square$

Proposition 2.5.7. $\mathrm{IC}(\mathbb{R}) \subseteq \mathrm{Bic}(\mathbb{R})$.

Proof. By Proposition 2.3.9 it suffices to show that if $f \in \mathrm{IC}(\mathbb{R})$, then f is uniformly continuous on every compact interval. If we fix some $[a,b]$, then by Proposition 2.5.4 we have that $f_{|[a,b]} \in \mathrm{IC}([a,b])$, hence by Proposition 2.5.6 we get that $f_{|[a,b]}$ is uniformly continuous on $[a,b]$. $\square$

Therefore, the concept of an inductive continuous function on A is a notion of continuity satisfying the following properties:
W1. If f is a uniformly continuous real-valued function on $[0,1]$, then $f \in \mathrm{IC}([0,1])$.
W2. If $f \in \mathrm{IC}(A)$ and $g \in \mathrm{IC}(B)$ such that $\mathrm{rng}(f) \subseteq \mathrm{dom}(g)$, then $g \circ f$ is in $\mathrm{IC}(A)$.
W3. If $f \in \mathrm{IC}([0,1])$, then f is uniformly continuous.
We cannot show within BISH that the inverse function $x \mapsto \frac{1}{x}$, for every $x > 0$ is in $\mathrm{IC}(0,+\infty)$ (W4), since, as Waaldijk showed within BISH in [98], the fact that a notion of continuity satisfies the properties W1-W4 is equivalent to the fan theorem, which is not accepted in BISH. The Bishop continuous functions in $\mathrm{Bic}(\mathbb{R})$ also share W1-W3, while, as we have already said, for the needs of TBS what is required is that the inverse function is in $\mathrm{Bic}([c,\infty))$, for every $c > 0$. Note that if $f \in \mathrm{Bic}(A)$ and $g \in \mathrm{Bic}(f(A))$, then $g \circ f \in \mathrm{Bic}(A)$, if $f(B)$ is included in a bounded subset of $f(A)$, where $B \subseteq^b A$. This happens, for example, if A is a locally compact subset of $\mathbb{R}$. For an interesting notion of constructive continuity with respect to W1-W4 see also [87].
Since it is immediate to see that $\mathrm{Bic}([a,b]) = C_u([a,b])$, we conclude that

$$\mathrm{Bic}([a,b]) = \mathrm{IC}([a,b]),$$

hence the initial seemingly ad hoc notion of Bishop continuity is in the case of compact intervals identical to a natural, inductive notion of continuity.
On the other hand, the notion of $\mathrm{IC}(\mathbb{R})$ seems practically not very useful. It is not clear that $|f|, \sqrt{f} \in \mathrm{IC}(\mathbb{R})$, if $f \in \mathrm{IC}(\mathbb{R})$; the problem in an inductive proof is in both examples the case of addition. One way out is to define the least set of inductive continuous functions $\mathrm{IC}^*(A,\Phi_0)$ which includes a given (rather small) set Φ_0 of elements of $\mathrm{Bic}(\mathbb{R})$, the identity and the constants, and it is closed under addition, multiplication, uniform limits and composition with the elements of $\mathrm{IC}(\mathbb{R},\Phi_0)$.

Definition 2.5.8. *If $A \subseteq \mathbb{R}$ and $\Phi_0 \subseteq \mathrm{Bic}(\mathbb{R})$, the set $\mathrm{IC}^*(A,\Phi_0) \subseteq \mathbb{F}(A)$ of Φ_0-inductive continuous functions on A is defined by the following inductive rules*

$$\frac{\phi_0 \in \Phi_0}{\phi_{0|A} \in \mathrm{IC}^*(A,\Phi_0)}\ , \quad \frac{}{\mathrm{id}_A \in \mathrm{IC}^*(A,\Phi_0)}\ , \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathrm{IC}^*(A,\Phi_0)}\ , \quad \frac{f,g \in \mathrm{IC}^*(A,\Phi_0)}{f+g \in \mathrm{IC}^*(A,\Phi_0)}\ ,$$

$$\frac{f,g \in \mathrm{IC}^*(A,\Phi_0)}{f \cdot g \in \mathrm{IC}^*(A,\Phi_0)}\ , \quad \frac{f \in \mathrm{IC}^*(A,\Phi_0), \phi \in \mathrm{IC}^*(\mathbb{R},\Phi_0)}{\phi \circ f \in \mathrm{IC}^*(A,\Phi_0)}\ , \quad \frac{(g \in \mathrm{IC}^*(A,\Phi_0),\ U(g,f,\epsilon))_{\epsilon>0}}{f \in \mathrm{IC}^*(A,\Phi_0)}\ .$$

If $|.| \in \Phi_0$, then by the added composition rule we have that if $f \in \mathrm{IC}^*(A, \Phi_0)$, then $|f| \in \mathrm{IC}^*(A, \Phi_0)$. It is easy to see that $\mathrm{IC}^*(A, \Phi_0)$ satisfies W1-W3, it is closed under restriction to subsets, as it is shown for $\mathrm{IC}(A)$ in Proposition 2.5.4, and $\mathrm{IC}^*(\mathbb{R}, \Phi_0) \subseteq \mathrm{Bic}(\mathbb{R})$. One could study the above notion of inductive continuous real function in order to have a uniform inductive and non ad hoc approach to the notion of a Bishop topology defined in the next chapter (see also section 8.3).
One could raise the objection that the meaning of continuity is hidden with definitions like the inductive ones, since inductive continuity of a real function is not a special case of a more general notion of continuity. To this objection one could say that within TBS continuity is captured by the general notion of a Bishop morphism (see section 3.6) and only the base case of $\mathbb{R}$ needs to be settled. Moreover, if equality, inequality, subsets and maybe other notions are defined for a given Bishop set X in a way specific to X, it is not that strange if continuity of functions of type $X \to Y$ are defined also in a way specific to X and Y.
The inductive definition of a set, or a predicate, goes beyond the notion of Bishop set presented in section 2.1. Myhill's system CST^*, developed in [67], formalizes such a notion of a set.

Chapter 3

Continuity as a primitive notion

> ... a constructive development of (some form of) general topology is at least a challenge and may well shed light even on aspects of the classical theory.
>
> D. S. Bridges and L. S. Vîţă 2002

Bishop believed that mathematics should have "numerical meaning" i.e., as he notes in [4], p.308,

> ... every mathematical theorem should admit an ultimate interpretation to the effect that certain finite computations within the set of positive integers will give certain results.

Constructive or computational topology tries to answer the following

Main Question: How much of the classical, abstract theory of topological spaces has computational content?

Even very elementary facts like the distance $d(x, A)$, where A is a non-empty subset of a metric space X and $x \in X$, cannot be calculated, in general, by a computer, since as Bishop notes in [5], p.343,

> ... there is no finite routine method to compute a rational approximation to $d(x, A)$ to within a prescribed limit of accuracy.

In a more advanced level, the search for effective computations in Algebraic Topology led rather recently to the new field of Effective Algebraic Topology (see [83], [80]). The notion of effective computation in this setting includes the use of classical logic and unbounded search. If we want to find though, really effective algorithms which specify step-by-step how to build an object, like the homotopy group $\pi_n X$, given a simply connected polyhedron X and some $n \geq 2$, we need to omit the principle of the excluded middle, since there is no effective procedure in general for deciding whether a proposition is true or false.

To answer the above main question within TBS, and in order to provide in the end really effective algorithms, we determine first a primitive notion of continuity, the Bishop topology, which by its definition suits to a genuine constructive study. Thus, instead of having a common space-structure on a set X and $\mathbb{R}$, that determines a posteriori which functions of type $X \to \mathbb{R}$ are continuous with respect to it, within TBS we start from a given class of "continuous" functions of type $X \to \mathbb{R}$ that determines a posteriori a space-structure on X. In this chapter we prove some basic properties of Bishop spaces and their morphisms which are necessary to the subsequent development of TBS. We provide many abstract and concrete examples of Bishop spaces and incorporating into TBS some older results of Bridges we study the morphisms between metric spaces seen as Bishop spaces. The space-structure $N(F)$ on some inhabited set X that is determined a posteriori by some Bishop topology F on X is the canonical neighborhood structure induced by F and studied in section 3.7. Although this space-structure is used in Chapter 5 to establish some correlations between TBS and the standard topology, its study is not a "real" part of TBS due to the set-theoretic character of a neighborhood space. The most important Bishop spaces are the inductively generated Bishop spaces studied in section 3.4. The induction principle that corresponds to these Bishop spaces is an important tool for proving results on them and at the same time for establishing their constructive character.

3.1 Bishop spaces

Definition 3.1.1. *A Bishop space is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$, a Bishop topology, or simply a topology, satisfies the following conditions:*
(BS_1) $a \in \mathbb{R} \to \overline{a} \in F$.
(BS_2) $f \in F \to g \in F \to f + g \in F$.
(BS_3) $f \in F \to \phi \in \mathrm{Bic}(\mathbb{R}) \to \phi \circ f \in F$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F \ni \phi \circ f"'] & \mathbb{R} \arrow[d, "\phi \in \mathrm{Bic}(\mathbb{R})"] \\ & \mathbb{R}. \end{tikzcd}$$

(BS_4) $f \in \mathbb{F}(X) \to U(F, f) \to f \in F$.

Bishop used the term *function space* for $\mathcal{F}$ and *topology* for F. Since the former is used in many different contexts, we prefer the term Bishop space for $\mathcal{F}$, while we use the latter, since the *topology of functions* F on X corresponds nicely to the standard *topology of opens* $\mathcal{T}$ on X. Note that Bishop didn't mention in his original definition, as Bridges later did in [19], that X is inhabited. A classical mathematician can read the previous definition just by replacing $\mathrm{Bic}(\mathbb{R})$ with $C(\mathbb{R})$, the pointwise continuous functions on X with values in the classical continuum. It is clear that if BS_3 is satisfied by some $F \subseteq \mathbb{F}(X)$, then BS_1

is equivalent to F being inhabited; if $f \in F$, then $\mathrm{Const}(X) = \{\overline{a} \circ f \mid \overline{a} \in \mathrm{Const}(\mathbb{R})\}$, and $\mathrm{Const}(\mathbb{R}) \subseteq \mathrm{Bic}(\mathbb{R})$. The other direction is trivial. The proof of the next equivalence requires the principle of countable choice, which is used freely by Bishop (see also Myhill's formalization [67] of BISH) and criticized later by Richman (see [78] and [85]).

Proposition 3.1.2. *BS_4 is equivalent to $f_n \subseteq F \to f \in \mathbb{F}(X) \to f_n \stackrel{u}{\to} f \to f \in F$*

Proof. We suppose BS_4 and that $f_n \stackrel{u}{\to} f$. If we fix some $\epsilon > 0$, then we just take $g = f_{n_0(\epsilon)}$, and the hypotheses of BS_4 are satisfied. For the converse we work as follows: by the main hypothesis of BS_4 and the corresponding principle of countable choice we get a sequence $f_m \subseteq F$ such that $\forall_m \forall_{x \in X}(|f_m(x) - f(x)| \leq \frac{1}{m})$. Next we fix some m and if we take $n_0 = m$, then for each $n \geq m$ we have that $\forall_{x \in X}(|f_n(x) - f(x)| \leq \frac{1}{n} \leq \frac{1}{m})$. □

Following Richman, we try to avoid the formulation of BS_4 as the closure of F under uniform limits. The next fundamental proposition expresses that a topology F on some X is an algebra over $\mathbb{R}$, and a lattice. It is proved both in [6] and in [19].

Proposition 3.1.3. *If F is a topology on X, then fg, λf, $-f$, $\max\{f,g\}$, $\min\{f,g\}$ and $|f| \in F$ for every $f, g \in F$ and $\lambda \in \mathbb{R}$.*

Proof. We use BS_2, BS_3 and the identities:

$$f \vee g := \max\{f, g\} = \frac{f + g + |f - g|}{2},$$

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2},$$

$$f \wedge g := \min\{f, g\} = -\max\{-f, -g\} = \frac{f + g - |f - g|}{2},$$

$$\lambda f = \overline{\lambda} f.$$

Since $|\mathrm{id}_{\mathbb{R}}| \in \mathrm{Bic}(\mathbb{R})$, where $\mathrm{id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$, in order to show for example that $\max\{f, g\}$ we use BS_3 to establish that $|g|, -g \in F$ and $\frac{1}{2}g \in F$, and then we use BS_2. □

The definition of a Bishop space has a "structure" that can be found in the definition of some other related notions of space. As we explain in section 3.6 the elements of a topology F on some X are the Bishop morphisms between the Bishop space $\mathcal{F} = (X, F)$ and the Bishop space of reals $\mathbb{R}$, described in section 3.3. Since the notion of a Bishop morphism is the function-theoretic analogue within TBS to the set-theoretic notion of a continuous function in standard general topology, a Bishop topology F is the function-theoretic analogue to the ring of continuous functions $C(X)$ on some topological space X. Proposition 3.1.3 reinforces this analogy. In other words, the definitional clauses BS_1-BS_4 express which properties should be satisfied by a notion of abstract continuity independently from any topological structure on X. Since all four of them are natural and expected, they form *a minimal collection of properties of continuity as a primitive notion*. In [6], p.74, Bishop commented on the definition of a function space saying that it

> ... should not be taken seriously. The purpose is merely to list a minimal number of properties that the set of all continuous functions in a topology should be expected to have. Other properties could be added; to find a complete list seems to be a nontrivial and interesting problem.

He included the same comment in [15], p.80, too. In this Thesis we take this definition seriously, and we hope to show that its study generates a fruitful and promising theory of constructive point-function topology. The issue of the "completeness" of this definition is more complex and depends on how one interprets completeness. Our remark before Theorem 5.5.13 suggests a property that could be added to this list of properties. The problem of the cartesian closure of the category of Bishop spaces **Bis** may also lead to some interesting more special notion of a Bishop space. An inductive version of a Bishop space based on the closure of a topology under composition with inductively defined real functions is discussed also in section 8.2.
Limit spaces are abstract spaces in which sequential convergence is a primitive notion. Considering convergence as a fundamental, or primitive, notion is a quite natural step, and is no coincidence that such spaces were introduced by Fréchet in [42], in the form of $\mathcal{L}$-spaces, before Hausdorff's notion of an abstract topological space was introduced in his work [48] of 1914. Although sequential convergence does not fully capture the notion of convergence in general topology, its importance to constructive mathematics, see [19], p.101, and to the computability theory at higher types, see [69], motivated our study in [76].
If $x \in X$, then (x) denotes the constant sequence x, but we write for simplicity $\lim(x, x)$ instead of $\lim(x, (x))$, and we denote the set of all strictly monotone sequences of type $\mathbb{N} \to \mathbb{N}$ by $\mathcal{S}$.

Definition 3.1.4. *A limit space, or a Kuratowski limit space, is a pair* $(X, \lim)$*, where* X *is an inhabited set, and* $\lim \subseteq X \times X^{\mathbb{N}}$ *satisfies the following conditions:*
(LS$_1$) $x \in X \to \lim(x, (x))$.
(LS$_2$) $\forall_{\alpha \in \mathcal{S}}(\lim(y, x_n) \to \lim(y, x_{\alpha(n)}))$,

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\alpha \in \mathcal{S}"] \arrow[dr, "(x_n) \circ \alpha"'] & \mathbb{N} \arrow[d, "(x_n)"] \\
& X.
\end{tikzcd}$$

(LS$_3$) (Urysohn's axiom) $\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}}(\lim(x, x_{\alpha(\beta(n))})) \to \lim(x, x_n)$.
We say that the limit space $(X, \lim)$ *has the uniqueness property if*

$$\forall_{x,y \in X} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \to \lim(y, x_n) \to x = y).$$

The "structural" analogy between Definitions 3.1.1 and 3.1.4 is obvious. An $\mathcal{L}$-space, or a *Fréchet limit space*, is a structure $(X, \lim \subseteq X \times X^{\mathbb{N}})$ satisfying LS_1, LS_2 and the uniqueness property.

Definition 3.1.5. *If F is a topology on X, the canonical limit relation on X induced by F is defined by*

$$\lim_F(x, x_n) :\leftrightarrow \forall_{f \in F}(f(x_n) \to f(x)).$$

The canonical limit relation of F trivially satisfies LS_1 and LS_2, while in Proposition 5.7.3 we show that if F is completely regular, then $(X, \lim_F)$ is a Fréchet limit space. The definition of a limit space has "influenced" the definition of many notions of space.

Definition 3.1.6. *A filter space, or a Choquet space, is a pair $\mathbb{F} = (X, \mathrm{Lim})$, where X is an inhabited set, and $\mathrm{Lim} \subseteq X \times \mathcal{F}(X)$ is a relation satisfying the following conditions:*

(FS_1) *If $x \in X$ and $F_x = \{A \subseteq X \mid x \in A\}$, then $\mathrm{Lim}(x, F_x)$.*
(FS_2) $F \subseteq G \to \mathrm{Lim}(x, F) \to \mathrm{Lim}(x, G)$.
(FS_3) $\forall_{G \supseteq F} \exists_{X \supsetneq H \supseteq G}(\mathrm{Lim}(x, H)) \to \mathrm{Lim}(x, F)$.

A convergence space is a pair $\mathbb{F} = (X, \mathrm{Lim})$ such that FS_1, FS_2 are satisfied together with

$(\mathrm{FS}_4)\mathrm{Lim}(x, F) \to \mathrm{Lim}(x, G) \to \mathrm{Lim}(x, F \cap G)$.

It is easy to see that $\mathrm{FS}_3 \to \mathrm{FS}_4$. In 1963, very close to the period in which Bishop started redeveloping constructive analysis, Spanier introduced in [88] the quasi-topological spaces and showed that their category is cartesian closed. He did so in order to overcome the fact that the category of topological spaces **Top** is not cartesian closed. Although the motivation behind the introduction of quasi-topological spaces and Bishop spaces was different, there is a similarity between the two notions. A quasi-topological space, although it expresses a kind of function-theoretic approach to topology, relies heavily on the set-theoretic character of the compact Hausdorff spaces and the continuous functions between them. We denote by **chTop** the category of compact Hausdorff spaces.

Definition 3.1.7. *A quasi-topological space is a structure $(X, Q(K, X)_{K \in \mathbf{chTop}})$, where for every $K, K', K_1, K_2 \in \mathbf{chTop}$ the set of functions $Q(K, X) \subseteq \mathbb{F}(K, X)$ satisfies the following conditions:*

(QT_1) $x \in X \to \overline{x} \in Q(K, X)$.
(QT_2) $f \in Q(K, X) \to g \in C(K', K) \to f \circ g \in Q(K', X)$,

$$\begin{tikzcd}
K' \arrow[r, "g"] \arrow[dr, "f \circ g \ni Q(K', X)"'] & K \arrow[d, "f \in Q(K, X)"] \\
 & X.
\end{tikzcd}$$

(QT_3) *If $g \in C(K', K)$ is a surjection, then $f \in Q(K, X) \leftrightarrow f \circ g \in Q(K', X)$.*
(QT_4) *If K is the disjoint union of K_1, K_2, then $f \in Q(K, X)$ if and only if $f_{|K_1} \in Q(K_1, X)$ and $f_{|K_2} \in Q(K_2, X)$.*

It is easy to see by QT_3 and QT_4 that if ${K_1}', {K_2}' \in \mathbf{chTop}$, $g_1 \in \mathbb{F}({K_1}', K), g_2 \in \mathbb{F}({K_2}', K)$ such that $K_1 = \mathrm{rng}(g_1), K_2 = \mathrm{rng}(g_2)$ form a partition of K, and $f \in \mathbb{F}(K, X)$, then

$$f \circ g_1 \in Q({K_1}', X) \to f \circ g_2 \in Q({K_2}', X) \to f \in Q(K, X).$$

Definition 3.1.8. *If $(X, Q(K,X)_{K\in\mathbf{chTop}})$ and $(Y, Q(K,Y)_{K\in\mathbf{chTop}})$ are quasi-topological spaces, a morphism between them is a mapping $h : X \to Y$, called a quasi-continuous function, satisfying*

$$\forall_{K\in\mathbf{chTop}}\forall_{f\in Q(K,X)}(h \circ f \in Q(K,Y)),$$

$$\begin{tikzcd}
X \arrow[r, "h"] & Y. \\
K \arrow[u, "Q(K,X) \ni f"] \arrow[ur, "h \circ f \in Q(K,Y)"'] &
\end{tikzcd}$$

There is a "symmetry" between the category of quasi-topological spaces and the category of Bishop spaces, as it is described in section 3.6. The topological structure on a quasi-topological space X is determined by abstract continuous functions to X, while on a Bishop space X this is determined by abstract continuous functions on X. As a result, a morphism between two Bishop spaces satisfies instead the corresponding closure under composition with the elements of the codomain Y.
As we saw, the definition of a Bishop space is one of the many definitions found in standard mathematics that determine the minimal properties of some abstract notion of convergence or continuity structure on some set X and share the following structure:

(i) Inclusion of the appropriate objects determined by the elements of the codomain of the functions studied (the elements of X in the case of a quasi-topological space, the elements of $\mathbb{R}$ in the case of Bishop spaces).

(ii) Inclusion of the closure under composition with an already given set of functions.

(iii) Closure under some appropriate notion of approximation.

Note that the condition BS_2 is a special closure condition due to the special algebraic structure of $\mathbb{R}$.

3.2 Examples of abstract Bishop spaces

If F_1, F_2 are topologies on X we say that F_1 is *smaller* or *coarser* than F_2, and F_2 is *larger* or *finer* than F_1, if $F_1 \subseteq F_2$.

(I) The sets $\mathrm{Const}(X)$ and $\mathbb{F}(X,\mathbb{R})$ are topologies on X. We call the former the *trivial* topology on X and the proof that $\mathrm{Const}(X)$ is a topology is similar to the proof of Proposition 3.4.8. We call $\mathbb{F}(X,\mathbb{R})$ the *discrete* topology on X. Clearly, for every topology F on some inhabited set X we have that

$$\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X).$$

Note that the converse to Proposition 3.1.3 does not hold generally; if we consider the topology $\mathrm{Const}(\mathbb{N})$ on $\mathbb{N}$ and the function $f(n) = 1$, if $n \neq 0$ and $f(0) = -1$, then $f^2 = |f| = \overline{1} \in \mathrm{Const}(\mathbb{N})$, while $f \notin \mathrm{Const}(\mathbb{N})$. Moreover, if $g(n) = -1$, if $n \neq 0$ and

$g(0) = 1$, then $f + g \in \mathrm{Const}(\mathbb{N})$, while $f, g \notin \mathrm{Const}(\mathbb{N})$. Clearly, $0f = \overline{0} \in \mathrm{Const}(\mathbb{N})$, while $f \notin \mathrm{Const}(\mathbb{N})$.

(II) If $x_1, x_2 \in X, A \subseteq X$ and $|A| \geq 2$, the following sets are topologies on X: :

$$\mathrm{Const}(A) := \{f \in \mathbb{F}(X) \mid f_{|A} \text{ is constant}\},$$

$$\mathrm{Const}(x_1, x_2) := \{f \in \mathbb{F}(X) \mid f(x_1) = f(x_2).\}$$

(III) The proof that $\mathbb{F}_b(X)$

$$\mathbb{F}_b(X) := \{f \in \mathbb{F}(X) \mid f \text{ is bounded}\}$$

is a topology is similar to the proof of Proposition 3.4.4. If $f \in \mathbb{F}_b(X)$ and the supremum $\sup\{|f(x)| \mid x \in X\}$ exists, which is not always the case constructively (e.g., take a sequence in $\{0, 1\}$ with at most one 1), its norm $||f||$ is defined by $||f|| := \sup\{|f(x)| \mid x \in X\}$. If X is finite, then $\mathbb{F}_b(X) = \mathbb{F}(X)$.

(IV) If $(X, F_1), (X, F_2)$ are Bishop spaces, then $(X, F_1 \cap F_2)$ is a Bishop space. Hence, if F is a topology on X, then $F(x_1, x_2) = F \cap \mathrm{Const}(x_1, x_2)$, $F(A) = F \cap \mathrm{Const}(A)$ and $F_b(X) = F \cap \mathbb{F}_b(X)$ are topologies on X.

(V) We get many examples of Bishop spaces through the notion of the *least* topology $\mathcal{F}(F_0)$ including some $F_0 \subseteq \mathbb{F}(X)$ that it is defined in section 3.4).

(VI) If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, their *product* $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$ is studied in section 4.1.

(VII) If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, the *pointwise exponential* Bishop space $\mathcal{F} \to \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \to G)$ is studied in section 4.3. The *limit exponential topology* $F \Rightarrow G$ on $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is larger than $F \to G$ and it is an example of an abstract topology not defined through the notion of the least topology, briefly studied in section 4.3. The path-exponential topology $F \xrightarrow{\gamma} G$ is another topology on $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ which is larger than the pointwise exponential and it is defined in section 7.1 through the notion of an F-path. The *dual* Bishop space is special case of the pointwise exponential topology, that it is defined in section 4.4.

(VIII) If $\mathcal{G} = (Y, G)$ is a Bishop space, X is an inhabited set and $\theta : X \to Y$, the *weak* topology $F(\theta)$ on X is studied in section 4.5.

(IX) If $\mathcal{F} = (X, F)$ is a Bishop space, Y is an inhabited set and $\phi : X \to Y$ is onto Y, the *quotient* topology G_ϕ on Y is studied in section 4.6.

(X) If $\mathcal{F} = (X, F)$ is a Bishop space and $Y \subseteq X$ is inhabited, the *relative* topology on Y is studied in section 4.7.

(XI) If $\mathcal{F}_i = (X, F_i)$ is a family of Bishop spaces indexed by some set I, their *supremum* $\bigvee_{i \in I} \mathcal{F}_i$ is defined in section 4.5.

(XII) If $\mathcal{G}_i = (Y, G_i)$ is a Bishop space, $e_i : X \to Y_i$, and $F(e_i)$ is the weak topology on X induced by e_i, for every $i \in I$, the *projective limit topology* $\mathrm{Lim}_I F(e_i)$ on X determined by the family $(\mathcal{G}_i, e_i)_{i \in I}$ is defined in section 4.5.

3.3 Examples of concrete Bishop spaces

(I) If X is a metric space, the set $C_p(X)$ of all weakly continuous functions of type $X \to \mathbb{R}$, as it is defined in [15], p.76, is the set of pointwise continuous ones. It is easy to see that the pair

$$\mathcal{W}(X) = (X, C_p(X))$$

is Bishop space. Bishop calls $C_p(X)$ the weak topology on X, but here we avoid this term, since we use it differently, using instead the term *pointwise* topology on X.

(II) It is easy to see that if X is a compact metric space, the set $C_u(X)$ of all uniformly continuous functions of type $X \to \mathbb{R}$ is a topology, called by Bishop the *uniform* topology on X. We denote this space by

$$\mathcal{U}(X) = (X, C_u(X)).$$

If $a, b \in \mathbb{R}$ such that $a < b$, it is immediate that $C_u([a,b]) = \mathrm{Bic}([a,b])$ and we use the notations

$$\mathcal{I}_{ab} = ([a,b], \mathrm{Bic}(a,b])),$$

$$\mathcal{I} = ([0,1], \mathrm{Bic}(0,1])).$$

Note that in [15], p.88, the uniform topology is defined on an arbitrary metric space, but, as it is correctly explained in [19], p.103, if a metric space is not compact, BS_3 may not be satisfied; e.g., if $X = \mathbb{R}$, then $\phi(x) = x^2 \in \mathrm{Bic}(\mathbb{R})$ and $\mathrm{id}_{\mathbb{R}} \in C_u(\mathbb{R})$, but $\phi \circ \mathrm{id}_{\mathbb{R}} = \phi \notin C_u(\mathbb{R})$.

(III) If (X, d) is a metric space and $x_0 \in X$, the function

$$d_{x_0} : X \to [0, +\infty),$$

$$d_{x_0}(x) := d(x, x_0),$$

for every $x \in X$, is uniformly continuous on X with modulus of continuity the identity $\mathrm{id}_{\mathbb{R}^+}$, for every $x_0 \in X$ (see [15], p.86). We define the *pointed* Bishop space $\mathcal{U}_0(X)$ on X by

$$\mathcal{U}_0(X) = (X, C_0(X)),$$

$$C_0(X) = \mathcal{F}(U_0(X)),$$

$$U_0(X) = \{d_{x_0} \mid x_0 \in X\},$$

and we call $C_0(X)$ the *pointed* topology on X. The importance of the pointed topology is stressed in section 3.8. If X is any metric space, by Remark 2.3.11 we have that

$$C_0(X) \subseteq C_u(X) \subseteq \mathrm{Bic}(X) \subseteq C_p(X),$$

although $C_u(X), \mathrm{Bic}(X)$ are not necessarily Bishop topologies on X.

(IV) If X is a locally compact metric space, then by Proposition 2.3.10 the structure $(X, \mathrm{Bic}(X))$ is a Bishop space. As a special case we get the the *Bishop space of reals*

$$\mathcal{R} = (\mathbb{R}, \mathrm{Bic}(\mathbb{R})).$$

In section 4.7 we address the question of for which subsets A of a locally compact metric space X the set $\mathrm{Bic}(A)$ is a topology on A.
(V) If $(X, \lim)$ is a limit space, a function $f : X \to \mathbb{R}$ is called lim-*continuous*, if

$$\lim(x, x_n) \to f(x_n) \stackrel{n}{\to} f(x),$$

for every $(x_n)_n \subseteq X$ and $x \in X$. It is straightforward to see that the set

$$F_{\lim} := \{f : X \to \mathbb{R} \mid f \text{ is } \lim\text{-continuous}\}$$

is a topology on X, that we call the *limit* topology on X.
(VI) Bridges showed in [19] that if X is a metric space, the set $B(X)$ of all B-continuous functions of type $X \to \mathbb{R}$ is a Bishop topology, and called the Bishop space $(X, B(X))$ *metrical*.

3.4 Inductively generated Bishop spaces

In [6] Bishop included two constructive notions of topological space, the set-based notion of neighborhood space (see section 3.7) and the function-based notion of function space. In [6], p.71, and in [15], p.77, he suggested to focus attention on Bishop spaces instead of on neighborhood spaces. The reason for that, and the main motivation behind the introduction of Bishop spaces, is that function-based concepts are more suitable to constructive study than set-based ones. Although a Bishop topology of functions F on X is a set of functions, the set-theoretic character of TBS is not as central as it seems. Helmut Schwichtenberg suggested to me the following argument against a set-theoretic "reading" of the notion of a Bishop topology: a Bishop topology can be seen as a predicate on the objects of type $X \to \mathbb{R}$ and not as a subset of $\mathbb{F}(X)$ in the strict set-theoretic sense. Even if we read though, the definition of a Bishop space in a set-theoretic way, as a standard mathematician would do, the essence of this concept is not set-theoretic. The reason is Bishop's inductive concept of the least topology found in [6], p.72, and in [15], p.78, generated by turning the definitional clauses of a Bishop space into inductive rules.

Definition 3.4.1. *The least topology $\mathcal{F}(F_0)$ generated by a set $F_0 \subseteq \mathbb{F}(X)$, called a subbase of $\mathcal{F}(F_0)$, is defined by the following inductive rules:*

$$\frac{f_0 \in F_0}{f_0 \in \mathcal{F}(F_0)}\,, \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathcal{F}(F_0)}\,, \quad \frac{f, g \in \mathcal{F}(F_0)}{f + g \in \mathcal{F}(F_0)}\,,$$

$$\frac{f \in \mathcal{F}(F_0),\ \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \mathcal{F}(F_0)}\,, \quad \frac{(g \in \mathcal{F}(F_0),\ U(g, f, \epsilon))_{\epsilon > 0}}{f \in \mathcal{F}(F_0)}\,.$$

Note that if F_0 is inhabited, then the rule of the inclusion of the constant functions is redundant to the rule of closure under composition with $\mathrm{Bic}(\mathbb{R})$. As in Definition 2.5.2 the most complex inductive rule above can be replaced by the rule

$$\frac{g_1 \in \mathcal{F}(F_0) \ \wedge \ U(g_1, f, \frac{1}{2}), \ g_2 \in \mathcal{F}(F_0) \ \wedge \ U(g_2, f, \frac{1}{2^2}), \ g_3 \in \mathcal{F}(F_0) \ \wedge \ U(g_3, f, \frac{1}{2^3}), \dots}{f \in \mathcal{F}(F_0)},$$

which has the "structure" of Brouwer's F-inference with countably many conditions in its premiss (see e.g., [64]). The above rules induce the following induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$:

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \to \\
&\forall_{a \in \mathbb{R}}(P(\overline{a})) \to \\
&\forall_{f,g \in \mathcal{F}(F_0)}(P(f) \to P(g) \to P(f+g)) \to \\
&\forall_{f \in \mathcal{F}(F_0)} \forall_{\phi \in \mathrm{Bic}(\mathbb{R})}(P(f) \to P(\phi \circ f)) \to \\
&\forall_{f \in \mathcal{F}(F_0)}(\forall_{\epsilon > 0} \exists_{g \in \mathcal{F}(F_0)}(P(g) \ \wedge \ U(g, f, \epsilon)) \to P(f)) \to \\
&\forall_{f \in \mathcal{F}(F_0)}(P(f)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. Hence, starting with a constructively acceptable subbase F_0 the generated least topology $\mathcal{F}(F_0)$ is a constructively graspable set of functions exactly because of the corresponding principle $\mathrm{Ind}_{\mathcal{F}}$. Despite the seemingly set-theoretic character of the notion of a Bishop space the core of TBS is the study of the inductively generated Bishop spaces. For example, since $\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, we get by the closure of $\mathcal{F}(\mathrm{id}_{\mathbb{R}})$ under BS_3 that

$$\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}}).$$

Moreover, most of the new Bishop spaces generated from old ones are defined through the inductive concept of least topology.

Definition 3.4.2. *A property P on $\mathbb{F}(X)$ is lifted from a subbase F_0 to the generated topology $\mathcal{F}(F_0)$, or P is $\mathcal{F}$-lifted, if*

$$\forall_{f_0 \in F_0}(P(f_0)) \to \forall_{f \in \mathcal{F}(F_0)}(P(f)).$$

Since Bishop did not pursue a constructive reconstruction of topology in [6], he didn't mention $\mathrm{Ind}_{\mathcal{F}}$, or some related lifted property. Apart from the notion of a Bishop space, Bishop introduced in [6], p.68, the inductive notion of the least algebra $\mathcal{B}(B_{0,F})$ of Borel sets generated by a given set $B_{0,F}$ of F-complemented subsets, where F is an arbitrary subset of $\mathbb{F}(X)$. Since this notion was central to the development of constructive measure theory in [6], Bishop explicitly mentioned there the corresponding induction principle $\mathrm{Ind}_{\mathcal{B}}$ and studied specific lifted properties in that setting. Brouwer's inductive definition of the countable ordinals in [28] and Bishop's inductive notion of Borel set were the main inductively defined classes of mathematical objects used in constructive mathematics which motivated the formal study of inductive definitions in the 60s and the 70s (see [31]). Since then the

use of inductive definitions in constructive mathematics and theoretical computer science became a common practice. In [14] Bishop and Cheng developed though, a reconstruction of constructive measure theory independently from the inductive definition of Borel sets, that replaced the old theory in [15].
In [7] Bishop, influenced by Gödel's Dialectica interpretation, discussed a formal system Σ that would "efficiently express" his informal system of constructive mathematics. Since the new measure theory was already conceived and the theory of Bishop spaces was not elaborated at all, Bishop found no reason to extend Σ to subsume inductive definitions. In [67] Myhill proposed instead the formal theory CST of sets and functions to codify [6]. He also took Bishop's inductive definitions at face value and showed that the existence and disjunction properties of CST persist in the extended with inductive definitions system CST*. Bishop's informal system of constructive mathematics BISH, inductive definitions included, is a system naturally connected to Martin-Löf's constructivism [62] and type theory [63].
Next we show some elementary facts on Bishop spaces with a given subbase.

Proposition 3.4.3. *Suppose that $F_0, F_1 \subseteq \mathbb{F}(X)$ and $\mathcal{F} = (X, F)$ a Bishop space.*
(i) $\mathcal{F}(F_0) \subseteq F \leftrightarrow F_0 \subseteq F$.
(ii) $F_0 \subseteq F_1 \rightarrow \mathcal{F}(F_0) \subseteq \mathcal{F}(F_1)$.
(iii) $\mathcal{F}(F_0) \cup \mathcal{F}(F_1) \subseteq \mathcal{F}(F_0 \cup F_1)$.
(iv) $\mathcal{F}(\mathcal{F}(F_0)) = \mathcal{F}(F_0)$.
(v) $\mathcal{F}(F_0 \cap F_1) \subseteq \mathcal{F}(F_0) \cap \mathcal{F}(F_1)$.
(vi) $\mathcal{F}(\emptyset) = \mathrm{Const}(X)$.

Proof. (i) The $(\rightarrow)$ direction follows trivially by $F_0 \subseteq \mathcal{F}(F_0) \subseteq F$. For the converse implication we use $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$. The case $f \in F_0$ is exactly our hypothesis. The constant functions $\overline{a}$ are by definition in F. Suppose next that $f_1, f_2 \in \mathcal{F}(F_0)$ such that $f_1, f_2 \in F$. By definition, $f_1 + f_2 \in F$. If $f \in \mathcal{F}(F_0)$ such that $f \in F$, then $\phi \circ f \in F$, for every $\phi \in \mathrm{Bic}(\mathbb{R})$. We use BS_4 on F and the inductive hypothesis to show the last clause.
(ii) By implication $(\leftarrow)$ of (i) we get $F_0 \subseteq F_1 \subseteq \mathcal{F}(F_1) \rightarrow \mathcal{F}(F_0) \subseteq \mathcal{F}(F_1)$.
(iii) We just use (ii), since $F_0, F_1 \subseteq F_0 \cup F_1$.
(iv) $\mathcal{F}(F_0) \subseteq \mathcal{F}(\mathcal{F}(F_0))$, and we get the converse inclusion by (i) and $\mathcal{F}(F_0) \subseteq \mathcal{F}(F_0)$.
(v) We just use (ii) and the trivial fact $F_0 \cap F_1 \subseteq F_0, F_1$.
(vi) We use (i) and the trivial fact $\emptyset \subseteq \mathrm{Const}(X)$. $\square$

Proposition 3.4.3(i) expresses that $\mathcal{F}(F_0)$ is the least Bishop topology including F_0. For simplicity we use the notation

$$\mathcal{F}(f_1, \ldots, f_n) := \mathcal{F}(\{f_1, \ldots, f_n\}).$$

Since $F \subseteq F$, we get by Proposition 3.4.3(i) that $F = \mathcal{F}(F)$. It is easy to find F_0, F_1 such that $\mathcal{F}(F_0) \cup \mathcal{F}(F_1) \subsetneq \mathcal{F}(F_0 \cup F_1)$. Take for example, $F_0 = \{f_0\}, F_1 = \{f_1\}$ such that $f_1 \notin \mathcal{F}(f_0)$ and $f_0 \notin \mathcal{F}(f_1)$ (we explain how to find such a pair of functions after proving Proposition 3.4.8). Then $\mathcal{F}(f_0) \cup \mathcal{F}(f_1) \subsetneq \mathcal{F}(f_0, f_1)$, since, if $f_0 + f_1 \in \mathcal{F}(f_0)$, then $f_1 \in \mathcal{F}(f_0)$, while if $f_0 + f_1 \in \mathcal{F}(f_1)$, then $f_0 \in \mathcal{F}(f_1)$. It is also easy to find F_0, F_1 such that $\mathcal{F}(F_0 \cap F_1) \subsetneq \mathcal{F}(F_0) \cap \mathcal{F}(F_1)$. E.g., $F_0 = \{f_0\}, F_1 = \{f_1\}$ such that $f_1 = f_0 + \overline{1}$ and $f_0 \notin \mathrm{Const}(X)$. Since $F_0 \cap F_1 = \emptyset$ we get that $\mathcal{F}(F_0 \cap F_1) = \mathrm{Const}(X)$, while $\mathcal{F}(F_0) \cap \mathcal{F}(F_1) = \mathcal{F}(F_0) = \mathcal{F}(F_1) \supsetneq \mathrm{Const}(X)$.

Proposition 3.4.4 ($\mathcal{F}$-lifting of boundedness)**.** *Suppose that $F_0 \subseteq \mathbb{F}(X)$, for some inhabited set X. If every $f_0 \in F_0$ is bounded, then every $f \in \mathcal{F}(F_0)$ is bounded.*

Proof. The constant functions and the elements of F_0 are bounded. If f_1, f_2 are bounded by M_1, M_2 respectively, then their sum is also bounded, since $|f_1(x)+f_2(x)| \leq |f_1(x)|+|f_2(x)| \leq M_1 + M_2$. If f is bounded and $|f(x)| \leq M$, for each x, then $-M \leq f(x) \leq M$ (see e.g., [92], p.263) and if $\phi \in \mathrm{Bic}(\mathbb{R})$ the quantities $\sup \phi, \inf \phi$ on $[-M, M]$ exist (see [15], p.38), which is more than $\phi \circ f$ being bounded. Suppose next that $U(g, f\epsilon)$ and g is bounded by some $M > 0$. If $x \in X$, then $|f(x)| \leq |f(x) - g(x)| + |g(x)| \leq \epsilon + M$. □

Definition 3.4.5. *(i) If X is an inhabited metric space and $f \in \mathbb{F}(X)$, we call f locally bounded, if it maps a bounded subset of X to a bounded subset of $\mathbb{R}$, or, equivalently, if the image of a bounded subset of X under f is included in a bounded subset of $\mathbb{R}$.*
(ii) If X is a set, we call a function $f : X \to \mathbb{R}$ totally bounded, if $f(X)$ is a totally bounded subset of $\mathbb{R}$.
(iii) If X is an inhabited metric space, we say that a function $f : X \to \mathbb{R}$ is locally totally bounded, if $f(A)$ is a totally bounded subset of $\mathbb{R}$, for every totally bounded subset A of X.

A bounded function is locally bounded, while it is routine to show that the set $\mathbb{F}_{lb}(A) = \{f \in \mathbb{F}(A) \mid f \text{ is locally bounded}\}$ is a topology on A.

Proposition 3.4.6 ($\mathcal{F}$-lifting of local boundedness)**.** *Suppose that $F_0 \subseteq \mathbb{F}(X)$, for some metric space X. If every $f_0 \in F_0$ is locally bounded, then every $f \in \mathcal{F}(F_0)$ is locally bounded.*

Proof. Clearly, the constant functions and the elements of F_0 are locally bounded. If $B \subseteq X$ is bounded and $f_1, f_2 \in \mathcal{F}(F_0)$ such that $f_1(B), f_2(B)$ are bounded subsets of $\mathbb{R}$, then $(f_1 + f_2)(B) = f_1(B) + f_2(B)$ is a bounded subset of $\mathbb{R}$ (it is easy to see that if A, B are bounded subsets of $\mathbb{R}$, then $A + B$ is a bounded subset of $\mathbb{R}$). If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f(B)$ is a bounded subset of $\mathbb{R}$, then $f(B) \subseteq K$, for some compact $K \subset \mathbb{R}$, therefore $\phi(f(B)) \subseteq \phi(K)$ is a bounded subset of $\mathbb{R}$. Suppose next that $|f(x) - g(x)| \leq \epsilon$, for every $x \in X$ and for some $g \in \mathcal{F}(F_0)$. If $M_{g,B}$ is a bound for $g(B)$, where B is bounded in X, we get that $f(B)$ is also bounded, since for every $x, y \in B$ we have that $|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \leq \epsilon + M_{g,B} + \epsilon$. □

Proposition 3.4.7 ($\mathcal{F}$-lifting of total boundedness)**.** *Suppose that $F_0 \subseteq \mathbb{F}(X)$, for some inhabited set X, and $A, B \subseteq \mathbb{R}$.*
(i) If A, B are totally bounded, then $A + B$ is totally bounded.
(ii) If f_0 is a totally bounded function, for every $f_0 \in F_0$, then f is a totally bounded function, for every $f \in \mathcal{F}(F_0)$.
(iii) If X is a metric space, and f_0 is a locally totally bounded function, for every $f_0 \in F_0$, then f is a locally totally bounded function, for every $f \in \mathcal{F}(F_0)$.

Proof. (i) Suppose that $\{a_1, \ldots, a_m\}$ is an $\frac{\epsilon}{2}$-approximation of A and $\{b_1, \ldots, b_l\}$ is an $\frac{\epsilon}{2}$-approximation of B. Then $\{a_i + b_j \mid i \in \{1, \ldots, m\}, j \in \{1, \ldots, l\}\}$ is an ϵ-approximation of $A + B$, since if $x = a + b \in A + B$, $|a - a_i| \leq \frac{\epsilon}{2}$ and $|b - b_j| \leq \frac{\epsilon}{2}$, we have that $|x - (a_i + b_j)| = |a - a_i + b - b_j| \leq |a - a_i| + |b - b_j| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
(ii) and (iii) We show only (ii) and for (iii) we work exactly as in the proof of (ii). For every $a \in \mathbb{R}$ the set $\overline{a}(X) = \{a\}$ is trivially totally bounded. If $f_1, f_2 \in \mathcal{F}(F_0)$ such that $f_1(X), f_2(X)$ are totally bounded, then $(f_1 + f_2)(X) = f_1(X) + f_2(X)$ is totally bounded by (i). If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f(X)$ is totally bounded, then $(\phi \circ f)(X) = \phi(f(X))$ is totally bounded by Proposition 4.2 in [15], p.94, and the fact that ϕ is uniformly continuous on the bounded set $f(X)$. Suppose next that $g \in \mathcal{F}(F_0)$ such that $\forall_{x \in X}(|g(x) - f(x)| \leq \frac{\epsilon}{3})$, and $\{g(x_1), \ldots, g(x_m)\}$ is an $\frac{\epsilon}{3}$-approximation of $g(X)$. Then $\{f(x_1), \ldots, f(x_m)\}$ is an ϵ-approximation of $f(X)$, since, if $|g(x) - g(x_i)| \leq \frac{\epsilon}{3}$, we have that

$$\begin{aligned}
|f(x) - f(x_i)| &= |f(x) - g(x) + g(x) - g(x_i) + g(x_i) - f(x_i)| \\
&\leq |f(x) - g(x)| + |g(x) - g(x_i)| + |g(x_i) - f(x_i)| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

□

Proposition 3.4.8 ($\mathcal{F}$-lifting of constancy). *Suppose that $F_0 \subseteq \mathbb{F}(X)$, for some X. If the restriction of every $f_0 \in F_0$ to some $A \subseteq X$ is constant, then the restriction of every $f \in \mathcal{F}(F_0)$ to A is constant.*

Proof. The case $f \in F_0$ is exactly our hypothesis, and trivially the constant functions are constant on A. It is also immediate to see that if $f_{1|A} = \overline{c_1}_{|A}$ and $f_{2|A} = \overline{c_2}_{|A}$, for some $c_1, c_2 \in \mathbb{R}$, then $(f_1 + f_2)_{|A} = \overline{(c_1 + c_2)}_{|A}$. Suppose next that $f = \phi \circ f'$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f' \in \mathcal{F}(F_0)$ such that ${f'}_{1|A} = \overline{c_1}_{|A}$. Then we have that $f(x) = \phi(f'(x)) = \phi(c_1)$, for every $x \in A$. Suppose next that $f \in \mathcal{F}(F_0)$ and $a_1, a_2 \in A$ such that $|f(a_1) - f(a_2)| = \epsilon > 0$. If $g \in \mathcal{F}(F_0)$ such that $U(g, f, \frac{\epsilon}{4})$ and $g_{|A}$ is constant, we have that

$$\begin{aligned}
|f(a_1) - f(a_2)| &\leq |f(a_1) - g(a_1)| + |g(a_1) - g(a_2)| + |g(a_2) - f(a_2)| \\
&\leq \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} \\
&= \frac{\epsilon}{2},
\end{aligned}$$

which is a contradiction. Hence, $f(a_1) = f(a_2)$. □

We can use the previous proposition to find functions f_0, f_1 such that $f_1 \notin \mathcal{F}(f_0)$ and $f_0 \notin \mathcal{F}(f_1)$. Consider a set X with at least four distinct points x_1, x_2, x_3, x_4. If $f_0(x_1) = f_0(x_2)$ and $f_0(x_3) \neq f_0(x_4)$, while $f_1(x_1) \neq f_1(x_2)$ and $f_1(x_3) = f_1(x_4)$, then $f_1 \notin \mathcal{F}(f_0)$, since then f_1 has to have the same values on x_3, x_4. Similarly we get that $f_0 \notin \mathcal{F}(f_1)$.
Next we show that the property of uniform continuity can also be lifted from a subbase to the whole space. We use this lifting property in many cases; for example, to provide a basic example of a codense Bishop space in section 7.1 and in the proof of Proposition 6.4.7.

Proposition 3.4.9 ($\mathcal{F}$-lifting of uniform continuity). *Suppose that (X, d) is a metric space and $F_0 \subseteq \mathbb{F}(X)$ such that every $f_0 \in F_0$ is bounded and uniformly continuous on X. Then every $f \in \mathcal{F}(F_0)$ is uniformly continuous on X.*

Proof. By the $\mathcal{F}$-lifting of boundedness all the elements of $\mathcal{F}(F_0)$ are bounded, so we need only to show that they are uniformly continuous on X. The case $f \in F_0$ is exactly our hypothesis, and the constant functions are uniformly continuous on X. It is immediate to see that if f_1, f_2 are uniformly continuous on X, then $f_1 + f_2$ is uniformly continuous on X. Suppose that $f = \phi \circ f'$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f' \in \mathcal{F}(F_0)$ such that f' is uniformly continuous on X with modulus of continuity $\omega_{f'}$. Since $f'(X)$ is a bounded subset of $\mathbb{R}$, we have that ϕ is uniformly continuous on $f'(X)$ with some modulus of continuity $\omega_{\phi,f'(X)}$. If $\epsilon > 0$, then for every $x_1, x_2 \in X$ we have that $d(x_1, x_2) \leq \omega_{f'}(\omega_{\phi,f'(X)}(\epsilon)) \rightarrow |f'(x_1) - f'(x_2)| \leq \omega_{\phi,f'(X)}(\epsilon)$, hence $|\phi(f'(x_1)) - \phi(f'(x_2))| \leq \epsilon$ i.e., f is uniformly continuous on X with modulus of continuity $\omega_f = \omega_{f'} \circ \omega_{\phi,f'(X)}$. Suppose next that $f \in \mathcal{F}(F_0)$ and $\forall_{\epsilon>0}\exists_{g\in\mathcal{F}(F_0)}(U(g, f, \epsilon) \wedge g$ is uniformly continuous on $X)$. If we consider such a function g for $\frac{\epsilon}{3}$, a standard $\frac{\epsilon}{3}$-argument shows that f is uniformly continuous on X with modulus of continuity $\omega_f(\epsilon) = \omega_g(\frac{\epsilon}{3})$. □

The proof of the next lifting is a simpler version of the previous proof and it is omitted.

Proposition 3.4.10 ($\mathcal{F}$-Lifting of pointwise continuity). *Suppose that (X, d) is a metric space, $Y \subseteq X$ and $F_0 \subseteq \mathbb{F}(Y)$ such that every $f_0 \in F_0$ is pointwise continuous on Y. Then every $f \in \mathcal{F}(F_0)$ is pointwise continuous on Y.*

Next we show that strong continuity is a lifted property using the Ex falso rule. This lifting is compatible to the fact that $\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}})$, where $\mathrm{id}_{\mathbb{R}}$ is trivially strongly continuous, and the fact that if X is a compact metric space with positive diameter, then $\mathcal{F}(U_0(X)) = C_u(X)$ (see Corollary 3.8.4). One can show that the elements of $U_0(X)$ are strongly continuous exactly as in the proof of their uniform continuity in [15], p.86.

Proposition 3.4.11 ($\mathcal{F}$-lifting of strong continuity). *Suppose that $\bowtie$ is a point-point apartness relation on the inhabited set X and that $\mathcal{F}(F_0))$ is a topology on X, for some $F_0 \subseteq \mathbb{F}(X)$. If every $f_0 \in F_0$ is strongly continuous, then every $f \in \mathcal{F}(F_0)$ is strongly continuous.*

Proof. If $a \in \mathbb{R}$, then $\overline{a}(x) \bowtie_{\mathbb{R}} \overline{a}(y)$ is false and the implication $\overline{a}(x) \bowtie_{\mathbb{R}} \overline{a}(y) \rightarrow x \bowtie y$ holds trivially with the use of the Ex falso rule. If $f_1(x) + f_2(x) \bowtie_{\mathbb{R}} f_1(y) + f_2(y)$, then by Proposition 2.2.7 we have that $f_1(x) \bowtie_{\mathbb{R}} f_1(y)$ or $f_2(x) \bowtie_{\mathbb{R}} f_2(y)$, and we apply the inductive hypothesis on f_1 or on f_2. If $\phi \in \mathrm{Bic}(\mathbb{R})$, then by Remark 2.3.12 we have that the hypothesis $\phi(f(x)) \bowtie_{\mathbb{R}} \phi(f(y))$ implies that $f(x) \bowtie_{\mathbb{R}} f(y)$, and our inductive hypothesis on f gives that $x \bowtie y$. Suppose next that $f \in \mathcal{F}(F_0)$ such that for every $\epsilon > 0$ there exists some $g \in \mathcal{F}(F_0)$ such that $U(g, f, \epsilon)$ and g is strongly continuous. If $f(x) \bowtie_{\mathbb{R}} f(y)$, then by Proposition 2.3.6 there exists some $g \in$ such that $g(x) \bowtie_{\mathbb{R}} g(y)$, and by the inductive hypothesis on g we get that $x \bowtie y$. □

Proposition 3.4.12 ($\mathcal{F}$-lifting of the limit relation)**.** *If $x \in X, x_n \in X^{\mathbb{N}}$ and $F_0 \subseteq \mathbb{F}(X)$, then*

$$\lim_{\mathcal{F}(F_0)} (x, x_n) \leftrightarrow \forall_{f_0 \in F_0}(f_0(x_n) \to f(x)).$$

Proof. We show only the last case of the inductive argument. If $f, g \in \mathcal{F}(F_0)$ such that $U(g, f, \frac{\epsilon}{3})$, then for every $n \geq n_{0,f}(\epsilon) := n_{0,g}(\frac{\epsilon}{3})$ we have that

$$\begin{aligned} |f(x_n) - f(x)| &\leq |f(x_n) - g(x_n)| + |g(x_n) - g(x)| + |g(x) - f(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

$\square$

3.5 Base of a Bishop space

In [67], p.377, Myhill wrote the following comment on the inductive definition of a least Bishop space generated by some given subbase:

> In Part I of this paper I maintained that in the arguments of Bishop's book transfinite inductive definitions played no essential role. One of them (Borel sets) was shown to be avoidable in Appendix D; another one was drawn to my attention by Dr. Zame of S.U.N.Y. at Buffalo. In [6], p.71, a function space is defined as a set X together with a set $F \subset X \to \mathbb{R}$ containing the constant functions and closed under $+$ and $\times$, composition with continuous functions and uniform limits. On p.73 the notion of a function-space generated by given functions $f : X \to \mathbb{R}$ is introduced, and it is used on p.74 to define the product of a sequence of function spaces. The natural way to formalize this is certainly by a transfinite inductive definition, but Dr. Zame showed us a simple trick to avoid this, namely to form the least function-space F containing the family $F_0 \subset X \to \mathbb{R}$ we first define F_1 by ordinary finite induction as the least family containing $F_0 \cup \{\text{constants}\}$ and closed under $+, \times$ and composition with continuous functions, and then define F as the set of uniform limits of sequences $g : \mathbb{N} \to F_1$. It is easy to see that F has the right closure properties.

Clearly, if F_0 is inhabited, the constant functions on X are generated by the composition of the constant functions $\mathrm{Const}(\mathbb{R})$ with F_0. We find that it is easy to see that F as defined above has the right closure properties, if the elements of $\mathrm{Bic}(\mathbb{R})$ were only uniformly continuous functions on $\mathbb{R}$. What we have managed to show is that F as defined above has the right closure properties, if the elements of F_0, therefore by the $\mathcal{F}$-lifting of boundedness the elements of $\mathcal{F}(F_0)$, are bounded. In our view, the situation in the general case is less easy than suggested by Myhill and it seems that transfinite inductive definitions do play a role in Bishop's book.

Definition 3.5.1. *If $\Phi \subseteq \mathbb{F}(X)$, for some inhabited set X, its uniform closure $\mathcal{U}(\Phi)$ is*

$$\mathcal{U}(\Phi) := \{f \in \mathbb{F}(X) \mid U(\Phi, f)\}.$$

A property P on $\mathbb{F}(X)$ is $\mathcal{U}$-lifted from Φ to its uniform closure $\mathcal{U}(\Phi)$, if

$$\forall_{\phi \in \Phi}(P(\phi)) \to \forall_{f \in \mathcal{U}(\Phi)}(P(\phi)).$$

Because of the condition BS_4, if P is an $\mathcal{F}$-lifted property, then P is a $\mathcal{U}$-lifted property. The converse is not true. For example, if we define

$$P(\phi) :\leftrightarrow |\phi(x) - \phi(y)| \geq c,$$

where $x, y \in X$ and $c > 0$, then because of the constant functions P is not $\mathcal{F}$-lifted from Φ to $\mathcal{F}(\Phi)$, but it is $\mathcal{U}$-lifted; if $f \in \mathcal{U}(\Phi)$, we consider some $\phi \in \Phi$ such that $U(\phi, f, \frac{\epsilon}{2})$. Since $c \leq |\phi(x) - \phi(y)| \leq |\phi(x) - f(x)| + |f(x) - f(y)| + |f(y) - \phi(y)|$, we get that $c - \epsilon \leq |f(x) - f(y)|$, and since $\epsilon > 0$ is arbitrary, we conclude that $c \leq |f(x) - f(y)|$.

Definition 3.5.2. *A subset Φ_0 of a topology F on X is a base of F, $\mathcal{U}(\Phi_0) = F$ i.e., if every element of F is arbitrarily close and uniformly approximated by some element of Φ_0.*

Note that the inhabitedness of F implies the inhabitedness of a base Φ_0; if $f \in F$ and $\epsilon > 0$, there exists by definition some $g \in \Phi_0$ such that $U(g, f, \epsilon)$. Since by BS_4 we have that $\mathcal{U}(\Phi_0) \subseteq F$, the set Φ_0 is a base of F if and only if $F \subseteq \mathcal{U}(\Phi_0)$. Definitions 3.5.2 and 3.4.1 are in complete analogy to the definitions of a base and subbase for a uniform structure on a set given in [44], p.217. This is not an accident, since, as we explain in section 8.3, one can associate to a Bishop space a natural notion of a uniform structure and conversely.

Definition 3.5.3. *We call a Bishop space $\mathcal{F}$ pseudo-compact, if $\mathcal{F} = \mathcal{F}_b$ i.e., if every element of F is bounded.*

Definition 3.5.4. *A pseudo-Bishop space is a pair $\mathcal{F}_0 = (X, F)$ satisfying conditions BS_1-BS_3 of the definition of a Bishop space. If $F_0 \subseteq \mathbb{F}(X)$, the least pseudo-Bishop space $\mathcal{F}_0(F_0)$ generated by F_0 is defined by the following inductive rules:*

$$\frac{f_0 \in F_0}{f_0 \in \mathcal{F}_0(F_0)}, \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathcal{F}_0(F_0)}, \quad \frac{f, g \in \mathcal{F}_0(F_0)}{f + g \in \mathcal{F}_0(F_0)}, \quad \frac{f \in \mathcal{F}_0(F_0),\ \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \mathcal{F}_0(F_0)}.$$

The above inductive rules induce the following induction principle $\mathrm{Ind}_{\mathcal{F}_0}$ on $\mathcal{F}_0(F_0)$

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \to \\
&\forall_{a \in \mathbb{R}}(P(\overline{a})) \to \\
&\forall_{f, g \in \mathcal{F}_0(F_0)}(P(f) \to P(g) \to P(f + g)) \to \\
&\forall_{f \in \mathcal{F}_0(F_0)}\forall_{\phi \in \mathrm{Bic}(\mathbb{R})}(P(f) \to P(\phi \circ f)) \to \\
&\forall_{f \in \mathcal{F}_0(F_0)}(P(f)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. The next proposition shows that there is a simple description of a base of a topology F with respect to a given subbase of F when the corresponding inductively generated Bishop space is pseudo-compact. If $\Phi \subseteq \mathrm{Bic}(\mathbb{R})$, $\Theta \subseteq \mathbb{F}(X)$ and $h \in \mathbb{F}(X)$, we use the notations

$$\Phi \circ \Theta := \{\phi \circ f \mid \phi \in \Phi, f \in \Theta\},$$

$$\Phi \circ h := \Phi \circ \{h\}.$$

Proposition 3.5.5. *If F_0 is a subset of $\mathbb{F}(X)$ such that every element of F_0 is bounded, then $\mathcal{F}_0(F_0)$ is a base of $\mathcal{F}(F_0)$.*

Proof. It is clear that $\mathcal{U}(\mathcal{F}_0(F_0)) \subseteq \mathcal{F}(F_0)$. Next we show inductively that $\mathcal{F}(F_0) \subseteq \mathcal{U}(\mathcal{F}_0(F_0))$. Clearly, $F_0 \subseteq \mathcal{F}_0(F_0)$ and $\mathrm{Const}(X) \subseteq \mathcal{F}_0(F_0)$, since $\mathrm{Const}(X) = \mathrm{Const}(\mathbb{R}) \circ F_0 \subseteq \mathrm{Bic}(\mathbb{R}) \circ F_0 \subseteq \mathcal{F}_0(F_0)$, and it is also here that we need F_0 to be inhabited. If $f_1, f_2 \in \mathcal{F}(F_0)$ such that $f_1, f_2 \in \mathcal{U}(\mathcal{F}_0(F_0))$ i.e., for every $\epsilon > 0$ there exist $g_1, g_2 \in \mathcal{F}_0(F_0)$ such that $U(g_1, f_1, \frac{\epsilon}{2})$ and $U(g_2, f_2, \frac{\epsilon}{2})$, then $U(g_1 + g_2, f_1 + f_2, \epsilon)$. Since $g_1 + g_2 \in \mathcal{F}_0(F_0)$ and $\epsilon > 0$ is arbitrary, we get that $f_1 + f_2 \in \mathcal{U}(\mathcal{F}_0(F_0))$. Suppose next that $f' = \phi \circ f$, where $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \mathcal{U}(\mathcal{F}_0(F_0))$. Since by the $\mathcal{F}$-lifting of boundedness every element of $\mathcal{F}(F_0)$ is bounded, let $M > 0$ such that $|f| \leq \overline{M}$. Without loss of generality we assume that $M > 1$. Also, there is no loss of generality, if we assume that for every bounded subset B of $\mathbb{R}$ and for every $\epsilon > 0$ the modulus $\omega_{\phi,B}(\epsilon) < 1$, since we may use the modulus $\omega^*_{\phi,B} = \omega_{\phi,B} \wedge \overline{\frac{1}{2}}$. From all these innocent assumptions we have that

$$\forall_{B \subseteq^b \mathbb{R}} \forall_{\epsilon > 0} (2M > M + 1 > M + \omega_{\phi,B}(\epsilon)).$$

We consider next the bounded subset $[-2M, 2M]$ of $\mathbb{R}$, and let $g \in \mathcal{F}_0(F_0)$ such that $U(g, f, \omega_{\phi,[-2M,2M]}(\epsilon))$ i.e., $\forall_{x \in X}(|g(x) - f(x)| \leq \omega_{\phi,[-2M,2M]}(\epsilon))$. Since

$$|g(x)| \leq |g(x) - f(x)| + |f(x)| \leq \omega_{\phi,[-2M,2M]}(\epsilon) + M < 1 + M < 2M,$$

for every $x \in X$, and since $|f| \leq \overline{M}$, we conclude that $g(x), f(x) \in [-2M, 2M]$, for every $x \in X$. Therefore, the hypothesis $U(g, f, \omega_{\phi,[-2M,2M]}(\epsilon))$ implies that $U(\phi \circ g, \phi \circ f, \epsilon)$. Since $\phi \circ g \in \mathcal{F}_0(F_0)$ and $\epsilon > 0$ is arbitrary, we get that $\phi \circ f \in \mathcal{U}(\mathcal{F}_0(F_0))$. Finally, we suppose that

$$\forall_{\epsilon > 0} \exists_{g \in \mathcal{F}(F_0)} (U(g, f, \epsilon) \wedge g \in \mathcal{U}(\mathcal{F}_0(F_0))).$$

Let $\epsilon > 0$, $g \in \mathcal{F}(F_0)$ such that $U(g, f, \frac{\epsilon}{2})$ and $h \in \mathcal{F}_0(F_0)$ such that $U(h, g, \frac{\epsilon}{2})$. Since

$$U(h, g, \frac{\epsilon}{2}) \to U(g, f, \frac{\epsilon}{2}) \to U(h, f, \epsilon),$$

$h \in \mathcal{F}_0(F_0)$ and $\epsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{U}(\mathcal{F}_0(F_0))$. $\square$

The next result shows the degree of iteration of the operator $\mathcal{U} \circ \mathcal{F}_0$ upon F_0 needed to capture the least Bishop space generated by some subbase F_0. Within the classical theory of ordinals we define the following function $\Phi : \mathrm{On} \to V$ by

$$\Phi_0 = F_0$$

$$\Phi_{\alpha+1} = \mathcal{U}(\mathcal{F}_0(\Phi_\alpha))$$
$$\Phi_\lambda = \bigcup_{\alpha<\lambda} \Phi_\alpha,$$

where λ is any limit ordinal.

Proposition 3.5.6 (CLASS). *If F_0 is a subset of $\mathbb{F}(X)$, then $\mathcal{F}(F_0) = \Phi_{\omega_1}$.*

Proof. It is clear that for every $\alpha < \omega_1$ we have that $\Phi_\alpha \subseteq \mathcal{F}(F_0)$, hence $\Phi_{\omega_1} \subseteq \mathcal{F}(F_0)$. We show inductively the converse inclusion

$$\mathcal{F}(F_0) \subseteq \bigcup_{\alpha<\omega_1} \Phi_\alpha.$$

Clearly, $F_0 = \Phi_0 \subseteq \Phi_{\omega_1}$, $\mathrm{Const}(X) = \mathrm{Const}(\mathbb{R}) \circ F_0 \subseteq \Phi_1$, and $\alpha < \beta \leq \omega_1 \to \Phi_\alpha \subseteq \Phi_\beta \subseteq \Phi_{\omega_1}$. If $f_1, f_2 \in \Phi_{\omega_1}$, there exist $\alpha_1, \alpha_2 < \omega_1$ such that $f_1 \in \Phi_{\alpha_1}$ and $f_2 \in \Phi_{\alpha_2}$. Without loss of generality we assume that $\alpha_1 \leq \alpha_2$, hence $f_1 + f_2 \in \Phi_{\alpha_2+1} \subseteq \Phi_{\omega_1}$. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \Phi_{\omega_1}$ i.e., $f \in \Phi_\alpha$, for some $\alpha < \omega_1$, then $\phi \circ f \in \Phi_{\alpha+1} \subseteq \Phi_{\omega_1}$. Suppose next that $U(g_n, f, \frac{1}{n})$, and $g_n \in \mathcal{F}(F_0) \cap \Phi_{\omega_1}$ i.e., $g_n \in \Phi_{\alpha_n}$, for some $\alpha_n < \omega_1$, for every $n \in \mathbb{N}$. If $A = \{\alpha_n \mid n \in \mathbb{N}\}$, then the ordinal $\alpha = \sup A = \bigcup_{n\in\mathbb{N}} \alpha_n$ is countable, therefore $\alpha < \omega_1$. Consequently, $f \in \Phi_{\alpha+1} \subseteq \Phi_{\omega_1}$. $\square$

By Proposition 3.5.5 we have that if $\mathcal{F}(F_0)$ is a pseudo-compact Bishop space, then $\mathcal{F}(F_0) = \mathcal{U}(\mathcal{F}_0(F_0) = \Phi_1$.

3.6 Bishop morphisms

Within the theory of Bishop spaces "continuity" is represented in an abstract, but very simple and purely function-theoretic way.

Definition 3.6.1. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a Bishop morphism, or simply a morphism, from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \to Y$ such that*

$$\forall_{g\in G}(g \circ h \in F)$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g \circ h"'] & Y \arrow[d, "g \in G"] \\ & \mathbb{R}. \end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$, or by $X \to Y$ when the topologies on X, Y are fixed, the set of morphisms from $\mathcal{F}$ to $\mathcal{G}$. The Bishop morphisms are the arrows in the category of Bishop spaces **Bis**, where $1_{\mathcal{F}} = \mathrm{id}_X$ and if $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $j \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$, where $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are given Bishop spaces, the composition $j \circ e : X \to Z$ is in $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$, since, if we fix some $h \in H$, we have for $g = h \circ j \in G$ that

$$h \circ (j \circ e) = (h \circ j) \circ e = g \circ e \in F.$$

Proposition 3.6.2. *If $\mathcal{F} = (X, F)$ is a Bishop space, then* $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$.

Proof. The condition BS_3 is rewritten as $\forall_{g \in \mathrm{Bic}(\mathbb{R})}(g \circ f \in F)$ i.e., $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{R})$, or $F \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{R})$. For the converse inclusion we fix $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{R})$. By the definition of a morphism we get that $\mathrm{id}_{\mathbb{R}} \circ h = h \in F$. □

Hence, a topology F is the set of morphisms from $\mathcal{F}$ to $\mathcal{R}$ in analogy to the fact that $C(X)$ is a set of continuous functions from X to $\mathbb{R}$. In our study the role of the topological space $(\mathbb{R}, \mathcal{T})$ in standard topology is played by the Bishop space $\mathcal{R}$. Moreover, the above simple fact in case of the topology $\mathrm{Bic}([a, b])$ on some compact interval $[a, b]$ of $\mathbb{R}$ reflects the uniform continuity of the Bishop morphisms between $\mathcal{I}_{ab}$ and $\mathcal{R}$ (see Corollary 5.4.5). This "classical" behavior is of course, due to the definition of $\mathrm{Bic}(\mathbb{R})$ and it is going to appear many times in this Thesis.
The next proposition includes some simple but fundamental facts about morphisms.

Proposition 3.6.3. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$, $\mathcal{F}_1 = (X, F_1)$, $\mathcal{F}_2 = (X, F_2)$ $\mathcal{G}_1 = (Y, G_1)$ and $\mathcal{G}_2 = (Y, G_2)$ are Bishop spaces. Then the following hold:*
(i) $\mathrm{Const}(X, Y) = \{\overline{y} \mid y \in Y\} \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$.
(ii) $G_1 \subseteq G_2 \to \mathrm{Mor}(\mathcal{F}, \mathcal{G}_2) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G}_1)$, *while* $G_1 \subsetneq G_2$ *doesn't imply that* $\mathrm{Mor}(\mathcal{F}, \mathcal{G}_2) \subsetneq \mathrm{Mor}(\mathcal{F}, \mathcal{G}_1)$. *Moreover, we have that*

$$\mathrm{Mor}(\mathcal{F}, (Y, \mathbb{F}(Y))) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F}, (Y, \mathrm{Const}(Y))).$$

(iii) $F_1 \subseteq F_2 \to \mathrm{Mor}(\mathcal{F}_1, \mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F}_2, \mathcal{G})$, *while* $F_1 \subsetneq F_2$ *doesn't imply that* $\mathrm{Mor}(\mathcal{F}_1, \mathcal{G}) \subsetneq \mathrm{Mor}(\mathcal{F}_2, \mathcal{G})$. *Moreover, we have that*

$$\mathrm{Mor}((X, \mathrm{Const}(X)), \mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \subseteq \mathrm{Mor}((X, \mathbb{F}(X)), \mathcal{G}).$$

Proof. (i) Since $g \circ \overline{y} = \overline{g(y)} \in \mathrm{Const}(X)$, we get that $g \circ \overline{y} \in F$, for every $g \in G$.
(ii) If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_2)$, then $\forall_{g_2 \in G_2}(g_2 \circ h \in F)$. Hence, $\forall_{g_1 \in G_1 \subseteq G_2}(g_1 \circ h \in F)$. The double inclusion is derived by applying this property on the double inclusion $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$. Next we observe that

$$(*) \quad \mathrm{Mor}(\mathcal{F}, (Y, \mathrm{Const}(Y))) = \mathbb{F}(X, Y),$$

since $\overline{a} \circ h \in \mathrm{Const}(X)$, for every $a \in \mathbb{R}$ and $h \in \mathbb{F}(X, Y)$. On the other hand, $h \in \mathrm{Mor}(\mathcal{F}, (Y, \mathbb{F}(Y))) \leftrightarrow \forall_{g \in \mathbb{F}(Y)}(g \circ h \in F)$. If $F = \mathbb{F}(X)$, then $\mathrm{Mor}((X, \mathbb{F}(X)), (Y, \mathbb{F}(Y))) = \mathbb{F}(X, Y)$, and we get the required equality for $G_1 = \mathrm{Const}(Y)$, $G_2 = \mathbb{F}(Y, \mathbb{R})$ and $\mathcal{F} = (X, \mathbb{F}(X))$. If we consider $F = \mathrm{Const}(X)$, then we take classically that

$$\mathrm{Mor}((X, \mathrm{Const}(X)), (Y, \mathbb{F}(Y))) = \mathrm{Const}(X, Y),$$

since, if a morphism $h \notin \mathrm{Const}(X, Y)$, then there exist (classically) $x_1, x_2 \in X$ such that $h(x_1) = y_1 \neq y_2 = h(x_2)$. If we take some $g \in \mathbb{F}(Y)$ such that $g(y_1) = 0$ and $g(y_2) = 1$, then $g \circ h \notin \mathrm{Const}(X)$. We give a constructive example of $\mathrm{Mor}((X, \mathrm{Const}(X)), \mathcal{G}) = \mathrm{Const}(X, Y)$ in Proposition 5.7.5.

(iii) If $h \in \mathrm{Mor}(\mathcal{F}_1, \mathcal{G})$, then $\forall_{g \in G}(g \circ h \in F_1 \subseteq F_2)$ i.e., $h \in \mathrm{Mor}(\mathcal{F}_2, \mathcal{G})$. The double inclusion is derived by applying this property on the double inclusion $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$. Next we observe that $\mathrm{Mor}((X, \mathbb{F}(X)), \mathcal{G}) = \mathbb{F}(X, Y)$, since $g \circ h \in \mathbb{F}(X)$, for each $g \in G$ and $h \in \mathbb{F}(X, Y)$. On the other hand, $h \in \mathrm{Mor}((X, \mathrm{Const}(X)), \mathcal{G}) \leftrightarrow \forall_{g \in G}(g \circ h \in \mathrm{Const}(X))$. If $G = \mathrm{Const}(Y)$, then by $(*)$ we have that $\mathrm{Mor}((X, \mathrm{Const}(X)), (Y, \mathrm{Const}(Y))) = \mathbb{F}(X, Y)$, and we get the required equality for $F_1 = \mathrm{Const}(X)$, $F_2 = \mathbb{F}(X)$ and $\mathcal{G} = (Y, \mathrm{Const}(X))$. $\square$

If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then h is lim-continuous i.e., $\lim_F(x, x_n) \to \lim_G(h(x), h(x_n))$, for every $x \in X$ and $(x_n)_n \in X^{\mathbb{N}}$, since $\lim_G(h(x), h(x_n)) \leftrightarrow \forall_{g \in G}(g(h(x_n)) \to g(h(x))) \leftrightarrow \forall_{g \in G}((g \circ h)(x_n) \to (g \circ h)(x))$, which is true by the hypothesis and the fact that $g \circ h \in F$. At first sight it seems that in order to show that a function $h : X \to Y$ is a morphism from (X, F) to $(Y, \mathcal{F}(G_0))$ we need to use $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(G_0)$. The $\mathcal{F}$-lifting of morphisms is a fundamental fact which shows that the situation is simpler.

Proposition 3.6.4 ($\mathcal{F}$-lifting of morphisms)**.** *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G}_0 = (Y, \mathcal{F}(G_0))$ are Bishop spaces. A function $h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_0)$ if and only if*

$$\forall_{g_0 \in G_0}(g_0 \circ h \in F)$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g_0 \circ h"'] & Y \arrow[d, "g_0 \in G_0"] \\ & \mathbb{R}. \end{tikzcd}$$

Proof. It suffices to show that $\forall_{g_0 \in G_0}(g_0 \circ h \in F) \to \forall_{g \in \mathcal{F}(G_0)}(g \circ h \in F)$. The case $g \in G_0$ is exactly our hypothesis. If $g = \overline{a}$, for some $a \in \mathbb{R}$, then $\overline{a} \circ h$ is the constant function a on X, which by BS_1 it is in F. If $g_1, g_2 \in \mathcal{F}(G_0)$ such that $g_1 \circ h, g_2 \circ h \in F$, and since $(g_1 + g_2) \circ h = (g_1 \circ h) + (g_2 \circ h)$, we get that $(g_1 + g_2) \circ h \in F$. If $g = \phi \circ g'$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$ and some $g' \in \mathcal{F}(G_0)$ such that $g' \circ h \in F$, we conclude that $g \circ h = (\phi \circ g') \circ h = \phi \circ (g' \circ h) \in F$. Next we fix $g \in \mathcal{F}(G_0)$ such that $\forall_{\epsilon > 0} \exists_{g' \in \mathcal{F}(G_0)}(g' \circ h \in F \ \wedge \ \forall_{y \in Y}(|g(y) - g'(y)| \leq \epsilon))$. We show that $g \circ h \in F$ using condition BS_4 of F. If $\epsilon > 0$, and since $|(g \circ h)(x) - (g' \circ h)(x)| \leq \epsilon$, for every $x \in X$, we conclude that $g \circ h \in F$. $\square$

Consequently, $h \in \mathrm{Mor}(\mathcal{F}, (Y, \mathcal{F}(g)))$ if and only if $g \circ h \in F$. Next we follow some standard categorical definitions[1].

Definition 3.6.5. *Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. We call h a monomorphism, or $h \in \mathrm{Mono}(\mathcal{F}, \mathcal{G})$, if*

$$\forall_{\mathcal{H}}(\forall_{e, j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})}(h \circ e = h \circ j \to e = j))$$

$$\begin{tikzcd} Z \arrow[r, shift left, "{e, j}"] \arrow[r, shift right] & X \arrow[r, "h"] & Y. \end{tikzcd}$$

[1]For all categorical concepts mentioned in this Thesis we refer to [1].

We call h an isomorphism between $\mathcal{F}$ and $\mathcal{G}$, if there is some $e \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ such that $e \circ h = \mathrm{id}_X$ and $h \circ e = \mathrm{id}_Y$. An isomorphism between $\mathcal{F}$ and $\mathcal{F}$ is called an automorphism of $\mathcal{F}$. We call h an epimorphism, if

$$\forall_{\mathcal{H}}(\forall_{e,j \in \mathrm{Mor}(\mathcal{G},\mathcal{H})}(e \circ h = j \circ h \to e = j))$$

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, "h"] \arrow[urr, "e \circ h"] \arrow[drr, "j \circ h"'] & Y \arrow[ur, "e"] \arrow[dr, "j"] & \\
 & & Z.
\end{tikzcd}$$

We denote by $\mathrm{Epi}(\mathcal{F}, \mathcal{G})$ *the set of the epimorphisms between the Bishop spaces $\mathcal{F}, \mathcal{G}$. We call h a set-epimorphism, if it is onto Y. We denote by* $\mathrm{setEpi}(\mathcal{F}, \mathcal{G})$ *the set of set-epimorphisms between the Bishop spaces $\mathcal{F}, \mathcal{G}$.*

As expected, $h \in \mathrm{Mono}(\mathcal{F}, \mathcal{G}) \leftrightarrow h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and h is $1{-}1$; it is trivial that if h is $1{-}1$, then $h \in \mathrm{Mono}(\mathcal{F}, \mathcal{G})$, while if $h \in \mathrm{Mono}(\mathcal{F}, \mathcal{G})$ and $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$, then for the functions $e = \overline{x_1}, j = \overline{x_2} \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ we get that $h \circ \overline{x_1} = h \circ \overline{x_2}$, hence $\overline{x_1} = \overline{x_2}$ i.e., $x_1 = x_2$. A morphism $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is an isomorphism if and only h is a monomorphism onto Y and $h^{-1} \in \mathrm{Mor}(\mathcal{G}.\mathcal{F})$ (see [15], p.17). Clearly, if h is an isomorphism, its inverse h^{-1} is also one.

Proposition 3.6.6. *Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that h is* 1-1 *and onto Y, then $h^{-1} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ if and only if $\forall_{f \in F} \exists_{g \in G}(f = g \circ h)$.*

Proof. If $h^{-1} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, then $\forall_{f \in F}(f \circ h^{-1} \in G)$. If $f \in F$, then we define $g = f \circ h^{-1}$, and we get that $f = g \circ h$. For the converse we have that $f \circ h^{-1} = (g \circ h) \circ h^{-1} = g$. $\square$

Clearly, $\mathrm{setEpi}(\mathcal{F}, \mathcal{G}) \subseteq \mathrm{Epi}(\mathcal{F}, \mathcal{G})$. We can show classically though, that it is impossible to have an epimorphism in **Bis** which is not a set-epimorphism; Suppose that there are Bishop spaces $\mathcal{G}' = (Y', G'), \mathcal{G} = (Y, G)$ and a function $i : Y' \to Y \in \mathrm{Epi}(\mathcal{G}', \mathcal{G})$ which is not onto Y, therefore there exists some $y_0 \in Y$ such that $y_0 \notin i(Y')$. Let X be a set containing at least two points and $\mathcal{F} = (X, \mathrm{Const}(X))$. If $e \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, then we define $i : Y \to X$ as follows: if $y \in i(Y')$, then $i(y) = e(y)$, and $i(y_0) \neq e(y_0)$ (this can trivially be done, since X contains at least two different elements). Since $\mathrm{Mor}(\mathcal{G}, (X, \mathrm{Const}(X))) = \mathbb{F}(Y, X)$ we get that $i \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$. By the definition of i we have that $e \circ i = j \circ i$, and at the same time $e \neq j$. If we restrict to the full subcategory **crBis** of the completely regular Bishop spaces, there are epimorphisms that are not set-epimorphisms (see Corollary 5.7.9). Because of these facts we find it appropriate to keep the distinction between epimorphisms and set-epimorphisms.
Due to Proposition 3.6.6 we give the following definition.

Definition 3.6.7. *We call a morphism $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ open, if*

$$\forall_{f \in F} \exists_{g \in G}(f = g \circ h).$$

Note that an isomorphism is trivially an open morphism, while there are open morphisms which are not isomorphisms (see e.g., the projection functions of Proposition 4.6.4). Next we prove inductively the *lifting of openness*, a fundamental fact that we use in the proof of Theorem 5.8.6 and it is crucial in proving that some concrete Bishop spaces with a given subbase are isomorphic (see e.g., Propositions 4.1.8, 4.1.11, 6.2.8 and 6.3.2). First we show a necessary well-definability lemma.

Lemma 3.6.8 (Well-definability lemma)**.** *Suppose that X, Y are inhabited sets, $h : X \to Y$ is onto Y, $\Theta \subseteq \mathbb{F}(Y)$ and $f : X \to \mathbb{R}$. If $f \in \mathcal{U}(\Theta \circ h)$, then the function*

$$f^{\#} : Y \to \mathbb{R},$$
$$f^{\#}(y) = f^{\#}(h(x)) := f(x),$$

for every $y \in Y$, is well-defined i.e.,

$$\forall_{x_1, x_2 \in X}(h(x_1) = h(x_2) \to f(x_1) = f(x_2)).$$

Proof. We fix $x_1, x_2 \in X$ such that $h(x_1) = h(x_2) = y_0$, and some $\epsilon > 0$. By our hypothesis on f there exists some $g : Y \to \mathbb{R} \in \Theta$ such that $\forall_{x \in X}(|(g \circ h)(x) - f(x)| \leq \frac{\epsilon}{2})$. Hence, $|g(h(x_1)) - f(x_1)| = |g(y_0) - f(x_1)| \leq \frac{\epsilon}{2}$ and $|g(h(x_2)) - f(x_2)| = |g(y_0) - f(x_2)| \leq \frac{\epsilon}{2}$. Consequently, $|f(x_1) - f(x_2)| \leq |f(x_1) - g(y_0)| + |g(y_0) - f(x_2)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary, we get that $|f(x_1) - f(x_2)| \leq 0$, which implies that $f(x_1) = f(x_2)$. □

Proposition 3.6.9 ($\mathcal{U}$-lifting of openness)**.** *Suppose that X, Y are inhabited sets, $h : X \to Y$ is onto Y, $\Phi \subseteq \mathbb{F}(X)$ and $\Theta \subseteq \mathbb{F}(Y)$. Then*

$$\forall_{\phi_0 \in \Phi} \exists_{\theta_0 \in \Theta}(\phi_0 = \theta_0 \circ h) \to \forall_{\phi \in \mathcal{U}(\Phi)} \exists_{\theta \in \mathcal{U}(\Theta)}(\phi = \theta \circ h).$$

Proof. If $\phi \in \mathcal{U}(\Phi)$, then $U(\phi_0, \phi, \epsilon) \leftrightarrow U(\theta_0 \circ h, \phi, \epsilon)$, for some $\theta_0 \in \Theta$, and for every $\epsilon > 0$. By the well-definability lemma we get that $\phi = \phi^{\#} \circ h$, and since h is onto Y the relation $U(\theta_0 \circ h, \phi^{\#} \circ h, \epsilon)$ on X implies $U(\theta_0, \phi^{\#}, \epsilon)$. Since this is the case for every $\epsilon > 0$, we get that $\phi^{\#} \in \mathcal{U}(\Theta)$. □

Proposition 3.6.10 ($\mathcal{F}$-lifting of openness)**.** *If $\mathcal{F} = (X, \mathcal{F}(F_0))$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$, then*

$$\forall_{f_0 \in F_0} \exists_{g \in G}(f_0 = g \circ h) \to \forall_{f \in \mathcal{F}(F_0)} \exists_{g \in G}(f = g \circ h).$$

Proof. If $f = f_0 \in F_0$, then we just use our premiss. Clearly, a constant function $\overline{a} : X \to \mathbb{R}$ is written as the composition $\overline{a} \circ h$, where we use the same notation for the constant function of type $Y \to \mathbb{R}$ with value a. If $f = f_1 + f_2$ such that $f_1 = g_1 \circ h$ and $f_2 = g_2 \circ h$, for some $g_1, g_2 \in G$, then $f = (g_1 + g_2) \circ h$, where $g_1 + g_2 \in G$ by BS_2. If $f = \phi \circ f'$, where $\phi \in \mathrm{Bic}(\mathbb{R})$, and there is some $g \in G$ such that $f' = g \circ h$, then $f = (\phi \circ g) \circ h$, where $\phi \circ g \in G$ by BS_3. Suppose next that $\epsilon > 0$ and $f' \in \mathcal{F}(F_0)$ such that $f' = g \circ h$, for some $g \in G$, and $\forall_{x \in X}(|f'(x) - f(x)| = |g(h(x)) - f(x)| \leq \epsilon)$. By the $\mathcal{U}$-lifting of openness we get that $f \in \mathcal{U}(G)$, where by the condition BS_4 we have that $\mathcal{U}(G) = G$. □

The above lifting does not require h to be a morphism, only a function onto Y. In most applications of the lifting of openness though, h is already a morphism. The next fact is used in the proof of the existence of an isomorphism between a completely regular Bishop space and its fixed ideals (Proposition 5.9.7).

Proposition 3.6.11. *If $\mathcal{F} = (X, F)$ is a Bishop space, and $e : X \to Y$ is a bijection, there is a unique topology G_F on Y such that e is an isomorphism between $\mathcal{F}$ and $\mathcal{G}_{\mathcal{F}} = (Y, G_F)$.*

Proof. We define $G_F = \{e_f \mid f \in F\}$, where for every $y \in Y$ we have that $e_f(y) = e_f(e(x)) = f(x)$. Clearly, $e_{\overline{a}}$ is the constant function $\overline{a}$ on Y, and it is easy to see that $e_{f_1} + e_{f_2} = e_{f_1+f_2}$ and that $\phi \circ e_f = e_{\phi \circ f}$, where $\phi \in \mathrm{Bic}(\mathbb{R})$. If $e_{f_n} \stackrel{u}{\to} g$, for some $g : Y \to \mathbb{R}$, we define $f = g \circ e$, and we get that $e_f = g$, since $e_f(y) = f(x) = g(y)$, for each $y \in Y$. Next we show that $f \in F$, hence $g \in G_F$; since $e_{f_n} \stackrel{u}{\to} g$ we have that for some $\epsilon > 0$ there is some n_0 such that $\forall_{n \geq n_0} \forall_{y \in Y}(|e_{f_n}(y) - e_f(y)| \leq \epsilon) \leftrightarrow \forall_{n \geq n_0} \forall_{x \in X}(|f_n(x) - f(x)| \leq \epsilon)$ i.e., $f_n \stackrel{u}{\to} f$ i.e., $f \in F$. Since $\forall_{f \in F}(e_f \circ e = f \in F)$, we get that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_{\mathcal{F}})$. Since $\forall_{f \in F} \exists_{e_f \in G_F}(f = e_f \circ e)$ we get that e is an open morphism, and consequently e is an isomorphism between $\mathcal{F}$ and $\mathcal{G}_{\mathcal{F}}$. Suppose next that G is a topology on Y such that e is an isomorphism between $\mathcal{F}$ and (Y, G). Since e is open with respect to G we have that $\forall_{f \in F} \exists_{g \in G}(f = g \circ e)$ i.e., if we fix some $f \in F$, we have that $f = g \circ e = e_f \circ e \to g = e_f \in G$ i.e., $G_F \subseteq G$. Since $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, if we fix some $g \in G$, then $g \circ e = f \in F$, and since $f = e_f \circ e$, we get that $g = e_f$ i.e., $G \subseteq G_F$. $\square$

The next definition follows the basic theory of $C(X)$.

Definition 3.6.12. *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the induced mapping $h^* : G \to F$ by h is defined by $g \mapsto h^*(g)$, where*

$$h^*(g) := g \circ h.$$

In the next proposition we show that h^* is an algebra and lattice homomorphism.

Proposition 3.6.13. *Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Then the following hold:*
(i) $h^*(\overline{a}) = \overline{a}$, and consequently $h^*(\mathrm{Const}(Y)) = \mathrm{Const}(X)$.
(ii) $h^*(g_1 + g_2) = h^*(g_1) + h^*(g_2)$.
(iii) $h^*(\phi \circ g) = \phi \circ h^*(g)$, for every $\phi \in \mathrm{Bic}(\mathbb{R})$.
(iv) If $g_n \stackrel{u}{\to} g$, then $h^*(g_n) \stackrel{u}{\to} h^*(g)$.
(v) $h^*(|g|) = |h^*(g)|$.
(vi) $h^*(f \vee g) = h^*(f) \vee h^*(g)$.
(vii) $h^*(f \wedge g) = h^*(f) \wedge h^*(g)$.
(viii) $h^*(f \cdot g) = h^*(f) \cdot h^*(g)$.

Proof. We only show (v) and (vi). If $x \in X$, then $(|g| \circ h)(x) = |g|(h(x)) = |g(h(x))| =$

$|(g \circ h)(x)| = |g \circ h|(x)$. Moreover,

$$\begin{aligned}
h^*(f \vee g) &= h^*(\frac{1}{2}(f + g + |f - g|)) \\
&= \frac{1}{2}h^*(f + g + |f - g|) \\
&= \frac{1}{2}(h^*(f) + h^*(g) + h^*(|f - g|)) \\
&= \frac{1}{2}(h^*(f) + h^*(g) + |h^*(f - g)|) \\
&= \frac{1}{2}(h^*(f) + h^*(g) + |h^*(f) - h^*(g)|) \\
&= h^*(f) \vee h^*(g).
\end{aligned}$$

□

Proposition 3.6.14. *Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$.*
(i) If h is open and onto Y, then $h^*(G)$ is a Bishop topology. Precisely, $h^*(G) = F$.
(ii) If h is 1–1 and onto Y, then h^* is 1–1.
(iii) If h is an isomorphism, then h^* is onto F.
(iv) If $g \in G$ is bounded, then $h^*(g)$ is bounded.
(v) If $g \in G$ and $g_{|h(A)} \in \mathrm{Const}(h(A))$, for some $A \subseteq X$, then $(h^*(g))_{|A} \in \mathrm{Const}(A)$.

Proof. (i) The first three clauses of the definition of a Bishop space are derived by Proposition 3.6.13(i), (ii) and (iii), respectively. If $h^*(g_n) \stackrel{u}{\to} f$, for some $f \in \mathbb{F}(X)$, then by BS_4 we get that $f \in F$, therefore $f = g \circ h$, for some $g \in G$. Since h is onto Y, we get the implication $g_n \circ h \stackrel{u}{\to} g \circ h \to g_n \stackrel{u}{\to} g$.
(ii) $h^*(g) = h^*(g') \leftrightarrow g \circ h = g' \circ h \to (g \circ h) \circ h^{-1} = (g' \circ h) \circ h^{-1} \leftrightarrow g = g'$.
(iii) Directly from Proposition 3.6.6.
(iv) and (v) Trivially by the definition of h^*. □

Proposition 3.6.15. *Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is an isomorphism. Then the following hold:*
(i) $h^*(G_b) = F_b$.
(ii) $h^*(G(h(A))) = F(A)$.

Proof. (i) By Proposition 3.6.14(iv) we have directly that $h^*(G_b) \subseteq F_b$. Suppose next that $f \in F_b$. By Proposition 3.6.14(iii) there exists some $g \in G$ such that $f = h^*(g) = g \circ h$ i.e., $g = f \circ h^{-1}$. If f is bounded by some $M > 0$, then $|g(y)| = |(f \circ h^{-1})(h(x))| = |f(x)| \leq M$.
(ii) The fact that $h^*(G(h(A))) \subseteq F(A)$ follows easily from Proposition 3.6.14(v). If $f \in F$ such that $f_{|A} = \overline{a}_{|A}$, for some $a \in \mathbb{R}$, then $f = h^*(g) = g \circ h$, for some $g \in G$ for which $g = f \circ h^{-1}$. Then $g(h(x)) = (f \circ h^{-1})(h(x)) = f(x) = a$, for each $x \in A$ i.e., $g \in G(h(A))$. □

Next we see that the existence of a subbase is preserved under isomorphisms.

Proposition 3.6.16. *If h is an isomorphism from (X, F) to $(Y, \mathcal{F}(G_0)$, then $F = \mathcal{F}(h^*(G_0))$.*

Proof. If we define $F_0 := h^*(G_0) \subseteq F$, it is immediate that $\mathcal{F}(F_0) \subseteq F$. For the converse inclusion it suffices to show that $\forall_{g \in \mathcal{F}(G_0)}(g \circ h \in \mathcal{F}(F_0))$, since every $f \in F$ is written as $g \circ h$, for some $g \in \mathcal{F}(G_0)$. If $g \in G_0$, then $g \circ h \in F_0 \subseteq \mathcal{F}(F_0)$. The cases of the constant function, addition and composition are trivial. If $g_n \stackrel{u}{\to} g$ such that $g_n \circ h \in \mathcal{F}(F_0)$, then $g \circ h \in \mathcal{F}(F_0)$, since $g_n \circ h \stackrel{u}{\to} g \circ h$. $\square$

Proposition 3.6.17. *Suppose that $\mathcal{F} = (X, \mathcal{F}(f))$, where f is 1−1 and onto $\mathbb{R}$. Then f is an isomorphism between $\mathcal{F}$ and $\mathcal{R}$.*

Proof. It suffices to show that $\forall_{f' \in \mathcal{F}(f)} \exists_{\phi \in \mathrm{Bic}(\mathbb{R})}(f' = \phi \circ f)$. We only show the last inductive step. Suppose that ${f'}_n \stackrel{u}{\to} f'$ such that there exists ϕ_n so that ${f'}_n = \phi_n \circ f$, therefore $\phi_n = {f'}_n \circ f^{-1}$. Since ${f'}_n \circ f^{-1} \stackrel{u}{\to} f' \circ f^{-1} = \phi$, we get that $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f' = \phi \circ f$. $\square$

Thus, Bishop spaces of the form $(X, \mathcal{F}(f))$, where f is 1−1 and onto $\mathbb{R}$, are just copies of $\mathcal{R}$. This result corresponds to the classical fact that an 1−1 continuous function $\mathbb{R} \to \mathbb{R}$ and onto $\mathbb{R}$ is a homeomorphism. The next proposition holds also for open mappings in the standard topological sense, except that the case (iii) requires that e_1 is also 1−1.

Proposition 3.6.18. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ are Bishop spaces and $e_1 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}), e_2 \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$. Then the following hold:*
(i) e_1, e_2 are open $\to e_2 \circ e_1$ is open.
(ii) $e_2 \circ e_1$ is open $\to e_1$ onto $Y \to e_2$ is open.
(iii) $e_2 \circ e_1$ is open $\to e_1$ is open.

Proof. (i) We need to show that $\forall_{f \in F} \exists_{h \in H}(f = h \circ (e_2 \circ e_1))$. We fix some $f \in F$. We denote (O_1) and (O_2) the openness hypothesis of e_1 and e_2, respectively. Then, there exist $g \in G$ and $h \in H$ such that $f \stackrel{(O_1)}{=} g \circ e_1 \stackrel{(O_2)}{=} (h \circ e_2) \circ e_1 = h \circ (e_2 \circ e_1)$.
(ii) We fix some $g \in G$. Since $e_1 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, we have that $g \circ e_1 \in F$, and since $e_2 \circ e_1$ is open, there exists some $h \in H$ such that $g \circ e_1 = h \circ (e_2 \circ e_1)$. Since e_1 is onto Y we get that for each $y \in Y$ $g(y) = g(e_1(x)) = (h \circ e_2)(e_1(x)) = (h \circ e_2)(y) \to g = h \circ e_2$.
(iii) We fix some $f \in F$. Since $e_2 \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$, we have that $h \circ e_2 \in G$, for each $h \in H$. Using the openness hypothesis of $e_2 \circ e_1$ there is $h \in H$ and $g = h \circ e_2 \in G$ such that $f = h \circ (e_2 \circ e_1) = (h \circ e_2) \circ e_1 = g \circ e_1$. $\square$

Proposition 3.6.19. *(i) If X, Y are sets and $f : X \to Y, g : Y \to X$ such that $g \circ f = \mathrm{id}_X$, then f is 1−1 and g is onto X.*
(ii) If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}), h' \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ such that $h' \circ h = \mathrm{id}_X$ and $h \circ h' = \mathrm{id}_Y$, then h is an isomorphism and $h' = h^{-1}$.

Proof. (i) If $x, x' \in X$, then $f(x) = f(x') \to g(f(x)) = g(f(x')) \leftrightarrow \mathrm{id}_X(x) = \mathrm{id}_X(x') \leftrightarrow x = x'$. Since $g(f(x)) = x$, we get that g is onto X.
(ii) By (i) h is 1−1 and onto Y, while h' is 1−1 and onto X. It suffices then to show that $h' = h^{-1}$ in order to conclude that h is an isomorphism; $h'(y) = h'(h(h^{-1}(y))) = h'h(h^{-1}(y)) = \mathrm{id}_X(h^{-1}(y)) = h^{-1}(y)$, for every $y \in Y$. $\square$

The next proposition is "inspired" by the fact that if X, Y are topological spaces, then $f : X \to Y$ is continuous if and only if $\forall_{A \subseteq X}(f(\overline{A}) \subseteq \overline{f(A)})$, or, if and only if $\forall_{B \subseteq Y}(\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}))$.

Proposition 3.6.20. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces, $h \in$ setEpi$(\mathcal{F}, \mathcal{G})$, $\Theta \subseteq G$ and $\Phi \subseteq F$.*
(i) $h^*(\mathcal{U}(\Theta)) \subseteq \mathcal{U}(h^*(\Theta))$.
(ii) If h is open and Θ is a base of G, then $h^*(\Theta)$ is a base of F.
(iii) $\mathcal{U}(h^{*-1}(\Phi)) \subseteq h^{*-1}(\mathcal{U}(\Phi))$.
(iv) If h is open and Φ is a base of F, then $h^{*-1}(\Phi)$ is a base of G.

Proof. (i) We fix some $g \in \mathcal{U}(\Theta)$ i.e., $\forall_{\epsilon>0}\exists_{\theta\in\Theta}(U(\theta, g, \epsilon))$, and let $h^*(g) = g \circ h$. By definition $h^*(\Theta) = \{\theta \circ h \mid \theta \in \Theta\}$. Since h is onto Y we have that $U(\theta, g, \epsilon) \to U(\theta \circ h, g \circ h, \epsilon)$, for every $\epsilon > 0$, therefore $h^*(g) \in \mathcal{U}(h^*(\Theta))$.
(ii) If h is open, then h^* is onto F, and by (i) we have that $F = h^*(G) = h^*(\mathcal{U}(\Theta)) \subseteq \mathcal{U}(h^*(\Theta)) \subseteq F$, therefore $\mathcal{U}(h^*(\Theta)) = F$.
(iii) Let $g \in \mathcal{U}(h^{*-1}(\Phi)) \leftrightarrow \forall_{\epsilon>0}\exists_{g'\in h^{*-1}(\Phi)}(U(g', g, \epsilon))$, therefore $\forall_{\epsilon}\exists_{g'\circ h\in\Phi}(U(g' \circ h, g \circ h, \epsilon))$. Hence, $g \circ h \in \mathcal{U}(\Phi)$ i.e., $g \in h^{*-1}(\mathcal{U}(\Phi))$.
(iv) Let $g \in G$. Since $g \circ h \in F$ we have that $\forall_{\epsilon>0}\exists_{\phi\in\Phi}(U(\phi, g \circ h, \epsilon))$. Since h is open we have that every ϕ in the previous formula can be written as $g_\phi \circ h$, where $g_\phi \in G$. Since then $\forall_{\epsilon>0}\exists_{g_\phi\in h^{*-1}(\Phi)}(U(g_\phi \circ h, g \circ h, \epsilon)) \leftrightarrow \forall_{\epsilon>0}\exists_{g_\phi\in h^{*-1}(\Phi)}(U(g_\phi, g, \epsilon))$, we get that $g \in \mathcal{U}(h^{*-1}(\Phi))$. $\square$

If $(X, F), (Y, G)$ are Bishop spaces, $h \in$ setEpi and Φ is a base of F such that $\forall_{\phi\in\Phi}\exists_{g\in G}(\phi = g \circ h)$, then by the $\mathcal{U}$-lifting of openness we get that h is open and by Proposition 3.6.20(iv) we get that the set $h^{*-1}(\Phi) = \{g \in G \mid g \circ h \in \Phi\}$ is a base of G.

3.7 The Neighborhood structure of a Bishop space

The notion of a neighborhood space is a set-theoretic constructive analogue to the classical notion of a topological space, introduced by Bishop in [6], p.69. In classical terms, a neighborhood space is a set with a base for a topology $\mathcal{T}_{\mathcal{N}}$ on X.

Definition 3.7.1. *A neighborhood space is a pair $\mathcal{N} = (X, N)$, where X is an inhabited set and N, a neighborhood structure on X, is a set of subsets of X satisfying:*
(NS_1) $\forall_{x\in X}\exists_{U\in N}(x \in U)$.
(NS_2) $\forall_{U,V\in N}\forall_{x\in X}(x \in U \cap V \to \exists_{W\in N}(x \in W \subseteq U \cap V))$.

Definition 3.7.2. *The canonical neighborhood space induced by a Bishop space $\mathcal{F} = (X, F)$ is the structure $\mathcal{N}(\mathcal{F}) = (X, N(F))$, where $N(F)$, the canonical neighborhood structure induced by F, is defined by*

$$N(F) := \{U(f) \mid f \in F\}$$

$$U(f) := [f > \overline{0}] = \{x \in X \mid f(x) > 0\}.$$

The proof that $N(F)$ is a neighborhood structure on X is based on the following facts:

$$a > 0 \to U(\overline{a}) = \mathbb{R},$$

$$U(f) \cap U(g) = U(f \wedge g),$$

where the last equality is due to Proposition 2.2.3(iv). The neighborhood structure $N(F)$ is the natural "space-structure" defined a posteriori on X by the topology F. A characteristic of the theory of neighborhood spaces is that the topological notions connected with it are defined positively i.e., negation is avoided as much as possible, in order to avoid the complications that result from the behavior of negation in intuitionistic logic. For example, a closed subset X in some neighborhood structure N on X is not defined as the complement of an open set in N. The following notions are defined in [15], p.75.

Definition 3.7.3. *If N is a neighborhood structure on some X, then $\mathcal{O} \subseteq X$ is open in N, a subset Y of X is closed in N and the closure of Y in N are defined, respectively, by*

$$\forall_{x \in \mathcal{O}} \exists_{U \in N} (x \in U \subseteq \mathcal{O}),$$

$$\forall_{x \in X} (\forall_{U \in N} (x \in U \to U \between Y) \to x \in Y),$$

$$\overline{Y} := \{x \in X \mid \forall_{U \in N} (x \in U \to U \between Y)\}.$$

Consequently, the openness of some $\mathcal{O} \subseteq X$, the closeness and the closure of some $Y \subseteq X$ in $N(F)$ are given by

$$\forall_{x \in \mathcal{O}} \exists_{f \in F} (U(f) \subseteq \mathcal{O}),$$

$$\forall_{x \in X} (\forall_{f \in F} (f(x) > 0 \to U(f) \between Y) \to x \in Y),$$

$$\overline{Y} := \{x \in X \mid \forall_{f \in F} (f(x) > 0 \to U(f) \between Y)\}.$$

We call the family $\mathcal{T}_{N(F)}$ of all open sets in $N(F)$ the *canonical topology* on X induced by F, and we denote by $C(N(F))$ the family of closed sets in $N(F)$. Clearly, $C(N(F))$ is closed under arbitrary intersections. We also denote by $C_p(N(F))$ the family of continuous functions in the classical sense from a Bishop space $\mathcal{F}$ to the *real neighborhood space* $(\mathbb{R}, N(\mathrm{Bic}(\mathbb{R})))$, and by $C_p(N(F), N(G))$ the family of continuous of type $X \to Y$ which are continuous in the classical sense with respect to the topologies induced by F and G, respectively.

Next we use the term open and closed mapping in the standard topological sense.

Proposition 3.7.4. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h : X \to Y$, then the following hold:*

(i) $\mathcal{O}$ is open in $N(\mathrm{Bic}(\mathbb{R}))$ if and only if it is open in the standard topology on $\mathbb{R}$.

(ii) The induced topology $\mathcal{T}_{N(F)}$ is the smallest topology $\mathcal{T}$ of opens on X such that $f \in C(X)$, for every $f \in F$.

(iii) If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $h \in C_p(N(F), N(G))$ and the inverse image of a closed set in $N(G)$ under h is closed in $N(F)$.

(iv) If h is an open set-epimorphism, then h is an open and closed mapping.

(v) The set-theoretic complement $U(f)^c$ of $U(f)$ is closed in $N(F)$, for every $f \in F$.

Proof. (i) If (a,b) is a standard basic open set, then $(a,b) = (a,+\infty) \cap (-\infty,b) = U(\mathrm{id}_{\mathbb{R}} - \overline{a}) \cap U(\overline{b} - \mathrm{id}_{\mathbb{R}}) = U((\mathrm{id}_{\mathbb{R}} - \overline{a}) \wedge (\overline{b} - \mathrm{id}_{\mathbb{R}})) \in N(F)$. If $U(\phi) \in N(\mathrm{Bic}(\mathbb{R})))$, then $U(\phi) = \phi^{-1}(0,+\infty)$, which is a open in the standard topology on $\mathbb{R}$, since by Remark 2.3.11 ϕ is pointwise continuous.

(ii) If $f \in F$ and $a < b \in \mathbb{R}$, then $f^{-1}(a,b) \in \mathcal{T}_{N(F)}$, since $f^{-1}(a,b) = f^{-1}(a,+\infty) \cap f^{-1}(-\infty,b) = U(f - \overline{a}) \cap U(-f + \overline{b}) = U((f - \overline{a}) \wedge (\overline{b} - f))$. If every $f \in F$ is continuous with respect to some topology $\mathcal{T}$ on X, then $f^{-1}(0,+\infty) \in \mathcal{T}$ and consequently $\mathcal{T}_{N(F)} \subseteq \mathcal{T}$.

(iii) If $\mathcal{O}$ is open in G and $h(x) = y \in U(g) \subseteq \mathcal{O}$, then $x \in U(g \circ h) \subseteq h^{-1}(\mathcal{O})$, since

$$\begin{aligned} h^{-1}(U(g)) &= \{x \in X \mid h(x) \in U(g)\} \\ &= \{x \in X \mid g(h(x)) > 0\} \\ &= U(g \circ h) \in N(F), \end{aligned}$$

Next we show that if $B \in C(N(G))$, then $h^{-1}(B) \in C(N(F))$. Our premiss amounts to $\forall_{y \in Y}(\forall_{g \in G}(g(y) > 0 \to U(g) \between B) \to y \in B)$ and we show that $\forall_{x \in X}(\forall_{f \in F}(f(x) > 0 \to U(f) \between h^{-1}(B)) \to x \in h^{-1}(B))$ as follows. We fix $x \in X$ and we suppose that $\forall_{f \in F}(f(x) > 0 \to \exists_{x' \in X}(f(x') > 0 \wedge h(x') \in B))$. We show that $\forall_{g \in G}(g(h(x)) > 0 \to \exists_{y' \in Y}(g(y') > 0 \wedge y' \in B)$. We fix $g \in G$ such that $g(h(x)) > 0 \leftrightarrow (g \circ h)(x) > 0$. By our hypothesis on x and on $g \circ h \in F$ we get the existence of some $x' \in X$ such that $g(h(x')) > 0$ and $y' = h(x') \in B$. Since B is closed in $N(G)$, we get that $y = h(x) \in B \leftrightarrow x \in h^{-1}(B)$. Since $x \in X$ is arbitrary, we conclude that $h^{-1}(B)$ is closed in $N(F)$.

(iv) We fix $f \in F$ and $g \in G$ such that $f = g \circ h$. The openness of h follows by

$$\begin{aligned} h(U(f)) &= \{h(x) \in Y \mid f(x) > 0\} \\ &= \{h(x) \in Y \mid g(h(x)) > 0\} \\ &= \{y \in Y \mid g(y) > 0\} \\ &= U(g) \in N(G), \end{aligned}$$

and then the image of an open set in $N(F)$ under h is trivially open in $N(G)$. Suppose next that A is closed in $N(F)$ i.e., $\forall_{x \in X}(\forall_{f \in F}(f(x) > 0 \to U(f) \between A) \to x \in A)$. We show that $\forall_{y \in Y}(\forall_{g \in G}(g(y) > 0 \to U(g) \between h(A)) \to y \in h(A))$. We fix some $y \in Y$, we suppose $\forall_{g \in G}(g(y) > 0 \to U(g) \between h(A))$ and we show that $y \in h(A)$. We fix some $x \in X$ such that $h(x) = y$, and we show that $\forall_{f \in F}(f(x) > 0 \to U(f) \between A)$. We fix some $f \in F$ such that $f(x) > 0$, and let $g \in G$ such that $f = g \circ h$. Since $g(h(x)) = g(y) > 0$, we get that $U(g) \between h(A)$ i.e., there exists some $y' = h(x')$, for some $x' \in A$ such that $g(h(x')) > 0$, therefore $f(x') > 0$ and $x' \in A$ i.e., $U(f) \between A$. By our initial supposition we get that $x \in A$, hence $h(x) = y \in h(A)$.

(v) If $f \in F$, then by definition we have that

$$\begin{aligned} U(f)^c &= \{x \in X \mid x \notin U(f)\} \\ &= \{x \in X \mid \neg(f(x) > 0)\} \\ &= \{x \in X \mid f(x) \leq 0\} \\ &= [f \leq \overline{0}]. \end{aligned}$$

We show that $\forall_{x\in X}(\forall_{f\in F}(f(x) > 0 \to U(f) \between [f \leq \overline{0}]) \to x \in [f \leq \overline{0}])$. We fix $x \in X$, we suppose that $\forall_{h\in F}(h(x) > 0 \to \exists_{z\in X}(h(z) > 0 \wedge f(z) \leq 0))$ and we show that $f(x) \leq 0$. Suppose that $f(x) > 0$. By the above hypothesis on f itself we get that $\exists_{z\in X}(f(z) > 0 \wedge f(z) \leq 0)$, which is absurd. Hence, $\neg(f(x) > 0)$, which implies that $f(x) \leq 0$. $\square$

If $\mathcal{N} = (X, N)$ and $\mathcal{M} = (Y, M)$ are neighborhood spaces, the continuous functions $C_p(N, M)$ in the standard sense are the arrows in the category of neighborhood spaces **Nbh**. If $Y = \mathbb{R}$ and $M = N(\mathrm{Bic}(\mathbb{R}))$ we denote the continuous functions $C_p(N, M)$ by $C_p(N)$.

Proposition 3.7.5. *The following mappings are covariant functors*

$$\nu : \mathbf{Bis} \to \mathbf{Nbh},$$

$$(X, F) \mapsto (X, N(F)),$$

$$h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \mapsto h \in C_p(N(F), N(G)),$$

$$\varpi : \mathbf{Nbh} \to \mathbf{Bis},$$

$$(X, N) \mapsto (X, C_p(N)),$$

$$h \in C_p(N, M) \mapsto h \in \mathrm{Mor}(\varpi(\mathcal{N}), \varpi(\mathcal{M})).$$

Proof. By our previous analysis and Proposition 3.7.4(iii) the mapping ν is well-defined, while it is trivial to see that $\nu(\mathrm{id}_X) = \mathrm{id}_{\nu(X)}$ and $\nu(h \circ j) = \nu(h) \circ \nu(j)$. Also, it is easy to see that if N is a neighborhood structure on X, then $C_p(N)$ is a Bishop topology on X and that $h \in \mathrm{Mor}((X, C_p(N)), (Y, C_p(M)))$, since the composition $g \circ h$ of the continuous functions $g \in C_p(Y)$ and h is a function in $C_p(N)$. The equalities $\varpi(\mathrm{id}_X) = \mathrm{id}_{\varpi(X)}$ and $\varpi(h \circ j) = \varpi(h) \circ \varpi(j)$ are trivially satisfied. $\square$

By Proposition 3.7.4(ii) we have that $F \subseteq C_p(N(F))$. The constructions $F \rightsquigarrow N(F) \rightsquigarrow C_p(N(F))$ can be such that $F \subsetneq C_p(N(F))$; consider the Bishop space $(\mathbb{N}, \mathcal{F}(F_0))$, where $F_0 = \{f_n \mid n \in \mathbb{N}\}$ and

$$f_n(m) = \begin{cases} 1 & , \text{if } m = n \\ 0 & , \text{ow}, \end{cases}$$

for every $n \in \mathbb{N}$. Clearly, $U(f_n) = \{n\}$ i.e., the induced topology $\mathcal{T}_{\mathcal{N}(F)}$ is the discrete topology on $\mathbb{N}$. Hence, $C_p(N(F)) = \mathbb{F}(\mathbb{N})$, while $\mathcal{F}(F_0) \subseteq \mathbb{F}_b(\mathbb{N}) \subsetneq \mathbb{F}(\mathbb{N})$, by Proposition 3.4.4 and the fact that every f_n is bounded.

Recall that a topological space is complete regular, if any point outside a given closed set and the closed set are separated by some element of $C(X)$. Next we show classically that the canonical topology of a Bishop space is always completely regular. A consequence of this fact is that it is not interesting to define a completely regular Bishop space as one the canonical topology of which is completely regular, since this is always the case. As we show in section 5.7 a completely regular Bishop space is defined differently. What is important in defining such a notion of a Bishop space is not how the induced topological spaces behave within **Top**, but how these Bishop spaces behave within **Bis**.

Proposition 3.7.6 (CLASS)**.** *If $\mathcal{F} = (X, F)$ is a Bishop space, the canonical topology on X induced by F is completely regular. Moreover, the separation of a point outside a closed set and the closed set is realized by some element of F.*

Proof. We fix $x \in X$ and $Y \subseteq X$ closed in $N(F)$ such that $x \notin Y$. Contraposing the implication $\forall_{f\in F}(f(x) > 0 \to U(f) \between Y) \to x \in Y$ in the definition of a closed set we get classically that there exists some $f \in F$ such that $f(x) > 0$ and $U(f) \cap Y = \emptyset$ i.e., $\forall_{y\in Y}(f(y) \leq 0)$ which implies that $\forall_{y\in Y}((f \vee \overline{0})(y) = 0)$. We also get $(f \vee \overline{0})(x) = f(x) > 0$ i.e., $f \vee \overline{0} \in F$ and it separates x from Y. $\square$

By the previous result, if we start from a neighborhood structure N which is not completely regular, then $N \neq N(C_p(N))$, since $N(C_p(N))$ is completely regular.

3.8 Metric spaces as Bishop spaces

In this section we study the morphisms between metric spaces seen as Bishop spaces. Although some results of this section are known, it is their interpretation within the theory of Bishop spaces which is of interest to us. We mainly revisit the work of Bridges on related issues providing slightly stronger formulations of his results.
Already in [6] Bishop proved constructively the Stone-Weierstrass theorem for compact metric spaces. In [15], p.105 and p.106, we find the following definitions.

Definition 3.8.1. *The least set $\mathcal{A}(\Phi)$ of real-valued functions on X including Φ and closed under addition, multiplication by reals and multiplication is the set*

$$\mathcal{A}(\Phi) := \{p \circ \vec{f} \mid p \in \mathrm{Pol}^*(N, n),\ \vec{f} = (f_1, \ldots, f_n), f_j \in \Phi, 1 \leq j \leq n, N, n \in \mathbb{N}, \},$$

$$\mathrm{Pol}^*(N, n) := \{p \in \mathrm{Pol}(N, n) \mid p(\vec{0}) = 0\},$$

$$\mathrm{Pol}(N, n) := \{p : \mathbb{R}^n \to \mathbb{R} \mid p(\vec{x}) = \Sigma_{0\leq i_1+\ldots+i_n\leq N} a_{i_1,\ldots i_n} x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}\},$$

where $\mathrm{Pol}(N, n)$ *are the polynomials of degree N in n variables and* $\mathrm{Pol}^*(N, n)$ *is the set of strict polynomials. A subset Φ of $C_u(X)$ is called separating, if*

$$\exists_{\delta:\mathbb{R}^+\to\mathbb{R}^+}\{[\forall_{\epsilon>0}\forall_{x,y\in X}(\rho(x, y) \geq \epsilon \to$$

$$\exists_{f\in\Phi}(\forall_{z\in X}(\rho(x, z) \leq \delta(\epsilon) \to |f(z)| \leq \epsilon) \wedge$$

$$\wedge\ \forall_{z\in X}(\rho(y, z) \leq \delta(\epsilon) \to |f(z) - 1| \leq \epsilon))]$$

$$\wedge\ \forall_{\epsilon>0}\forall_{y\in X}\exists_{f\in\Phi}(\forall_{z\in X}(\rho(y, z) \leq \delta(\epsilon) \to |f(z) - 1| \leq \epsilon)\}.$$

Note that Bishop used an explicit rather than an inductive definition of $\mathcal{A}(\Phi)$, while his concept of a separating subset of $C_u(X)$ is highly technical.

Theorem 3.8.2 (Stone-Weierstrass theorem for compact metric space (SWM))**.** *If G is a separating set of $C_u(X)$ on a compact metric space X, then $\mathcal{A}(G)$ is dense in $C_u(X)$.*

Through SWM Bishop proved the following important corollary, which shows the central role of $U_0(X) = \{d_{x_0} \mid x_0 \in X\}$ as a subset of $C_u(X)$. If $A \subseteq C_u(X)$, then $\overline{A}$ denotes the closure of A within $C_u(X)$, and it is also called the *uniform closure* of A.

Corollary 3.8.3. *If X is a compact metric space with positive diameter, then $\mathcal{A}(U_0(X))$ is dense in $C_u(X)$ i.e., $\overline{\mathcal{A}(U_0(X))} = C_u(X)$.*

The previous corollary implies the following two facts within Bishop spaces. Condition BS_4 is crucial for the first.

Corollary 3.8.4. *If X is a compact metric space with positive diameter, then $\mathcal{F}(U_0(X)) = C_u(X)$.*

Proof. By the $\mathcal{F}$-lifting of uniform continuity for the bounded functions of $U_0(X)$ we get that $\mathcal{F}(U_0(X)) \subseteq C_u(X)$. For the converse inclusion and by the previous corollary it suffices to show that $\overline{\mathcal{A}(U_0(X))} \subseteq \mathcal{F}(U_0(X))$. Clearly, $U_0(X) \subseteq \mathcal{F}(U_0(X))$ and $\mathcal{A}(U_0(X)) \subseteq \mathcal{F}(U_0(X))$, since $\mathcal{F}(U_0(X))$ is a Bishop space, therefore it is closed under addition and multiplication by Proposition 3.1.3, while it is closed under multiplication by reals by BS_3. We get that $\overline{\mathcal{A}(U_0(X))} \subseteq \mathcal{F}(U_0(X))$ by BS_4. $\square$

Corollary 3.8.5. *If Y is a compact metric space with positive diameter and $\mathcal{F} = (X, F)$ is a Bishop space, then $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{U}(Y))$ if and only if $\forall_{g \in U_0(Y)}(g \circ h \in F)$.*

Proof. Directly from the previous corollary and the $\mathcal{F}$-lifting of morphisms. $\square$

A proof of Proposition 3.8.6 is found in [19], pp.102-103, although our formulation is a bit stronger and our proof of the less trivial implication is simpler than the already simple proof of Bridges. The elegant constructive proof of Lemma 3.8.7 is found in the proof of the backward uniform continuity theorem in [20], p.32. Another proof of it can be found in [17], pp.179-180. Proposition 3.8.10 appears in [17] and as an exercise in [15], p.123 and in [20], p.46. We include all proofs here because we want to emphasize the refinement we introduce on the premiss of these propositions. Namely, it suffices to suppose that uniform continuity is preserved under composition with the elements of $U_0(X)$, rather than of the whole $C_u(Y)$. This is in accordance with the importance of $U_0(X)$ mentioned above. If X, Y are metric spaces, we denote by $C_p(X, Y)$ the set of the pointwise continuous functions from X to Y.

Proposition 3.8.6. *If X, Y are metric spaces and $h : X \to Y$, then the following are equivalent:*
(i) $h \in \mathrm{Mor}(\mathcal{W}(X), \mathcal{W}(Y))$.
(ii) $\forall_{g \in U_0(Y)}(g \circ h \in C_p(X))$.
(iii) $h \in C_p(X, Y)$.

Proof. (i) $\to$ (ii) By definition $h \in \mathrm{Mor}(\mathcal{W}(X), \mathcal{W}(Y)) \leftrightarrow \forall_{g \in C_p(Y)}(g \circ h \in C_p(X))$, and (ii) follows, since $U_0(Y) \subseteq C_p(X)$.
(ii) $\to$ (iii) We fix some $x_0 \in X$ and we show that $d(x, x_0) \leq \delta_{h,x_0}(\epsilon) \to d(h(x), h(x_0)) \leq \epsilon$, for every $x \in X$. By hypothesis we have that $d_{h(x_0)} \circ h \in C_p(X)$, therefore for each

$x \in X$ we get that $d(x, x_0) \leq \delta_{d_{h(x_0)} \circ h}(\epsilon) \rightarrow |(d_{h(x_0)} \circ h)(x) - (d_{h(x_0)} \circ h)(x_0)| \leq \epsilon$, and since $|(d_{h(x_0)} \circ h)(x) - (d_{h(x_0)} \circ h)(x_0)| = |d(h(x_0), h(x)) - d(h(x_0), h(x_0))| = |d(h(x_0), h(x))| = d(h(x_0), h(x))$, we conclude that $\delta_{h,x_0}(\epsilon) = \delta_{d_{h(x_0)} \circ h}(\epsilon)$.
(iii) $\rightarrow$ (i) The implication $h \in C_p(X) \rightarrow \forall_{g \in C_p(Y)}(g \circ h \in C_p(X))$ is trivial. □

Lemma 3.8.7. *Suppose that X is a metric space and Y is a totally bounded metric space. If $h : X \rightarrow Y$ is such that $\forall_{g \in U_0(Y)}(g \circ h \in C_u(X))$, then h is a uniformly continuous function with modulus of continuity given by*

$$\omega(\epsilon) := \min\{\omega_{d_{y_1} \circ h}(\frac{\epsilon}{3}), \ldots, \omega_{d_{y_m} \circ h}(\frac{\epsilon}{3})\},$$

where $\{y_1, \ldots, y_m\}$ is an $\frac{\epsilon}{3}$-approximation of Y.

Proof. We show that $\forall_{x_1, x_2 \in X}(d(x_1, x_2) \leq \omega_h(\epsilon) \rightarrow d(h(x_1), h(x_2)) \leq \epsilon)$. If $\{y_1, \ldots, y_m\}$ is an $\frac{\epsilon}{3}$-approximation of Y, and since $d_{y_i} \circ h \in C_u(X)$, for every $i \in \{1, \ldots, m\}$, we have that

$$d(x_1, x_2) \leq \omega_{d_{y_i} \circ h}(\frac{\epsilon}{3}) \rightarrow |d_{y_i}(h(x_1)) - d_{y_i}(h(x_2))| = |d(y_i, h(x_1)) - d(y_i, h(x_2))| \leq \frac{\epsilon}{3},$$

for each $x_1, x_2 \in X$. It suffices to show that for each $z_1, z_2 \in Y$

$$\forall_i(|d_{y_i}(z_1) - d_{y_i}(z_2)| \leq \frac{\epsilon}{3}) \rightarrow d(z_1, z_2) \leq \epsilon.$$

Consider $j \in \{1, \ldots, m\}$ such that $d(z_1, y_j) = d_{y_j}(z_1) \leq \frac{\epsilon}{3}$. Since by hypothesis $|d_{y_j}(z_1) - d_{y_j}(z_2)| \leq \frac{\epsilon}{3}$, we conclude that $d(z_2, y_j) \leq \frac{2\epsilon}{3}$; if $a, b > 0$ such that $|a - b| \leq \frac{\epsilon}{3}$ and $a \leq \frac{\epsilon}{3}$, then $b \leq \frac{2\epsilon}{3}$: if $b > \frac{2\epsilon}{3}$, then $|a - b| = b - a > \frac{2\epsilon}{3} - a \geq \frac{2\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}$, which leads to a contradiction[2]. Since $\neg(b > \frac{2\epsilon}{3}) \rightarrow b \leq \frac{2\epsilon}{3}$, we are done. Finally we get that $d(z_1, z_2) \leq d(z_1, y_j) + d(y_j, z_2) \leq \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$. □

Theorem 3.8.8 (Backward uniform continuity theorem (BUCT))**.** *Suppose that X is a metric space, Y is a compact metric space, and $h : X \rightarrow Y$. If F is a topology on X such that $F \supseteq C_u(X)$, then the following are equivalent:*
(i) $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{U}(Y))$ such that $\forall_{g \in U_0(Y)}(g \circ h \in C_u(X))$.
(ii) h is uniformly continuous.

Proof. (i) $\rightarrow$ (ii) Since a compact metric space is locally compact, we use the previous lemma to get the uniform continuity of h.
(ii) $\rightarrow$ (i) It is trivial that $\forall_{g \in C_u(Y)}(g \circ h \in C_u(X) \subseteq F)$. □

We use BUCT in the proof of Proposition 7.3.5. The next corollary is mentioned also in [19] (as a corollary of Proposition 16). Here we derive it from the previous theorem.

Corollary 3.8.9. *If X and Y are compact metric spaces, then $h : X \rightarrow Y \in \mathrm{Mor}(\mathcal{U}(X), \mathcal{U}(Y))$ if and only if h is uniformly continuous.*

[2]Clearly, $x > a \wedge a \geq x \rightarrow \perp$, since $x > x \rightarrow 0 \in \mathbb{R}^+$ i.e., there is some n such that $0 > \frac{1}{n}$.

Proof. Trivially from BUCT, since the topology on X is exactly $C_u(X)$. $\square$

The proof-idea of Lemma 3.8.7 is found already in [17] in the proof of the part (ii) of the next proposition. Because of the more complex character of the proof of BUCT in [17], the proof of part (iii) of the next proposition makes use of the constructive version of the Tietze extension theorem given by Bishop in [15], p.120. The use of the above proof of Lemma 3.8.7 reveals the role of $U_0(X)$ and simplifies that proof which avoids the use of Tietze theorem; it is trivial that every element of $U_0(K)$ is extended to an element of $U_0(X)$, if $K \subseteq X$.

Proposition 3.8.10. *Suppose that X is a compact metric space, Y is a metric space, and $h : X \to Y$ such that $\forall_{g \in U_0(Y)}(g \circ h \in C_u(X))$. Then the following hold:*
(i) h is pointwise continuous and $h(X)$ is bounded.
(ii) h is uniformly continuous if and only if $h(X)$ is totally bounded.
(iii) If Y is locally compact, then h is uniformly continuous.

Proof. (i) We fix some $x_0 \in X$ and we show that $\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x\in X}(d(x,x_0) \leq \delta \to d(h(x),h(x_0)) \leq \epsilon)$. We fix $\epsilon > 0$ and by the uniform continuity of $d_{h(x_0)} \circ h$ on X we have that

$$d(x,x') \leq \omega_{d_{h(x_0)}\circ h}(\epsilon) \to |d_{h(x_0)}(h(x)) - d_{h(x_0)}(h(x'))| \leq \epsilon,$$

for every $x, x' \in X$. If $x' = x_0$ we get that

$$d(x,x_0) \leq \omega_{d_{h(x_0)}\circ h}(\epsilon) \to |d_{h(x_0)}(h(x)) - d_{h(x_0)}(h(x_0))| \leq \epsilon.$$

Since $|d_{h(x_0)}(h(x)) - d_{h(x_0)}(h(x_0))| = |d(h(x_0),h(x)) - d(h(x_0),h(x_0))| = |d(h(x_0),h(x))| = d(h(x_0),h(x))$, we conclude that the required δ is $\omega_{d_{h(x_0)}\circ h}(\epsilon)$.
To show that $h(X)$ is bounded we fix $x_1, x_2 \in X$ and let x_0 inhabit X. Since $d_{h(x_0)} \circ h$ is uniformly continuous on the totally bounded X, there is a bound M of $d_{h(x_0)}(h(X))$ and

$$\begin{aligned} d(h(x_1),h(x_2)) &\leq d(h(x_1),h(x_0)) + d(h(x_0),h(x_2)) \\ &= d_{h(x_0)}(h(x_1)) + d_{h(x_0)}(h(x_2)) \\ &\leq M + M \\ &= 2M. \end{aligned}$$

(ii) If h is uniformly continuous, then $h(X)$ is totally bounded, since X is totally bounded. Therefore, the hypothesis $\forall_{g\in U_0(Y)}(g \circ h \in C_u(X))$ implies that $\forall_{g\in U_0(h(X))}(g \circ h \in C_u(X))$ and by Lemma 3.8.7 on $h : X \to h(X)$ we get that h is uniformly continuous on X.
(iii) By (i) we know that $h(X)$ is a bounded subset of Y, therefore it is included in some compact $K \subseteq Y$. By Lemma 3.8.7 it suffices to show that $\forall_{k\in K}(d_k \circ h \in C_u(X))$. Each function $d_k : K \to [0,+\infty)$ has the obvious uniform continuous extension $d_{k,Y} : Y \to [0,+\infty)$, where $d_{k,Y}(y) = d(k,y)$, and for which we have by hypothesis that $d_{k,Y} \circ h = d_k \circ h \in C_u(X)$. $\square$

Note that by Proposition 3.8.6 if $h \in \mathrm{Mor}(\mathcal{U}(Y), F(Y))$, where $F(Y)$ is a topology on Y such that $U_0(Y) \subseteq F(Y)$, we get that $h \in C_p(X, Y)$. Hence, the pointwise continuity of h in part (i) of the previous proposition is proved also by Proposition 3.8.6, and the compactness hypothesis is not relevant. Actually, our above proof of the pointwise continuity of h is the same as the proof of Proposition 3.8.6.
It is trivial that if X is a compact metric space, Y is a metric space and $h : X \to Y$ is uniformly continuous, then $\forall_{g \in C_u(Y)}(g \circ h \in C_u(X))$. It is an open question if the converse, which is known as the forward uniform continuity theorem (FUCT), is true within BISH. It is true in CLASS and in INT, since in both these varieties of constructive mathematics UCT holds (by Proposition 3.8.10 (i) we get that h is pointwise continuous and then UCT applies). In [17] Bridges considered FUCT as a "possible constructive substitute for UCT", which in its general form says that a pointwise continuous mapping of a compact Hausdorff space into a uniform space is uniformly continuous. Bridges also suggested that "it is unlikely that [FUCT] will prove to be essentially non-constructive". He based his suggestion on Proposition 3.8.10(iii) and in the intuitionistic validity of FUCT under the hypothesis that Y is separable (see [94]; 13.1.6). Much later in [19] Bridges proved that the so-called antithesis of Specker's theorem (AS) implies FUCT over BISH[3]. Actually, the proof of Bridges shows that we can again weaken our premiss in the expected way. What is required in Bridges's proof is that

$$f_k(y) = \min\{d_{h(a_i)}(y) \mid \in \{1, \ldots, k\}\}$$

is uniformly continuous, and then $f_k \circ h$ is uniformly continuous (actually B-continuous). But it is clear that

$$(f \wedge g) \circ h = (f \circ h) \wedge (g \circ h),$$

since $(f \wedge g)(h(x)) = \min\{f(h(x)), g(h(x))\} = [(f \circ h) \wedge (g \circ h)](x)$, for every $x \in X$. Thus, it is only the uniform continuity of $g \circ h$, where $g \in U_0(Y)$, that it is actually used in the proof. In complete accordance with all the previous results we write FUCT as follows.

Theorem 3.8.11 (Forward uniform continuity theorem (FUCT), (AS)). *Suppose that X is a compact metric space, Y is a metric space and $h : X \to Y$. If $\forall_{g \in U_0(Y)}(g \circ h \in C_u(X))$, then h is uniformly continuous.*

Hence, if h is a morphism between a compact metric space endowed with the uniform topology and a metric space endowed with a topology larger than the pointed one such that $\forall_{g \in U_0(Y)}(g \circ h \in C_u(X))$, then h is uniformly continuous. I.e., in this case the abstract notion of Bishop morphism captures the expected property of uniform continuity.
Using AS Bridges also showed in [19] over BISH that a morphism between metrical Bishop

[3] In RUSS Specker's theorem asserts the existence of an increasing sequence of rational numbers in [0, 1] that is eventually bounded away from every point of [0, 1], while in CLASS and INT the antithesis of Specker's theorem asserts that every sequence in a metric space X that is eventually bounded away from every point of a compact subset K is eventually bounded away from K. As it is noted by Bridges, AS is classically equivalent to the Bolzano-Weierstrass property for K, and in [3] it was shown that if $X = \mathbb{R}$ and $K = [0, 1]$, then AS is constructively equivalent to Brouwer's fan theorem for c-bars.

spaces is B-continuous, hence again the notion of Bishop morphism captures B-continuity, which, according to Proposition 2.4.5(i), is identical to uniform continuity on compact subsets.
Since $\mathbb{Z}$ is a locally compact subset of $\mathbb{R}$, then by Proposition 4.7.15(iii) we have that $\mathrm{Bic}(\mathbb{Z})$ is a topology on $\mathbb{Z}$. In relation to Proposition 3.8.10 (iii) we can note that if $e : [a, b] \to \mathbb{Z}$, then $e \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{Z})$ if and only if e is constant; by Proposition 6.2.1 we have that $\mathbb{F}(\mathbb{Z}, \mathbb{R}) = \mathrm{Bic}(\mathbb{Z}) = \mathcal{F}(\mathrm{id}_{\mathbb{Z}})$. By Proposition 2.3.3 we get that if $e : [a, b] \to \mathbb{Z}$, then $e \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{Z}) \leftrightarrow \mathrm{id}_{\mathbb{Z}} \circ e \in \mathrm{Bic}([a, b]) \leftrightarrow e \in \mathrm{Bic}([a, b]) \leftrightarrow e$ is uniformly continuous on $[a, b]$, which is equivalent to e being constant.
In this Thesis we use FUCT in the proof of Proposition 7.4.6.

Chapter 4

New Bishop spaces from old

In this chapter we study the product Bishop topology, the weak and the relative topology, and we introduce the pointwise exponential topology, which corresponds to the classical topology of the pointwise convergence, the dual of a Bishop space, as a special case of the pointwise exponential topology, and the quotient Bishop spaces. Moreover, we show a theorem of Stone-Weierstrass type for pseudo-compact Bishop spaces.

4.1 Product Bishop spaces

The product of Bishop spaces was defined already in [6], p.73. Here we show that the defined product satisfies the universal property of a product, and we show some of its properties necessary to the rest of this Thesis. The fact that the product of Bishop spaces is given with respect to a certain subbase is essential to all related proofs. If π_1, π_2 are the projection functions on $X \times Y$ to X and Y, respectively, we use the notations

$$F \circ \pi_1 := \{f \circ \pi_1 \mid f \in F\}, \quad G \circ \pi_2 := \{g \circ \pi_2 \mid g \in G\}.$$

Definition 4.1.1. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, their product is the pair $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where*

$$\begin{aligned} F \times G &:= \mathcal{F}(\{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}) \\ &= \{f \circ \pi_1 \mid f \in F\} \vee \{g \circ \pi_2 \mid g \in G\} \\ &= (F \circ \pi_1) \vee (G \circ \pi_2). \end{aligned}$$

Proposition 4.1.2. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, their product $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$ satisfies the universal property of products:*

$$\forall_{\mathcal{H} \in \mathbf{Bis}} \forall_{f_1 \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})} \forall_{f_2 \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})} \exists !_{h \in \mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})} (f_1 = \pi_1 \circ h \ \wedge \ f_2 = \pi_2 \circ h)$$

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dashed, "h"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y. \end{tikzcd}$$

Proof. The fact that $\mathcal{F} \times \mathcal{G}$ is a Bishop space is immediate by its definition. To prove that it satisfies the universal property of products we show first that the standard projection functions are in $\mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{G})$, respectively, and then that they satisfy the universal property for products. We show that $\pi_1 \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{F})$ and for π_2 we work similarly. We need to show that $\forall_{f \in F}(f \circ \pi_1 \in F \times G)$, which holds by definition. For the universal property we fix some $\mathcal{H} = (Z, H) \in \mathbf{Bis}$ and some $f_1 \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$, $f_2 \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$ i.e., $\forall_{f \in F}(f \circ f_1 \in H)$ and $\forall_{g \in G}(g \circ f_2 \in H)$. We define

$$h(z) := (f_1(z), f_2(z)),$$

for every $z \in Z$. The facts that $f_1 = \pi_1 \circ h$ and $f_2 = \pi_2 \circ h$ follow automatically. Next we show that $h \in \mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$. By Proposition 3.6.4 it suffices to show that $\forall_{j \in \{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}}(j \circ h \in H)$. Suppose first that $j = f \circ \pi_1$, for some $f \in F$. Then, $(f \circ \pi_1)(h(z)) = (f \circ \pi_1)(f_1(z), f_2(z)) = f(f_1(z))$ i.e., $(f \circ \pi_1) \circ h = f \circ f_1 \in H$, by our hypothesis on f_1. Similarly, if $j = g \circ \pi_2$, for some $g \in G$, then $(g \circ \pi_2)(h(z)) = (g \circ \pi_2)(f_1(z), f_2(z)) = g(f_2(z))$ i.e., $(g \circ \pi_2) \circ h = g \circ f_2 \in H$, by our hypothesis on f_2. The uniqueness of h satisfying the above property follows immediately. $\square$

In [6], p.73, Definition 4.1.1 is extended naturally to the countably infinite case. More generally we give the following definition.

Definition 4.1.3. *If I is a given set and $\mathcal{F}_i = (X_i, F_i)$ is a Bishop space, for every $i \in I$, their I-product, or simply their product, is the pair $\prod_{i \in I} \mathcal{F}_i = (\prod_{i \in I} X_i, \prod_{i \in I} F_i)$,, where*

$$\begin{aligned}\prod_{i \in I} F_i &:= \mathcal{F}(\bigcup_{i \in I}\{f \circ \pi_i \mid f \in F_i\}) \\ &= \bigvee_{i \in I}\{f \circ \pi_i \mid f \in F_i\} \\ &= \bigvee_{i \in I}(F_i \circ \pi_i).\end{aligned}$$

The I-product of the same Bishop space $\mathcal{F} = (X, F)$ is denoted by $\mathcal{F}^I = (X^I, F^I)$. An I-product of $\mathcal{R}$ is called a Euclidean Bishop space, while an I-product of $\mathbf{2} = (2, \mathbb{F}(2))$ is called a Boolean Bishop space.

Working as above, it is easy to see that $\prod_{i \in I} \mathcal{F}_i$ satisfies the corresponding universal property. Actually, all results of this section extend to arbitrary products of Bishop spaces. Next we show that the product topology is even simpler in the presence of subbases.

Proposition 4.1.4. *Suppose that X, Y are inhabited, $F_0 \subseteq \mathbb{F}(X)$ and $G_0 \subseteq \mathbb{F}(Y)$. Then*

$$\mathcal{F}(F_0) \times \mathcal{F}(G_0) = \mathcal{F}(\{f_0 \circ \pi_1 \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\}).$$

$$F \times \mathcal{F}(G_0) = \mathcal{F}(\{f \circ \pi_1 \mid f \in F\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\}).$$

Proof. (i) We need to show that $\mathcal{F}(\{f\circ\pi_1 \mid f \in \mathcal{F}(F_0)\}\cup\{g\circ\pi_2 \mid g \in \mathcal{F}(G_0)\}) = \mathcal{F}(\{f_0\circ\pi_1 \mid f_0 \in F_0\}\cup\{g_0\circ\pi_2 \mid g_0 \in G_0\})$. The inclusion ($\supseteq$) follows from Proposition 3.4.3(ii). For the inclusion ($\subseteq$) it suffices to show by Proposition 3.4.3(i) that $\{f\circ\pi_1 \mid f \in \mathcal{F}(F_0)\}\cup\{g\circ\pi_2 \mid g \in \mathcal{F}(G_0)\} \subseteq \mathcal{F}(\{f_0\circ\pi_1 \mid f_0 \in F_0\}\cup\{g_0\circ\pi_2 \mid g_0 \in G_0\})$. First we show that $\forall_{f\in\mathcal{F}(F_0)}(f\circ\pi_1 \in \mathcal{F}(B))$, where $B := \{f_0\circ\pi_1 \mid f_0 \in F_0\}\cup\{g_0\circ\pi_2 \mid g_0 \in G_0\}$. If $f = f_0$, for some $f_0 \in F_0$, then $f_0\circ\pi_1 \in B \subseteq \mathcal{F}(B)$. If $\overline{a} \in \mathrm{Const}(X)$, then $\overline{a}\circ\pi_1 \in \mathrm{Const}(X\times Y) \subseteq \mathcal{F}(B)$. Next we suppose that $f = f_1 + f_2$, where $f_1, f_2 \in \mathcal{F}(F_0)$ and $f_1\circ\pi_1, f_2\circ\pi_1 \in \mathcal{F}(B)$. Since $(f_1+f_2)\circ\pi_1 = (f_1\circ\pi_1)+(f_2\circ\pi_1)$, we conclude that $f\circ\pi_1 \in \mathcal{F}(B)$. If $f \in \mathcal{F}(F_0)$ such that $f\circ\pi_1 \in \mathcal{F}(B)$ and $\phi \in \mathrm{Bic}(\mathbb{R})$, then $(\phi\circ f)\circ\pi_1 \in \mathcal{F}(B)$, since $(\phi\circ f)\circ\pi_1 = \phi\circ(f\circ\pi_1)$. The last step of the inductive argument is proved easily using the inductive hypothesis and the condition BS_4 of $\mathcal{F}(B)$. Similarly we show that $\forall_{g\in\mathcal{F}(G_0)}(g\circ\pi_2 \in \mathcal{F}(B))$. For the second equality we show similarly inductively that $\{f\circ\pi_1 \mid f \in F\}\cup\{g\circ\pi_2 \mid g \in \mathcal{F}(G_0)\} \subseteq \mathcal{F}(\{f\circ\pi_1 \mid f \in F\}\cup\{g_0\circ\pi_2 \mid g_0 \in G_0\}) \leftrightarrow \{g\circ\pi_2 \mid g \in \mathcal{F}(G_0)\} \subseteq \mathcal{F}(\{f\circ\pi_1 \mid f \in F\}\cup\{g_0\circ\pi_2 \mid g_0 \in G_0\})$. □

Since $\mathrm{id}_{\mathbb{R}}$ generates $\mathrm{Bic}(\mathbb{R})$, the product topology $\mathrm{Bic}(\mathbb{R})\times\mathrm{Bic}(\mathbb{R}))$ takes then the form

$$\begin{aligned}\mathrm{Bic}(\mathbb{R})\times\mathrm{Bic}(\mathbb{R}) &= \mathcal{F}(\{\mathrm{id}_{\mathbb{R}}\circ\pi_1\}\cup\{\mathrm{id}_{\mathbb{R}}\circ\pi_2\})\\ &= \mathcal{F}(\mathrm{id}_{\mathbb{R}}\circ\pi_1, \mathrm{id}_{\mathbb{R}}\circ\pi_2)\\ &= \mathcal{F}(\pi_1, \pi_2).\end{aligned}$$

It is somewhat remarkable that a notion of a "Euclidean" dimension appears so soon in the development of TBS. The generalization to arbitrary Euclidean Bishop spaces is clear.

Proposition 4.1.5. *If (X, F) and (Y, G) Bishop spaces, then $F\times\mathrm{Const}(Y) = F\circ\pi_1$ and $\mathrm{Const}(X)\times G = G\circ\pi_2$.*

Proof. We prove the equality $F\times\mathrm{Const}(Y) = F\circ\pi_1$ and for the other we work similarly. By definition we have that $F\times\mathrm{Const}(Y) = \mathcal{F}((F\circ\pi_1)\cup(\mathrm{Const}(Y)\circ\pi_2)) = \mathcal{F}((F\circ\pi_1)\cup\mathrm{Const}(X\times Y)) = \mathcal{F}(F\circ\pi_1)$. We show inductively that $\mathcal{F}(F\circ\pi_1) \subseteq F\circ\pi_1$. It is trivial that $F\circ\pi_1$ and $\mathrm{Const}(X\times Y)$ are included to $F\circ\pi_1$. If $f', g' \in \mathcal{F}(F\circ\pi_1)$ such that $f' = f\circ\pi_1$ and $g' = g\circ\pi_1$, for some $f, g \in F$, then $f' + g' = (f\circ\pi_1) + (g\circ\pi_1) = (f+g)\circ\pi_1$, and $f+g \in F$. Similarly, if $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f' \in \mathcal{F}(F\circ\pi_1)$ such that $f' = f\circ\pi_1$, for some $f \in F$, then $\phi\circ f' = \phi\circ(f\circ\pi_1) = (\phi\circ f)\circ\pi_1$ and $\phi\circ f \in F$. Suppose that $f' \in \mathcal{F}(F\circ\pi_1)$ such that for every $\epsilon > 0$ there is some $g' \in \mathcal{F}(F\circ\pi_1)$ such that $U(g', f', \epsilon)$ and $g' = g\circ\pi_1$, for some $g \in F$. We fix some $x \in X$ and we show that $\forall_{y,y'\in Y}(f'(x,y) = f'(x,y'))$; if $y, y' \in Y$, we suppose that $f'(x,y) \bowtie_{\mathbb{R}} f'(x,y') \leftrightarrow |f'(x,y) - f'(x,y')| = \epsilon_0 > 0$. If $g' = g\circ\pi_1$, for some $g \in F$, such that $U(g', f', \frac{\epsilon_0}{4})$, we have that

$$\begin{aligned}\epsilon_0 &\leq |f'(x,y) - g(x)| + |g(x) - f'(x,y')|\\ &= |f'(x,y) - g'(x,y))| + |g'(x,y') - f'(x,y')|\\ &\leq \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4}\\ &= \frac{\epsilon_0}{2},\end{aligned}$$

which is absurd. By the tightness of $\bowtie_{\mathbb{R}}$ we get that $f'(x,y) = f'(x,y')$. If we define the map $f : X \to \mathbb{R}$ by $f(x) = f'(x, y_0)$, where y_0 inhabits Y, we get by the previous fact that f is well defined and $f' = f \circ \pi_1$, since $(f \circ \pi_1)(x,y) = f(x) = f'(x, y_0) = f'(x,y)$, for every $(x,y) \in X \times Y$. Moreover, $U(g', f', \epsilon) \to U(g, f, \epsilon)$, since $|g(x) - f(x)| = |g'(x, y_0) - f'(x, y_0)| \leq \epsilon$, for every $x \in X$. Hence, $f \in F$ and $f' \in F \circ \pi_1$. $\square$

The next propositions show the similarity between the Bishop and the topological product.

Proposition 4.1.6. *If $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are Bishop spaces, the following hold:*
(i) $F \times G$ is the least topology on $X \times Y$ which turns the projections π_1, π_2 into morphisms.
(ii) $j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$ if and only if $\pi_1 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ and $\pi_2 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1 \circ j"'] \arrow[d, "j"] \arrow[dr, "\pi_2 \circ j"] & \\ X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y. \end{tikzcd}$$

Proof. (i) We have already shown that π_1, π_2 are morphisms with respect to the product of Bishop spaces. If K is a topology on $X \times Y$ such that $\pi_1, \pi_2 \in \mathrm{Mor}(\mathcal{K}, \mathcal{F} \times \mathcal{G})$, then $f \circ \pi_1 \in K$ and $g \circ \pi_2 \in K$, for each $f \in F$ and $g \in G$, respectively. Then we get what we want by Proposition 3.4.3(i). For (ii) by Proposition 3.6.4 we have that

$$\begin{aligned} j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G}) &\leftrightarrow \forall_{h \in \{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}} (h \circ j \in H) \\ &\leftrightarrow \forall_{f \in F}((f \circ \pi_1) \circ j \in H) \ \wedge \ \forall_{g \in G}((g \circ \pi_2) \circ j \in H) \\ &\leftrightarrow \forall_{f \in F}(f \circ (\pi_1 \circ j) \in H) \ \wedge \ \forall_{g \in G}(g \circ (\pi_2 \circ j) \in H) \\ &\leftrightarrow \pi_1 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F}) \ \wedge \ \pi_2 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{G}). \end{aligned}$$

$\square$

One direction of Proposition 4.1.6(ii) follows from the universal property of the product. In contrast to the product topology of opens, the projection mappings are not in general open functions.

Corollary 4.1.7. *If $\mathcal{H} = (Z, H), \mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces and $h_1 : Z \to X$, $h_2 : Z \to Y$, then the function*

$$h : Z \to X \times Y,$$

$$z \mapsto (h_1(z), h_2(z)),$$

is in $\mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$ if and only if $h_1 \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ and $h_2 \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$.

Proof. Immediately by Proposition 4.1.6(ii), since $h_1 = \pi_1 \circ h$ and $h_2 = \pi_2 \circ h$. $\square$

Proposition 4.1.8. *Suppose that $\mathcal{F}_n = (X_n, F_n)$ and $\mathcal{G}_n = (Y_n, F_n)$ are two sequences of Bishop spaces and $h_n : X_n \to Y_n$ are given functions. Then the function*

$$\prod_n f_n : \prod_n X_n \to \prod_n Y_n,$$

$$(x_n)_n \mapsto (f_n(x_n))_n,$$

is in $\mathrm{Mor}(\prod_n \mathcal{F}_n, \prod_n \mathcal{G}_n)$ *if and only if* $f_n \in \mathrm{Mor}(\mathcal{F}_n, \mathcal{G}_n)$*, for every* n*. Moreover, if every* f_n *is an open set-epimorphism, then* $\prod_n f_n$ *is an open set-epimorphism.*

Proof. The fact that $\prod_n f_n$ is a morphism is derived easily by Corollary 4.1.7. It is also easy to see that the surjectivity of every f_n implies the surjectivity of $\prod_n f_n$. Since $\prod_n F_n = \bigvee_n (F_n \circ \pi_n)$, by the $\mathcal{F}$-lifting of openness it suffices to show that

$$\forall_k \forall_{f \in F_k} \exists_{g \in \prod_n G_n} (f \circ \pi_k = g \circ \prod_n f_n).$$

We fix k and $f \in F_k$. By the openness of f_k there is some $g \in G_k$ such that $f = g \circ f_k$. We get that $f \circ \pi_k = (g \circ \pi_k) \circ \prod_n f_n$, since $[(g \circ \pi_k) \circ \prod_n f_n]((x_n)_n) = g(\pi_k((f_n(x_n))_n)) = g(f_k(x_k)) = (g \circ f_k)(x_k) = f(x_k) = (f \circ \pi_k)((x_n)_n)$. $\square$

Proposition 4.1.9. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, $A \subseteq X$ and $B \subseteq Y$.*
(i) If $F = F_b$ and $G = G_b$, then $F \times G = (F \times G)_b$.
(ii) If $F = F(A)$ and $G = G(B)$, then $F \times G = (F \times G)(A \times B)$.

Proof. (i) By Proposition 3.4.4 it suffices to show that every element of $F \circ \pi_1 \cup G \circ \pi_2$ is bounded. Since every $f \in F$ and $g \in G$ is bounded, this holds automatically.
(ii) By Proposition 3.4.8 it suffices to show that every element of the union $F \circ \pi_1 \cup G \circ \pi_2$ restricted to $A \times B$ is constant. Since for every $f \in F$ we have that $f_{|A} = \overline{c_1}_{|A}$, for some $c_1 \in \mathbb{R}$, and for each $g \in G$ we have that $g_{|B} = \overline{c_2}_{|B}$, for some $c_2 \in \mathbb{R}$, we get that $(f \circ \pi_1)(a, b) = f(a) = c_1$ and $(g \circ \pi_2)(a, b) = g(b) = c_2$, for each $(a, b) \in A \times B$. $\square$

Next we show a proposition necessary in the proof of some elementary properties of the homotopic relation between morphisms (see Chapter 7).

Proposition 4.1.10. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ $\mathcal{G}' = (Y', G')$ and $\mathcal{H} = (Z, H)$ are Bishop spaces such that $\theta : Y \to Y'$ is an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. If $e : X \times Y \to Z$ is in* $\mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$*, then the mapping*

$$e' : X \times Y' \to Z,$$

$$e'(x, y') = e'(x, \theta(y)) := e(x, y),$$

for every $y' \in Y'$, is in $\mathrm{Mor}(\mathcal{F} \times \mathcal{G}', \mathcal{H})$.

Proof. We define a morphism $\Theta : X \times Y' \to X \times Y$ by $\Theta(x, y') = \Theta(x, \theta(y)) := (x, y)$, and we show that $\Theta \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}', \mathcal{F} \times \mathcal{G})$ i.e.,

$$\forall_{f \in F}((f \circ \pi_1) \circ \Theta \in F \times G') \wedge \forall_{g \in G}((g \circ \pi_2) \circ \Theta \in F \times G').$$

If we fix some $f \in F$, we have that $((f \circ \pi_1) \circ \Theta)(x, y') = (f \circ \pi_1)(\Theta(x, y')) = (f \circ \pi_1)(x, y) = f(x) = (f \circ \pi_1)(x, y')$, where $\theta(y) = y' = f(x)$ i.e., $(f \circ \pi_1) \circ \Theta = f \circ \pi_1 \in F \times G'$. Similarly, if we fix some $g \in G$, we have that $((g \circ \pi_2) \circ \Theta)(x, y') = (g \circ \pi_2)(\Theta(x, y')) = (g \circ \pi_2)(x, y) = g(y) = ((g \circ \theta^{-1}) \circ \pi_2)(x, y')$, where $\theta(y) = y' = g(y)$ i.e., $(g \circ \pi_2) \circ \Theta = (g \circ \theta^{-1}) \circ \pi_2 \in F \times G'$, since $g \circ \theta^{-1} \in G'$. Hence, $e' \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}', \mathcal{H})$ as a composition of the following morphisms

$$\begin{tikzcd}
X \times Y \arrow[r, "e'"] \arrow[d, "\Theta"'] & Z. \\
X \times Y' \arrow[ur, "e"'] &
\end{tikzcd}$$

□

The next proposition is used in the proof of Proposition 6.3.2.

Proposition 4.1.11. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are isomorphic Bishop spaces, and I is an index set. Then $\prod_{i \in I} \mathcal{F}$ is isomorphic to $\prod_{i \in I} \mathcal{G}$.*

Proof. Suppose that $e : X \to Y$ is an isomorphism between $\mathcal{F}$ and $\mathcal{G}$. The mapping

$$E : \prod_{i \in I} X \to \prod_{i \in I} Y,$$

$$(x_i)_{i \in I} \mapsto (e(x_i))_{i \in I},$$

is clearly a bijection. By definition $\prod_{i \in I} F = \mathcal{F}(\{f \circ \pi_i \mid f \in F, i \in I\})$, $\prod_{i \in I} G = \mathcal{F}(\{g \circ \pi_i \mid g \in G, i \in I\})$ and $E \in \mathrm{Mor}(\prod_{i \in I} \mathcal{F}, \prod_{i \in I} \mathcal{G}) \leftrightarrow \forall_{i \in I}((g \circ \pi_i) \circ E \in F) \leftrightarrow \forall_{i \in I}(g \circ (\pi_i \circ E) \in F) \leftrightarrow \forall_{i \in I}(g \circ e \in F)$, which is the case since $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Since e is open there is some $g \in G$ such that $f = g \circ e$. The equalities $(f \circ \pi_i)((x_i)_{i \in I}) = ((g \circ e) \circ \pi_i)((x_i)_{i \in I}) = g(e(x_i))$, and $((g \circ \pi_i) \circ E)((x_i)_{i \in I}) = (g \circ \pi_i)((e(x_i))_{i \in I}) = g(e(x_i))$, show that $f \circ \pi_i = (g \circ e) \circ \pi_i = (g \circ \pi_i) \circ E$ and by the $\mathcal{F}$-lifting of openness we conclude that E is also open. □

The first part of the next proposition is used in the proof of Proposition 5.2.4, while its third is used to show that $\Phi_t \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ in section 7.4.

Proposition 4.1.12. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ are Bishop spaces, $x \in X, y \in Y$, $\phi : X \times Y \to \mathbb{R} \in F \times G$ and $\Phi : X \times Y \to Z \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$.*
(i) $i_x : Y \to X \times Y$, $y \mapsto (x, y)$, and $i_y : X \to X \times Y$, $x \mapsto (x, y)$, are open morphisms.
(ii) $\phi_x : Y \to \mathbb{R}$, $y \mapsto \phi(x, y)$, and $\phi_y : X \to \mathbb{R}$, $x \mapsto \phi(x, y)$, are in G and F, respectively.
(iii) $\Phi_x : Y \to Z$, $y \mapsto \Phi(x, y)$, and $\Phi_y : X \to Z$, $x \mapsto \Phi(x, y)$, are in $\mathrm{Mor}(\mathcal{G}, \mathcal{H})$ and $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$, respectively.

Proof. (i) We show it only for i_y. By the $\mathcal{F}$-lifting of morphisms we have that $i_y \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G}) \leftrightarrow \forall_{f \in F}((f \circ \pi_1) \circ i_y \in F) \wedge \forall_{g \in G}((g \circ \pi_2) \circ i_y \in F)$. If $f \in \overline{F}$, then $(f \circ \pi_1) \circ i_y = f$, which shows also that i_y is open, while if $g \in G$, then $(g \circ \pi_2) \circ i_y = \overline{g(y)} \in F$.
(ii) We show it only for ϕ_y. We have that $\phi_y = \phi \circ i_y$, since $(\phi \circ i_y)(x) = \phi(x, y) = \phi_y(x)$, for each $x \in X$. Since $i_y \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G})$ and $\phi \in F \times G$, we get that $\phi \circ i_y = \phi_y \in F$.
(iii) The proof is similar to the proof of (ii), Actually, (ii) is a special case of (iii). $\square$

4.2 A Stone-Weierstrass theorem for pseudo-compact Bishop spaces

According to the classical Stone-Weierstrass theorem, if $(X, \mathcal{T})$ is a compact Hausdorff topological space and $A \subseteq C(X)$ such that

(i) A separates the points of X i.e., $\forall_{x,y \in X}(x \neq y \rightarrow \exists_{f \in A}(f(x) \neq f(y)))$,
(ii) $\mathrm{Const}(X) \subseteq A$,
(iii) A is a subalgebra of $C(X)$,

then the uniform closure of A is $C(X)$. A slightly stronger formulation of this theorem is the following (for an explanation see [68]):

Suppose that $(X, \mathcal{T})$ is a compact Hausdorff topological space and $A \subseteq C(X)$ is a subalgebra of $C(X)$. Then A is uniformly dense in $C(X)$ if and only if

(i) A separates the points of X.
(ii) A contains a function f which is bounded away from zero i.e.,

$$\inf\{|f(x)| \mid x \in X\} > 0.$$

As a corollary of the Stone-Weierstrass theorem, if X, Y are compact Hausdorff topological spaces, $f \in C(X \times Y)$ and $\epsilon > 0$, there are functions $g_1, \ldots, g_n \in C(X)$ and $h_1, \ldots, h_n \in C(Y)$ such that

$$U\left(\sum_{i=1}^{n}(g_i \circ \pi_1)(h_i \circ \pi_2), f, \epsilon\right)$$

i.e., a function of the form

$$f_\epsilon(x, y) = \sum_{i=1}^{n} g_i(x) h_i(y)$$

is uniformly ϵ-close to f. In the general case, if $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of compact Hausdorff topological spaces and $\mathcal{T}$ is the product topology on $X = \prod_{i \in I} X_i$, then for every $f \in C(X)$ and $\epsilon > 0$, there exists $\phi \in \Sigma(X)$ such that $U(\phi, f, \epsilon)$, where

$$\Sigma_0(X) := \{\prod_{j \in J}(f_j \circ \pi_j) \mid J \subseteq^{\mathrm{fin}} I, f_j \in C(X_j), j \in J\},$$

$$\Sigma(X) := \{\sum_{k=1}^{n} \phi_k \mid n \in \mathbb{N}, \phi_k \in \Sigma_0(X), 1 \leq k \leq n\}.$$

For a proof of both these facts by the Stone-Weierstrass theorem see [16], p.314. Next we translate these two corollaries into TBS and we show them constructively.

Remark 4.2.1. *If $F_0 \subseteq \mathbb{F}(X)$, then $\mathcal{U}(\mathcal{U}(F_0)) = \mathcal{U}(F_0)$.*

Proof. Clearly, $\mathcal{U}(F_0) \subseteq \mathcal{U}(\mathcal{U}(F_0))$. For the converse inclusion we fix some $f \in \mathcal{U}(\mathcal{U}(F_0))$ and we show that $f \in \mathcal{U}(F_0)$. If $g \in \mathcal{U}(F_0)$ such that $U(g, f, \frac{\epsilon}{2})$, there exists some $f_0 \in F_0$ such that $U(f_0, g, \frac{\epsilon}{2})$. Therefore, $U(f_0, f, \epsilon)$, and since $\epsilon > 0$ is arbitrary, we get $f \in \mathcal{U}(F_0)$. □

If (X, F) and (Y, G) are Bishop spaces, we define the subset $F \oplus G$ of $F \times G$ by

$$F \oplus G := \{\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2) \mid n \in \mathbb{N}, f_i \in F, g_i \in G, 1 \leq i \leq n\}.$$

Lemma 4.2.2. *Suppose that (X, F) and (Y, G) are pseudo-compact Bishop spaces.*
(i) The set $F \oplus G$ includes $F \circ \pi_1, G \circ \pi_2, \mathrm{Const}(X \times Y)$, and it is closed under addition, multiplication by reals and multiplication.
(ii) If p is a real polynomial and $\theta \in F \oplus G$, then $p \circ \theta \in F \oplus G$.
(iii) $\mathcal{U}(F \oplus G)$ is closed under addition, multiplication by reals and multiplication.
(iv) If p is a real polynomial and $f \in \mathcal{U}(F \oplus G)$, then $p \circ f \in \mathcal{U}(F \oplus G)$.

Proof. (i) If $f \in F$ and $g \in G$, then $f \circ \pi_1 = (f \circ \pi_1)(\overline{1} \circ \pi_2) \in F \oplus G$ and $g \circ \pi_2 = (\overline{1} \circ \pi_1)(g \circ \pi_2) \in F \oplus G$. If $\overline{a} \in \mathrm{Const}(X \times Y)$, then $\overline{a} = \overline{a} \circ \pi_1 \in F \oplus G$, where we use the same notation for the map $\overline{a} \in \mathrm{Const}(X) \subseteq F$. If $\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2), \sum_{j=1}^{m} (f_j{}' \circ \pi_1)(g_j{}' \circ \pi_2) \in F \oplus G$, then their sum is equal to $\sum_{k=1}^{n+m} (f_k{}'' \circ \pi_1)(g_k{}'' \circ \pi_2) \in F \oplus G$, where $f''{}_k = f_k$, if $1 \leq k \leq n$, $f''{}_k = f_j{}'$, if $n + 1 \leq k = n + j \leq n + m$, and $g''{}_k$ is defined similarly. The closure of $F \oplus G$ under multiplication by some real λ is shown by

$$\begin{aligned}
\lambda(\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2)) &= \sum_{i=1}^{n} \lambda(f_i \circ \pi_1)(g_i \circ \pi_2) \\
&= \sum_{i=1}^{n} (\lambda f_i \circ \pi_1)(g_i \circ \pi_2) \in F \oplus G.
\end{aligned}$$

If $f_i, f_j{}' \in F$ and if $(x, y) \in X \times Y$, we have that

$$[(f_i \circ \pi_1)(f_j{}' \circ \pi_2)](x, y) = f_i(x) f_j{}'(x) = (f_i f_j{}' \circ \pi_1)(x, y).$$

Similarly we show that if $g_i, g_j{}' \in G$, then $(g_i \circ \pi_2)(g_j{}' \circ \pi_2) = g_i g_j{}' \circ \pi_2$. The closure of $F \oplus G$ under multiplication follows by the equalities

$$\begin{aligned}
\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2) \sum_{j=1}^{m} (f_j{}' \circ \pi_1)(g_j{}' \circ \pi_2) &= \sum_{j=1}^{m} [\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2)](f_j{}' \circ \pi_1)(g_j{}' \circ \pi_2) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} (f_i \circ \pi_1)(f_j{}' \circ \pi_1)(g_i \circ \pi_2)(g_j{}' \circ \pi_2) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} (f_i f_j{}' \circ \pi_1)(g_i g_j{}' \circ \pi_2) \in F \oplus G,
\end{aligned}$$

since F and G are rings, $\sum_{i=1}^{n}(f_i f_j{}' \circ \pi_1)(g_i g_j{}' \circ \pi_2) \in F \oplus G$ by the definition of $F \oplus G$, and $F \oplus G$ is closed under addition.
(ii) It follows immediately by (i).
(iii) Suppose that $f_1, f_2 \in \mathcal{U}(F \oplus G)$ and $\theta_1, \theta_2 \in F \oplus G$ such that $U(\theta_1, f_1, \frac{\epsilon}{2})$ and $U(\theta_2, f_2, \frac{\epsilon}{2})$, for some $\epsilon > 0$. Since then we get $U(\theta_1 + \theta_2, f_1 + f_2, \epsilon)$, and $\theta_1 + \theta_2 \in F \oplus G$ by (i), we conclude that $f_1 + f_2 \in \mathcal{U}(F \oplus G)$. For the closure of $\mathcal{U}(F \oplus G)$ under multiplication by reals we fix some $\lambda \in \mathbb{R}$ and $f \in \mathcal{U}(F \oplus G)$. Hence, there is some $\theta \in F \oplus G$ such that $U(\theta, f, \frac{\epsilon}{|\lambda|+\sigma})$, where $\epsilon, \sigma > 0$. Since

$$|\lambda\theta(x, y) - \lambda f(x, y)| = |\lambda||\theta(x, y) - f(x, y)| \leq |\lambda| \frac{\epsilon}{|\lambda| + \sigma} < 1\epsilon = \epsilon,$$

we get that $U(\lambda\theta, \lambda f, \epsilon)$ and by (i) $\lambda\theta \in F \oplus G$. For the closure of $\mathcal{U}(F \oplus G)$ under multiplication it suffices again to show that $f \in \mathcal{U}(F \oplus G) \to f^2 \in \mathcal{U}(F \oplus G)$. Since $F \oplus G \subseteq F \times G$ and $F \times G$ by BS_4 is closed under the $\mathcal{U}$-operator, we get that $\mathcal{U}(F \oplus G) \subseteq F \times G$. Therefore, the elements of $\mathcal{U}(F \oplus G)$ are bounded functions. Let $M_f > 0$ be a bound for f and without loss of generality we take $M_f > 1$, $\epsilon \leq 1$ and we show that $U(\theta', f^2, \epsilon)$, for some $\theta' \in F \oplus G$ (note that if $U(\theta', f^2, \epsilon \wedge 1)$, then $U(\theta', f^2, \epsilon)$, for arbitrary $\epsilon > 0$). Consider $\theta \in F \oplus G$ such that $U(\theta, f, \frac{\epsilon}{3M_f})$, hence, for every $(x, y) \in X \times Y$, we have that

$$|\theta(x, y)| \leq |\theta(x, y) - f(x, y)| + |f(x, y)| \leq \frac{\epsilon}{3M_f} + M_f < \epsilon + M_f \leq 1 + M_f < 2M_f,$$

$$\begin{aligned}
|\theta^2(x, y) - f^2(x, y)| &= |\theta(x, y) - f(x, y)||\theta(x, y) + f(x, y)| \\
&\leq |\theta(x, y) - f(x, y)|(|\theta(x, y)| + |f(x, y)|) \\
&\leq |\theta(x, y) - f(x, y)|(2M_f + M_f) \\
&= |\theta(x, y) - f(x, y)|3M_f \\
&\leq \frac{\epsilon}{3M_f} 3M_f \\
&= \epsilon.
\end{aligned}$$

Hence $U(\theta^2, f^2, \epsilon)$, and since by (i) $\theta^2 \in F \oplus G$ we get that $f^2 \in \mathcal{U}(F \oplus G)$.
(iv) It follows immediately by (iii). $\square$

Note that the pseudo-compactness hypothesis is used in the previous proof only for the cases (iii) and (iv). We need to add here that the pseudo-compact topological spaces do not behave like the pseudo-compact Bishop spaces. For example, the product of two pseudo-compact topological spaces may not be pseudo-compact.

Theorem 4.2.3. *If (X, F) and (Y, G) are pseudo-compact Bishop spaces, then $F \oplus G$ is a base of $F \times G$.*

Proof. Since $F \times G = \mathcal{F}(F \circ \pi_1 \cup G \circ \pi_2)$ and by the pseudo-compactness of F and G the elements of $F \circ \pi_1 \cup G \circ \pi_2$ are bounded, by Proposition 3.5.5 we get that $\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2)$

is a base of $F \times G$. We show that $\mathcal{U}(F \oplus G) = \mathcal{U}(\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2)) = F \times G$. Clearly, $F \oplus G \subseteq F \times G \to \mathcal{U}(F \oplus G) \subseteq F \times G$. Thus, we need to show that $\mathcal{U}(\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2)) \subseteq \mathcal{U}(F \oplus G)$, and for that it suffices to show that $\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2) \subseteq \mathcal{U}(F \oplus G)$, since by Remark 4.2.1 we get then that $\mathcal{U}(\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2)) \subseteq \mathcal{U}(\mathcal{U}(F \oplus G)) = \mathcal{U}(F \oplus G)$. Using the induction principle $\mathrm{Ind}_{\mathcal{F}_0}$ on $\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2)$ we show the required inclusion

$$\mathcal{F}_0(F \circ \pi_1 \cup G \circ \pi_2) \subseteq \mathcal{U}(F \oplus G).$$

By Lemma 4.2.2(i) we have that $F \circ \pi_1, G \circ \pi_2, \mathrm{Const}(X \times Y) \subseteq F \oplus G \subseteq \mathcal{U}(F \oplus G)$. If $f, g \in \mathcal{U}(F \oplus G)$, then by Lemma 4.2.2(iii) we have that $f + g \in \mathcal{U}(F \oplus G)$. Finally, if $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \mathcal{U}(F \oplus G)$, we show that $\phi \circ f \in \mathcal{U}(F \oplus G)$. Let $M_f > 0$ be a bound of f. Since $|f| \leq M_f$ we get that $\phi \circ f = \phi_{|[-M_f, M_f]} \circ f$, where by the definition of $\mathrm{Bic}(\mathbb{R})$ the function $\phi_{|[-M_f, M_f]}$ is uniformly continuous on $[-M_f, M_f]$. By the Weierstrass approximation theorem there exists a sequence of real polynomials $(p_n)_{n \in \mathbb{N}}$ such that $p_n \xrightarrow{u} \phi_{|[-M_f, M_f]}$. Therefore, $(p_n \circ f) \xrightarrow{u} \phi_{|[-M_f, M_f]} \circ f = \phi \circ f$. By Lemma 4.2.2(iv) we have that $p_n \circ f \in \mathcal{U}(F \oplus G)$, for every $n \in \mathbb{N}$. Since $\mathcal{U}(F \oplus G)$ is closed under uniform limits, we conclude that $\phi \circ f \in \mathcal{U}(F \oplus G)$. $\square$

If Θ is a base of $F \times G$, by Proposition 4.1.12(ii) the following sets are subsets of F and G, respectively:

$$\Theta_Y := \{\theta_y \mid \theta \in \Theta, y \in Y\}, \text{ where } \theta_y(x) = \theta(x, y) \in F,$$

$$\Theta_X := \{\theta_x \mid \theta \in \Theta, x \in X\}, \text{ where } \theta_x(y) = \theta(x, y) \in G.$$

Remark 4.2.4. *If (X, F) and (Y, G) are Bishop spaces and Θ is a base of $F \times G$, then Θ_Y is a base of F and Θ_X is a base of G.*

Proof. We only show that Θ_Y is a base of F, since the proof for Θ_X is similar. If $f \in F$ and $\epsilon > 0$, we find $\theta_y \in \Theta_Y$ such that $U(\theta_y, f, \epsilon)$. Since $f \circ \pi_1 \in F \times G$, there exists $\theta \in \Theta$ such that $U(\theta, f \circ \pi_1, \epsilon) \leftrightarrow \forall_{(x,y) \in X \times Y}(|\theta(x, y) - f(x)| \leq \epsilon)$. If y_0 inhabits Y, we consider the function θ_{y_0}, for which we get that $\forall_{x \in X}(|\theta_{y_0}(x) - f(x)| \leq \epsilon) \leftrightarrow U(\theta_{y_0}, f, \epsilon)$. $\square$

If $\mathcal{F}_n = (X_n, F_n)$ is a sequence of Bishop spaces, we define the subset $\bigoplus_{n \in \mathbb{N}} F_n$ of $\prod_{n \in \mathbb{N}} F_n$ by

$$\Sigma_0 := \{\prod_{i=1}^{n}(f_k \circ \pi_k) \mid n \in \mathbb{N}, f_k \in F_k, 1 \leq k \leq n\},$$

$$\bigoplus_{n \in \mathbb{N}} F_n = \{\sum_{j=1}^{m} \phi_j \mid m \in \mathbb{N}, \phi_j \in \Sigma_0, 1 \leq j \leq m\}.$$

The following proposition is proved exactly like Lemma 4.2.2 and Theorem 4.2.3.

Proposition 4.2.5. *Suppose that $\mathcal{F}_n = (X_n, F_n)$ is a sequence of pseudo-compact Bishop spaces and $\mathcal{F} = (X, F)$, where $X = \prod_{n \in \mathbb{N}} X_n$ and $F = \prod_{n \in \mathbb{N}} F_n$.*
(i) The set $\bigoplus_{n \in \mathbb{N}} F_n$ includes $\mathrm{Const}(X)$, $F_k \circ \pi_k$, for every $k \in \mathbb{N}$, and it is closed under addition, multiplication by reals and multiplication.

(ii) If p is a real polynomial and $\theta \in \bigoplus_{n\in\mathbb{N}} F_n$, then $p \circ \theta \in \bigoplus_{n\in\mathbb{N}} F_n$.
(iii) $\mathcal{U}(\bigoplus_{n\in\mathbb{N}} F_n)$ is closed under addition, multiplication by reals and multiplication.
(iv) If p is a real polynomial and $f \in \mathcal{U}(\bigoplus_{n\in\mathbb{N}} F_n)$, then $p \circ f \in \mathcal{U}(\bigoplus_{n\in\mathbb{N}} F_n)$.
(v) The set $\bigoplus_{n\in\mathbb{N}} F_n$ is a base of F.

The similarity of the proof of Proposition 4.2.5 to the proof of Lemma 4.2.2 and Theorem 4.2.3 suggests the following generalization, which we call the Stone-Weierstrass theorem for pseudo-compact Bishop spaces. Its proof is an abstract version of the proofs of Lemma 4.2.2 and Theorem 4.2.3.

Theorem 4.2.6 (Stone-Weierstrass theorem for pseudo-compact Bishop spaces)**.** *Suppose that $\mathcal{F} = (X, \mathcal{F}(F_0))$ is a Bishop space such that every element of F_0 is bounded. If $\Theta \subseteq F$ such that:*
(i) $F_0 \subseteq \Theta$,
(ii) $\mathrm{Const}(X) \subseteq \Theta$ and
(iii) Θ is closed under addition, multiplication by reals and multiplication,
then Θ is a base of $\mathcal{F}(F_0)$.

Proof. First we show that $\mathcal{U}(\Theta)$ is closed under addition, multiplication by reals and multiplication. Suppose that $f_1, f_2 \in \mathcal{U}(\Theta)$ and $\theta_1, \theta_2 \in \Theta$ such that $U(\theta_1, f_1, \frac{\epsilon}{2})$ and $U(\theta_2, f_2, \frac{\epsilon}{2})$, for some $\epsilon > 0$. Since then we get $U(\theta_1 + \theta_2, f_1 + f_2, \epsilon)$, and $\theta_1 + \theta_2 \in \Theta$ by hypothesis, we conclude that $f_1 + f_2 \in \mathcal{U}(\Theta)$. For the closure of $\mathcal{U}(\Theta)$ under multiplication by reals we fix some $\lambda \in \mathbb{R}$ and $f \in \mathcal{U}(\Theta)$. Hence, there is some $\theta \in \Theta$ such that $U(\theta, f, \frac{\epsilon}{|\lambda|+\sigma})$, where $\epsilon, \sigma > 0$. Since

$$|\lambda\theta(x) - \lambda f(x)| = |\lambda||\theta(x) - f(x)| \leq |\lambda|\frac{\epsilon}{|\lambda| + \sigma} < 1\epsilon = \epsilon,$$

we get that $U(\lambda\theta, \lambda f, \epsilon)$ and by hypothesis $\lambda\theta \in F \oplus G$. For the closure of $\mathcal{U}(\Theta)$ under multiplication it suffices to show that $f \in \mathcal{U}(\Theta) \rightarrow f^2 \in \mathcal{U}(\Theta)$. Since $\Theta \subseteq F$ and F by BS_4 is closed under the $\mathcal{U}$-operator, we get that $\mathcal{U}(\Theta) \subseteq F$. Therefore, the elements of $\mathcal{U}(\Theta)$ are bounded functions. Let $M_f > 0$ be a bound for f and without loss of generality we take $M_f > 1$, $\epsilon \leq 1$ and we show that $U(\theta', f^2, \epsilon)$, for some $\theta' \in F \oplus G$. Consider $\theta \in \Theta$ such that $U(\theta, f, \frac{\epsilon}{3M_f})$, hence, for every $x \in X$, we have that

$$|\theta(x)| \leq |\theta(x) - f(x)| + |f(x)| \leq \frac{\epsilon}{3M_f} + M_f < \epsilon + M_f \leq 1 + M_f < 2M_f,$$

$$\begin{aligned}
|\theta^2(x) - f^2(x)| &= |\theta(x) - f(x)||\theta(x) + f(x)| \\
&\leq |\theta(x) - f(x)|(|\theta(x)| + |f(x)|) \\
&\leq |\theta(x) - f(x)|(2M_f + M_f) \\
&= |\theta(x) - f(x)|3M_f \\
&\leq \frac{\epsilon}{3M_f}3M_f \\
&= \epsilon.
\end{aligned}$$

Hence $U(\theta^2, f^2, \epsilon)$, and since by hypothesis $\theta^2 \in \Theta$ we get that $f^2 \in \mathcal{U}(\Theta)$. By Proposition 3.5.5 we have that $\mathcal{F}_0(F_0)$ is a base of F. We show that $\mathcal{U}(\Theta) = \mathcal{U}(\mathcal{F}_0(F_0)) = F$. Clearly, $\Theta \subseteq F \to \mathcal{U}(\Theta) \subseteq F$. Thus, we need to show that $\mathcal{U}(\mathcal{F}_0(F_0)) \subseteq \mathcal{U}(\Theta)$, and for that it suffices to show that

$$\mathcal{F}_0(F_0) \subseteq \mathcal{U}(\Theta).$$

Using the induction principle $\mathrm{Ind}_{\mathcal{F}_0}$ on $\mathcal{F}_0(F_0)$ we show the required inclusion $\mathcal{F}_0(F_0) \subseteq \mathcal{U}(\Theta)$. By hypothesis we have that $F_0, \mathrm{Const}(X) \subseteq \Theta \subseteq \mathcal{U}(\Theta)$. If $f, g \in \mathcal{U}(\Theta)$, we have shown that $f + g \in \mathcal{U}(\Theta)$. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \mathcal{U}(\Theta)$, we show that $\phi \circ f \in \mathcal{U}(\Theta)$. Let $M_f > 0$ be a bound of f. Since $|f| \leq \overline{M_f}$ we get that $\phi \circ f = \phi_{|[-M_f, M_f]} \circ f$, where by the definition of $\mathrm{Bic}(\mathbb{R})$ the function $\phi_{|[-M_f, M_f]}$ is uniformly continuous on $[-M_f, M_f]$. By the Weierstrass approximation theorem there exists a sequence of real polynomials $(p_n)_{n\in\mathbb{N}}$ such that $p_n \stackrel{u}{\to} \phi_{|[-M_f, M_f]}$. Therefore, $(p_n \circ f) \stackrel{u}{\to} \phi_{|[-M_f, M_f]} \circ f = \phi \circ f$. We get that $p_n \circ f \in \mathcal{U}(\Theta)$, for every $n \in \mathbb{N}$, since Θ is closed under multiplication by reals as it is closed under multiplication and includes $\mathrm{Const}(X)$, therefore $\mathcal{U}(\Theta)$ is closed under addition, multiplication by reals and multiplication. Since $\mathcal{U}(\Theta)$ is closed under uniform limits, we conclude that $\phi \circ f \in \mathcal{U}(\Theta)$. $\square$

According to Corollary 5.4.4 we have that $\mathrm{Bic}([a, b]) = \mathcal{F}(\mathrm{id}_{[a,b]})$, where $\mathrm{id}_{[a,b]}$ is bounded. By Theorem 4.2.6 if $\mathrm{id}_{[a,b]} \in \Theta \subseteq \mathrm{Bic}([a, b])$, therefore Θ separates the points of $[a, b]$, if $\mathrm{Const}([a, b]) \subseteq \Theta$ and if Θ is closed under addition, multiplication by reals and multiplication, then Θ is a base of $\mathrm{Bic}([a, b])$. It is clear that in this case Θ includes the polynomials on $[a, b]$, which clearly satisfy the above properties. In other words, the Weierstrass approximation theorem follows from Theorem 4.2.6, and since it is used in its proof, they are equivalent. Moreover, the Weierstrass approximation theorem determines the least $\Theta \subseteq \mathrm{Bic}([a, b])$ satisfying the above properties.

The epigraph at the beginning of this chapter expressed the hope that a constructive development of some form of general topology may shed some light even on aspects of the classical theory, and, in our view, this is the case with Theorem 4.2.3 and its generalization in Theorem 4.2.6. While in the classical theory one refers to compact topological spaces and uses the Stone-Weierstrass theorem, in TBS we refer to pseudo-compact Bishop spaces and we use only the Weierstrass approximation theorem[1]. The classical version of the Stone-Weierstrass theorem for compact topological spaces requires that "Θ separates the points of X", which is not used here. On the other hand we use the hypothesis that Θ includes a subbase. The importance of Theorem 4.2.6 is related to the problem of showing constructively a Stone-Weierstrass theorem for a suitable notion of compactness within TBS (see also section 6.6). A notion of a compact Bishop space should include that of pseudo-compactness, as it is the case for the notion of 2-compact Bishop space (Proposition 6.5.5), and the Stone-Weierstrass theorem for pseudo-compact Bishop spaces holds automatically for such a notion of compactness.

Theorem 4.2.6 somehow corresponds to Nel's theorem on the approximation of bounded continuous functions (see [68]).

[1] The Weierstrass approximation theorem is in [15] a corollary of the Stone-Weierstrass theorem, but it can be proved without using the whole machinery of the proof of the Stone-Weierstrass theorem.

Theorem 4.2.7 (Nel 1968)**.** *If X is a topological space and A is a subalgebra of $C^*(X)$ such that*
(i) A separates disjoint zero sets, and
(ii) A contains a function which is bounded away from zero,
then A is uniformly dense in $C^*(X)$.

Recall that if $(X_1, d_1), \ldots, (X_n, d_n)$ are metric spaces, the finite product metric σ_n on the product $\prod_{i=1}^n X_i$ is defined by

$$\sigma_n((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \sum_{i=1}^n d_i(x_i, y_i).$$

Theorem 4.2.8. *If (X, d) and (Y, ρ) are compact metric spaces, then*

$$C_u(X) \times C_u(Y) = C_u(X \times Y),$$

where $C_u(X) \times C_u(Y)$ is the Bishop product and $C_u(X \times Y)$ is the uniform topology on $X \times Y$ with the product metric σ_2.

Proof. First we show that $C_u(X) \times C_u(Y) \subseteq C_u(X \times Y)$. By definition $C_u(X) \times C_u(Y) = \mathcal{F}((C_u(X) \circ \pi_1) \cup (C_u(Y) \circ \pi_2))$. It is clear that every element of $(C_u(X) \circ \pi_1) \cup (C_u(Y) \circ \pi_2)$ is bounded. It is also uniformly continuous on $X \times Y$ with $\omega_{f \circ \pi_1} = \omega_f$ and $\omega_{g \circ \pi_2} = \omega_g$, since, for example, for the case of $f \circ \pi_1$ we have that

$$\sigma_2((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2) \leq \omega_f(\epsilon) \rightarrow d(x_1, x_2) \leq \omega_f(\epsilon), \text{ and}$$

$$|((f \circ \pi_1)(x_1, y_1) - ((f \circ \pi_1)(x_2, y_2)| = |f(x_1) - f(x_2)| \leq \epsilon.$$

By the $\mathcal{F}$-lifting of uniform continuity (Proposition 3.4.9) we get that every element of $C_u(X) \times C_u(Y)$ is an element of $C_u(X \times Y)$. For the converse inclusion we use a special case of the Corollary 5.15, found in [15], p.108. According to it,

$$C_u(X \times Y) = \mathcal{U}(C_u(X) \oplus C_u(Y)),$$

and since $C_u(X) \oplus C_u(Y) \subseteq \mathcal{F}((C_u(X) \circ \pi_1) \cup (C_u(Y) \circ \pi_2))$, we conclude that $C_u(X \times Y) \subseteq C_u(X) \times C_u(Y)$. $\square$

The above result is extended to the infinite case. Recall that, if $(X_n, d_n)_{n \in \mathbb{N}}$ is a sequence of metric spaces, where every d_n is bounded by 1, the infinite product metric σ_∞ on the product $\prod_{i \in \mathbb{N}} X_i$ is defined by

$$\sigma_\infty((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) := \sum_{i \in \mathbb{N}} \frac{d_i(x_i, y_i)}{2^i}.$$

Theorem 4.2.9. *If $(X_n, d_n)_{n\in\mathbb{N}}$ is a sequence of compact metric spaces, where every d_n is bounded by 1, then*

$$\prod_{i\in\mathbb{N}} C_u(X_i) = C_u(\prod_{i\in\mathbb{N}} X_i),$$

where $\prod_{i\in\mathbb{N}} C_u(X_i)$ is the infinite Bishop product and $C_u(\prod_{i\in\mathbb{N}} X_i)$ is the uniform topology on $\prod_{i\in\mathbb{N}} X_i$ with the product metric σ_∞.

Proof. First we show that $\prod_{i\in\mathbb{N}} C_u(X_i) \subseteq C_u(\prod_{i\in\mathbb{N}} X_i)$. By definition $\prod_{i\in\mathbb{N}} C_u(X_i) = \mathcal{F}(\bigcup_{i\in\mathbb{N}}(C_u(X_i) \circ \pi_i))$. It is clear that every element $f \circ \pi_i$ of $C_u(X_i) \circ \pi_i$ is bounded. It is also uniformly continuous on $\prod_{i\in\mathbb{N}} X_i$ with

$$\omega_{f\circ\pi_i} = \frac{\omega_f}{2^i},$$

$$\sigma_\infty((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \leq \frac{\omega_f(\epsilon)}{2^i} \to \frac{d_i(x_i, y_i)}{2^i} \leq \frac{\omega_f(\epsilon)}{2^i} \to d_i(x_i, y_i) \leq \omega_f(\epsilon), \text{ and}$$
$$|((f \circ \pi_i)((x_i)_{i\in\mathbb{N}}) - ((f \circ \pi_i)((y_i)_{i\in\mathbb{N}})| = |f(x_i) - f(y_i)| \leq \epsilon.$$

By the $\mathcal{F}$-lifting of uniform continuity we get that every element of $\prod_{i\in\mathbb{N}} C_u(X_i)$ is an element of $C_u(\prod_{i\in\mathbb{N}} X_i)$. For the converse inclusion we use exactly the Corollary 5.15, found in [15], p.108. According to it,

$$C_u(\prod_{i\in\mathbb{N}} X_i) = \mathcal{U}(\bigoplus_{i\in\mathbb{N}} C_u(X_i)),$$

and since $\bigoplus_{i\in\mathbb{N}} C_u(X_i) \subseteq \mathcal{F}(\bigcup_{i\in\mathbb{N}}(C_u(X_i) \circ \pi_i))$, we get that $C_u(\prod_{i\in\mathbb{N}} X_i) \subseteq \prod_{i\in\mathbb{N}} C_u(X_i)$. $\square$

The last two theorems reflect that the Bishop product "captures" uniform continuity, reinforcing our conviction that TBS is a fruitful approach to the problem of constructivising topology, as this was formulated by Bishop himself (see section 1.2).

4.3 Pointwise exponential Bishop space

We introduce here the pointwise exponential Bishop space $\mathcal{F} \to \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \to G)$, which corresponds to the point-open topology within **Top**. It seems that the category of Bishop spaces **Bis** behaves like the category of topological spaces **Top** with respect to the cartesian closure property.

Definition 4.3.1. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, the pointwise exponential Bishop space, or simply p-exponential Bishop space, $\mathcal{F} \to \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \to G)$, or $\mathcal{F} \to \mathcal{G} = (X \to Y, F \to G)$, is defined, for every $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, by*

$$F \to G := \mathcal{F}(\{e_{x,g} \mid x \in X, g \in G\}),$$

$$e_{x,g} : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R}$$
$$e_{x,g}(h) = g(h(x)).$$

This is a quite small topology on $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ generated by the set of the simplest mappings $\mathrm{Mor}(\mathcal{F},\mathcal{G}) \to \mathbb{R}$ one can think of. The definition of $F \to G$ gives the impression that the topology F is somehow "missing", but it is "found" in the fact that $h \in \mathrm{Mor}(\mathcal{F},\mathcal{G})$ and consequently $g \circ h \in F$. If, for example, we consider the topologies $F_1 \subseteq F_2$ on X, then $\mathrm{Mor}(\mathcal{F}_1,\mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F}_2,\mathcal{G})$ and every $e_{x,g} \in F_1 \to G$ is extended to the corresponding function in $F_2 \to G$, but all the elements of $F_2 \to G$ act on $\mathrm{Mor}(\mathcal{F}_2,\mathcal{G})$. It is true though, that F doesn't strongly influence $F \to G$. First we show the expected simplification of the p-exponential topology when a basis G_0 is given for G.

Proposition 4.3.2 ($\mathcal{F}$-lifting of the p-exponential topology)**.** *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, \mathcal{F}(G_0))$ are Bishop spaces, then $F \to \mathcal{F}(G_0) = \mathcal{F}(E_0)$, where*

$$E_0 := \{e_{x,g_0} \mid x \in X, g_0 \in G_0\},$$

$$e_{x,g_0}(h) = g_0(h(x)),$$

for every $h \in \mathrm{Mor}(\mathcal{F},\mathcal{G})$.

Proof. We show that

$$\{e_{x,g} \mid x \in X, g \in \mathcal{F}(G_0)\} = \bigcup_{x\in X} \{e_{x,g} \mid g \in \mathcal{F}(G_0)\} \subseteq \mathcal{F}(E_0).$$

We fix some $x \in X$ and we show that $\forall_{g\in\mathcal{F}(G_0)}(e_{x,g} \in \mathcal{F}(E_0))$. If $g \in G_0$, there is nothing to prove. If $\overline{a} \in \mathcal{F}(G_0)$, then $e_{x,\overline{a}}(h) = \overline{a}(h(x)) = a$ i.e., $e_{x,\overline{a}}$ is the constant function a which is by definition in $\mathcal{F}(E_0)$. If $e_{x,g_1}, e_{x,g_2} \in \mathcal{F}(E_0)$, then $e_{x,g_1+g_2} = e_{x,g_1} + e_{x,g_2} \in \mathcal{F}(E_0)$, since $e_{x,g_1+g_2}(h) = (g_1 + g_2)(h(x)) = g_1(h(x)) + g_2(h(x)) = e_{x,g_1}(h) + e_{x,g_2}(h) = (e_{x,g_1} + e_{x,g_2})(h)$, for each $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}(G_0))$. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $e_{x,g} \in \mathcal{F}(E_0)$, then $e_{x,\phi\circ g} = \phi \circ e_{x,g} \in \mathcal{F}(E_0)$, since $e_{x,\phi\circ g}(h) = (\phi\circ g)(h(x)) = \phi(g(h(x))) = \phi(e_{x,g}(h))$, for every $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}(G_0))$. Suppose next that $e_{x,g_n} \in \mathcal{F}(E_0)$. We show that $g_n \xrightarrow{u} g \to e_{x,g_n} \xrightarrow{u} e_{x,g}$. The premiss means that $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_{y\in Y}(|g_n(y) - g(y)| \leq \epsilon)$, while to show the conclusion $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_{h\in\mathrm{Mor}(\mathcal{F},\mathcal{F}(G_0))}(|e_{x,g_n}(h) - e_{x,g}(h)| \leq \epsilon)$ we just fix some $\epsilon > 0$, and using the $n_0(\epsilon)$ of the premiss we get that $|e_{x,g_n}(h) - e_{x,g}(h)| = |g_n(h(x)) - g(h(x))| \leq \epsilon$. By the condition BS_4 of $\mathcal{F}(E_0)$ we conclude that $e_{x,g} \in \mathcal{F}(E_0)$. □

Consequently, if X is a metric space, F is a topology on X, and Y is a compact metric space with positive diameter, then

$$F \to \mathcal{F}(U_0(Y)) = \mathcal{F}(\{e_{x,d_y} \mid x \in X, y \in Y\}),$$

$$e_{x,d_y}(h) = d_y(h(x)) = d(y, h(x)),$$

for every $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{U}(Y))$. The p-exponential topology is called pointwise because it behaves like the the classical topology of the pointwise convergence on $\mathbb{F}(X, Y)$. An indication of this common behavior is the pointwise character of the limit relation on

$\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ induced by $F \to G$; if $h, h_n \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, for every $n \in \mathbb{N}$, by the $\mathcal{F}$-lifting of the limit relation (Proposition 3.4.12 we have that

$$\begin{aligned} \lim_{F \to G}(h, h_n) &\leftrightarrow \forall_{x \in X} \forall_{g \in G}(e_{x,g}(h_n) \to e_{x,g}(h)) \\ &\leftrightarrow \forall_{x \in X} \forall_{g \in G}(g(h_n(x)) \to g(h(x))) \\ &\leftrightarrow \forall_{x \in X}(\lim_{G}(h(x), h_n(x))). \end{aligned}$$

Moreover, in Proposition 4.7.10 we show that the topology $F \to G$ on $X \to Y$ coincides with the topology of a subspace of the product $\prod_{x \in X} G_x$, where $G_x = G$, for every $x \in X$.

Definition 4.3.3. *If* **Cat** *is a an abstract category with carrier class $\mathcal{C}$, then* **Cat** *is cartesian closed, if*
(i) There exists a terminal object $1 \in \mathcal{C}$ i.e., $\forall_{X \in \mathcal{C}} \exists!_{f : X \to 1}(f \in \mathrm{Mor}(X, 1))$.
(ii) For every $A, B \in \mathcal{C}$ there exists the product $A \times B \in \mathcal{C}$ satisfying the universal property of products.
(iii) For every $B, C \in \mathcal{C}$ there exist the exponential $B \to C \in \mathcal{C}$ and an arrow $\varepsilon_{B,C}$ such that for any object $A \in \mathcal{C}$ and arrow $f : A \times B \to C$ there is a unique arrow $\Lambda(f) : A \to (B \to C)$ such that $\varepsilon_{B,C} \circ (\Lambda(f) \times 1_B) = f$

$$\begin{tikzcd} B \to C & (B \to C) \times B \arrow[r, "\varepsilon_{B,C}"] & C \\ A \arrow[u, "\Lambda(f)"] & A \times B \arrow[u, "\Lambda(f) \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

If $\mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are Bishop spaces, the *evaluation function* is the mapping

$$\mathrm{ev}_{\mathcal{G},\mathcal{H}} : (Y \to Z) \times Y \to Z,$$

$$\mathrm{ev}_{\mathcal{G},\mathcal{H}}(\theta, y) := \theta(y),$$

for every $\theta \in Y \to Z$ and $y \in Y$. If $\mathcal{F} = (X, F)$ is a Bishop space and $\psi : X \times Y \to Z \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$ the (p-exponential) *transpose* $\Lambda(\psi)$ of ψ is the function

$$\Lambda(\psi) : X \to (Y \to Z),$$

$$x \mapsto \Lambda(\psi)(x),$$

$$\Lambda(\psi)(x)(y) := \psi(x, y),$$

for every $x \in X$ and $y \in Y$.

Proposition 4.3.4. *Suppose that $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ and $\mathcal{F} = (X, F)$ are Bishop spaces.*
(i) The terminal object of **Bis** *is $1 = (\{x_0\}, \mathrm{Const}(\{x_0\}))$.*
(ii) $\Lambda(\psi) \in \mathrm{Mor}(\mathcal{F}, \mathcal{G} \to \mathcal{H})$.
(iii) $\mathrm{ev}_{\mathcal{G},\mathcal{H}} \circ (\Lambda(\psi) \times \mathrm{id}_Y) = \psi$.
(iv) $\Lambda(\psi)$ is the unique element of $\mathrm{Mor}(\mathcal{F}, \mathcal{G} \to \mathcal{H})$ satisfying (iii).

Proof. (i) First we remark that $\mathrm{Const}(\{x_0\}) = \mathbb{F}(\{x_0\})$ and the only function $h : X \to \{x_0\}$ is the constant map $x \mapsto x_0$. Then, $h \in \mathrm{Mor}(\mathcal{F}, 1) \leftrightarrow \forall_{g \in \mathrm{Const}(\{x_0\},\mathbb{R})}(g \circ h \in F) \leftrightarrow \forall_{a \in \mathbb{R}}(\overline{a} \circ h \in F)$, which is the case, since $\overline{a} \circ h$ is the constant map on X with value a.
(ii) Since $G \to H = \mathcal{F}(\{e_{y,h} \mid y \in Y, h \in H\})$, by the $\mathcal{F}$-lifting of morphisms we need only to show that $\forall_{y \in Y}\forall_{h \in H}(e_{y,h} \circ \Lambda(\psi) \in F)$. We fix $y \in Y$ and $h \in H$, and we have that $(e_{y,h} \circ \Lambda(\psi))(x) = e_{y,h}(\Lambda(\psi)(x)) = h(\Lambda(\psi)(x)(y) = h(\psi(x, y))$. If we consider the map $i_y : X \to X \times Y$, where $x \mapsto (x, y)$, then we have that $[(h \circ \psi) \circ i_y](x) = (h \circ \psi)(x, y) = h(\psi(x, y))$. Hence, we get that $e_{y,h} \circ \Lambda(\psi) = (h \circ \psi) \circ i_y \in F$, since $h \circ \psi \in F \times G$, by the hypothesis $\psi \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$, and the fact that $i_y \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G})$.
(iii) For every $(x, y) \in X \times Y$ we have that $[\mathrm{ev}_{\mathcal{G},\mathcal{H}} \circ (\Lambda(\psi) \times \mathrm{id}_Y)](x, y) = \mathrm{ev}_{\mathcal{G},\mathcal{H}}((\Lambda(\psi) \times \mathrm{id}_Y)(x, y)) = \mathrm{ev}_{\mathcal{G},\mathcal{H}}(\Lambda(\psi)(x), \mathrm{id}_Y(y)) = \mathrm{ev}_{\mathcal{G},\mathcal{H}}(\Lambda(\psi)(x), y) = \Lambda(\psi)(x)(y) = \psi(x, y)$.
(iv) If $M(\psi) \in \mathrm{Mor}(\mathcal{F}, \mathcal{G} \to \mathcal{H})$ such that $\mathrm{ev}_{\mathcal{G},\mathcal{H}} \circ (M(\psi) \times \mathrm{id}_Y) = \psi$, then $[\mathrm{ev}_{\mathcal{G},\mathcal{H}} \circ (M(\psi) \times \mathrm{id}_Y)](x, y) = \psi(x, y) \leftrightarrow \mathrm{ev}_{\mathcal{G},\mathcal{H}}((M(\psi) \times \mathrm{id}_Y)(x, y) = \psi(x, y) \leftrightarrow \mathrm{ev}_{\mathcal{G},\mathcal{H}}(M(\psi)(x), y) = \psi(x, y) \leftrightarrow M(\psi)(x)(y) = \psi(x, y) \leftrightarrow M(\psi)(x)(y) = \Lambda(\psi)(x)(y)$. Since x, y are arbitrary, we conclude that $M = \Lambda$. $\square$

The main problem in proving that **Bis** is cartesian closed with the p-exponential is that we cannot show that $\mathrm{ev}_{\mathcal{G},\mathcal{H}} \in \mathrm{Mor}((\mathcal{G} \to \mathcal{H}) \times \mathcal{G}, \mathcal{H})$ i.e., $\forall_{h \in H}(h \circ \mathrm{ev}_{\mathcal{G},\mathcal{H}} \in (G \to H) \times G)$. By Proposition 4.1.4(ii) we have that $(G \to H) \times G = \mathcal{F}(\{\phi \circ \pi_1 \mid \phi \in G \to H\} \cup \{g \circ \pi_2 \mid g \in G\}) = \mathcal{F}(\{e_{y,h} \circ \pi_1 \mid y \in Y, h \in H\} \cup \{g \circ \pi_2 \mid g \in G\})$. If $h \in H$, we get that $h \circ \mathrm{ev}_{\mathcal{G},\mathcal{H}} = e_{y,h} \circ \pi_1 \in (G \to H) \times G$, since $(h \circ \mathrm{ev}_{\mathcal{G},\mathcal{H}})(\theta, y) = h(\mathrm{ev}_{\mathcal{G},\mathcal{H}}(\theta, y)) = h(\theta(y)) = e_{y,h}(\theta) = (e_{y,h} \circ \pi_1)(\theta, y)$, but this expression of $h \circ \mathrm{ev}_{\mathcal{G},\mathcal{H}}$ is not independent from y.

Proposition 4.3.5. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are Bishop spaces, then the transpose function*

$$\Lambda : [(X \times Y) \to Z] \to [X \to (Y \to Z)],$$

$$\psi \mapsto \Lambda(\psi),$$

$$\Lambda(\psi)(x)(y) = \psi(x, y),$$

for every $x \in X$ and $y \in Y$, is in $\mathrm{Mor}((\mathcal{F} \times \mathcal{G}) \to \mathcal{H}, \mathcal{F} \to (\mathcal{G} \to \mathcal{H}))$. Moreover, if $\phi : X \to (Y \to Z) \in \mathrm{Mor}(\mathcal{F}, \mathcal{G} \to \mathcal{H})$, then $\Lambda(\mathrm{ev}_{\mathcal{G},\mathcal{H}} \circ (\phi \times \mathrm{id}_Y)) = \phi$.

Proof. By the definition of the exponential topology and by Proposition 4.3.2 we have that $(F \times G) \to H = \mathcal{F}(\{e_{(x,y),h} \mid (x, y) \in X \times Y, h \in H\})$, $G \to H = \mathcal{F}(\{e_{y,h} \mid y \in Y, h \in H\})$, and $F \to (G \to H) = \mathcal{F}(\{e_{x,e_{y,h}} \mid x \in X, y \in Y, h \in H\})$. By the $\mathcal{F}$-lifting of morphisms we have that $\Lambda \in \mathrm{Mor}((\mathcal{F} \times \mathcal{G}) \to \mathcal{H}, \mathcal{F} \to (\mathcal{G} \to \mathcal{H}))$ if and only if $\forall_{x \in X}\forall_{y \in Y}\forall_{h \in H}(e_{x,e_{y,h}} \circ \Lambda \in (F \times G) \to H)$. If we fix $x \in X, y \in Y$ and $h \in H$ we have that

$$\begin{aligned}
(e_{x,e_{y,h}} \circ \Lambda)(\psi) &= e_{x,e_{y,h}}(\Lambda(\psi)) \\
&= e_{y,h}(\Lambda(\psi)(x)) \\
&= h(\Lambda(\psi)(x)(y)) \\
&= h(\psi(x, y)) \\
&= e_{(x,y),h}(\psi)
\end{aligned}$$

i.e., $e_{x,e_{y,h}} \circ \Lambda = e_{(x,y),h} \in (F \times G) \to H$. Moreover, if $\phi : X \to (Y \to Z)$, then $\phi \times \mathrm{id}_Y : X \times Y \to [(Y \to Z) \times Y]$, where $(x,y) \mapsto (\phi(x), y)$. We define $\psi : (X \times Y) \to Z$ by $\psi(x,y) = \mathrm{ev}_{\mathcal{G},\mathcal{H}}((\phi \times \mathrm{id}_Y)(x,y)) = \mathrm{ev}_{\mathcal{G},\mathcal{H}}(\phi(x), y) = \phi(x)(y)$. Since $\Lambda(\psi)(x)(y) = \psi(x,y) = \phi(x)(y)$, we conclude that $\Lambda(\psi) = \phi$. $\square$

If we try to show the continuity of the composition map $T : (X \to Y) \times (Y \to Z) \to (X \to Z)$, defined by $(\phi, \theta) \mapsto \theta \circ \phi$, we have the same problem with the continuity of the evaluation map. As in **Top** we can only show that T is always continuous in each argument separately.

Proposition 4.3.6. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are Bishop spaces.*
(i) The mapping $j : Y \to (X \to Y)$, defined by $y \mapsto \overline{y}$, where $\overline{y}$ is the constant map with value y, is an open monomorphism.
(ii) The composition map T is continuous in each argument separately.

Proof. (i) By definition $j \in \mathrm{Mor}(\mathcal{G}, \mathcal{F} \to \mathcal{G}) \leftrightarrow \forall_{x \in X} \forall_{g \in G}(e_{x,g} \circ j \in G)$. But $e_{x,g} \circ j = g$, since $e_{x,g}(j(y)) = e_{x,g}(\overline{y}) = g(\overline{y}(x)) = g(y)$, for each $y \in Y$. This fact also shows that j is open. Clearly, j is $1{-}1$.
(ii) The proof is straightforward. $\square$

Next we define another topology on $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ based on the definition of a limit relation on the set of lim-continuous continuous functions between two limit spaces (see [61] for a classical treatment and [76] for a constructive treatment of the subject) and the fact that the lim-continuous functions of type $X \to \mathbb{R}$, where $(X, \lim)$ is a limit space, is a Bishop topology on X, as it is mentioned in section 3.3.

Definition 4.3.7. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, the exponential limit relation on* $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ *is defined by*

$$\lim_{\Rightarrow}(h, h_n) :\leftrightarrow \forall_{x \in X} \forall_{(x_n)_n \in X^{\mathbb{N}}}(\lim_{F}(x, x_n) \to \lim_{G}(h(x), h_n(x_n))$$

and the limit exponential topology $F \Rightarrow G$ is the limit topology $F_{\lim_{\Rightarrow}}$ i.e.,

$$F \Rightarrow G = \{e : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R} \mid \forall_{h \in \mathrm{Mor}(\mathcal{F},\mathcal{G})} \forall_{(h_n)_n \subseteq \mathrm{Mor}(\mathcal{F},\mathcal{G})}(\lim_{\Rightarrow}(h, h_n) \to (e(h_n) \to e(h)))\}.$$

Proposition 4.3.8. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, $F \to G \subseteq F \Rightarrow G$ and $\lim_{\Rightarrow} \subseteq \lim_{\mathcal{F} \Rightarrow \mathcal{G}}$.*

Proof. For the first inclusion it suffices to show that $e_{x,g} \in F \Rightarrow G$, for every $x \in X$ and $g \in G$. Suppose that $\lim_{\Rightarrow}(h, h_n)$ for some $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $(h_n)_n \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. By the definition of $\lim_{\Rightarrow}(h, h_n)$ we have that

$$\begin{aligned} \lim_{F}(x, x) &\to \lim_{G}(h(x), h_n(x)) \\ &\leftrightarrow \forall_{g \in G}(g(h_n(x)) \to g(h(x))) \\ &\leftrightarrow \forall_{g \in G}(e_{x,g}(h_n) \to e_{x,g}(h)). \end{aligned}$$

By the definition of $F \Rightarrow G$, if $\lim_{\Rightarrow}(h, h_n)$, then $e(h_n) \to e(h)$, for every $e \in F \Rightarrow G$ i.e., $\lim_{F \Rightarrow G}(h, h_n)$. $\square$

The difference between $\lim_{F \to G}$ and $\lim_{F \Rightarrow G}$ indicates that generally $F \to G \subsetneq F \Rightarrow G$.

4.4 The dual Bishop space

In this section we define a natural topology on a given Bishop topology, and we show that it is a special case of the p-exponential topology.

Definition 4.4.1. *If $\mathcal{F} = (X, F)$ is a Bishop space, the dual Bishop space of $\mathcal{F}$ is the space $\mathcal{F}^* = (F, F^*)$, where*

$$F^* := \mathcal{F}(\{\hat{x} \mid x \in X\})$$
$$\hat{x} : F \to \mathbb{R}$$
$$\hat{x}(f) = f(x),$$

for every $f \in F$ and $x \in X$.

Proposition 4.4.2. *If F is a topology on an inhabited set X, then $F^* = F \to \mathrm{Bic}(\mathbb{R})$.*

Proof. By Proposition 3.6.2 we have that $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$, and by the $\mathcal{F}$-lifting of the exponential topology we get that

$$F \to \mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\{e_{x,g} \mid x \in X, g \in \mathrm{Bic}(\mathbb{R}\}) = \mathcal{F}(\{e_{x,\mathrm{id}_{\mathbb{R}}} \mid x \in X\}) = F^*,$$
$$e_{x,\mathrm{id}_{\mathbb{R}}}(f) = \mathrm{id}_{\mathbb{R}}(f(x)) = f(x) = \hat{x}(f).$$

$\square$

We define the *double dual* of a Bishop space $\mathcal{F}$ to be the dual of the dual of $\mathcal{F}$, namely $\mathcal{F}^{**} = (F^*, F^{**})$, where

$$F^{**} := \mathcal{F}(\{\hat{f} \mid f \in F\}),$$
$$\hat{f} : F^* \to \mathbb{R},$$
$$\hat{f}(\phi) = \phi(f),$$

for every $\phi \in F^*$ and $f \in F$. Next we show the "continuity" of the $*$-operation. Its injectivity is shown in Proposition 5.8.10 under the hypothesis of complete regularity.

Proposition 4.4.3. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces.*
(i) $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to h^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*)$.
(ii) The map $* : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*)$, $h \mapsto h^*$, is in $\mathrm{Mor}(\mathcal{F} \to \mathcal{G}, \mathcal{G}^* \to \mathcal{F}^*)$.

Proof. (i) By the $\mathcal{F}$-lifting of morphisms $h^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*) \leftrightarrow \forall_{x \in X}(\hat{x} \circ h^* \in G^*)$. But $\hat{x} \circ h^* = \widehat{h(x)} \in G^*$, since $(\hat{x} \circ h^*)(g) = \hat{x}(h^*(g)) = \hat{x}(g \circ h) = (g \circ h)(x) = g(h(x)) = \widehat{h(x)}(g)$.
(ii) By Proposition 4.3.2 $G^* \to F^* = \mathcal{F}(\{e_{g,\hat{x}} \mid g \in G, x \in X\})$. Thus, $* : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*) \leftrightarrow \forall_{g \in G} \forall_{x \in X}(e_{g,\hat{x}} \circ * \in F \to G)$. But $e_{g,\hat{x}} \circ * = e_{x,g} \in F \to G$, since $(e_{g,\hat{x}} \circ *)(h) = e_{g,\hat{x}}(h^*) = \hat{x}(h^*(g)) = \hat{x}(g \circ h) = g(h(x) = e_{x,g}(h)$, for every $h \in X \to Y$. $\square$

Proposition 4.4.4. *If $\mathcal{F} = (X, F)$ is a Bishop space, then the operations $+ : F \times F \to F$, $f, g \mapsto f + g$, and $\cdot : F \times F \to F$, $f, g \mapsto f \cdot g$, are in $\mathrm{Mor}(\mathcal{F}^* \times \mathcal{F}^*, \mathcal{F}^*)$, while the scalar multiplication $s : \mathbb{R} \times F \to F$, $(\lambda, f) \mapsto \lambda f$, is in $\mathrm{Mor}(\mathcal{R} \times \mathcal{F}^*, \mathcal{F}^*)$.*

Proof. It is clear that for every $x \in X$ the following equalities hold: $\hat{x} \circ + = (\hat{x} \circ \pi_1) + (\hat{x} \circ \pi_2) \in F^* \times F^*$, $\hat{x} \circ \cdot = (\hat{x} \circ \pi_1) \cdot (\hat{x} \circ \pi_2) \in F^* \times F^*$, and $\hat{x} \circ s = (\mathrm{id}_{\mathbb{R}} \circ \pi_1) \cdot (\hat{x} \circ \pi_2) \in \mathrm{Bic}(\mathbb{R}) \times F^*$. $\square$

Proposition 4.4.5. *If (X, d) is a metric space and $F \supseteq C_u(X)$ is a topology on X, then the function $d^* : X \to C_u(X)$ defined by $x \mapsto d_x$ is in $\mathrm{Mor}(\mathcal{F}, \mathcal{U}(X)^*)$.*

Proof. Since $\mathcal{U}(X)^* = (C_u(X), C_u(X)^*)$ and $C_u(X) \subseteq F$, it suffices to show that $\forall_{x \in X}(\hat{x} \circ d^* \in C_u(X))$. Since $(\hat{x} \circ d^*)(y) = \hat{x}(d_y) = d_y(x) = d(y, x) = d_x(y)$, it is easy to see that the uniform continuity of d_x implies the uniform continuity of $\hat{x} \circ d^*$ and that $\omega_{\hat{x} \circ d^*} = \omega_{d_x}$. $\square$

If X is a compact metric space, then the sup-norm and sup-metric on $C_u(X)$ are defined by

$$||f||_\infty := \sup\{|f(x)| \mid x \in X\}$$

$$d_\infty(f, g) := ||f - g||_\infty,$$

where the $\sup\{|f(x)| \mid x \in X\}$ exists because $f(X)$ is totally bounded and the supremum of a totally bounded subset of $\mathbb{R}$ always exists (see [15], p.38). It is easy to see that every function $\hat{x} : C_u(X) \to \mathbb{R}$ is uniformly continuous; if $f, g \in C_u(X)$ we have that

$$d_\infty(f, g) \leq \epsilon \to |\hat{x}(f) - \hat{x}(g)| = |f(x) - g(x)| \leq d_\infty(f, g) \leq \epsilon.$$

The set $\hat{x}(C_u(X))$ is an unbounded subset of $\mathbb{R}$, since the set $\hat{x}(\mathrm{Const}(X))$ is unbounded, therefore we cannot use the $\mathcal{F}$-lifting of uniform continuity to conclude that every $\phi : C_u(X) \to \mathbb{R} \in C_u(X)^*$ is a uniformly continuous function.

4.5 Weak Bishop spaces

Definition 4.5.1. *If $\mathcal{G} = (Y, G)$ is a Bishop space, X is an inhabited set and $\theta : X \to Y$, the weak topology $F(\theta)$ on X induced by θ is defined by*

$$F(\theta) := \mathcal{F}(F_0(\theta)),$$

$$F_0(\theta) := \{g \circ \theta \mid g \in G\}.$$

The space $\mathcal{F}(\theta) = (X, F(\theta))$ is called the weak Bishop space on X induced by θ.

The second part of the next proposition is proved already in [19], while the third shows how the induced topology is simplified if a subbase of G is given.

Proposition 4.5.2. *Suppose that $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ are Bishop spaces, X is an inhabited set, $\theta : X \to Y$, $G_0 \subseteq \mathbb{F}(Y, \mathbb{R})$, and $F(\theta)$ is the weak topology on X induced by θ.*
(i) $F(\theta)$ is the least topology on X which makes θ a morphism.
(ii) $h \in \mathrm{Mor}(\mathcal{H}, \mathcal{F}(\theta))$ if and only if $\theta \circ h \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$.
(iii) If $G = \mathcal{F}(G_0)$, then $F(\theta) = \mathcal{F}(F'_0(\theta)$, where $F'_0(\theta) := \{g_0 \circ \theta \mid g_0 \in G_0\}$.

Proof. (i) By Proposition 3.6.4 we have that $\theta \in \mathrm{Mor}(\mathcal{F}(\theta), \mathcal{G}) \leftrightarrow \forall_{g \in G}(g \circ \theta \in F(\theta))$, which clearly is the case, since $g \circ \theta \in F_0(\theta)$. Clearly, every topology $\mathcal{F}$ on X such that $\theta \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ includes $F_0(\theta)$, therefore it includes $F(\theta)$.
(ii) By Proposition 3.6.4 we have that $h \in \mathrm{Mor}(\mathcal{H}, \mathcal{F}(\theta)) \leftrightarrow \forall_{g \in G}((g \circ \theta) \circ h \in H) \leftrightarrow \forall_{g \in G}(g \circ (\theta \circ h) \in H) \leftrightarrow \theta \circ h \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$.
(iii) It suffices to show that $F_{0,\theta} \subseteq \mathcal{F}({F'}_0(\theta)) \leftrightarrow \forall_{g \in \mathcal{F}(G_0)}(g \circ \theta \in \mathcal{F}({F'}_0(\theta)))$. This follows from the standard inductive argument based on the equalities $(g_1 + g_2) \circ \theta = (g_1 \circ \theta) + (g_2 \circ \theta)$, $(\phi \circ g) \circ \theta = \phi \circ (g \circ \theta)$, for every $\phi \in \mathrm{Bic}(\mathbb{R})$, and the implication $g_n \xrightarrow{u} g \to g_n \circ \theta \xrightarrow{u} g \circ \theta$. $\square$

The next proposition shows that if θ is a surjection, the topology $F(\theta)$ is even simpler. This fact is used in the proof of Proposition 5.5.6 and it is related to the definition of the quotient topology (section 4.6).

Proposition 4.5.3. *Suppose that $\mathcal{G} = (Y, G)$ is a Bishop space, X is an inhabited set and $\theta : X \to Y$ is a surjection. Then $F(\theta) = F_0(\theta)$.*

Proof. By definition for every $f_0 \in F_0(\theta)$ there exists some $g \in G$ such that $f_0 = g \circ \theta$. Since θ is by hypothesis a set-epimorphism, by the $\mathcal{F}$-lifting of openness we get that for every $f \in F(\theta)$ there exists some $g \in G$ such that $f = g \circ \theta$. Hence, $F_0(\theta) \subseteq F(\theta) \subseteq F_0(\theta)$. $\square$

Definition 4.5.4. *If $\mathcal{F}_i = (X, F_i)$ is a family of Bishop spaces indexed by some set I, their supremum Bishop space is the structure $\bigvee_{i \in I} \mathcal{F}_{i \in I} = (X, \bigvee_{i \in I} F_i)$, where*

$$\bigvee_{i \in I} F_i := \mathcal{F}(\bigcup_{i \in I} F_i).$$

If for every $i \in I$, $F_i = \mathcal{F}(F_{0,i})$, for some $F_{0,i} \subseteq \mathbb{F}(X_i, \mathbb{R})$, we get that

$$\bigvee_{i \in I} \mathcal{F}(F_{0,i}) = \mathcal{F}(\bigcup_{i \in I} \mathcal{F}(F_{0,i})) = \mathcal{F}(\bigcup_{i \in I} F_{0,i}) = \bigvee_{i \in I} F_{0,i}.$$

Since the $\supseteq$-inclusion is trivial, it suffices to show that $\bigcup_{i \in I} \mathcal{F}(F_{0,i}) \subseteq \mathcal{F}(\bigcup_{i \in I} F_{0,i})$. This follows from the fact that for each $i \in I$ we have that $F_{0,i} \subseteq \bigcup_{i \in I} F_{0,i}$, hence $\mathcal{F}(F_{0,i}) \subseteq \mathcal{F}(\bigcup_{i \in I} F_{0,i})$, therefore $\bigcup_{i \in I} \mathcal{F}(F_{0,i}) \subseteq \mathcal{F}(\bigcup_{i \in I} F_{0,i})$.

Definition 4.5.5. *If $\mathcal{G}_i = (Y, G_i)$ is a Bishop space, $e_i : X \to Y_i$, and $F(e_i)$ is the weak topology on X induced by e_i, for every $i \in I$, the projective limit topology $\mathrm{Lim}_I F(e_i)$ on X determined by the families $\{\mathcal{G}_i \mid i \in I\}$ and $\{e_i \mid i \in I\}$ is defined by*

$$\mathrm{Lim}_I F(e_i) = \bigvee_{i \in I} F(e_i) = \mathcal{F}(\bigcup_{i \in I} F e_i)),$$

and the corresponding Bishop space is denoted by $\mathrm{Lim}_I \mathcal{F}(e_i)$.

By the previous remark we have that

$$\mathrm{Lim}_I F(e_i) = \mathcal{F}(\bigcup_{i \in I} \{g_i \circ e_i \mid g_i \in G_i\}).$$

The next proposition corresponds to the classical fact that the topology of a completely regular topological space is precisely the projective limit topology determined by the family of all continuous maps $f : X \to [0,1]$ (see [40], p.159). Within TBS $[0,1]$ is replaced by $\mathbb{R}$.

Proposition 4.5.6. *A topology F is the projective limit topology determined by F.*

Proof. If we take F as index-set and $Y_f = \mathbb{R}$, for every $f \in F$, we have that $F(f) = \mathcal{F}(\{\phi \circ f \mid \phi \in \mathrm{Bic}(\mathbb{R})\}) \subseteq F$ and $\mathrm{Lim}_F F(f) = \bigvee_{f \in F} F(f) = \mathcal{F}(\bigcup_{f \in F} F(f)) \subseteq F$. Since $f \in F \to f = \mathrm{id}_{\mathbb{R}} \circ f \in F(f)$, we get that $F = \mathrm{Lim}_F F(f)$. □

The next proposition is also proved in [19], p.103.

Proposition 4.5.7. *If X is an inhabited set and $\mathrm{Lim}_I F(e_i)$ is the projective limit topology on X determined by $\{\mathcal{G}_i \mid i \in I\}, E = \{e_i \mid i \in I\}$, and $\mathcal{H} = (Z, H)$ is a Bishop space, then $e : Z \to X \in \mathrm{Mor}(\mathcal{H}, \mathrm{Lim}_I \mathcal{F}(e_i))$ if and only if $e_i \circ e \in \mathrm{Mor}(\mathcal{H}, \mathcal{G}_i)$, for every $i \in I$.*

Proof. By the last description of $\mathrm{Lim}_I F(e_i)$ we have that

$$\begin{aligned} e \in \mathrm{Mor}(\mathcal{H}, \mathrm{Lim}_I \mathcal{F}(e_i)) &\leftrightarrow \forall_{i \in I} \forall_{g_i \in G_i} ((g_i \circ e_i) \circ e \in H) \\ &\leftrightarrow \forall_{i \in I} \forall_{g_i \in G_i} (g_i \circ (e_i \circ e) \in H) \\ &\leftrightarrow \forall_{i \in I} (e_i \circ e \in \mathrm{Mor}(\mathcal{H}, \mathcal{G}_i)). \end{aligned}$$

□

4.6 Quotient Bishop spaces

Definition 4.6.1. *If $\mathcal{F} = (X, F)$ is a Bishop space, Y is an inhabited set and $\phi : X \to Y$ is onto Y, the quotient topology G_ϕ on Y is defined by*

$$G_\phi := \{g \in \mathbb{F}(Y) \mid g \circ \phi \in F\}.$$

We call $\mathcal{G}_\phi = (Y, G_\phi)$ the quotient Bishop space of Y with respect to ϕ.

As it is noted in [58], "the standard construction of quotient spaces in (classical) topology uses full separation and power sets". Although Ishihara and Palmgren showed in [58] "how to make this construction using only the generalized predicative methods available in constructive type theory and constructive set theory", it is far more complex than the notion of quotient topology within Bishop spaces.
The next proposition shows the analogy between the basic properties of the quotient topology of functions and the quotient topology of open sets.

Proposition 4.6.2. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{H} = (Z, H)$ are Bishop spaces, Y is an inhabited set, G is a topology on Y, and $\phi : X \to Y$.*
(i) G_ϕ is the largest topology on Y which makes ϕ a morphism.
(ii) A function $h : Y \to Z \in \mathrm{Mor}(\mathcal{G}_\phi, \mathcal{H})$ if and only if $h \circ \phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$.
(iii) If ϕ is an open morphism from $\mathcal{F}$ to (Y, G), then $G = G_\phi$.

Proof. (i) The fact that G_ϕ is a topology on Y is straightforward; e.g., the condition BS_4 is proved as follows: if $g_n \stackrel{u}{\to} g$ such that $g_n \circ \phi \in F$, then it is easy to see that $g_n \circ \phi \stackrel{u}{\to} g \circ \phi$, hence $g \in G_\phi$. Consider G to be a topology on Y which makes ϕ a morphism from $\mathcal{F}$ to $\mathcal{G}$. By definition $g \circ \phi \in F$, for every $g \in G$, therefore $G \subseteq G_\phi$.
(ii) We have that $h \circ \phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \leftrightarrow \forall_{j \in H}(j \circ (h \circ \phi) \in F) \leftrightarrow \forall_{j \in H}((j \circ h) \circ \phi \in F) \leftrightarrow \forall_{j \in H}(j \circ h \in G_\phi) \leftrightarrow h \in \mathrm{Mor}(\mathcal{G}_\phi, \mathcal{H})$.
(iii) By (i) we have that $G \subseteq G_\phi$. We show that $G_\phi \subseteq G$. If $g \in G_\phi$, then $g \circ \phi \in F$, while by the hypothesis of openness on ϕ there exists $g' \in G$ such that $f = g' \circ \phi$. Since ϕ is onto Y, every $y \in Y$ is equal to $\phi(x)$, for some $x \in X$. Hence, $g'(y) = g'(\phi(x)) = f(x) = g(\phi(x)) = g(y)$ i.e., $g \in G$. $\square$

As a direct rephrasing of (iii), there is only one topology, G_ϕ, with respect to which ϕ is an open morphism. Although parts (i) and (ii) of the previous proposition are independent from the onto Y hypothesis on ϕ, this is necessary for the proof of (iii). In order to have a notion of a quotient topology of functions symmetric to the quotient topology of open sets we include this hypothesis in our definition, although we will use the notation G_ϕ for the corresponding Bishop topology, even if ϕ is not onto Y.
The converse to Proposition 4.6.2(iii) is not generally true i.e., a quotient mapping need not be open. Consider, for example, a setoid $(X, =)$, a decidable $Y \subseteq X$ inhabited by some y_0 such that $X \setminus Y$ contains at least two elements $y_1 \neq y_2$, and a topology F on X containing some f such that $f(y_1) \neq f(y_2)$. If $\phi : X \to Y$ is defined by $\phi(x) = x$, if $x \in Y$, and $\phi(x) = y_0$, otherwise, then for every $g : Y \to \mathbb{R}$ we get that $(g \circ \phi)(x) = g(x)$, if $x \in Y$, and $(g \circ \phi)(x) = g(y_0)$, otherwise. Thus, ϕ is not open since the aforementioned $f \in F$ cannot be written as $g \circ \phi$, as the latter is constant on $X \setminus Y$, while f is not.

Definition 4.6.3. *If $\mathcal{F}, \mathcal{G}$ are Bishop spaces, we call a function $e : F \to G$ a partial isometry, if $||e(f)||$ exists whenever $||f||$ exists and if it exists, then $||e(f)|| = ||f||$. We also call $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ strongly open, if $\forall_{f \in F} \exists!_{g \in G}(f = g \circ h)$.*

The next proposition describes a big class of quotient maps which are strongly open. Note that we do not use the axiom of choice, only the previous notation, since we first define the function ρ from f and then we show the corresponding uniqueness[2].

Proposition 4.6.4. *Let $\mathcal{F} = (X, F)$ be a Bishop space and $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2 \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2))$. If $\pi : X \to X/{\sim}$ is the function $x \mapsto [x]_\sim$ and $\mathcal{F}/{\sim} = (X/{\sim}, G_\pi)$ is the quotient Bishop space, then π is a strongly open morphism from $\mathcal{F}$ to $\mathcal{F}/{\sim}$, and the function*

$$\rho : F \to G_\pi,$$
$$f \mapsto \rho(f),$$
$$\rho(f)([x]_\sim) := f(x),$$

for every $f \in F$ and every $x \in X$, is a ring and a lattice homomorphism, and a partial isometry onto G_π.

[2]One can use Myhill's axiom in order to show that a strongly open morphism induces a function $\rho : F \to G$ such that $\forall_{f \in F}(f = \rho(f) \circ h)$.

Proof. Clearly, $\rho(f)$ is by the definition of the equivalence relation $\sim$ a well-defined function. In order to show that $\rho(f) \in G_\pi$ we need to show that $\rho(f) \circ \pi \in F$, which is true, since $(\rho(f) \circ \pi)(x) = \rho(f)(\pi(x)) = \rho(f)([x]_\sim) = f(x)$, for every $x \in X$, therefore $\rho(f) \circ \pi = f \in F$. Hence $\rho(f)$ satisfies the openness condition $f = \rho(f) \circ \pi$ for π. To show the strong openness of π, if $g \in G_\pi$ such that $f = g \circ \pi$, then since π is onto $X/{\sim}$ we have $g(\pi(x)) = f(x) = \rho(f)(\pi(x))$, for every $x \in X$, therefore $g = \rho(f)$. The fact that ρ is a partial isometry follows directly from the fact that $|\rho(f)([x]_\sim)| = |f(x)|$. To show that ρ is onto G_π we fix $g : X/{\sim} \to \mathbb{R}$ such that $g \circ \pi = f \in F$. Since $\rho(f)(\pi(x)) = f(x) = g(\pi(x))$ and π is onto $X/{\sim}$, we conclude that $\rho(f) = g$. To show that ρ is a ring and a lattice homomorphism we use the equalities:

$$\rho(f \diamond g)([x]_\sim) = (f \diamond g)(x) = f(x) \diamond g(x) = \rho(f)([x]_\sim) \diamond \rho(g)([x]_\sim),$$

where $\diamond \in \{+, \cdot, \wedge, \vee\}$. $\square$

The above result is used in section 5.8. Note that a similar proposition for continuous functions requires the restriction to compact topological spaces (see [36], p.148). Clearly, π is an open morphism which in general is not an isomorphism.

Proposition 4.6.5. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $\phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\theta \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ such that $\phi \circ \theta = \mathrm{id}_Y$, then ϕ is a quotient map.*

Proof. We know already that $G \subseteq G_\phi$. For the converse inclusion we fix some $g : Y \to \mathbb{R}$ such that $g \circ \phi \in F$. Since $\theta \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ we have that $(g \circ \phi) \circ \theta \in G$. Since $(g \circ \phi) \circ \theta = g \circ (\phi \circ \theta) = g \circ \mathrm{id}_Y = g$, we conclude that $g \in G$. $\square$

Proposition 4.6.6. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $\phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that ϕ is onto Y, then ϕ is a quotient map if and only if $\forall_{\mathcal{H}=(Z,H)} \forall_{h:Y \to Z}(h \circ \phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \to h \in \mathrm{Mor}(\mathcal{G}, \mathcal{H}))$.*

Proof. Necessity follows immediately from Proposition 4.6.2(ii). For the converse it suffices to show that $G_\phi \subseteq G$. We fix some $g : Y \to \mathbb{R}$ such that $g \circ \phi \in F$. By Proposition 3.6.2 we have that $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$ and the hypothesis $g \circ \phi \in F$ is written as $g \circ \phi \in \mathrm{Mor}(\mathcal{F}, \mathcal{R})$. By (ii) we have that $g \in \mathrm{Mor}(\mathcal{G}, \mathcal{R})$, and since $\mathrm{Mor}(\mathcal{G}, \mathcal{R}) = G$, we conclude that $g \in G$. $\square$

Proposition 4.6.7. *If $\mathcal{F} = (X, F)$ is a Bishop space, Y, Z are inhabited sets, $\phi : X \to Y$, $\psi : X \to Z$ are surjections, and $\phi \times \psi : X \to Y \times Z$ is defined by $x \mapsto (\phi(x), \psi(x))$, for every $x \in X$, then $G_\phi \times G_\psi \subseteq G_{\phi \times \psi}$, where this inclusion can be strict.*

Proof. Because of Proposition 4.6.2(i) we prove the required inclusion by showing that $\phi \times \psi \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_\phi \times \mathcal{G}_\psi)$. By Proposition 4.1.6(ii) we have that $\phi \times \psi \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_\phi \times \mathcal{G}_\psi)$ if and only if $\pi_1 \circ (\phi \times \psi) \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_\phi)$ and $\pi_2 \circ (\phi \times \psi) \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_\psi)$, as follows from the equalities $\pi_1 \circ (\phi \times \psi) = \phi$ and $\pi_2 \circ (\phi \times \psi) = \psi$, respectively. To give an example of a strict inclusion we consider an infinite inhabited set X with a decidable equality and a topology F of bounded functions on X. Since $F_{\mathrm{id}_X} = \{f : X \to \mathbb{R} \mid f \circ \mathrm{id}_X = f \in F\} = F$, by Proposition 4.1.9(i) we know that $F^2_{\mathrm{id}_X}$ is also a topology of bounded functions. Since

$\mathrm{id}_X^2 = \mathrm{id}_X \times \mathrm{id}_X : X \to X \times X$, defined by $x \mapsto (x, x)$, for every $x \in X$, and $F_{\mathrm{id}_X^2} = \{j : X \times X \to \mathbb{R} \mid j \circ \mathrm{id}_X^2 \in F\}$, if we define $j(x, x) = f(x)$, for some $f \in F$ and $j(x, y) = g(y)$, for $y \neq x$ and g is an unbounded function on X, therefore $g \notin F$. Since j itself is unbounded, we conclude that $j \notin F^2_{\mathrm{id}_X}$, where by definition $j \in F_{\mathrm{id}_X^2}$. $\square$

4.7 Relative Bishop spaces

Definition 4.7.1. *If $\mathcal{F} = (X, F)$ is a Bishop space and $Y \subseteq X$ is inhabited, the relative topology on Y is defined by*

$$F_{|Y} := \mathcal{F}(G_{0,Y}),$$

$$G_{0,Y} := \{f_{|Y} \mid f \in F\}.$$

We call the corresponding Bishop space $\mathcal{F}_{|Y} = (Y, F_{|Y})$ the relative Bishop space of $\mathcal{F}$ on Y.

In the extreme cases $F = \mathrm{Const}(X)$, or $F = \mathbb{F}(X)$ the corresponding subbases are also Bishop spaces. Note also that in general $\mathbb{F}_b(X)_{|Y} \subseteq \mathbb{F}_b(Y)$, and we get equality only if a bounded function on Y can be extended to a bounded function on X (this extendability will be studied in section 5.5). In [19], p.109, a *function subspace* is defined as the weak Bishop space generated by the canonical topological embedding $i : Y \to X$, in accordance with the general definition of a subset given in [15]. If we consider $i = \mathrm{id}_Y$, then the weak topology F^{id_Y} induced on Y by id_Y has as a subbase the set $F_{0,\mathrm{id}_Y} = \{f \circ \mathrm{id}_Y \mid f \in F\} = G_{0,Y}$, since $f \circ \mathrm{id}_Y = f_{|Y}$, therefore the two definitions coincide.
The next proposition expresses the standard simplification of the definition of the relative topology in case a subbase of the initial space is given.

Proposition 4.7.2. *If $\mathcal{F} = (X, \mathcal{F}(F_0))$ is a Bishop space, then the relative topology on some inhabited $Y \subseteq X$ is given by*

$$\mathcal{F}(F_0)_{|Y} := \mathcal{F}(G_{00,Y}),$$

$$G_{00,Y} := \{f_{0|Y} \mid f_0 \in F_0\}.$$

Proof. Since $G_{00,Y} \subseteq G_{0,Y}$ already, it suffices to show that $G_{0,Y} \subseteq \mathcal{F}(G_{00,Y})$ if and only if $\forall_{f \in \mathcal{F}(F_0)}(f_{|Y} \in \mathcal{F}(G_{00,Y}))$. If $f = f_0 \in F_0$, then $f_{0|Y} \in G_{00,Y}$. If $f_1, f_2 \in F$ such that $f_{1|Y}, f_{2|Y} \in \mathcal{F}(G_{00,Y})$, then $f_{1|Y} + f_{1|Y} = (f_1 + f_2)_{|Y} \in \mathcal{F}(G_{00,Y})$. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \mathcal{F}(F_0)$ such that $f_{|Y} \in \mathcal{F}(G_{00,Y})$, then $\phi \circ f_{|Y} = (\phi \circ f)_{|Y} \in \mathcal{F}(G_{00,Y})$. Finally, if $f_n \subseteq \mathcal{F}(F_0)$ such that $f_{n|Y} \in \mathcal{F}(G_{00,Y})$, for each n, and $f_n \stackrel{u}{\to} f$, for some $f \in \mathcal{F}(F_0)$, then $f_{n|Y} \stackrel{u}{\to} f_{|Y} \in \mathcal{F}(G_{00,Y})$, using BS_4 property of $\mathcal{F}(G_{00,Y})$. $\square$

Since $\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}})$, if $Y \subseteq \mathbb{R}$, then $\mathrm{Bic}(\mathbb{R})_{|Y} = \mathcal{F}((\mathrm{id}_{\mathbb{R}})_{|Y}) = \mathcal{F}(\mathrm{id}_Y)$. Clearly, id_Y is a morphism which is not in general open.

Proposition 4.7.3. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{H} = (Z, H)$ are Bishop spaces, and $Y \subseteq X, B \subseteq Z$ are inhabited.*
(i) $F_{|Y}$ is the smallest topology G on Y satisfying the property $\mathrm{id}_Y \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$.
(ii) If $e : X \to B$, then $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \leftrightarrow e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_{|B})$. If e is open as a morphism from $\mathcal{F}$ to $\mathcal{H}$, then it is open as a morphism from $\mathcal{F}$ to $\mathcal{H}_{|B}$.
(iii) If $e : X \to Z$, then $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \to e_{|Y} \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H})$.
(iv) If $e : X \to Z$ is an isomorphism, then $e_{|Y}$ is an isomorphism between $\mathcal{F}_{|Y}$ and $\mathcal{H}_{|e(Y)}$, while $e_{|X\setminus Y}$ is an isomorphism between $\mathcal{F}_{|(X\setminus Y)}$ and $\mathcal{H}_{|e(X\setminus Y)}$.

Proof. (i) $\mathrm{id}_Y \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{F}) \leftrightarrow f \circ \mathrm{id}_Y \in F_{|Y}$, for every $f \in F$, which holds, since $f \circ \mathrm{id}_Y = f_{|Y} \in G_{0,Y}$. If G is a topology on Y such that $\mathrm{id}_Y \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, then $f \circ \mathrm{id}_Y \in G$, for every $f \in F$ i.e., $G_{0,Y} \subseteq G$, therefore $F_{|Y} \subseteq G$.
(ii) By Proposition 3.6.4 we have that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \leftrightarrow \forall_{h\in H}(h \circ e \in F) \leftrightarrow \forall_{h\in H}(h_{|B} \circ e \in F) \leftrightarrow \forall_{h\in H_{|B}}(h \circ e \in F) \leftrightarrow e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_{|B})$. Suppose that $\forall_{f\in F}\exists_{h\in H}(f = h \circ e)$. Since $h \circ e = h_{|B} \circ e$ and $h \in H \to h_{|B} \in H_{|B}$, we get that $\forall_{f\in F}\exists_{h'\in H_{|B}}(f = h' \circ e)$, where $h' = h_{|B}$.
(iii) As in (ii), we have that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \leftrightarrow \forall_{h\in H}(h \circ e \in F) \to \forall_{h\in H}((h \circ e)_{|Y} \in F_{|Y}) \leftrightarrow \forall_{h\in H}(h \circ e_{|Y} \in F_{|Y}) \leftrightarrow e_{|Y} \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H})$.
(iv) By (iii) we have that $e_{|Y}$ is a morphism from $\mathcal{F}_{|Y}$ to $\mathcal{H}$, while by (ii) we get that it is a morphism from $\mathcal{F}_{|Y}$ to $\mathcal{H}_{|e(Y)}$. Since e^{-1} is also an isomorphism, we conclude similarly that $e^{-1}_{|e(Y)} = (e_{|Y})^{-1}$ is a morphism from $\mathcal{H}_{|e(Y)}$ to $\mathcal{F}_{|Y}$. For the second part it suffices to show, because of the first part, that $e(X \setminus Y) = Z \setminus e(Y)$. First we show that $e(X \setminus Y) \subseteq Z \setminus e(Y)$ i.e., $x \notin Y \to e(x) \notin e(Y)$; Suppose that $e(x) \in e(Y)$ i.e., there exists $y \in Y$ such that $e(x) = e(y)$. Since e is $1{-}1$ we conclude $x = y \in Y$, which is a contradiction. For the converse inclusion we fix some $z \in Z \setminus e(Y)$. Since e is onto Z there exists some $x \in X$ such that $e(x) = z$. We show that $x \notin Y$; if $x \in Y$, then $e(x) = z \in e(Y)$, which is a contradiction. □

Proposition 4.7.4. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h : X \to Y$, then the function $h' : X \to \mathrm{Gr}(h) = \{(x, h(x)) \mid x \in X\} \subseteq X \times Y$, defined by $x \mapsto (x, h(x))$, for every $x \in X$, is an isomorphism between $\mathcal{F}$ and $(\mathrm{Gr}(h), (F \times G)_{|\mathrm{Gr}(h)})$ if and only if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$.*

Proof. By Proposition 4.7.3(ii) we have that $h' \in \mathrm{Mor}(\mathcal{F}, (\mathcal{F}\times\mathcal{G})_{|\mathrm{Gr}(h)}) \leftrightarrow h' \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G}) \leftrightarrow \forall_{f\in F}((f \circ \pi_1) \circ h' \in F) \wedge \forall_{g\in G}((g \circ \pi_2) \circ h' \in F) \leftrightarrow \forall_{g\in G}(g \circ h \in F) \leftrightarrow h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, since $(f \circ \pi_1) \circ h' = f$, for every $f \in F$. This fact also shows that h' is open, and $(g \circ \pi_2) \circ h' = g \circ h$, for every $g \in G$. It is clear that h' is $1{-}1$. □

Proposition 4.7.5. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $A \subseteq X$, $B \subseteq Y$ are inhabited, then $(F \times G)_{|A\times B} = F_{|A} \times G_{|B}$.*

Proof. By Proposition 4.7.2 we have that $(F \times G)_{|A\times B} = \mathcal{F}(\{h_{|A\times B} \mid h \in F \times G\}) = \mathcal{F}(\{(f \circ \pi_1)_{|A\times B} \mid f \in F\} \cup \{(g \circ \pi_2)_{|A\times B} \mid g \in G\}) = \mathcal{F}(\{f_{|A} \circ \pi_1 \mid f \in F\} \cup \{g_{|B} \circ \pi_2 \mid g \in G\}) = F_{|A} \times G_{|B}$. □

Definition 4.7.6. *If $\mathcal{F}_i = (X_i, F_i)$ is a family of Bishop spaces indexed by some inhabited set I and $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$, then the slice $S(x; j)$ through x parallel to x_j, where $j \in I$, is the set*

$$S(x; j) := X_j \times \prod_{i \neq j} \{x_i\} \subseteq \prod_{i \in I} X_i$$

of all I-tuples where all components other the j-component are the ones of x, while the j-component ranges over X_j.

The next fact is used in the proof of Proposition 5.6.7.

Proposition 4.7.7. *If $\mathcal{F}_i = (X_i, F_i)$ is a family of Bishop spaces indexed by some inhabited set I and $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$, then the function*

$$s_j : X_j \to S(x; j),$$

$$x_j \mapsto x_j \times \prod_{i \neq j} \{x_i\},$$

for every $x_j \in X_j$, where $S(x; j)$ is the slice through x parallel to x_j, is an isomorphism between $\mathcal{F}_j$ and $\mathcal{S}(x; j) = (S(x; j), F(x; j))$, where $F(x; j) = (\prod_{i \in I} F_i)_{|S(x;j)}$.

Proof. It is clear that s_j is 1−1 and onto $S(x; j)$. By Propositions 4.7.3(ii) and 4.1.6(ii)

$$\begin{aligned} s_j \in \mathcal{S}(x; j) &\leftrightarrow s_j \in \mathrm{Mor}(\mathcal{F}_j, \prod_{i \in I} \mathcal{F}_i) \\ &\leftrightarrow \forall_{i \in I}(\pi_i \circ s_j \in \mathrm{Mor}(\mathcal{F}_j, \mathcal{F}_i)) \\ &\leftrightarrow \forall_{i \in I}(\pi_i \circ s_j \in \mathrm{Mor}(\mathcal{F}_j, \mathcal{F}_{i|(\pi_i \circ s_j)(X_j)})) \end{aligned}$$

which is true, since $\pi_j \circ s_j = \mathrm{id}_{X_j}$ and $(\pi_j \circ s_j)(X_j) = X_j$, while $\pi_i \circ s_j = \overline{x_i}$ and $(\pi_i \circ s_j)(X_j) = \{x_i\}$, if $i \neq j$. Next we show that $s_j^{-1} : S(x; j) \to X_j \in \mathrm{Mor}(\mathcal{S}(x; j), \mathcal{F}_j)$ i.e., $\forall_{f \in F_j}(f \circ s_j^{-1} \in F(x; j))$. By Proposition 4.7.5 we have that

$$\begin{aligned} F(x; j) &= (\prod_{i \in I} F_i)_{|(X_j \times \prod_{i \neq j} \{x_i\})} \\ &= F_{j|X_j} \times \prod_{i \in I} F_{i|\{x_i\}} \\ &= F_j \times \prod_{i \in I} F_{i|\{x_i\}} \\ &= F(\{f \circ \pi_j \mid f \in F_j\} \cup \bigcup_{i \neq j} \{\overline{a} \circ \pi_i \mid a \in \mathbb{R}\}). \end{aligned}$$

Since $(f \circ s_j^{-1})(x_j \times \prod_{i \neq j} \{x_i\}) = f(x_j) = (f \circ \pi_j)(x_j \times \prod_{i \neq j} \{x_i\})$, we conclude that $f \circ s_j^{-1} = f \circ \pi_j \in F(x; j)$, for every $f \in F_j$. $\square$

Definition 4.7.8. *If (X, F) is a Bishop space and $Y \subseteq X$ is inhabited, a retraction of X onto Y is a function $r : X \to Y$ such that $r(y) = y$, for every $y \in Y$, and $r \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}_{|Y})$. In this case Y is called a retract of X.*

In section 6.2 we explain why the Cantor space endowed with the product topology on $(2, \mathbb{F}(2))$ is a retract of the Baire space endowed with the product topology on $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$.

Proposition 4.7.9. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space and $A \subseteq X$ is inhabited. If $r : X \to A$ is a retraction of X onto A, then r is a quotient map.*

Proof. It suffices to show that $F_{|A} = G_r$. Since G_r is the largest topology such that r is a morphism, we get $F_{|A} \subseteq G_r$. If $g \in G_r$, then $(g \circ r)_{|A} = g \in F_{|A}$ i.e., $F_{|A} \supseteq G_r$ $\square$

Proposition 4.7.10. *If (X, F) and (Y, G) are Bishop spaces, the pointwise exponential topology $F \to G$ on $X \to Y$ coincides with the topology of a subspace of the product $\prod_{x \in X} G_x$, where $G_x = G$, for every $x \in X$.*

Proof. We consider the Bishop space $(X \to Y, (\prod_{x \in X} G)_{|X \to Y})$, where

$$\prod_{x \in X} G = \mathcal{F}(\bigcup_{x \in X} \{g \circ \pi_x \mid g \in G\}),$$

$$(\prod_{x \in X} G)_{|X \to Y} = \mathcal{F}(\bigcup_{x \in X} \{(g \circ \pi_x)_{|X \to Y} \mid g \in G\}),$$

$$(g \circ \pi_x)(h) = g(\pi_x(h)) = g(h(x)) = e_{x,g}(h),$$

for every $h \in X \to Y$. Consequently, $g \circ \pi_x = e_{x,g}$ and $(\prod_{x \in X} G)_{|X \to Y} = F \to Y$. $\square$

Next follows a basic lemma of constructive analysis which, without the uniqueness property, is shown in [15], pp.91-2, while the uniqueness property is included in [71], p.238.

Lemma 4.7.11. *If $D \subseteq X$ is a dense subset of the metric space X, Y is a complete metric space, and $f : D \to Y$ is uniformly continuous with modulus of continuity ω, there exists a unique uniform continuous extension $g : X \to Y$ of f with modulus of continuity $\frac{1}{2}\omega$.*

Corollary 4.7.12. *If X is a compact metric space and D is dense in X, then $C_u(D)$ is a topology on D such that $C_u(D) = C_u(X)_{|D} = \{f_{|D} \mid f \in C_u(X)\}$.*

Proof. Clearly, $\{f_{|D} \mid f \in C_u(X)\} \subseteq C_u(D)$, and by Lemma 4.7.11 we get the inverse inclusion, hence $C_u(D) = \{f_{|D} \mid f \in C_u(X)\}$. We get all the required equalities, if we show that $C_u(D)$ is a topology on D. Clearly, $\mathrm{Const}(D) \subseteq C_u(D)$, and if $f, g \in C_u(D)$, then $f + g \in C_u(D)$. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $g \in C_u(D)$, then by Lemma 4.7.11 $g = f_{|D}$, for some $f \in C_u(D)$, therefore $\phi \circ g = \phi \circ f_{|D} = (\phi \circ f)_{|D} \in C_u(D)$, since $\phi \circ f \in C_u(X)$. The standard $\frac{\epsilon}{3}$-argument shows that $U(C_u(D), g) \to g \in C_u(D)$. $\square$

The next lemma is a very useful generalization of Lemma 4.7.11.

Lemma 4.7.13. *Suppose that X is an inhabited metric space, $D \subseteq X$ is dense in X and Y is a complete metric space. If $f : D \to Y$ is uniformly continuous on every bounded subset of D, then there exists a unique extension $g : X \to Y$ of f which is uniformly continuous on every bounded subset of X with modulus of continuity*

$$\omega_{g,B}(\epsilon) = \frac{1}{2}\omega_{f,B\cap D}(\epsilon),$$

for every inhabited, bounded and metric-open subset B of X. Moreover, if f is bounded by some $M > 0$, then g is also bounded by M.

Proof. Because of our definition of a bounded subset of a metric space it suffices to show the existence of some g which is uniformly continuous on every inhabited, bounded and metric-open subset B of X. If $x \in B$, there exists some $\epsilon > 0$, such that $\mathcal{B}(x, \epsilon) \subseteq B$. Since X is inhabited and D is dense, D is also inhabited and the set $B \cap D$ is a bounded subset of D. If we fix some $x \in B$ and $\mathcal{B}(x, \epsilon) \subseteq B$, for some $\epsilon > 0$, then $\mathcal{B}(x, \epsilon) \cap D \subseteq B \cap D$, and by the density of D there exists a sequence $d_n \subseteq D$ such that $d_n \to x$. Without loss of generality we may assume that $(d_n)_n \subseteq \mathcal{B}(x, \epsilon) \cap D$. If $\epsilon' > 0$ and $\omega_{f,B\cap D}$ is the modulus of continuity of f on $B \cap D$, we have that $\rho(d_n, d_m) \leq \omega_{f,B\cap D}(\epsilon')$, for every $n \geq n_0(\omega_{f,B\cap D}(\epsilon'))$, hence $\rho(f(d_n), f(d_m)) \leq \epsilon'$ i.e., the sequence $(f(d_n))_n$ is a Cauchy sequence in Y. By completeness of Y there exists some $y \in Y$ such that $f(d_n) \to y$. We define

$$g(x) := y,$$

and we show that g doesn't depend on d_n and on B. For the latter suppose that $B' \supseteq \mathcal{B}(x, \epsilon^*)$, for some $\epsilon^* > 0$. Then we take the ball $\mathcal{B}(x, \min\{\epsilon, \epsilon^*\}) \subseteq B \cap B'$ and we consider $d_n \subseteq (B \cap B') \cap D$. By uniqueness of the limit in Y we get that $g(x)$ is uniquely determined. For the former we work as in [15], p.91. If $x_1, x_2 \in B$, such that $\rho(x_1, x_2) \leq \frac{1}{2}\omega_{f,B\cap D}(\epsilon')$, then we can assume without loss of generality that there are sequences $d_n^{(1)}, d_n^{(2)} \subseteq B \cap D$ such that $d_n^{(1)} \to x_1$ and $d_n^{(2)} \to x_2$, respectively. This is true exactly because there are $\epsilon_1, \epsilon_2 > 0$ such that $\mathcal{B}(x_1, \epsilon_1) \subseteq B$ and $\mathcal{B}(x_2, \epsilon_2) \subseteq B$. For every

$$n \geq n_0 = \max\{n_0(d_n^{(1)}, \frac{1}{4}\omega_{f,B\cap D}(\epsilon')), n_0(d_n^{(2)}, \frac{1}{4}\omega_{f,B\cap D}(\epsilon'))\}$$

we get that

$$\begin{aligned}
\rho(d_n^{(1)}, d_n^{(2)}) &\leq \rho(d_n^{(1)}, x_1) + \rho(x_1, x_2) + \rho(x_2, d_n^{(2)}) \\
&\leq \frac{1}{4}\omega_{f,B\cap D}(\epsilon') + \frac{1}{2}\omega_{f,B\cap D}(\epsilon') + \frac{1}{4}\omega_{f,B\cap D}(\epsilon') \\
&= \omega_{f,B\cap D}(\epsilon').
\end{aligned}$$

Hence, $\rho(f(d_n^{(1)}), f(d_n^{(2)})) \leq \epsilon'$. Since, $\epsilon' \geq \rho(g(x_1), g(x_2)) = \lim \rho(f(d_n^{(1)}), f(d_n^{(2)}))$, we conclude that, if $x_1 = x_2$, then $g(x_1) = g(x_2)$ i.e., g is a well-defined function, and also uniformly continuous on B with $\omega_{g,B}(\epsilon') = \frac{1}{2}\omega_{f,B\cap D}(\epsilon')$. If $d \in D$, then $g(d) = f(d)$, since

we may consider the constant sequence $d_n = d$, for every n, converging to d. The uniqueness of g is proved as in [71], p.238.
If $|f(d)| \leq M$, for every $d \in D$, and if $d_n \to x$, for some $x \in X$, then $|g(x)| \leq |g(x) - f(d_n)| + |f(d_n)| \leq \epsilon + M$, for every $n \geq n_0(\epsilon)$. Since $\epsilon > 0$ is arbitrary, we get that $|g(x)| \leq M$, and since $x \in X$ is arbitrary, we conclude that $|g(x)| \leq M$, for every $x \in X$. $\square$

Definition 4.7.14. *An inhabited subset A of a locally compact metric space X is a Bishop subset of X, if $\mathrm{Bic}(A)$ is a Bishop topology on A.*

The next proposition offers a generalization of Corollary 4.7.12.

Proposition 4.7.15. *Suppose that X is a locally compact metric space, $A \subseteq X$ is inhabited, $D \subseteq X$ is dense in X, and $a, b \in \mathbb{R}$ such that $a < b$.*
(i) If $\mathrm{Bic}(A) \subseteq \mathbb{F}_{lb}(A)$, then A is a Bishop subset of X.
(ii) (a, b) is a Bishop subset of $\mathbb{R}$.
(iii) If A is locally compact, then A is a Bishop subset of X.
(iv) If $f \in \mathbb{F}(A)$ is extended to some $g \in \mathrm{Bic}(X)$, then f is locally bounded.
(v) D is a Bishop subset of X, and $\mathrm{Bic}(D) = \mathrm{Bic}(X)_{|D} = \{f_{|D} \mid f \in \mathrm{Bic}(X)\}$.

Proof. (i) The proof is similar to the proof that $\mathrm{Bic}(X)$ is a topology, if X is a locally compact metric space. We show only BS_3; if $\phi \in \mathrm{Bic}(\mathbb{R})$, $f \in \mathrm{Bic}(A)$, and B is a bounded subset of A, then $f(B)$ is a bounded subset of $\mathbb{R}$, and $\phi \circ f$ is uniformly continuous on B with modulus of continuity $\omega_{\phi \circ f, B} = \omega_{f,B} \circ \omega_{\phi, f(B)}$.
(ii) If $\phi \in \mathrm{Bic}((a, b))$, then f is uniformly continuous on (a, b), and since (a, b) is totally bounded $f((a, b))$ is bounded. Hence, ϕ is locally bounded, and we use (i).
(iii) If $f \in \mathrm{Bic}(A)$ and B is a bounded subset of A, B is included in a compact subset K of A, hence $f(B) \subseteq f(K)$, which is bounded, since K is totally bounded i.e., $\mathrm{Bic}(A) \subseteq \mathbb{F}_{lb}(A)$.
(iv) If B is a bounded subset of A, it is a bounded subset of X, hence it is included in a compact subset K of X. Consequently, $f(B) = g(B) \subseteq g(K)$, therefore it is bounded.
(v) By Lemma 4.7.13 there is some $g \in \mathrm{Bic}(X)$ which extends $f \in \mathrm{Bic}(D)$, hence by (iii) every $f \in \mathrm{Bic}(D)$ is locally bounded, and by (i) we get that D is a Bishop subset of X. By definition $\mathrm{Bic}(X)_{|D} = \mathcal{F}(\{f_{|D} \mid f \in \mathrm{Bic}(X)\})$. Clearly, $\{f_{|D} \mid f \in \mathrm{Bic}(X)\} \subseteq \mathrm{Bic}(D)$. Since $\mathrm{Bic}(D)$ is a Bishop topology on D and $\mathrm{Bic}(X)_{|D}$ is the least topology including $\{f_{|D} \mid f \in \mathrm{Bic}(X)\}$, we get that $\mathrm{Bic}(X)_{|D} \subseteq \mathrm{Bic}(D)$. For the converse inclusion we fix some $g \in \mathrm{Bic}(D)$ and by Lemma 4.7.13 there is some $f \in \mathrm{Bic}(X)$ such that $f_{|D} = g$ i.e., $\mathrm{Bic}(D) \subseteq \{f_{|D} \mid f \in \mathrm{Bic}(X)\} \subseteq \mathrm{Bic}(X)_{|D}$. $\square$

Chapter 5

Apartness in Bishop spaces

> ... computational success depends on the ability to distinguish objects rather than to show that they are close.
>
> D. S. Bridges and L.S. Vîţă 2002

Among the various approaches to constructive topology, TBS is mostly connected to the theory of apartness spaces, as it is developed by Bridges and Vîţă in [27]. The recognition that separating points is more important for the needs of topology rather than showing their nearness was behind the development of the theory of apartness spaces. A Bishop topology on X induces a natural notion of point-point apartness and of set-set apartness relation on X. These notions of "internal inequality" of points or of "internal separation" separation of subsets are crucial to the translation of parts of the classical theory of the rings of continuous functions to TBS. We introduce the Hausdorff Bishop spaces with respect to a given apartness relation, we study the zero sets of a Bishop space and we prove the Urysohn lemma for them. We translate the basic classical theory of embeddings of rings of continuous functions into TBS and we prove the Urysohn extension theorem for Bishop spaces. The tightness of the point-point apartness relation induced by some topology F on X guarantees that F determines the equality of X. We introduce the completely regular Bishop spaces, and we prove, among other facts, the Tychonoff embedding theorem which characterises them.

5.1 The canonical apartness relations

Definition 5.1.1. *The canonical point-point apartness relation on X induced by some topology F on X is defined, for every $x_1, x_2 \in X$, by*

$$x_1 \bowtie_F x_2 :\leftrightarrow \exists_{f \in F}(f(x_1) \bowtie_{\mathbb{R}} f(x_2)).$$

The apartness $\bowtie_F$ was introduced by Bridges in [19], and together with the canonical set-set apartness and the canonical point-set apartness induced by F is one of the most important and fruitful notions in TBS.

Proposition 5.1.2. *The apartness relations $\bowtie_{\mathbb{R}}$ and $\bowtie_{\mathrm{Bic}(\mathbb{R})}$ on $\mathbb{R}$ are equal.*

Proof. If $a \bowtie_{\mathrm{Bic}(\mathbb{R})} b$, then $\phi(a) \bowtie_{\mathbb{R}} \phi(b)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$. If $a, b \in I$, where I is an interval, by Remark 2.3.12(i) we have that $a \bowtie_{\mathbb{R}} b$. If $a \bowtie_{\mathbb{R}} b$, then $\mathrm{id}_{\mathbb{R}}(a) \bowtie_{\mathbb{R}} \mathrm{id}_{\mathbb{R}}(b)$. $\square$

The sufficiency condition of the following equivalence is already proved in [19], p.102.

Proposition 5.1.3. *If F is a topology on X, then the canonical apartness $\bowtie_F$ induced by F is tight if and only if $\forall_{x_1,x_2 \in X}(\forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2)$.*

Proof. We fix $x_1, x_2 \in X$ and we suppose that $\forall_{f \in F}(f(x_1) = f(x_2))$. In order to show that $x_1 = x_2$ it suffices to show that $\neg(x_1 \bowtie x_2)$. Suppose that $x_1 \bowtie x_2$. By definition there exists some $f \in F$ such that $f(x_1) \bowtie f(x_2)$. Our initial hypothesis implies that for that f we have that $f(x_1) = f(x_2)$, therefore by the first clause in the definition of the apartness relation we reach $\perp$ i.e., we have proved $\neg(x_1 \bowtie x_2)$. For the converse we suppose that $\neg(x_1 \bowtie x_2)$. If $f \in F$, we show that $f(x_1) = f(x_2)$; since the apartness relation on $\mathbb{R}$ is tight, it suffices to prove that $\neg(f(x_1) \bowtie f(x_2))$. If we suppose $f(x_1) \bowtie f(x_2)$, we get by definition that $x_1 \bowtie x_2$, and consequently by our initial hypothesis we take $\perp$. Since f is arbitrary we conclude that $\forall_{f \in F}(f(x_1) = f(x_2))$, therefore $x_1 = x_2$. $\square$

Definition 5.1.4. *A topology F on some X is called pointed, if*

$$\forall_{x \in X} \exists_{f \in F}(\{x\} = \{y \in X \mid f(y) = 0\}).$$

Proposition 5.1.5. *If F is a pointed topology on X, the canonical apartness $\bowtie_F$ is tight.*

Proof. If $x, y \in X$, $\forall_{f \in F}(f(x) = f(y))$, and $f_x \in F$ such that $\{x\} = \{y \in X \mid f_x(y) = 0\}$, then $f_x(y) = f_x(x) = 0$, therefore $y \in f^{-1}(\{0\}) = \{x\}$ i.e., $y = x$. $\square$

Since for every $a \in \mathbb{R}$ the polynomial $x - a \in \mathrm{Bic}(\mathbb{R})$, we conclude that $\mathrm{Bic}(\mathbb{R})$ is a pointed topology and by previous proposition we get another proof of $\bowtie_{\mathbb{R}}$ being tight. If (X, d) is a metric space then any topology F on X including $U_0(X)$ is pointed, since $\{x\} = d_x^{-1}(\{0\})$, hence its canonical apartness is tight. A trivial example of a pointed topology on some abstract inhabited set X with a decidable equality is the one with subbase $F_0 = \{f_x \mid x \in X\}$, where $f_x(y) = 1$, if $y \neq x$ and $f_x(y) = 0$, if $y = x$. The next corollary is a useful characterization of tightness in the presence of a subbase.

Corollary 5.1.6. *If $\mathcal{F}(F_0)$ is a topology on X, the following are equivalent:*
(i) $\forall_{x_1,x_2 \in X}(\forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2)$.
(ii) $\forall_{x_1,x_2 \in X}(\forall_{f_0 \in F_0}(f_0(x_1) = f_0(x_2)) \to x_1 = x_2)$.

Proof. Direction (ii) $\to$ (i) is trivial. For the converse we fix $x_1, x_2 \in X$ and we suppose that $\forall_{f_0 \in F_0}(f_0(x_1) = f_0(x_2))$. By Proposition 3.4.8 the property that every $f_0 \in F_0$ is constant on $\{x_1, x_2\}$ is lifted to F. Hence, the hypothesis of (i) is satisfied, and $x_1 = x_2$. $\square$

Definition 5.1.7. *The canonical set-set apartness relation on X and the canonical point-set apartness relation on X induced by F are defined, for every $A, B \subseteq X$ and $x \in X$, respectively, by*

$$A \bowtie_F B :\leftrightarrow \exists_{f \in F} \forall_{a \in A} \forall_{b \in B} (f(a) = 0 \wedge f(b) = 1).$$

$$x \bowtie_F B :\leftrightarrow \exists_{f \in F} \forall_{b \in B} (f(x) = 0 \wedge f(b) = 1).$$

Clearly, the set-set apartness relation $\bowtie_F$ extends the canonical point-set apartness $\bowtie_F$, which extends the canonical point-point apartness $\bowtie_F$, where we use for simplicity the same notation for all of them. The set-set apartness was defined already in [19], where Bridges showed that it satisfies the axioms of a set-set apartness relation and the Efremovič condition. If $x \in X$ and $B \subseteq X$, we write $x \bowtie_F B$ instead of $\{x\} \bowtie_F B$. If $f \in F$ separates A, B, we may also write $A \bowtie_f B$.

Proposition 5.1.8. *If $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces, $A, B, A_1, A_2, B_1, B_2 \subseteq X$, $f_1, f_2 \in F$, $C, D \subseteq Y$ and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then the following hold:*
(i) If $A_1 \bowtie_{f_1} B_1$ and $A_2 \bowtie_{f_2} B_2$, then $(A_1 \cup A_2) \bowtie_{f_1 \cdot f_2} (B_1 \cap B_2)$.
(ii) If there exists some $f \in F$ such that $f(A) \leq 0$ and $f(B) \geq 1$, then $A \bowtie_g B$ and $g \in F$.
(iii) If $A \bowtie_F B$, then $A \bowtie_f B$, for some $f \in F$ such that $f \geq \overline{0}$.
(iv) If $A \bowtie_F B$ and $C \bowtie_G D$, then $A \times C \bowtie_{F \times G} B \times D$.
(v) If $C \bowtie_G D$, then $h^{-1}(C) \bowtie_F h^{-1}(D)$.

Proof. (i) and (v) are immediate.
(ii) It suffices to consider $g = (\overline{0} \vee f) \wedge \overline{1} \in F$.
(iii) If $A \bowtie_g B$, then $A \bowtie_{g \vee \overline{0}} B$.
(iv) If $f \in F$ such that $f(A) = 0, f(B) = 1$, and $g \in G$ such that $g(C) = 0, g(D) = 1$, then $h(A \times C) = 0$ and $h(B \times D) = 1$, where

$$h = \frac{(f \circ \pi_1)^2 + (g \circ \pi_2)^2}{2} \in F \times G.$$

. ∎

Proposition 5.1.9. *Suppose that (X, F) is a Bishop space and Φ_0 is a base of F. Then the following hold:*
(i) $x \bowtie_F y \to x \bowtie_{\Phi_0} y$, for every $x, y \in X$.
(ii) If F separates the points of X, then Φ_0 separates the points of X.

Proof. (i) If $x \bowtie_f y$ and $U(\theta, f, \frac{\epsilon_0}{4})$, where $f \in F$, $\theta \in \Phi_0$ and $\epsilon_0 = |f(x) - f(y)|$, then

$$\begin{aligned} \epsilon_0 &\leq |f(x) - \theta(x)| + |\theta(x) - \theta(y)| + |\theta(y) - f(y)| \leftrightarrow \\ \epsilon_0 &\leq \frac{\epsilon_0}{4} + |\theta(x) - \theta(y)| + \frac{\epsilon_0}{4} \leftrightarrow \\ 0 &< \frac{\epsilon_0}{2} \leq |\theta(x) - \theta(y)|. \end{aligned}$$

(ii) If $x, y \in X$ such that $\forall_{\theta \in \Phi} (\theta(x) = \theta(y))$, it suffices to show that $\forall_{f \in F} (f(x) = f(y))$. We fix some $f \in F$ and by the tightness of $a \bowtie_{\mathbb{R}} b$ it suffices to show that $\neg(f(x) \bowtie_{\mathbb{R}} f(y))$. If

$f(x) \bowtie_{\mathbb{R}} f(y)$, then we set $\epsilon_0 = |f(x) - f(y)| > 0$. If $\theta \in \Phi_0$ such that $U(\theta, f, \frac{\epsilon_0}{4})$, we work as in the case (i), where the term $|\theta(x) - \theta(y)|$ by our hypothesis vanishes, and we reach the required absurdity. $\square$

Proposition 5.1.10. *Suppose that (X, F) is a Bishop space, $n \geq 2$, $a_1, \ldots, a_n \in \mathbb{R}$ and $A_1, \ldots, A_n \subseteq X$ such that $A_i \bowtie_F A_j$, for every $i \neq j$. Then there exists some $f_{n,\vec{A},\vec{a}} \in F$ such that $f_{n,\vec{A},\vec{a}}(A_i) = a_i$, for every i.*

Proof. We consider the $(n-1) + (n-2) + \ldots + 1$ functions $f_{ij} \in F$ such that $f_{ij}(A_i) = 0$ and $f_{ij}(A_j) = 1$, for every $i < j$. The function $f_{n,\vec{A},\vec{a}}$ on X, defined by

$$f_{n,\vec{A},\vec{a}}(x) := \sum_{i=1}^{n} a_i B_i(x),$$

$$B_i(x) := \prod_{i<j\leq n} (1 - f_{ij}) \prod_{1\leq k<i} f_{ki},$$

is in F and if $x_j \in A_j$, then $B_i(x_j) = 1$, if $j = i$ and $B_i(x_j) = 0$, if $j \neq i$. $\square$

5.2 Hausdorff Bishop spaces

In standard topology a functionally Hausdorff topological space X is a space in which $C(X)$ is a point-separating family. Using the notion of a point-point apartness relation we generalize this notion within TBS.

Definition 5.2.1. *If $\mathcal{F} = (X, F)$ is a Bishop space and $\bowtie$ is a given apartness relation on X, we say that F separates a pair of $\bowtie$-distinct points $x_1 \bowtie x_2$ of X, if $x_1 \bowtie_F x_2$, and F is $\bowtie$-Hausdorff, if it separates every pair of $\bowtie$-distinct points of X i.e., if $\bowtie \subseteq \bowtie_F$.*

Clearly, if F_1, F_2 are topologies on X such that $F_1 \subseteq F_2$, then F_2 is $\bowtie_{F_1}$-Hausdorff. Also, F is $\bowtie$-Hausdorff if and only if F_b is $\bowtie$-Hausdorff; let $x_1 \bowtie_X x_2$ and $a = f(x_1) < f(x_2) = b$, for some $f \in F$. Then the function $h := (f \vee \overline{a}) \wedge \overline{b}$ is in F_b, and $h(x_1) = a$, $h(x_2) = b$. If F is $\bowtie$-Hausdorff, then the tightness of $\bowtie$ implies the tightness of $\bowtie_F$. The converse is not general true; e.g., $\bowtie_{\mathrm{Const}(X)} = \emptyset$, and it is not tight, since for all $(x_1, x_2) \in X^2$ we have that $\neg(x_1 \bowtie_{\mathrm{Const}(X)} x_2)$, which does not imply in general that $x_1 = x_2$. Since $\neq$ is the largest apartness relation, but constructively it is not in general tight, a $\neq$-Hausdorff topology F satisfies the implication $x_1 \neq x_2 \to f(x_1) \bowtie_{\mathbb{R}} f(x_2)$, therefore $f(x_1) \neq f(x_2)$, for every $x_1, x_2 \in X$.

Definition 5.2.2. *If $\bowtie$ is a point-point apartness relation on X, we call a neighborhood structure N on X $\bowtie$-Hausdorff, if for every $\bowtie$-distinct pair $x_1 \bowtie x_2$ of X there are $U_1, U_2 \in N$ such that $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.*

Clearly, a $\neq$-Hausdorff neighborhood structure is the classical Hausdorff one. Although the definition of a $\bowtie$-Hausdorff neighborhood structure N is not completely positively formulated, the next result shows that if $N = N(F)$, for some topology F on X, it is constructively equivalent to the positively defined $\bowtie$-Hausdorff property of F.

Proposition 5.2.3. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space and $\bowtie$ is an apartness relation on X. Then, F is $\bowtie$-Hausdorff if and only if $N(F)$ is $\bowtie$-Hausdorff.*

Proof. We fix $x_1 \bowtie x_2$ in X and let $f \in F$ be such that $f(x_1) \bowtie_{\mathbb{R}} f(x_2)$. Suppose that $f(x_1) < f(x_2)$ (the case $f(x_1) > f(x_2)$ is treated similarly). If $c \in \mathbb{R}$ such that $f(x_1) < c < f(x_2)$, then $x_1 \in f^{-1}(-\infty, c) = U_1 \in N(F)$, $x_2 \in f^{-1}(c, \infty) = U_2 \in N(F)$, and $U_1 \cap U_2 = \emptyset$. For the converse we fix $x_1, x_2 \in X$ and we suppose that there exist $f_1, f_2 \in F$ such that $x_1 \in U(f_1)$, $x_2 \in U(f_2)$ and $U(f_1) \cap U(f_2) = U(f_1 \wedge f_2) = \emptyset$. We show that

$$[(f_1 \wedge f_2) \vee f_1](x_1) \bowtie_{\mathbb{R}} [(f_1 \wedge f_2) \vee f_1](x_2).$$

By our suppositions we have that $f_1(x_1) > 0$, $f_2(x_2) > 0$, and $(f_1 \wedge f_2)(x_1), (f_1 \wedge f_2)(x_2) \leq 0$, since $(f_1 \wedge f_2) \leq \overline{0}$; if there was some $x \in X$ such that $(f_1 \wedge f_2)(x) > 0$, then x would inhabit $U(f_1 \wedge f_2)$. Next we show that $f_2(x_1) \leq f_1(x_1)$; if $f_2(x_1) > f_1(x_1)$, then $f_2(x_1) \geq f_1(x_1)$, $f_1(x_1) \geq f_1(x_1)$, therefore $f_1(x_1) \wedge f_2(x_1) \geq f_1(x_1) > 0$, which is impossible. Similarly we get that $f_1(x_2) \leq f_2(x_2)$. Hence, $[(f_1 \wedge f_2) \vee f_1](x_2) = f_1(x_2) \leq 0$ and $[(f_1 \wedge f_2) \vee f_1](x_1) = f_1(x_1) > 0$. $\square$

Proposition 5.2.4. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces and $\bowtie_X, \bowtie_Y$ are given apartness relations on X and Y, respectively.*
(i) If e is an isomorphism between $\mathcal{F}$ and $\mathcal{H}$ which is strongly continuous, and $\mathcal{F}$ is $\bowtie_X$-Hausdorff, then $\mathcal{G}$ is $\bowtie_Y$-Hausdorff.
(ii) If $A \subseteq X$ and F is $\bowtie_X$-Hausdorff, then $F_{|Y}$ is $\bowtie_A$-Hausdorff.
(iii) F is $\bowtie_X$-Hausdorff and G is $\bowtie_Y$-Hausdorff if and only if $F \times H$ is $\bowtie_{X \times Y}$-Hausdorff.
(iv) If G is $\bowtie_Y$-Hausdorff and $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that it preserves apartness, then F is $\bowtie_X$-Hausdorff.
(v) $\mathcal{G}$ is $\bowtie_Y$-Hausdorff if and only if $\mathcal{F} \to \mathcal{G}$ is $\underset{\to}{\bowtie}$-Hausdorff.

Proof. (i) Suppose that $y_1, y_2 \in Y$ such that $y_1 \bowtie_Y y_2$. Since e is onto Y, there are $x_1, x_2 \in X$ such that $e(x_1) = y_1$ and $e(x_2) = y_2$. Since e is strongly continuous, we get that $x_1 \bowtie_X x_2$, therefore there is some $f \in F$ such that $f(x_1) \bowtie_{\mathbb{R}} f(x_2)$. Since e is open, there is some $g \in G$ such that $f = g \circ e$, and $f(x_1) \bowtie_{\mathbb{R}} f(x_2) \leftrightarrow (g \circ e)(x_1) \bowtie_{\mathbb{R}} (g \circ e)(x_2) \leftrightarrow g(y_1) \bowtie_{\mathbb{R}} g(y_2)$.
(ii) Suppose that $a_1, a_2 \in A$ such that $a_1 \bowtie_A a_2$. Since there exists some $f \in F$ such that $f(a_1) \bowtie_{\mathbb{R}} f(a_2)$, then $f_{|A} \in F_{|A}$ and $f_{|A}(a_1) \bowtie_{\mathbb{R}} f_{|A}(a_2)$.
(iii) If $(x_1, y_1) \bowtie_{X \times Y} (x_2, y_2)$, then $x_1 \bowtie_X x_2$ or $y_1 \bowtie_Y y_2$. In the first case there exists $f \in F$ such that $f(x_1) \bowtie_{\mathbb{R}} f(x_2)$, and consequently $(f \circ \pi_1)(x_1, y_1) \bowtie_{\mathbb{R}} (f \circ \pi_1)(x_2, y_2)$. If $y_1 \bowtie_Y y_2$, we work similarly. For the converse we only show that the premiss implies that F is $\bowtie_X$-Hausdorff. If y_0 inhabits Y, we consider the map $i_{y_0} : X \to X \times Y$, $x \mapsto (x, y_0)$ which is an isomorphism between $\mathcal{F}$ and $(i(X), (F \times \mathcal{G})_{|i(X)})$. Then, $(F \times \mathcal{G})_{|i(X)})$ is $\bowtie_{i_{y_0}(X)}$-Hausdorff by (ii), and by (i) F is also $\bowtie_X$-Hausdorff, since $i_{y_0}^{-1}$ is strongly continuous.
(iv) We fix $x_1 \bowtie_X x_2$ in X and by hypothesis $e(x_1) \bowtie_Y e(x_2)$ in Y. Since G is $\bowtie_Y$-Hausdorff, there is some $g \in G$ such that $g(e(x_1)) \bowtie_{\mathbb{R}} g(e(x_2))$. By definition of a morphism we have that $g \circ e \in F$ i.e., F is $\bowtie_X$-Hausdorff.
(v) We fix $h_1, h_2 \in F \to G$ and we suppose that $h_1(x) \bowtie_Y h_2(x)$, for some $x \in X$.

Then, there exists some $g \in G$ such that $g(h_1(x)) \bowtie_{\mathbb{R}} g(h_2(x))$, which is equivalent to $e_{x,g}(h_1) \bowtie_{\mathbb{R}} e_{x,g}(h_2)$. For the converse we suppose that $F \to G$ is a $\underset{\to}{\bowtie}$-Hausdorff topology on $X \to Y$, we fix $y_1, y_2 \in Y$ such that $y_1 \bowtie_Y y_2$, and we show that there exists some $g \in G$ such that $g(y_1) \bowtie_{\mathbb{R}} g(y_2)$. If $\overline{y_1}, \overline{y_2} \in \mathrm{Const}(X,Y) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, and since $\overline{y_1} \underset{\to}{\bowtie} \overline{y_2}$, $\overline{y_1}(x_0) = y_1 \bowtie_Y y_2 = \overline{y_2}(x_0)$, for some x_0 which inhabits X, we get by our $\underset{\to}{\bowtie}$-Hausdorff hypothesis on $F \to G$ the existence of some $e \in F \to G$ such that $e(\overline{y_1}) \bowtie_{\mathbb{R}} e(\overline{y_2})$. It suffices to show that

$$\forall_{e \in F \to G}(e(\overline{y_1}) \bowtie_{\mathbb{R}} e(\overline{y_2}) \to \exists_{g \in G}(g(y_1) \bowtie_{\mathbb{R}} g(y_2))).$$

For that we use $\mathrm{Ind}_{\mathcal{F}}$ on $F \to G = \mathcal{F}(\{e_{x,g} \mid x \in X, g \in G\})$. If $a \in \mathbb{R}$, then $\overline{a}(\overline{y_1}) \bowtie_{\mathbb{R}} \overline{a}(\overline{y_1})$ is false and the required implication holds trivially with the use of the Ex falso rule. If we consider some function of the form $e_{x,g}$ we have that

$$e_{x,g}(\overline{y_1}) \bowtie_{\mathbb{R}} e_{x,g}(\overline{y_1}) \leftrightarrow g(\overline{y_1}(x)) \bowtie_{\mathbb{R}} g(\overline{y_2}(x)) \leftrightarrow g(y_1) \bowtie_{\mathbb{R}} g(y_2).$$

If $e_1(\overline{y_1}) + e_2(\overline{y_1}) \bowtie_{\mathbb{R}} e_1(\overline{y_2}) + e_2(\overline{y_2})$, then $e_1(\overline{y_1}) \bowtie_{\mathbb{R}} e_1(\overline{y_2})$ or $e_2\overline{y_1}) \bowtie_{\mathbb{R}} e_2(\overline{y_2})$, and we apply the inductive hypothesis on e_1 or on e_2. If $\phi \in \mathrm{Bic}(\mathbb{R})$, then since ϕ is strongly continuous we have that the hypothesis $\phi(e(\overline{y_1})) \bowtie_{\mathbb{R}} \phi(e(\overline{y_2}))$ implies that $e(\overline{y_1}) \bowtie_{\mathbb{R}} e(\overline{y_2})$, and our inductive hypothesis on e gives us the required $g \in G$. Suppose next that for every $\epsilon > 0$ exists $j \in F \to G$ such that $(U(j, e, \epsilon))$ and j satisfies our inductive hypothesis. Then, if $e(\overline{y_1}) \bowtie_{\mathbb{R}} e(\overline{y_2})$, there exists some $\epsilon > 0$ and some $j \in F \to G$ such that $U(j, e, \epsilon)$ and $j(\overline{y_1}) \bowtie_{\mathbb{R}} j(\overline{y_2})$. By the inductive hypothesis on j we get the required $g \in G$. $\square$

The hypothesis in Proposition 5.2.4(iv) of e preserving apartness is not trivial, since there are Bishop spaces $\mathcal{F}, \mathcal{G}$ such that $\mathrm{Mor}(\mathcal{F}, \mathcal{G}) = \mathrm{Const}(X, Y)$, and consequently no morphism from $\mathcal{F}$ to $\mathcal{G}$ preserves apartness. E.g., if $\mathcal{F} = (X, \mathrm{Const}(X))$ and $\mathcal{G} = (Y, \mathbb{F}(Y))$ such that $=_Y$ is decidable and $\bowtie_Y$ is tight, then $\mathrm{Mor}(\mathcal{F}, \mathcal{G}) = \mathrm{Const}(X, Y)$, since, if $x_1, x_2 \in X$ and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $\neg(h(x_1) \bowtie_Y h(x_2))$; if $h(x_1) \bowtie_Y h(x_2)$, then we can define constructively some $g \in \mathbb{F}(Y)$ such that $g \circ h \notin \mathrm{Const}(X)$.
The next result generalizes a fact known for appropriate subalgebras of $C(X)$ (see [36], p.148), while its case $n = 2$ and $a_1 = 0$, $a_2 = 1$ is proved in [19], p.102.

Proposition 5.2.5. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space, $\bowtie$ is an apartness relation on X, and F is $\bowtie$-Hausdorff. If $n \geq 2$, $x_1, \ldots, x_n$ are pairwise $\bowtie$-distinct points of X and $a_1, \ldots, a_n \in \mathbb{R}$, there exists some $f_{n,\vec{x},\vec{a}} \in F$ such that $f_{n,\vec{x},\vec{a}}(x_i) = a_i$, for every i.*

Proof. We consider the $(n-1)+(n-2)+\ldots+1$ functions $f_{ij} \in F$ such that $f_{ij}(x_i) \bowtie_{\mathbb{R}} f_{ij}(x_j)$, for every $i < j$. The function $f_{n,\vec{x},\vec{a}}$ on X,

$$f_{n,\vec{x},\vec{a}}(x) := \sum_{i=1}^{n} a_i A_i(x),$$

$$A_i(x) := \prod_{k=i+1}^{n} \frac{f_{ik}(x) - f_{ik}(x_k)}{f_{ik}(x_i) - f_{ik}(x_k)} \prod_{k=1}^{i-1} \frac{f_{ki}(x_k) - f_{ki}(x)}{f_{ki}(x_k) - f_{ki}(x_i)},$$

is in F and $A_i(x_j) = 1$, if $j = i$, $A_i(x_j) = 0$, if $j \neq i$, and $f_{n,\vec{x},\vec{a}}(x_i) = a_i$, for every i. Since $x_1 \bowtie_F x_2 \leftrightarrow \exists_{f \in F}(f(x_1) = 0 \wedge f(x_2) = 1)$, the function $f_{n,\vec{A},\vec{a}}$ of Proposition 5.1.10, where $A_i = \{x_i\}$, also works. $\square$

In the previous proposition the a_i's need not be unequal. Hence, if $C = A \cup B$ is a finite set of pairwise $\bowtie$-distinct elements of X, there is some $f \in F$ such that $f(A) = a$ and $f(B) = b$, for any $a \bowtie_{\mathbb{R}} b$ in $\mathbb{R}$ i.e., $A \bowtie_F B$. Proposition 5.2.5 cannot be extended to infinite sequences $(x_n)_{n\in\mathbb{N}}$ of pairwise $\bowtie$-distinct elements of X and infinite sequences $(a_n)_{n\in\mathbb{N}}$ of reals. E.g., if $x_0 = 0$ and $x_n = \frac{1}{n}$, for every $n \neq 0$, while $a_n = n$, for every n, then $\mathrm{Bic}(\mathbb{R})$ is trivially $\bowtie_{\mathbb{R}}$-Hausdorff, but there is no $\phi \in \mathrm{Bic}(\mathbb{R})$ such that $\phi(x_n) = a_n$, since ϕ is pointwise continuous at 0.

5.3 The zero sets of a Bishop space

A final technical comment: although condition F_4 holds for the various examples of function space structures that we have dealt with in the foregoing, we have not actually made use of it to prove any substantial results about function spaces in general. Nevertheless, that condition will surely play a role in any more advanced work on function spaces that is carried out, by me or anyone else, in the future.

Douglas S. Bridges, 2012

Here we study within Bishop spaces the classical notion of a zero set of some $f \in C(X)$ (see Gillman and Jerison [44]).

Definition 5.3.1. *If F is a topology on X, the F-zero sets is the collection*

$$Z(F) := \{\zeta(f) \mid f \in F\},$$

$$\zeta(f) := [f = \overline{0}].$$

We call $\zeta(f)$ the zero set of f.

If $\zeta : F \to Z(F)$ is defined by $f \mapsto \zeta(f)$, then $\zeta(\overline{0}) = X$ and $\zeta(\overline{1}) = \emptyset$. Since $\zeta(f) = \zeta(|f| \wedge \overline{1})$, we get that $Z(F) = Z(F_b)$, while since $\zeta(f) = \bigcap_{n\in\mathbb{N}} |f|^{-1}(-1, \frac{1}{n})$, every F-zero set is a G_δ set in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$.

Proposition 5.3.2. *If F is a topology on X and $f \in F$, then $\zeta(f)$ is closed in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$.*

Proof. We fix $x \in X$, $f \in F$ and, according to the definition of a closed set in $\mathcal{T}_{\mathcal{N}(F)}$, we suppose that $\forall_{g \in F}(g(x) > 0 \to \exists_{y \in X}(g(y) > 0 \wedge f(y) = 0))$. We need to show that $f(x) = 0$. If $f(x) > 0$, then applying the above premiss on f we get that $\exists_{y \in X}(f(y) > 0 \wedge f(y) = 0))$, which is a contradiction, i.e., we get that $\neg(f(x) > 0)$, which implies that $f(x) \leq 0$. If $f(x) < 0$, then $-f(x) > 0$, therefore applying our hypothesis on $-f \in F$ we get that $\exists_{y \in X}((-f)(y) > 0 \wedge f(y) = 0 = (-f)(y)))$, which is a contradiction. Hence, $f(x) \geq 0$. □

Proposition 5.3.3. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space and $f \in F$.*
(i) $[f \geq \overline{0}] = \zeta(f \wedge \overline{0})$ and $[f \leq \overline{0}] = \zeta(f \vee \overline{0})$.
(ii) $Z(F)$ is closed under finite and countably infinite intersections.

Proof. (i) For the first equality we have that $(f \wedge \overline{0})(x) = 0 \leftrightarrow \min\{f(x), 0\} = 0 \leftrightarrow f(x) \geq 0$, where the last equivalence is due to the following properties of $\mathbb{R}$: $\min\{x, y\} \leq y$ and $x \geq 0 \to y \geq 0 \to \min\{x, y\} \geq 0$ (Proposition 2.2.3). For the second equality we have that $(f \vee \overline{0})(x) = 0 \leftrightarrow \max\{f(x), 0\} = 0 \leftrightarrow f(x) \leq 0$. The last equivalence is due to the following properties of $\mathbb{R}$: $\max\{x, y\} \geq x$ and $x \leq z \to y \leq z \to \max\{x, y\} \leq z$ (Proposition 2.2.3).
(ii) To show the closure of $Z(F)$ under the finite intersections we use the identities

$$\zeta(\sum_{i=1}^{n} f_i^2) = \zeta(\sum_{i=1}^{n} |f_i|) = \bigcap_{i=1}^{n} \zeta(f_i),$$

which are justified by Proposition 2.2.5(iii). For the infinite case we define for a sequence of zero sets $(\zeta(f_n))_n$, where $f_n \in F$, for every n, the functions

$$g_n := |f_n| \wedge \overline{2^{-n}} \geq 0.$$

The function

$$g := \sum_{n=1}^{\infty} g_n$$

is well-defined, since $|g_n| = g_n \leq 2^{-n}$ and the comparison test, proved in [15], p.32, implies that the convergence of the series $\sum_{i=1}^{\infty} 2^{-n}$ gives the convergence of the series $\sum_{n=1}^{\infty} g_n(x)$, for every $x \in X$. It also satisfies the equalities

$$\zeta(g) = \bigcap_{n=1}^{\infty} \zeta(g_n) = \bigcap_{n=1}^{\infty} \zeta(f_n),$$

since by Proposition 2.2.5(iv) we have that

$$g(x) = 0 \leftrightarrow \forall_n(g_n(x) = 0) \leftrightarrow \forall_n(|f_n(x)| = 0) \leftrightarrow \forall_n(f_n(x) = 0).$$

To prove that $g \in F$, and since $g_n \in F$, it suffices to show by BS_4 that

$$h_n = \sum_{i=1}^{n} g_i \xrightarrow{u} g,$$

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall x\in X(|g(x)-h_n(x)| = |\sum_{n=1}^{\infty}g_n(x)-\sum_{n=1}^{n}g_i(x)| = |\sum_{n=n_0+1}^{\infty}g_n(x)|\leq\epsilon).$$

Since $\sum_{n=1}^{\infty}2^{-n}=1$ and

$$|\sum_{n=n_0+1}^{\infty}g_n(x)| = \sum_{n=n_0+1}^{\infty}g_n(x)\leq\sum_{n=n_0+1}^{\infty}2^{-n}\leq\epsilon,$$

for some n_0, we reach our conclusion. □

Consequently, if $f,g,h\in F$, we have that $[f\geq g]=\zeta((f-g)\wedge\overline{0})$, $[f\leq g]=\zeta((f-g)\vee\overline{0})$, $[f=g]=\zeta(f-g)$, and $[h\leq f\leq g]\in Z(F)$. The previous proof is based on the corresponding proof for $C(X)$ in [44], p.16. It is important though, that we made sure that all necessary properties of $\mathbb{R}$ to the proof are constructively valid.

Definition 5.3.4. *If F is a topology on X, we say that $B\subseteq X$ is F-closed, if B is the intersection of a family of F-zero sets i.e., there is an inhabited $F_0\subseteq F$ such that $B=\bigcap Z(F_0)$. We denote the set of F-closed sets by $C(F)$.*

By Proposition 5.3.2 and the closure of $C(N(F))$ under arbitrary intersections we get that

$$Z(F)\subseteq C(F)\subseteq C(N(F)).$$

Definition 5.3.5. *If F is a topology on X, $a,b,a_1,\ldots,a_n\in\mathbb{R}$, $f,g\in F$, $x\in X$, $A\subseteq X$, $B=\bigcap Z(F_0)$ and $C=\bigcap Z(G_0)\in C(F)$, we define*

$$\mathrm{ID}(a_1,\ldots,a_n):\leftrightarrow\prod_{i=1}^{n}a_i=0\rightarrow\bigvee_{i=1}^{n}(a_i=0),$$

$$\mathrm{Sep}(a,b):\leftrightarrow|a|+|b|>0,$$

$$\mathrm{Sep}(\zeta(f),\zeta(g)):\leftrightarrow\forall_{x\in X}(\mathrm{Sep}(f(x),g(x))),$$

$$\mathrm{Usep}(\zeta(f),\zeta(g)):\leftrightarrow\exists_{c>0}(|f|+|g|\geq\overline{c}),$$

$$\mathrm{Away}(x,\zeta(f)):\leftrightarrow f(x)\bowtie_{\mathbb{R}}0,$$

$$\mathrm{Away}(A,\zeta(f)):\leftrightarrow\forall_{a\in A}(\mathrm{Away}(x,\zeta(f))),$$

$$\mathrm{Away}(x,B):\leftrightarrow\exists_{f_0\in F_0}(\mathrm{Away}(x,\zeta(f_0))),$$

$$\mathrm{Away}(A,B):\leftrightarrow\forall_{a\in A}(\mathrm{Away}(a,B)),$$

$$\mathrm{SEP}(B,C):\leftrightarrow\exists_{f_0\in F_0}\exists_{g_0\in G_0}(\mathrm{Usep}(\zeta(f_0),\zeta(g_0))),$$

and we say that $\zeta(f)$ and $\zeta(g)$ are separate, if $\mathrm{Sep}(\zeta(f),\zeta(g))$, $\zeta(f)$ and $\zeta(g)$ are uniformly separate, if $\mathrm{Usep}(\zeta(f),\zeta(g))$, and B,C are separate F-closed sets, if $\mathrm{SEP}(B,C)$.

These notions are positive version of non-intersecting zero sets or F-closed sets. Clearly, we have that

$$\mathrm{Usep}(\zeta(f), \zeta(g)) \to \mathrm{Usep}(\zeta(f), \zeta(g)),$$
$$\mathrm{Away}(x, \zeta(f)) \to x \bowtie_F \zeta(f),$$
$$\mathrm{Away}(x, B) \to x \bowtie_F B,$$
$$\mathrm{SEP}(B, C) \to \mathrm{Away}(B, C) \wedge \mathrm{Away}(C, B).$$

The set $Z(F)$ is classically closed under finite unions, but the inclusion

$$\zeta(fg) \subseteq \zeta(f) \cup \zeta(g)$$

requires the constructively not acceptable property $\forall_{a,b\in\mathbb{R}}(\mathrm{ID}(a, b))$. What we have constructively is only that $\zeta(fg) \supseteq \zeta(f) \cup \zeta(g)$. This is in analogy to the fact that constructively the arbitrary intersection of closed subsets of $\mathbb{R}$ is a closed set, while the fact that $[0, 1] \cup [1, 2]$ is closed can also be "dismissed" by a Brouwerian counterexample (see Lemma 7.3.1 and the proof of Proposition 7.3.2). Thus, the translation of the classical theory of zero sets into TBS has its limitations. It is an important corollary of Theorem 5.4.9 though, that there is a plethora of pairs of F-zero sets satisfying the equality $\zeta(fg) = \zeta(f) \cup \zeta(g)$, therefore $\zeta(f) \cup \zeta(g) \in Z(F)$.

Proposition 5.3.6. *(i)* $\forall_{a\in\mathbb{R}}(0 < a \to \forall_{b\in\mathbb{R}}(\mathrm{ID}(a, b)))$.
(ii) $\forall_{n\geq 2}\forall_{a_1\ldots,a_n\in\mathbb{R}}(\mathrm{ID}(|a_1|, \ldots, |a_n|) \leftrightarrow \mathrm{ID}(a_1, \ldots, a_n))$.
(iii) $\forall_{a,b\in\mathbb{R}}(\mathrm{Sep}(a, b) \to \mathrm{ID}(a, b))$.
(iv) $\forall_{n\geq 2}(\forall_{a_1\ldots,a_n\in\mathbb{R}}(\forall_{i<j\in\{1,\ldots,n\}}(\mathrm{Sep}(a_i, a_j)) \to \mathrm{ID}(a_1, \ldots, a_n)))$.

Proof. (i) We fix $a, b \in \mathbb{R}$ such that $a > 0$ and $ab = 0$. If $b < 0$, then $ab < 0$, hence $b \geq 0$. If $b > 0$, then $ab > 0$, hence $b \leq 0$. Hence, $b = 0$, and $\mathrm{ID}(a, b)$.
(ii) Trivially by $|a_1| \ldots |a_n| = 0 \leftrightarrow |a_1 \ldots a_n| = 0 \leftrightarrow a_1 \ldots a_n = 0$.
(iii) If $0 < |a|+|b|$, then by the constructive trichotomy we have that $0 < |a|$, or $|a| < |a|+|b|$. If $0 < |a|$, then by (i) $\mathrm{ID}(|a|, |b|)$, and by (ii) we get that $\mathrm{ID}(a, b)$. If $|a| < |a| + |b|$, then $0 < |b|$, hence by (i) $\mathrm{ID}(|b|, |a|)$, therefore by (ii) $\mathrm{ID}(b, a)$.
(iv) If $n = 2$, then we are reduced to (iii). If $\forall_{a_1\ldots,a_n\in\mathbb{R}}(\forall_{i<j\in\{1,\ldots,n\}}(\mathrm{Sep}(a_i, a_j)) \to \mathrm{ID}(a_1, \ldots, a_n))$, we show that

$$\forall_{a_1\ldots,a_{n+1}\in\mathbb{R}}(\forall_{i<j\in\{1,\ldots,n+1\}}(\mathrm{Sep}(a_i, a_j)) \to \mathrm{ID}(a_1, \ldots, a_{n+1})).$$

We fix $a_1 \ldots, a_{n+1} \in \mathbb{R}$ such that $\mathrm{Sep}(a_i, a_j)$, for every $i < j \in \{1, \ldots, n + 1\}$, and we show that

$$\forall_{b_1\ldots,b_{n+1}\in\mathbb{R}}(\forall_{i<j\in\{1,\ldots,n+1\}}(\mathrm{Sep}(b_i, b_j)) \to \exists_i(b_i > 0) \to \mathrm{ID}(b_1, \ldots, b_{n+1})).$$

If $b_i > 0$, then by (i) $\mathrm{ID}(b_i, \prod_{j\neq i} b_j)$ i.e., $\prod_{j\neq i} b_j = 0$. By our inductive hypothesis on the n-many reals $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n+1}$ we get $\mathrm{ID}(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n+1})$, hence $\mathrm{ID}(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_{n+1})$. Since $0 < |a_1|+|a_2|$, either $0 < |a_1|$, or $|a_1 < |a_1|+|a_2|$. If $0 < |a_1|$, then by the previous intermediate fact we get that $\mathrm{ID}(|a_1|, |a_2|, \ldots, |a_{n+1}|)$, and by (ii) we conclude that $\mathrm{ID}(a_1, \ldots, a_{n+1})$. If $|a_1 < |a_1| + |a_2|$, then $0 < |a_2|$, therefore $\mathrm{ID}(|a_2|, |a_1|, |a_3|, \ldots, |a_{n+1}|)$ i.e., $\mathrm{ID}(|a_1|, |a_2|, \ldots, |a_{n+1}|)$, hence by (ii) we get $\mathrm{ID}(a_1, \ldots, a_{n+1})$. □

Recall that $x \Join_d y \leftrightarrow d(x,y) > 0$ is the canonical apartness relation on a metric space X.

Corollary 5.3.7. *(i) If F is a topology on X and $f_1, \ldots, f_n \in F$ such that* $\mathrm{Sep}(\zeta(f_i), \zeta(f_j))$*, for every $i < j \in \{1, \ldots, n\}$, we have that*

$$\bigcup_{i=1}^{n} \zeta(f_i) = \zeta(\prod_{i=1}^{n} f_i) \in Z(F).$$

(ii) If F is a topology on a metric space X such that $F \supseteq \{d_x \mid x \in X\}$, and if $Y = \{x_1, \ldots, x_n\} \subseteq X$ such that $x_i \Join_d x_j$, for every $i \neq j$, then $Y \in Z(F)$.
(iii) If $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}$ such that $a_i \Join_{\mathbb{R}} a_j$, for every $i \neq j$, then $A \in Z(\mathrm{Bic}(\mathbb{R}))$.

Proof. (i) By Proposition 5.3.6(iv) we have that the hypothesis $\mathrm{Sep}(\zeta(f_i(x)), \zeta(f_j(x)))$, for every $i < j$, implies that $\mathrm{ID}(f_1(x), \ldots, f_n(x))$, hence $\zeta(\prod_{i=1}^{n} f_i) \subseteq \bigcup_{i=1}^{n} \zeta(f_i)$.
(ii) Since $0 < d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) = |d(x_i, x)| + |d(x, x_j)| = |d_{x_i}(x)| + |d_{x_j}(x)|$, we conclude that $\mathrm{Usep}(\zeta(d_{x_i}), \zeta(d_{x_j}))$. By (i) we have that $\bigcup_{i=1}^{n} \zeta(d_{x_i}) = Y \in Z(F)$.
(iii) If $a \in \mathbb{R}$, then the function d_a, where $d_a(x) = |x - a|$, is in $\mathrm{Bic}(\mathbb{R})$, since it is uniformly continuous on $\mathbb{R}$, hence by (ii) we get that $A \in Z(\mathrm{Bic}(\mathbb{R}))$. $\square$

As in the case of $C(X)$, $Z(F)$ is not closed under countable unions; e.g., $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \zeta(x - q)$, while it is not even in $C(N(\mathrm{Bic}(\mathbb{R})))$. If $A \subseteq X$ and $f \in F$, where F is a topology on X, we get immediately that

$$\zeta(f) \cap A = \zeta(f_{|A}),$$

while if G is a topology on Y and $g \in G$, then

$$\zeta(f) \times \zeta(g) = \zeta((f \circ \pi_1)^2 + (g \circ \pi_2)^2) \in Z(F \times G),$$

$$((f \circ \pi_1)^2 + (g \circ \pi_2)^2)(x,y) = 0 \leftrightarrow f(x) = 0 \wedge g(y) = 0 \leftrightarrow (x,y) \in \zeta(f) \times \zeta(g).$$

The next proposition follows immediately from the previous definitions.

Proposition 5.3.8. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is onto Y.*
(i) $h^{-1}(\zeta(g)) = \zeta(g \circ h) \in Z(F)$, for every $g \in G$, and if h is open, then $h(\zeta(f)) = \zeta(g) \in Z(G)$, where $f \in F$ and $g \in G$ such that $f = g \circ h$.
(ii) $y_1 \Join_G y_2 \rightarrow [h = \overline{y_1}] \Join_F [h = \overline{y_1}]$, for every $y_1, y_2 \in Y$.
(iii) If $\mathrm{Sep}(\zeta(g_1), \zeta(g_2))$, then $\mathrm{Sep}(h^{-1}(\zeta(g_1)), h^{-1}(\zeta(g_2)))$, for every $g_1, g_2 \in G$.
(iv) If $\mathrm{Usep}(\zeta(g_1), \zeta(g_2))$, then $\mathrm{Usep}(h^{-1}(\zeta(g_1)), h^{-1}(\zeta(g_2)))$, for every $g_1, g_2 \in G$.
(v) If $C \in C(G)$, then $h^{-1}(C) \in C(F)$.
(vi) If $\mathrm{SEP}(C, D)$, then $\mathrm{SEP}(h^{-1}(C), h^{-1}(D))$, for every $C, D \in C(G)$.

5.4 The Urysohn lemma for the zero sets of a Bishop space

The classical Urysohn lemma expresses that two closed and disjoint subsets of a normal space X are separated by some continuous function $f : X \to [0, 1]$, while the classical Urysohn lemma for $C(X)$-zero sets expresses that the disjoint zero sets of any topological space X are separated by some $f \in C(X)$ (see [44], p.17). Here we prove a constructive version of the latter for Bishop spaces, replacing the negative hypothesis of disjointness by the positive and stronger condition of separation. This version entails a positive form of the former for separated F-closed sets[1]. In order for the classical proof of the Urysohn lemma for zero sets to work constructively, we show a property of F of independent interest (Theorem 5.4.8) which follows here from the constructive Tietze theorem, found in Bishop and Bridges [15], p.120[2]. Recall that the classical Tietze theorem for a metric space X expresses the continuous extendability of a continuous and bounded real-valued function from a closed subset of X to the whole space, while the general version of Tietze theorem expresses the same extendability in a normal space and requires the Urysohn lemma.
Using the definition of a continuous function on a locally compact metric space, given in [15], p.110, Bishop's formulation of Tietze theorem becomes as follows.

Theorem 5.4.1 (Tietze theorem for metric spaces)**.** *Let Y be a locally compact subset of a metric space X and $I \subset \mathbb{R}$ an inhabited compact interval. Let $f : Y \to I$ be uniformly continuous on the bounded subsets of Y. Then there exists a function $g : X \to I$ which is uniformly continuous on the bounded subsets of X, and which satisfies $g(y) = f(y)$, for every $y \in Y$.*

Corollary 5.4.2. *If Y is a locally compact subset of $\mathbb{R}$ and $g : Y \to I \in \mathrm{Bic}(Y)$, where $I \subset \mathbb{R}$ is an inhabited compact interval, then there exists a function $\phi : \mathbb{R} \to I \in \mathrm{Bic}(\mathbb{R})$ which satisfies $\phi(y) = g(y)$, for every $y \in Y$.*

Corollary 5.4.3. *Suppose that Y is a totally bounded and closed subset of $\mathbb{R}$. Then* $\mathrm{Bic}(Y)$ *is a topology on Y such that* $\mathrm{Bic}(Y) = \mathcal{F}(\mathrm{id}_Y) = \mathrm{Bic}(\mathbb{R})_{|Y}$

Proof. Since a totally bounded subset of a metric space is located (see [15], p.95) and a closed and located subset of a locally compact space is locally compact (see [15], p.110), we get that Y is a locally compact subset of $\mathbb{R}$, therefore $\mathrm{Bic}(Y)$ is a topology on Y. Since Y is bounded, the subbase id_Y of $\mathcal{F}(\mathrm{id}_Y)$ is bounded and uniformly continuous on Y, hence by the $\mathcal{F}$-lifting of uniform continuity all elements of $\mathcal{F}(\mathrm{id}_Y)$ are uniformly continuous on Y, therefore $\mathcal{F}(\mathrm{id}_Y) \subseteq \mathrm{Bic}(Y)$. For the converse inclusion we fix some $g \in \mathrm{Bic}(Y)$, and since Y itself is bounded, g is uniformly continuous on Y. Since Y is totally bounded,

[1] A constructive treatment of the Urysohn lemma in an apartness space is given by Bridges and Diener in [26], where they also show that the general Urysohn lemma implies the weak law of excluded middle, $\neg P \vee \neg\neg P$.

[2] We also include a straightforward proof of the Urysohn lemma for the zero sets of Bishop spaces which does not involve the Tietze theorem and it was suggested to us by an anonymous referee.

the supremum and infimum of g exist (see [15], p.94) i.e., $g : Y \to [\inf g, \sup g]$. By Corollary 5.4.2 there exists a function $\phi : \mathbb{R} \to [\inf g, \sup g] \in \mathrm{Bic}(\mathbb{R})$ such that $\phi(y) = g(y)$, for every $y \in Y$. By BS_3 we get that $\phi \circ \mathrm{id}_Y = \phi_{|Y} = g \in \mathcal{F}(\mathrm{id}_Y)$. By our remark on the relative topology given with a subbase we have that $\mathrm{Bic}(\mathbb{R})_{|Y} = \mathcal{F}((\mathrm{id}_{\mathbb{R}})_{|Y}) = \mathcal{F}(\mathrm{id}_Y)$, therefore $\mathrm{Bic}(\mathbb{R})_{|Y} = \mathrm{Bic}(Y)$. □

Corollary 5.4.4. *If $a, b \in \mathbb{R}$ such that $a < b$,* $\mathrm{Bic}([a,b]) = \mathcal{F}(\mathrm{id}_{[a,b]}) = \mathrm{Bic}(\mathbb{R})_{|[a,b]}$.

Proof. Since $[a,b]$ is totally bounded and closed, we use the previous corollary. □

Corollary 5.4.5 (Uniform continuity theorem for morphisms)**.** *If $a, b \in \mathbb{R}$ such that $a < b$, then $f : [a,b] \to \mathbb{R} \in \mathrm{Mor}(\mathcal{R}_{|[a,b]}, \mathcal{R})$ if and only if f is uniformly continuous on $[a,b]$.*

Proof. Since $\mathrm{Bic}(\mathbb{R})_{|[a,b]} = \mathrm{Bic}([a,b])$, by the $\mathcal{F}$-lifting of morphisms $f \in \mathrm{Mor}(\mathcal{R}_{|[a,b]}, \mathcal{R})$ if and only if $\mathrm{id}_{\mathbb{R}} \circ f = f \in \mathrm{Bic}([a,b])$. Hence, f is uniformly continuous on $[a,b]$. The converse follows immediately. □

The next corollary is used in the proof of Proposition 7.3.5.

Corollary 5.4.6. *Suppose that (X,d) is a locally compact metric space and K is an inhabited compact subset of X. Then*

$$C_u(K) = \mathrm{Bic}(X)_{|K} = \{f_{|K} \mid f \in \mathrm{Bic}(X)\}.$$

Proof. First we show that $\mathrm{Bic}(X)_{|K} \subseteq C_u(K)$; if $f \in \mathrm{Bic}(X)$ and since K is a bounded subset of X, we get by the definition of $\mathrm{Bic}(X)$ that $f_{|K}$ is uniformly continuous, therefore $f \in C_u(K)$. Since every element $f_{|K}$ of the subbase of $\mathrm{Bic}(X)_{|K}$ is bounded, by the $\mathcal{F}$-lifting of uniform continuity we get that $\mathrm{Bic}(X)_{|K} \subseteq C_u(K)$. For the converse inclusion we fix some $g \in C_u(K)$, and since K is also a locally compact subset of X, then by the Tietze theorem for metric spaces we get that the function $g : K \to g(K) \subseteq I$, where I is an inhabited compact interval of $\mathbb{R}$, which is trivially uniformly continuous on every bounded subset of K, has an extension $h \in \mathrm{Bic}(X)$. □

The next proposition is found in Bishop and Bridges [15], p.39.

Proposition 5.4.7. *If I is an interval of reals and $f \in \mathrm{Bic}(I)$ is bounded away from 0 on every compact subinterval J of I - that is, if $|f(x)| \geq c$ for all x in J and some $c > 0$ - then f^{-1} is uniformly continuous on the compact subsets of I.*

Theorem 5.4.8. *Suppose that (X,F) is a Bishop space and $f \in F$ such that $f \geq \overline{c}$, for some $c > 0$. Then, $\frac{1}{f} \in F$.*

Proof. If $c > 0$, the interval $[c, +\infty)$ is a locally compact subset of $(\mathbb{R}, d_{\mathbb{R}})$, where $d_{\mathbb{R}}(x,y) := |x - y|$, since a bounded subset of $[c, +\infty)$ is bounded above; we define a bounded subset of a metric space to be an inhabited set, hence if $B \subseteq [c, +\infty)$ is bounded and x_0 inhabits B, then $|x| \leq |x - x_0| + |x_0| \leq M(B) + |x_0| = M$, where $M(B)$ is a bound of B, for every $x \in B$. Without loss of generality we can take $M > c$, therefore B is included in the compact subset $[c, M]$ of $[c, +\infty)$. Next we consider the inverse function $^{-1} : [c, +\infty) \to [0, \frac{1}{c}]$, $x \mapsto \frac{1}{x}$, which is uniformly continuous on the bounded subsets of $[c, +\infty)$; the identity function x is bounded away from 0 on every compact subinterval of $[c, +\infty)$, since there is a common c for which $|x| = x \geq c$, for every x in the compact subinterval, and since it is uniformly continuous on $[c, +\infty)$, we use Proposition 5.4.7 to conclude that x^{-1} is uniformly continuous on the compact subsets of $[c, +\infty)$, therefore it is uniformly continuous on the bounded subsets of $[c, +\infty)$. Since the range of $^{-1}$ is included in the inhabited compact interval $[0, \frac{1}{c}]$, by Corollary 5.4.2 there exists a function $\phi : \mathbb{R} \to [0, \frac{1}{c}]$ such that $\phi(x) = \frac{1}{x}$, for every $x \in [c, +\infty)$, and $\phi \in \mathrm{Bic}(\mathbb{R})$. If $f \in F$ such that $f \geq \overline{c}$, then by BS_3 the function $\phi \circ f \in F$. Since $\forall_{x \in X}(\phi(f(x)) = \frac{1}{f(x)})$, we conclude that $\phi \circ f = \frac{1}{f} \in F$. $\square$

Consequently, if $f \in F$, then $\frac{1}{f \vee \overline{c}} \in F$, for every $c > 0$. What classically we have for free, namely that if $f \in C(X)$ such that $f(x) \neq 0$ for every $x \in X$, then $\frac{1}{f} \in C(X)$, constructively and positively is reformulated as above. If we consider the trivial topology $\mathrm{Const}(X)$ we get directly by the definition of the inverse operation in $\mathbb{R}$ that, if $\overline{a} \geq \overline{c} > \overline{0}$, then $\frac{1}{\overline{a}} = \overline{\frac{1}{a}} \in \mathrm{Const}(X)$. It is easy to see that the induced by some $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ mapping $h^* : G \to F$ satisfies $h^*(\frac{1}{g}) = \frac{1}{h^*(g)}$, if $\frac{1}{g} \in G$.

Theorem 5.4.9 (Urysohn lemma for F-zero sets)**.** *Suppose that (X, F) is a Bishop space and A, B are subsets of X. Then,*

$$A \bowtie_F B \leftrightarrow \exists_{f,g \in F}(A \subseteq \zeta(f) \ \wedge \ B \subseteq \zeta(g) \ \wedge \ \mathrm{Usep}(\zeta(f), \zeta(g))).$$

Proof. Suppose first that $A \bowtie_F B$ i.e., there exists some $f \in F$ such that $f(A) = 0$ and $f(B) = 1$. Then, $A \subseteq [f \leq \overline{\frac{1}{3}}] = \zeta((f - \overline{\frac{1}{3}}) \vee \overline{0})$ and $B \subseteq [f \geq \overline{\frac{2}{3}}] = \zeta((f - \overline{\frac{2}{3}}) \wedge \overline{0})$. Since in $\mathbb{R}$ we have that $x \geq 0 \to |x| = x$ and $x \leq 0 \to |x| = -x$ we show that

$$|(f - \overline{\frac{1}{3}}) \vee \overline{0}| + |(f - \overline{\frac{2}{3}}) \wedge \overline{0}| = (f - \overline{\frac{1}{3}}) \vee \overline{0} - [(f - \overline{\frac{2}{3}}) \wedge \overline{0}] \geq \overline{\frac{1}{3}}.$$

Using the property $x \geq a \to b \geq y \to x - y \geq a - b$ of reals we have that

$$(f - \overline{\frac{1}{3}}) \vee \overline{0} \geq f - \overline{\frac{1}{3}} \geq f - \overline{\frac{2}{3}} \geq (f - \overline{\frac{2}{3}}) \wedge \overline{0} \ \to$$

$$(f - \overline{\frac{1}{3}}) \vee \overline{0} - [(f - \overline{\frac{2}{3}}) \wedge \overline{0}] \geq f - \overline{\frac{1}{3}} - (f - \overline{\frac{2}{3}}) = \overline{\frac{1}{3}}.$$

For the converse, if $|f| + |g| \geq \overline{c}$, for some $c > 0$, and since $|f| + |g| \in F$, by Theorem 5.4.8 we get that $\frac{1}{|f|+|g|} \in F$. Hence, the function h defined as

$$h := \frac{|f|}{|f| + |g|} \in F,$$

satisfies $h(A) = 0$ and $h(B) = 1$. $\square$

Proof. (elementary) If $A \bowtie_f B$ and without loss of generality $\overline{0} \leq f \leq \overline{1}$, then f and $\overline{1} - f$ have the desired properties; this is based on the fact that if $x \in \mathbb{R}$ such that $x \geq 0$, then $|x| + |1 - x| \geq 1$, since if $|x| + |1 - x| < 1$, then $|x| < 1$ and $|x| + |1 - x| = x + 1 - x = 1$, which is a contradiction. Conversely, consider the function $h = |\frac{1}{c}|f| \wedge \overline{1}|$. If $f(x) = 0$, then $h(x) = 0$. If $g(x) = 0$, then $c \leq |f(x)| + |g(x)| = |f(x)|$, and consequently for such x we have $h(x) = 1$. □

Consequently, any two uniformly separate F-zero sets are separated by some function in F. Conversely, whenever A, B are separated in F, then A, B are included in two uniformly separate zero sets $\zeta(f), \zeta(g)$, therefore there is a plethora of pairs of F for which $\zeta(f) \cup \zeta(g) \in Z(F)$. The next proposition shows that the Urysohn lemma for F-zero sets guarantees the existence of many strongly continuous functions.

Corollary 5.4.10. *If F is a topology on X, $f \in F$ and $A, B \subseteq X$, then*

$$f(A) \bowtie_{\mathrm{Bic}(\mathbb{R})} f(B) \to A \bowtie_F B.$$

Proof. By the Urysohn lemma for $\mathrm{Bic}(\mathbb{R})$-zero sets, if $f(A) \bowtie_{\mathrm{Bic}(\mathbb{R})} f(B)$, there are $\phi_A, \phi_B \in \mathrm{Bic}(\mathbb{R})$ and some $c > 0$ such that $f(A) \subseteq \zeta(\phi_A), f(B) \subseteq \zeta(\phi_B)$ and $|\phi_A| + |\phi_B| \geq \overline{c}$. Hence, $A \subseteq f^{-1}(\zeta(\phi_A)) = \zeta(\phi_A \circ f)$, $B \subseteq f^{-1}(\zeta(\phi_B)) = \zeta(\phi_B \circ f)$ and $|\phi_A \circ f| + |\phi_B \circ f| \geq \overline{c}$. By BS_3 we have that $\phi_A \circ f, \phi_B \circ f \in F$, therefore by the Urysohn lemma for F-zero sets we conclude that $A \bowtie_F B$. □

If F is a topology on some X and $x \in X$, we say that x is *G_δ-point*, if $\{x\}$ is a G_δ set in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$, which is clearly equivalent to the existence of a sequence $(f_n)_n$ in F such that $\{x\} = \bigcap_{n \in \mathbb{N}} U(f_n)$. Hence, the notion of a G_δ-point is completely determined by F.

Proposition 5.4.11. *If F is a topology on X and $x \in X$, then x is a G_δ-point if and only if $\{x\} \in Z(F)$.*

Proof. The sufficiency follows from the fact that every F-zero set is G_δ in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$. If x is a G_δ point, we have that $\{x\} = \bigcap_{n \in \mathbb{N}} U(f_n)$, for some $(f_n)_n$ in F. Since $U(f) = U(f \vee \overline{0})$, without loss of generality we can assume that $f_n \geq \overline{0}$, for every $n \in \mathbb{N}$. Hence, $f_n(x) = \sigma_n > 0$, and consequently $x \bowtie_{f_n} \zeta(f_n)$. As in the proof of Theorem 5.4.9, we have that $\zeta(f_n) \subseteq \zeta(g_n)$ and $\{x\} \subseteq \zeta(h_n)$, where

$$g_n = (f_n - \overline{\frac{\sigma_n}{3}}) \vee \overline{0},\ h_n = (f_n - \overline{\frac{2\sigma_n}{3}}) \wedge \overline{0},\ |g_n| + |h_n| \geq \overline{\frac{\sigma_n}{3}}.$$

Next we show that $\zeta(h_n) \subseteq U_{f_n}$; if $z \in X$ such that $h_n(z) = 0$, then $f_n(z) \geq \frac{2\sigma_n}{3} > 0$, hence $z \in U(f_n)$. Thus, $\{x\} \subseteq \bigcap_{n \in \mathbb{N}} \zeta(h_n) \subseteq \bigcap_{n \in \mathbb{N}} U(f_n) = \{x\}$, therefore $\{x\} = \bigcap_{n \in \mathbb{N}} \zeta(h_n) = \zeta(h)$, for some $h \in F$, by Proposition 5.3.3(ii). □

Note that if we don't use above the explicit form of g_n, h_n given in the proof of the Urysohn lemma, we cannot conclude constructively from $\neg(f_n(z) \leq 0)$ that $f_n(z) > 0$, since this property of the ordering of reals implies Markov's principle.

A Hausdorff topological space X is completely regular if and only if $Z(C(X))$ is a base for the closed sets i.e., every closed set is the intersection of a family of $C(X)$-zero sets, or equivalently

$$\forall_{B \text{ closed } \subseteq X} \forall_{x \notin B} \exists_{f \in C(X)} (B \subseteq \zeta(f) \wedge x \notin \zeta(f)).$$

Next follows a positive formulation of this relation between $Z(F)$ and $C(F)$.

Proposition 5.4.12. *If F is a topology on X and $B \in C(F)$, then*

$$\forall_{x \in X}(\mathrm{Away}(x, B) \to \exists_{f \in F}(B \subseteq \zeta(f) \wedge \mathrm{Away}(x, \zeta(f)))).$$

Proof. If $B = \bigcap Z(F_0)$, for some $F_0 \subseteq F$, and $f_0 \in F_0$ such that $f_0(x) \bowtie_{\mathbb{R}} 0$, then $x \bowtie_F \zeta(f_0)$. By the Urysohn lemma for F-zero sets there exist $f, g \in F$ and $c > 0$ such that $g(x) = 0$, $\zeta(f_0) \subseteq \zeta(f)$ and $|f| + |g| \geq \overline{c}$. Hence, $B \subseteq \zeta(f_0) \subseteq \zeta(f)$ and $|f(x)| + |g(x)| \geq c \leftrightarrow |f(x)| \geq c \leftrightarrow f(x) \bowtie_{\mathbb{R}} 0 \leftrightarrow \mathrm{Away}(x, \zeta(f))$. □

Proposition 5.4.13 (Urysohn lemma for separate F-closed sets)**.** *If F is a topology on X and $B, C \in C(F)$ such that* $\mathrm{SEP}(B, C)$*, then $B \bowtie_F C$.*

Proof. Suppose that $B = \bigcap Z(F_0), C = \bigcap Z(G_0)$, for some $F_0, G_0 \subseteq F$, and $f_0 \in F_0$, $g_0 \in G_0$ such that $\mathrm{Sep}(\zeta(f_0), \zeta(g_0))$. Then, $B \subseteq \zeta(f_0)$, $C \subseteq \zeta(g_0)$, and the Urysohn lemma for F-zero sets implies that $B \bowtie_F C$. □

Note that if $B, C \in C(F)$ and $f, g \in F$ such that $\mathrm{Usep}(\zeta(f), \zeta(g))$, then $\mathrm{SEP}(B \cap \zeta(f), C \cap \zeta(g))$, and consequently $B \cap \zeta(f) \bowtie_F C \cap \zeta(g)$. The next proposition is an adaptation of a result found in [44], p.17. Its formulation involves the negative notion of the complement $X \setminus A$ of a subset A of X and we use it in the classical proof of Proposition 5.4.15.

Proposition 5.4.14. *If $A, B \subseteq X$ and F is a topology on X, then*

$$A \bowtie_F B \to \exists_{f,g \in F}(A \subseteq X \setminus \zeta(f) \subseteq \zeta(g) \subseteq X \setminus B).$$

Proof. By definition there exists some $f \in F$ such that $f(A) = 0$ and $f(B) = 1$. We show that the F-zero sets $[f \geq \overline{\frac{1}{3}}]$ and $[f \leq \overline{\frac{1}{3}}]$ satisfy the inclusions

$$A \subseteq X \setminus [f \geq \overline{\frac{1}{3}}] \subseteq [f \leq \overline{\frac{1}{3}}] \subseteq X \setminus B.$$

The first inclusion is easy; if $f(a) = 0$, for some $a \in A$, then $a \notin [f \geq \overline{\frac{1}{3}}]$. The second inclusion follows from the property of reals $\neg(a \geq b) \to a \leq b$; if $a > b$, then $a \geq b$, which is absurd, and consequently $a \leq b$. The last inclusion is again easy; if $f(x) \leq \frac{1}{3}$, then $x \notin B$, since $f(B) = 1$. □

Proposition 5.4.15 (CLASS)**.** *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space such that F is $\neq$-Hausdorff, and $\mathcal{T}$ is a topology on X such that $\mathcal{T} \supseteq \mathcal{T}_{\mathcal{N}(F)}$. If K, L are disjoint compact subset of X with respect to $\mathcal{T}$ and $x \in X$ such that $x \notin K$, then there exists $f \in F$ which separates x and K, and some $g \in F$ which separates K and L.*

Proof. Suppose first that K is compact subset of X in $\mathcal{T}$ and $x \notin K$. Since the canonical set-set apartness relation induced by F extends the canonical apartness relation induced by F, we have that every pair of sets $\{x\}, \{y\}$, where $y \in K$, are F-separated, hence by Proposition 5.4.14 there are zero sets $\zeta(f_{x,y}), \zeta(g_{x,y})$ such that

$$\{y\} \subseteq X \setminus \zeta(f_{x,y}) \subseteq \zeta(g_{x,y}) \subseteq X \setminus \{x\}.$$

Since K is compact and an F-zero set is closed in $\mathcal{T}$ (an element of F is continuous with respect to $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$ and a trivial argument with nets shows that its zero set is closed in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$), therefore its complement is open in $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$. Since $K \subseteq \bigcup_{y \in K}\{y\} \subseteq \bigcup_{y \in K} X \setminus \zeta(f_{x,y})$, there exist $y_1, \ldots, y_N \in K$ such that

$$K \subseteq \bigcup_{i=1}^{N} X \setminus \zeta(f_{x,y_i}) = X \setminus \bigcap_{i=1}^{N} \zeta(f_{x,y_i}) = X \setminus \zeta(f),$$

where $0 \leq f = |f_1| + \ldots + |f_n| \in F$. This means that $f(K) \neq 0$. We also have that $f(x) = 0$, since by their definition $X \setminus \zeta(fx, y_i) \subseteq X \setminus \{x\} \to \{x\} \subseteq \zeta(f_{x,y_i})$, for every $i \in \{1, \ldots, N\}$, hence $\{x\} \subseteq \bigcap_{i=1}^{N} \zeta(f_{x,y_i}) = \zeta(f)$. We can find $m, M \in \mathbb{R}$ such that $f(K) \subseteq [m, M]$, since f is continuous with respect to $\mathcal{T}$, and we show that $m > 0$. Since $f \geq 0$, we know that $m \geq 0$, therefore it suffices to exclude the case $m = 0$; If $m = 0$, there is a sequence $k_n \subseteq K$ such that $f(k_n) \leq \frac{1}{n}$, for every n. Since K is compact, there is a subnet k_{n_μ} of k_n and some $k \in K$ such that $k_{n_\mu} \xrightarrow{\mu} k$. By the continuity of f we have that $f(k_{n_\mu}) \xrightarrow{\mu} f(k) = 0$, since the initial sequence $f(k_n)$ converges to 0. But then there is some $k \in K$ such that $k \in \zeta(f)$, which is absurd. If we define

$$h := \overline{\frac{2}{m}}(f \wedge \overline{\frac{m}{2}}) \in F,$$

we get that $h(x) = \frac{2}{m} f(x) = 0$ and $h(k) = \frac{2}{m}\frac{m}{2} = 1$, for every $k \in K$.
If K, L are disjoint compact subset of X with respect to $\mathcal{T}$, the previous case implies that the pair (k, L) is F-separated, for every $k \in K$. Hence there are zero sets $\zeta(f_{k,L}), \zeta(g_{k,L})$ such that $\{k\} \subseteq X \setminus \zeta(f_{k,L}) \subseteq \zeta(g_{k,L}) \subseteq X \setminus L$, and the proof continues exactly as in the previous case. □

It is known that if $C(X)$ separates the points of any topological space $(X, \mathcal{T})$, then any disjoint pair (x, K) or (K, L), as above, is separated by $C(X)$. What the previous proposition shows though, is that however larger is $\mathcal{T}$ from $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$, it is always F that separates the disjoint pairs (x, K) or (K, L). This is interesting because $\mathcal{T}_{\mathcal{N}(\mathcal{F})}$ is the smallest topology with respect to which F is a set of continuous functions.

5.5 Embeddings of Bishop spaces

In this section we develop the basic theory of embeddings of Bishop spaces in parallel to the basic classical theory of embeddings of rings of continuous functions, as it is presented in the first chapter of [44]. Its content is included in [75].

If $\mathcal{G}, \mathcal{F}$ are Bishop spaces, the notions "$\mathcal{G}$ is embedded in $\mathcal{F}$" and "$\mathcal{G}$ is bounded-embedded in $\mathcal{F}$" are the translations into TBS of the classical notions "Y is C-embedded in X" and "Y is C^*-embedded in X", for some $Y \subseteq X$ and a given topology of open sets $\mathcal{T}$ on X (see [44], p.17).

Definition 5.5.1. *If F is a topology on X, $f \in F$ and $a, b \in \mathbb{R}$ such that $a \leq b$, we say that a, b bound f, if $\forall_{x \in X}(a \prec f(x) \prec b)$, where $\prec \in \{<, \leq\}$.*

Definition 5.5.2. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$, then*
(i) $\mathcal{G}$ is embedded in $\mathcal{F}$, if $\forall_{g \in G}\exists_{f \in F}(f_{|Y} = g)$.
(ii) $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, if $\mathcal{G}_b$ is embedded in $\mathcal{F}_b$.
(iii) $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, if $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, and for every $g \in G_b$, if a, b bound g, then a, b bound some extension f of g in F_b.
(iv) $\mathcal{G}$ is dense-embedded in $\mathcal{F}$, if $\forall_{g \in G}\exists!_{f \in F}(f_{|Y} = g)$.
(v) $\mathcal{G}$ is dense-bounded-embedded in $\mathcal{F}$ and $\mathcal{G}$ is dense-full bounded-embedded in $\mathcal{F}$ are defined similarly to (iv).
(vi) $\mathcal{F}$ extends $\mathcal{G}$, if $\forall_{f \in F}(f_{|Y} \in G)$.

Clearly, (X, G) is embedded in (X, F) if and only if $G \subseteq F$. Note that Definition 5.5.2(vi) is necessary, since a topology F on some X does not necessarily behave like $C(X)$, where every $f \in C(X)$ restricted to Y belongs to $C(Y)$. By the definition of the relative Bishop space we get immediately that $\mathcal{F}$ extends $\mathcal{F}_{|Y}$. Clearly, if $\mathcal{G}$ is embedded in $\mathcal{F}$, then $\mathcal{G}'$ is embedded in $\mathcal{F}$, where $\mathcal{G}' = (Y, G')$ and $G' \subseteq G$.

Proposition 5.5.3. *Suppose that $Y \subseteq X$ and $\bowtie$ is a point-point apartness relation on X. Then the following hold:*
(i) $(Y, \mathrm{Const}(Y))$ is embedded in every Bishop space (X, F).
(ii) If $\forall_{x \in X}(x \in Y \vee x \notin Y)$, then $(Y, \mathbb{F}(Y))$ is embedded in $(X, \mathbb{F}(X))$.
(iii) If $Y = \{x_1, \ldots, x_n\}$, where $x_i \bowtie x_j$, for every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, and F is a topology on X which is $\bowtie$-Hausdorff, then $(Y, \mathbb{F}(Y))$ is full bounded-embedded in (X, F).
(iv) $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ is full bounded-embedded in $(\mathbb{Q}, \mathrm{Bic}(\mathbb{Q}))$.
(v) If $X = \mathbb{R}$ and Y is locally compact, then $(Y, \mathrm{Bic}(Y))$ is bounded-embedded in $\mathcal{R}$.
(vi) If X is a locally compact metric space and Y is dense in X, then $(Y, \mathrm{Bic}(Y))$ is dense-embedded and dense-bounded-embedded in $(X, \mathrm{Bic}(X))$.
(vii) If F is a topology on X and Y is a retract of X, then $\mathcal{F}_{|Y}$ is embedded in $\mathcal{F}$.

Proof. (i) and (ii) are trivial. To show (iii) we fix some $g \in \mathbb{F}(Y)$ and let $g(x_i) = a_i$, for every i. If we consider the $(n-1) + (n-2) + \ldots + 1$ functions $f_{ij} \in F$ such that $f_{ij}(x_i) \bowtie_{\mathbb{R}} f_{ij}(x_j)$, for every $i < j$, then the function f on X, defined by

$$f(x) := \sum_{i=1}^{n} a_i A_i(x),$$

$$A_i(x) := \prod_{k=i+1}^{n} \frac{f_{ik}(x) - f_{ik}(x_k)}{f_{ik}(x_i) - f_{ik}(x_k)} \prod_{k=1}^{i-1} \frac{f_{ki}(x_k) - f_{ki}(x)}{f_{ki}(x_k) - f_{ki}(x_i)},$$

is in F and $A_i(x_j) = 1$, if $j = i$, $A_i(x_j) = 0$, if $j \neq i$. Hence, f extends g, and clearly $(Y, \mathbb{F}(Y))$ is full-bounded embedded in (X, F). We need the $\bowtie$-Hausdorff condition on F so that $(f_{ij}(x_i) - f_{ij}(x_j)) \bowtie_{\mathbb{R}} 0$ and $(f_{ij}(x_i) - f_{ij}(x_j))^{-1}$ is well-defined, for every $i < j$.
(iv) If q is a rational such that $q \geq 0$, there is a unique $n \in \mathbb{N}$ such that $q \in [n, n+1)$. If $g : \mathbb{N} \to \mathbb{R}$, we define $\phi^*(q) = \gamma_n(q)$, where $\gamma_n : \mathbb{Q} \cap [n, n+1) \to \mathbb{R}$ is defined by

$$\gamma_n(q) = (g(n+1) - g(n))q + (n+1)g(n) - g(n+1)n$$

i.e., $\gamma_n(\mathbb{Q} \cap [n, n+1))$ is the set of the rational values in the linear segment between $g(n)$ and $g(n+1)$. Clearly, $\phi^*(n) = g(n)$. Next we define $\phi^*(q) = g(0)$, for every $q < 0$. To show that $\phi^* \in \mathrm{Bic}(\mathbb{Q})$, and since ϕ^* is constant on $\mathbb{Q}_-$, it suffices to show that $\phi^* \in \mathrm{Bic}(\mathbb{Q}_+)$. For that we fix a bounded subset (B, q_0, M) of $\mathbb{Q}_+$, where without loss of generality $M \in \mathbb{N}$. Since $B \subseteq \mathcal{B}(q_0, M)$, we have that $B \subseteq [n, N]$, where $n, N \in \mathbb{N}$, $n < N$, $q_0 - M \in [n, n+1)$ and $q_0 + M \in [N, N+1)$. Each γ_i is uniformly continuous on $[i, i+1) \cap \mathbb{Q}$ with modulus of continuity

$$\omega_i(\epsilon) = \frac{\epsilon}{|g(i+1) - g(i)| + 1},$$

for every $\epsilon > 0$. Hence, ϕ^* is uniformly continuous on B with modulus of continuity

$$\omega_{\phi^*, B}(\epsilon) = \min\{\omega_i(\epsilon) \mid n \leq i \leq N\},$$

for every $\epsilon > 0$. If g is bounded, then by its definition ϕ^* is also bounded and if a, b bound g, then a, b bound ϕ^*.
(v) If $M > 0$ such that $f(Y) \subseteq [-M, M]$, then we use Corollary 5.4.2.
(vi) Since $\mathbb{R}$ is a complete metric space, we use Lemma 4.7.13.
(vii) We show first that r is a quotient map i.e., $F_{|Y} = G_r = \{g : Y \to \mathbb{R} \mid g \circ r \in F\}$. By the definition of $r \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}_{|Y})$, we have that $\forall_{g \in F_{|Y}}(g \circ r \in F)$ i.e., $F_{|Y} \subseteq G_r$. For that we can also use the fact that the quotient topology G_r is the largest topology such that r is a morphism. If $g \in G_r$, then $(g \circ r)_{|Y} = g \in F_{|Y}$ i.e., $F_{|Y} \supseteq G_r$. Hence, if $g \in F_{|Y} = G_r$, the function $g \circ r \in F$ extends g. $\square$

Proposition 5.5.4. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$. If $\mathcal{G}$ is embedded in $\mathcal{F}$, then $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$.*

Proof. We show that if $g \in G_b$ such that $\exists_{f \in F}(f_{|Y} = g)$, then $\exists_{f \in F_b}(f_{|Y} = g)$; if f extends g and $|g| \leq M$, then $h = (-\overline{M} \vee f) \wedge \overline{M} \in F_b$ and $h_{|Y} = g$. Hence, $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, if $\forall_{g \in G_b}\exists_{f \in F}(f_{|Y} = g)$. Since $G_b \subseteq G$ and G is embedded in $\mathcal{F}$, we conclude that $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$. $\square$

There are trivial counterexamples to the converse of the previous proposition; if Y is an unbounded locally compact subset of $\mathbb{R}$, then by Proposition 5.5.3(v) $(Y, \mathrm{Bic}(Y))$ is full bounded-embedded in $\mathcal{R}_b$, while $(Y, \mathrm{Bic}(Y))$ is not embedded in $\mathcal{R}_b$, since $\mathrm{id}_Y \in \mathrm{Bic}(Y)$ and any extension of id_Y is an unbounded function.

Proposition 5.5.5. *Suppose that $Z \subseteq Y \subseteq X$ and $\mathcal{H} = (Z, H)$, $\mathcal{G} = (Y, G)$, $\mathcal{F} = (X, F)$ are Bishop spaces such that $\mathcal{F}$ extends $\mathcal{G}$ and $\mathcal{G}$ is embedded in $\mathcal{F}$. Then $\mathcal{H}$ is embedded in $\mathcal{F}$ if and only if $\mathcal{H}$ is embedded in $\mathcal{G}$.*

Proof. If $\forall_{h \in H} \exists_{f \in F}(f_{|Z} = h)$, we show that $\forall_{h \in H} \exists_{g \in G}(g_{|Z} = h)$. We fix $h \in H$, and if we restrict some $f \in F$ which extends h to Y, we get the required extension of h in G. For the converse we fix some $h \in H$ and first we extend it to some $g \in G$. Then g is extended to some $f \in F$, since $\mathcal{G}$ is embedded in $\mathcal{F}$. $\square$

The next three propositions show how the embedding of $\mathcal{G}$ to $\mathcal{F}$ generates new embeddings under the presence of certain morphisms.

Proposition 5.5.6. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ and $\mathcal{H} = (B, H)$ are Bishop spaces, where $B \subseteq Y$. If $\mathcal{H}$ is embedded in $\mathcal{G}$ and $e \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$, then the weak Bishop space $\mathcal{F}(e_{|A})$ on $A = e^{-1}(B)$ induced by $e_{|A}$ is embedded in $\mathcal{F}$.*

Proof. Since $e : X \to Y$ is onto Y, we have that $e_{|A} : A \to B$ is onto B and $e_{|A} \in \mathrm{setEpi}(\mathcal{F}(e_{|A}), \mathcal{H})$, therefore by Proposition 4.5.3 we have that $F(e_{|A}) = \{h \circ e_{|A} \mid h \in H\}$. If we fix some $h \circ e_{|A} \in F(e_{|A})$, where $h \in H$, then, since $\mathcal{H}$ is embedded in $\mathcal{G}$, there is some $g \in G$ such that $g_{|B} = h$. Since $e \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, we get that $g \circ e \in F$. If $a \in A$, then $(g \circ e)(a) = g(b) = h(b)$, where $b = e(a)$. Since $(h \circ e_{|A})(a) = h(e(a)) = h(b)$, we get that $(g \circ e)_{|A} = h \circ e_{|A}$ i.e., $\mathcal{F}(e_{|A})$ is embedded in $\mathcal{F}$. $\square$

Proposition 5.5.7. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ and $\mathcal{H} = (Z, H)$ are Bishop spaces, where $Y \subseteq X$. If $\mathcal{G}$ is embedded in $\mathcal{F}$ and $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$ is open, then the quotient Bishop space $\mathcal{G}_{e_{|Y}} = (e(Y), G_{e_{|Y}})$ is embedded in $\mathcal{H}$.*

Proof. Let $g' : e(Y) \to \mathbb{R} \in G_{e_{|Y}}$ i.e., $g' \circ e_{|Y} \in G$. Since $\mathcal{G}$ is embedded in $\mathcal{F}$, there exists some $f \in F$ such that $f_{|Y} = g' \circ e_{|Y}$. Since e is open, there exists some $h \in H$ such that $f = h \circ e$. We show that $h_{|e(Y)} = g'$; if $b = e(y) \in e(Y)$, for some $y \in Y$, then $h(b) = h(e(y)) = f(y) = (g' \circ e_{|Y})(y) = g'(e(y)) = g'(b)$. $\square$

Next we translate within TBS the classical fact that if an element of $C(X)$ carries a subset of X homeomorphically onto a closed set S in $\mathbb{R}$, then S is C-embedded in X (see [44], p.20).

Proposition 5.5.8. *Suppose that A is a locally compact subset of $\mathbb{R}$, $\mathcal{F} = (X, F)$ is a Bishop space, $Y \subseteq X$ and $f \in F$ such that $f_{|Y} : Y \to A$ is an isomorphism between $\mathcal{F}_{|Y}$ and $(A, \mathrm{Bic}(A)_b)$. Then $\mathcal{F}_{|Y}$ is embedded in $\mathcal{F}$.*

Proof. Since $f_{|Y}$ is an isomorphism between $\mathcal{F}_{|Y}$ and $(A, \mathrm{Bic}(A)_b)$ its inverse θ is an isomorphism between and $(A, \mathrm{Bic}(A)_b)$ and $\mathcal{F}_{|Y}$. We fix some $g \in F_{|Y}$. Since $\theta \in \mathrm{Mor}((A, \mathrm{Bic}(A)_b), \mathcal{F}_{|Y})$, we have that $g \circ \theta \in \mathrm{Bic}(A)_b$. By Corollary 5.4.2 there exists some $\phi \in \mathrm{Bic}(\mathbb{R})$ which extends $g \circ \theta$. By BS_3 we have that $\phi \circ f \in F$ and for every $y \in Y$ we have that $(\phi \circ f)(y) = ((g \circ \theta) \circ f)(y) = (g \circ (\theta \circ f))(y) = (g \circ (\theta \circ f_{|Y}))(y) = (g \circ \mathrm{id}_{|Y})(y) = g(y)$. $\square$

If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ is C^*-embedded in X, then if Y is also C-embedded in X, it is (completely) separated in $C(X)$ from every $C(X)$-zero set disjoint from it (see [44], pp.19-20). If we add within TBS a positive notion of disjointness between Y and $\zeta(f)$ though, we avoid the corresponding hypothesis of $\mathcal{G}$ being embedded in $\mathcal{F}$.

Definition 5.5.9. *If F is a topology on X, $f \in F$ and $Y \subseteq X$, we say that Y and $\zeta(f)$ are separated,* $\mathrm{Sep}(Y, \zeta(f))$, *if*
$$\forall_{y \in Y}(|f(y)| > 0),$$
and Y and $\zeta(f)$ are uniformly separated, $\mathrm{Usep}(Y, \zeta(f))$, *if*
$$\exists_{c>0}\forall_{y \in Y}(|f(y)| \geq c).$$

Clearly, $\mathrm{Usep}(Y, \zeta(f)) \to \mathrm{Sep}(Y, \zeta(f))$. If for some $f, g \in F$ we consider the condition $|f| + |g| \geq \overline{c}$ that appears in Theorem 5.4.9, we get that it implies $\mathrm{Usep}(\zeta(g), \zeta(f))$ and $\mathrm{Usep}(\zeta(f), \zeta(g))$.

Proposition 5.5.10. *Suppose that $\mathcal{F} = (X, F)$, $f \in F$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, $\mathcal{F}$ extends $\mathcal{G}$, and $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$. Then*
$$\mathrm{Usep}(Y, \zeta(f)) \to Y \bowtie_F \zeta(f).$$

Proof. Since $|f| \in F$ and $\mathcal{F}$ extends $\mathcal{G}$, we have that $|f|_{|Y} \in G$, and $|f|_{|Y} \geq \overline{c}$. By Theorem 5.4.8 we get that $\frac{1}{|f|_{|Y}} \in G$. Since $\overline{0} < \frac{1}{|f|_{|Y}} \leq \overline{\frac{1}{c}}$, we actually have that $\frac{1}{|f|_{|Y}} \in G_b$. Since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there exists $h \in F$ such that $h_{|Y} = \frac{1}{|f|_{|Y}}$. Since $|h| \in F$ satisfies $|h|_{|Y} = \frac{1}{|f|_{|Y}}$ too, we suppose without loss of generality that $h \geq \overline{0}$. If we define $g := h|f|$, then $g \in F$, $g(y) = h(y)|f(y)| = \frac{1}{|f(y)|}|f(y)| = 1$, for every $y \in Y$, and $g(x) = h(x)|f(x)| = h(x)0 = 0$, for every $x \in \zeta(f)$. □

Since the sets $U(f)$, where $f \in F$, are basic open sets in the induced neighborhood structure on X by F, we give the following definition.

Definition 5.5.11. *If F is a topology on X and $Y \subseteq X$, we call Y a uniform G_δ-set, if there exists a sequence $(f_n)_n$ in F such that*
$$Y = \bigcap_{n \in \mathbb{N}} U(f_n) \quad \textit{and} \quad \forall_{n \in \mathbb{N}}(\mathrm{Usep}(Y, \zeta(f_n))).$$

Corollary 5.5.12. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, and $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$. If Y is a uniform G_δ-set, then Y is an F-zero set.*

Proof. Suppose that $Y = \bigcap_{n \in \mathbb{N}} U(f_n)$ and $\forall_{y \in Y}(|f_n(y)| \geq c_n)$, where $c_n > 0$, for every $n \in \mathbb{N}$. Since $U(f) = U(f \vee \overline{0})$ and $\mathrm{Usep}(Y, \zeta(f)) \to \mathrm{Usep}(Y, \zeta(f \vee \overline{0}))$, we assume without loss of generality that $f_n \geq \overline{0}$, for every $n \in \mathbb{N}$. By the proof of Proposition 5.5.10 we have that there is a function $h_n \in F$ such that $h_n \geq \overline{0}$, $(h_n f_n)(Y) = 1$ and $(h_n f_n)(\zeta(f_n)) = 0$, for every $n \in \mathbb{N}$. Therefore, $Y \subseteq \zeta(g_n)$, where $g_n = (h_n f_n - \overline{\frac{2}{3}}) \wedge \overline{0}$, for every $n \in \mathbb{N}$. Next we show that $\zeta(g_n) \subseteq U(f_n)$, for every $n \in \mathbb{N}$. Since $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$ and according to the the proof of Proposition 5.5.10, $\overline{0} < \frac{1}{f_{n|Y}} \leq \overline{\frac{1}{c_n}}$, we get that $\overline{0} < h_n \leq \overline{\frac{1}{c_n}}$. If $z \in X$ such that $g_n(z) = 0$, then $h_n(z)f_n(z) \geq \frac{2}{3}$, and since $h_n(z) > 0$, we conclude that
$$f_n(z) \geq \frac{2}{3h_n(z)} > 0.$$

Thus, we have that

$$Y \subseteq \bigcap_{n\in\mathbb{N}} \zeta(g_n) \subseteq \bigcap_{n\in\mathbb{N}} U(f_n) = Y,$$

which implies that $Y = \bigcap_{n\in\mathbb{N}} \zeta(g_n) = \zeta(g)$, for some $g \in F$, since $Z(F)$ is closed within BISH under countably infinite intersections. $\square$

Note that without the condition of $\mathcal{G}$ being full bounded-embedded in $\mathcal{F}$ in the previous proposition we can only show that $\neg(f_n(z) = 0)$. Although $f_n(z) \geq 0$, we cannot infer within BISH that $f_n(z) > 0$; the property of the reals $\forall_{x,y\in\mathbb{R}}(\neg(x \geq y) \to x < y)$ is equivalent to Markov's principle (MP) (see [20], p.14 and [25], p.28), and it is easy to see that this property is equivalent to $\forall_{x\in\mathbb{R}}(x \geq 0 \to \neg(x = 0) \to x > 0)$.
Next we translate to TBS the classical result that if Y is C^*-embedded in X such that Y is (completely) separated from every $C(X)$-zero set disjoint from it, then Y is C-embedded in X. Constructively, it is not as easy as in the classical case to show that the expected positive formulation of the previous condition suffices to provide an inverse to Proposition 5.5.4. The reason is that if (X, F) is an arbitrary Bishop space, it is not certain that $\tan \circ f \in F$, for some $f : X \to (-\frac{\pi}{2}, \frac{\pi}{2}) \in F$ (note that $\tan^{-1} = \arctan \in \mathrm{Bic}(\mathbb{R})$).
Recall that if $\Phi_1\Phi_2 \subseteq \mathbb{F}(X)$, we denote by $\Phi_1 \vee \Phi_2$ the least topology including them. The proof of the more interesting case of the next theorem is in BISH + MP.

Theorem 5.5.13. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, $a > 0$,*

$$e : (-a, a) \to \mathbb{R},$$

$$e^{-1} : \mathbb{R} \to (-a, a) \in \mathrm{Bic}(\mathbb{R}),$$

$$\mathcal{F}(a) = (X, F(a)),$$

$$F(a) = F \vee \{e \circ f \mid f \in F \wedge f(X) \subseteq (-a, a)\}.$$

(i) If $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, then $\mathcal{G}$ is embedded in $\mathcal{F}(a)$.
(ii) (MP) *If $\forall_{f\in F}(\mathrm{Sep}(Y, \zeta(f)) \to Y \bowtie_F \zeta(f))$ and $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, then $\mathcal{G}$ is embedded in $\mathcal{F}(a)$.*

Proof. We fix some $g \in G$. Since $e^{-1} \in \mathrm{Bic}(\mathbb{R})$, by the condition BS_3 we have that $e^{-1} \circ g : Y \to (-a, a) \in G_b$. Since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there is some $f \in F_b$ such that $f_{|Y} = e^{-1} \circ g$.
(i) If $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, then we have that $f : X \to (-a, a)$. Hence, $e \circ f \in F(a)$, and $(e \circ f)_{|Y} = e \circ f_{|Y} = e \circ (e^{-1} \circ g) = g$.
(ii) As we have already shown within BISH, $[|f| \geq \overline{a}] = \{x \in X \mid |f|(x) \geq a\} = \zeta(f^*)$, where $f^* = (|f| - \overline{a}) \wedge \overline{0} \in F$. If $y \in Y$, then

$$|f^*(y)| = |(|f(y)| - a) \wedge 0| = |(|(e^{-1} \circ g)(y)| - a) \wedge 0| = ||(e^{-1} \circ g)(y)| - a| = a - |(e^{-1} \circ g)(y)| > 0,$$

since $|(e^{-1} \circ g)(y)| \in [0, a)$ (if $-a < x < a$, then $|x| < a$). Since $\mathrm{Sep}(Y, \zeta(f^*))$, by our hypothesis there exists some $h \in F$ such that $0 \leq h \leq 1$, $h(Y) = 1$ and $h(\zeta(f^*)) = 0$. There is no loss of generality if we assume that $0 \leq h \leq 1$, since if $h \in F$ separates Y and $\zeta(f^*)$, then $|h| \wedge \overline{1} \in F$ separates them too. We define

$$J := f \cdot h \in F.$$

If $y \in Y$, we have that $J(y) = f(y)h(y) = f(y)$. Next we show that

$$\forall_{x \in X}(\neg(|J(x)| \geq a)).$$

If $x \in X$ such that $|J(x)| \geq a$, then $|f(x)| \geq |f(x)||h(x)| = |j(x)| \geq a$, therefore $x \in \zeta(f^*)$. Consequently, $h(x) = 0$, and $0 = |J(x)| \geq a > 0$, which leads to a contradiction. Because of MP we get that $\forall_{x \in X}(|J(x)| < a))$, in other words, $J : X \to (-a, a)$. Hence $e \circ J \in F(a)$, and $(e \circ J)_{|Y} = e \circ J_Y = e \circ f = e \circ (e^{-1} \circ g) = g$. $\square$

5.6 The Urysohn extension theorem for Bishop spaces

In this section we prove the Urysohn extension theorem within TBS. Its content is included in [75].

The Urysohn extension theorem for Bishop spaces is an adaptation of Urysohn's theorem that any closed set in a normal topological space is C^*-embedded (see [44], p.266) and, as Gillman and Jerison note in [44], p.18, it is "the basic result about C^*-embedding". According to it, a subspace Y of a topological space X is C^*-embedded in X if and only if any two (completely) separated sets in Y are (completely) separated in X. Here we call Urysohn extension theorem the appropriate translation to TBS of the non-trivial classical sufficiency condition.

The next proposition is the translation to TBS of the trivial classical necessity condition. Note that the hypothesis that $\mathcal{F}$ extends $\mathcal{G}$ which is found in the Urysohn extension theorem for Bishop spaces (Theorem 5.6.3) is not necessary to the following proof of its inverse.

Proposition 5.6.1. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$. If $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, then*

$$\forall_{A,B \subseteq Y}(A \bowtie_{G_b} B \to A \bowtie_{F_b} B).$$

Proof. If $A, B \subseteq Y$ such that A, B are separated by some $g \in G_b$, and since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there is some $f \in F_b$ extending g, therefore f separates A and B. $\square$

Next we translate to TBS the proof of the classical Urysohn extension theorem, showing that it can be carried out in BISH. First we need a simple lemma. Note that using MP one shows immediately that

$$\neg(x \leq -q) \to \neg(x \geq q) \to |x| < q,$$

where $x \in \mathbb{R}$ and $q \in \mathbb{Q}$. Without using MP, but completely within BISH, we show that under the same hypotheses one gets that $|x| \leq q$, which is what we need in order to get a constructive proof of the Urysohn extension theorem.

Lemma 5.6.2. $\forall_{q\in\mathbb{Q}}\forall_{x\in\mathbb{R}}(\neg(x \leq -q) \to \neg(x \geq q) \to |x| \leq q)$.

Proof. We fix some $q \in \mathbb{Q}$, $x = (x_n)_n \in \mathbb{R}$ and we suppose that $\neg(x \leq -q)$ and $\neg(x \geq q)$. Since $|x| = (\max\{x_n, -x_n\})_{n\in\mathbb{N}}$, we show that $q \geq |x| \leftrightarrow q - |x| \geq 0 \leftrightarrow \forall_n(q - \max\{x_n, -x_n\} \geq -\frac{1}{n})$. If we fix some $n \in \mathbb{N}$, and since $x_n \in \mathbb{Q}$, we consider the following case distinction.
(i) $x_n \geq 0$: Then $q - \max\{x_n, -x_n\} = q - x_n$ and we get that $q - x_n < -\frac{1}{n} \to x_n - q > \frac{1}{n} \to x > q \to x \geq q \to \perp$, by our second hypothesis. Hence, $q - x_n \geq -\frac{1}{n}$.
(ii) $x_n \leq 0$: Then $q - \max\{x_n, -x_n\} = q + x_n$ and we get that $q + x_n < -\frac{1}{n} \to -q - x_n > \frac{1}{n} \to -q > x \to -q \geq x \to \perp$, by our first hypothesis. Hence, $q + x_n \geq -\frac{1}{n}$. □

Theorem 5.6.3 (Urysohn extension theorem for Bishop spaces)**.** *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$ and $\mathcal{F}$ extends $\mathcal{G}$. If $\forall_{A,B\subseteq Y}(A \bowtie_{G_b} B \to A \bowtie_{F_b} B)$, then $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$.*

Proof. We fix some $g \in G_b$, and let $|g| \leq \overline{M}$, for some natural $M > 0$. In order to find an extension of g in F_b we define a sequence $(g_n)_{n\in\mathbb{N}^+}$, such that $g_n \in G_b$ and

$$|g_n| \leq \overline{3r_n},$$

$$r_n := \frac{M}{2}(\frac{2}{3})^n,$$

for every $n \in \mathbb{N}^+$. For $n = 1$ we define $g_1 = g$, and we have that $|g_1| \leq \overline{M} = \overline{3r_1}$. Suppose next that we have defined some $g_n \in G_b$ such that $|g_n| \leq \overline{3r_n}$. We consider the sets

$$A_n = [g_n \leq \overline{-r_n}] = \{y \in Y \mid g_n(y) \leq -r_n\},$$

$$B_n = [g_n \geq \overline{r_n}] = \{y \in Y \mid g_n(y) \geq r_n\}.$$

Clearly, $g_n^*(A_n) = -r_n$ and $g_n^*(B_n) = r_n$, where $g_n^* = (\overline{-r_n} \vee g_n) \wedge \overline{r_n} \in G_b$. Since $g_n^*(A_n) \bowtie_{\mathbb{R}} g_n^*(B_n)$, we get that $A_n \bowtie_{G_b} B_n$, therefore there exists some $f \in F_b$ such that $A_n \bowtie_f B_n$. Without loss of generality we assume that $f_n(A_n) = -r_n$, $f_n(B_n) = r_n$ and $|f_n| \leq \overline{r_n}$. Next we define

$$g_{n+1} := g_n - f_{n|Y} \in G_b,$$

since $\mathcal{F}$ extends $\mathcal{G}$. If $y \in A_n$ we have that

$$|g_{n+1}(y)| = |(g_n - f_{n|Y})(y)| = |g_n(y) - (-r_n)| = |g_n(y) + r_n| \leq 2r_n,$$

since $-3r_n \leq g_n(y) \leq -r_n \to -2r_n \leq g_n(y) + r_n \leq 0$. If $y \in B_n$ we have that

$$|g_{n+1}(y)| = |(g_n - f_{n|Y})(y)| = |g_n(y) - r_n| = g_n(y) - r_n \leq 2r_n,$$

since $r_n \leq g_n(y) \leq 3r_n \to 0 \leq g_n(y) - r_n \leq 2r_n$. Next we show that

$$\forall_{y\in Y}(|g_{n+1}(y)| \leq 2r_n).$$

We fix some $y \in Y$ and we suppose that $|g_{n+1}(y)| > 2r_n$. This implies that $y \notin A_n \cup B_n$, since if $y \in A_n \cup B_n$, then by the previous calculations we get that $|g_{n+1}(y)| \leq 2r_n$, which contradicts our hypothesis. Hence we have that $\neg(g_n(y) \leq -r_n)$ and $\neg(g_n(y) \geq r_n)$. By Lemma 5.6.2 we get that $|g_n(y)| \leq r_n$, therefore $|g_{n+1}(y)| \leq |g_n(y)|+|f_n(y)| \leq r_n+r_n = 2r_n$, which contradicts our assumption $|g_{n+1}(y)| > 2r_n$. Thus we get that $|g_{n+1}(y)| \leq 2r_n$ and since y is arbitrary we get

$$|g_{n+1}| \leq \overline{2r_n} = \overline{3r_{n+1}}.$$

By the condition BS_4 the function

$$f := \sum_{n=1}^{\infty} f_n \in F,$$

since the partial sums converge uniformly to f. Note that the infinite sum is well-defined by the Weierstrass comparison test (see [15], p.32). Note also that

$$(f_1 + \ldots + f_n)_{|Y} = (g_1 - g_2) + (g_2 - g_3) + \ldots + (g_n - g_{n+1}) = g_1 - g_{n+1}.$$

Since $r_n \overset{n}{\to} 0$ we get $g_{n+1} \overset{n}{\to} 0$, and we conclude that $f_{|Y} = g_1 = g$. Note that f is also bounded by M, since

$$|f| = |\sum_{n=1}^{\infty} f_n| \leq \sum_{n=1}^{\infty} |f_n| \leq \sum_{n=1}^{\infty} \frac{M}{2}(\frac{2}{3})^n = \frac{M}{2}\sum_{n=1}^{\infty}(\frac{2}{3})^n = \frac{M}{2}2 = M.$$

$\square$

The main hypothesis of the Urysohn extension theorem

$$\forall_{A,B\subseteq Y}(A \bowtie_{G_b} B \to A \bowtie_{F_b} B)$$

requires quantification over the power set of Y, therefore it is against the common practice of predicative constructive mathematics. It is clear though by the above proof that we do not need to quantify over all the subsets of Y, but only over the ones which have the form of A_n and B_n. If we replace the initial main hypothesis by the following

$$\forall_{g,g'\in G_b}\forall_{a,b\in\mathbb{R}}([g \leq \overline{a}] \bowtie_{G_b} [g' \geq \overline{b}] \to [g \leq \overline{a}] \bowtie_{F_b} [g' \geq \overline{b}]),$$

we get a stronger form of the Urysohn extension theorem, since this is the least condition in order the above proof to work. Actually, this stronger formulation of the Urysohn extension theorem applies to the classical setting too. A slight variation of the previous new main hypothesis, which is probably better to use, is

$$\forall_{g,g'\in G_b}(\zeta(g) \bowtie_{G_b} \zeta(g') \to \zeta(g) \bowtie_{F_b} \zeta(g')),$$

since the sets of the form A_n and B_n are G_b-zero sets.

Definition 5.6.4. *If (X, F) is a Bishop space and $Y \subseteq X$ is inhabited, we say that Y is a Urysohn subset of X, if*

$$\forall_{g,g' \in (F_{|Y})_b}(\zeta(g) \bowtie_{(F_{|Y})_b} \zeta(g') \to \zeta(g) \bowtie_{F_b} \zeta(g')).$$

Next follows an immediate corollary of Theorem 5.6.3 and of the previous remark.

Corollary 5.6.5. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space, $Y \subseteq X$ is a Urysohn subset of X and $g : Y \to \mathbb{R}$ is in $(F_{|Y})_b$. Then there exists $f : X \to \mathbb{R}$ in F_b which extends g.*

An absolute retract for normal topological spaces is a space that can be substituted for $\mathbb{R}$ in the formulation of the Tietze theorem, according to which a continuous real-valued function on a closed subset of a normal topological space has a continuous extension (see [40], p.151).

Definition 5.6.6. *If Q is a property on sets, a Bishop space $\mathcal{H} = (Z, H)$ is called an absolute retract with respect to Q, or $\mathcal{H}$ is $\mathrm{AR}(Q)$, if for every Bishop space $\mathcal{F} = (X, F)$ and $Y \subseteq X$ inhabited we have that*

$$Q(Y) \to \forall_{e \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H})} \exists_{e^* \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})}(e^*_{|Y} = e).$$

Clearly, Corollary 5.6.5 says that $\mathcal{R}$ is AR(Urysohn). The next proposition shows that there exist many absolute retracts. In particular, the products $\mathcal{R}^n, \mathcal{R}^\infty$ are AR(Urysohn).

Proposition 5.6.7. *Suppose that $\mathcal{H}_i = (Z_i, H_i)$ is a Bishop space, for every $i \in I$. Then $\prod_{i \in I} \mathcal{H}_i$ is $\mathrm{AR}(Q)$ if and only if $\mathcal{H}_i$ is $\mathrm{AR}(Q)$, for every $i \in I$.*

Proof. Suppose that $Y \subseteq X$ such that $Q(Y)$ and that $\mathcal{H}_i$ is $\mathrm{AR}(Q)$, for each $i \in I$. By Proposition 4.1.6(ii) we have that

$$e : Y \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I}) \leftrightarrow \forall_{i \in I}(\pi_i \circ e \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H}_i))$$
$$\to \forall_{i \in I}(\exists_{e^*_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)}(e^*_{i\,|Y} = \pi_i \circ e))$$

We define $e^* : X \to \prod_{i \in I} Z_i$ by $x \mapsto (e^*_i(x))_{i \in I}$. It is clear that $e^*(y) = e^*_i(y))_{i \in I} = ((\pi_i \circ e)(y))_{i \in I} = e(y)$ and $e^* \in \mathrm{Mor}(\mathcal{F}, \prod_{i \in I} \mathcal{H}_{i \in I})$, by Proposition 4.1.6(ii) and the fact that $e^*_i = \pi_i \circ e^* \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)$, for every $i \in I$. For the converse direction suppose that $\prod_{i \in I} \mathcal{H}_i$ is $\mathrm{AR}(Q)$ and $e_i : Y \to Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H}_i)$. If we fix $z = (z_i)_{i \in I} \in \prod_{i \in I} Z_i$, then by Proposition 4.7.7 the function

$$s_i : Z_i \to S(z; i) = Z_i \times \prod_{j \neq i} \{z_j\} \subseteq \prod_{i \in I} Z_i$$

$$z_i \mapsto z_i \times \prod_{j \neq i} \{z_j\}$$

is an isomorphism between $\mathcal{H}_i$ and the slice space $\mathcal{S}(z; i) = (S(z; i), H(z; i))$, where $H(z; i) = (\prod_{i \in I} H_i)_{|S(z;i)}$. Hence, the mapping $s_i \circ e_i : Y \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I})$. By our hypothesis there exists some $e^* : X \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I})$ which extends $s_i \circ e_i$. Thus, $\pi_i \circ e^* : X \to Z_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)$, for every $i \in I$. But $\pi_i \circ e^* = e_i$, since for every $y \in Y$ we have that $(\pi_i \circ e^*)(y) = \pi(e^*(y)) = \pi_i((s_i \circ e_i)(y)) = \pi_i(e_i(y) \times \prod_{j \neq i} \{z_j\}) = e_i(y)$. $\square$

5.7 Completely regular Bishop spaces

In this section we introduce the completely regular Bishop spaces and we present some of their basic properties. Most of its content is included in [74].
A completely regular topological space $(X, \mathcal{T})$ is one in which any pair (x, B), where B is closed and $x \notin B$, is separated by some $f \in C(X, [0, 1])$. A completely regular and T_1-space satisfies classically the property

$$\forall_{x_1,x_2 \in X}(\forall_{f \in C(X)}(f(x_1) = f(x_2)) \to x_1 = x_2).$$

The importance and the "sufficiency" of the completely regular topological spaces in the theory of $C(X)$ is provided by the Stone-Čech theorem according to which, for every topological space X there exists a completely regular space ρX and a continuous mapping $\tau : X \to \rho X$ such that the induced function $g \mapsto \tau^*(g)$, where $\tau^*(g) = g \circ \tau$, is a ring isomorphism between $C(\rho X)$ and $C(X)$ (see [44], p.41).
In this section we define a notion of a completely regular Bishop space and we prove some fundamental results on them which justify our definition.

Definition 5.7.1. *We call a Bishop space $\mathcal{F} = (X, F)$ completely regular and its topology F a completely regular topology, if its canonical point-point apartness relation $\bowtie_F$ is tight i.e.,*

$$\forall_{x_1,x_2 \in X}(\forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2).$$

If $\mathcal{F}$ is completely regular, the equality of X is determined by F. Since $\bowtie_{\mathrm{Bic}(\mathbb{R})} \leftrightarrow \bowtie_{\mathbb{R}}$ and $\bowtie_{\mathbb{R}}$ is tight, $\mathcal{R}$ is completely regular, while if X has at least two points, then $\mathrm{Const}(X)$ is not completely regular. We denote by **crBis** the full subcategory of completely regular Bishop spaces of **Bis** i.e., the morphisms between two Bishop spaces in **crBis** are the morphisms between them in **Bis**. The next proposition says that if F is a topology on X such that $\bowtie_F$ is tight, then its restriction $\bowtie_{F_b}$ to F_b^2 is also tight.

Proposition 5.7.2. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space. Then $\mathcal{F}$ is completely regular if and only if $\mathcal{F}_b$ is completely regular.*

Proof. If $\mathcal{F}_b$ is completely regular, then $\mathcal{F}$ is trivially completely regular. We suppose next that F is completely regular and we fix $x, y \in X$ such that $\forall_{f \in F_b}(f(x) = f(y))$. In order to show that $x = y$ it suffices to show that $\forall_{f \in F}(f(x) = f(y))$. We fix $f \in F$ and we suppose that $f(x) = a$ and $f(y) = b$. Using the simple properties of reals $a \leq |a| \to a < |a| + 1$ and $-a \leq |-a| = |a| \to a \geq -|a| \to a > -(|a| + 1)$, we get that $m = -(|a| + |b| + 1) < a, b < |a| + |b| + 1 = M$, and consequently, if $g := (f \vee \overline{m}) \wedge \overline{M} \in F_b$, then $g(x) = g(y)$. Since $g(x) = f(x)$ and $g(y) = f(y)$, we conclude that $f(x) = f(y)$. □

Proposition 5.7.3. *If $\mathcal{F} = (X, F)$ is a completely regular Bishop space, then $(X, \lim_F)$ is a Fréchet limit space.*

Proof. It suffices to show that $(X, \lim_F)$ has the uniqueness property. Suppose that $\lim_F(x, x_n)$ and $\lim_F(x, x_n)$ i.e., $\forall_{f \in F}(f(x_n) \to f(x))$ and $\forall_{f \in F}(f(x_n) \to f(y))$, respectively. If $f \in F$, by the uniqueness of the sequential convergence in $\mathbb{R}$ we get that $f(x) = f(y)$. Since this is the case for every $f \in F$ and $\mathcal{F}$ is completely regular, we get that $x = y$. □

Proposition 5.7.4. *If $\mathcal{F} = (X, F)$ is a completely regular Bishop space and $x_0 \in X$, then $\{x_0\}$ is closed in $N(F)$.*

Proof. We show that $\forall_{x \in X}(\forall_{f \in F}(f(x) > 0 \to U(f) \between \{x_0\}) \to x \in \{x_0\})$. We fix $x \in X$ and we suppose that $\forall_{f \in F}(f(x) > 0 \to f(x_0) > 0)$. If $x \bowtie_F x_0$, we can find $f \in F$ such that $f(x_0) = 0$ and $f(x) = 1$, which contradicts our hypothesis i.e., $\neg(x \bowtie_F x_0)$, which by the tightness of $\bowtie_F$ implies that $x = x_0$. $\square$

Proposition 5.7.5. *Suppose that Y is inhabited, Y_0 is an inhabited subset of Y, and $\mathcal{H} = (Z, H)$ is a completely regular Bishop space. Then the following hold:*
(i) $\mathrm{Mor}((Y, \mathrm{Const}(Y)), \mathcal{H}) = \mathrm{Const}(Y, Z)$.
(ii) If $e, j \in \mathrm{Mor}((Y, \mathrm{Const}(Y)), \mathcal{H})$ and $e_{|Y_0} = j_{|Y_0}$, then $e = j$.

Proof. (i) It suffices to show that if $f \in \mathrm{Mor}((Y, \mathrm{Const}(Y)), \mathcal{H})$, then $f \in \mathrm{Const}(Y, Z)$. We fix some $y_1, y_2 \in Y$. By the definition of a morphism we have that $h \circ f \in \mathrm{Const}(Y)$, hence $h(f(y_1)) = h(f(y_1))$, for every $h \in H$. Since $\mathcal{H}$ is completely regular, we get that $f(y_1) = f(y_2)$. Since y_1, y_2 were arbitrary, we conclude that f is constant.
(ii) By (i) we have that $e, j \in \mathrm{Const}(Y, Z)$, therefore if $y_0 \in Y_0$, then $e = \overline{e(y_0)} = e_{|Y_0} = j_{|Y_0} = \overline{j(y_0)} = j$. $\square$

Proposition 5.7.6. *Suppose that $\mathcal{F} = (X, F)$ is a completely regular Bishop space, $D \subseteq [a, b]$ is dense, and $h_1, h_2 : [a, b] \to X \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F})$. Then, $h_{1|D} = h_{2|D} \to h_1 = h_2$.*

Proof. Since $f \circ h_i \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F}) \leftrightarrow f \circ h_i$ is uniformly continuous, we show first that $\forall_{f \in F}(f \circ h_1 = f \circ h_2)$. We fix $f \in F, x \in X$ and some $d \in D$ such that

$$|x - d| \leq \min\{\omega_{f \circ h_1}(\frac{\epsilon}{2}), \omega_{f \circ h_2}(\frac{\epsilon}{2})\}.$$

Since $h_1(d) = h_2(d)$, we get that

$$\begin{aligned} |(f \circ h_1)(x) - (f \circ h_2)(x)| &\leq |(f \circ h_1)(x) - (f \circ h_1)(d)| + |(f \circ h_1)(d) - (f \circ h_2)(x)| \\ &= |(f \circ h_1)(x) - (f \circ h_1)(d)| + |(f \circ h_2)(d) - (f \circ h_2)(x)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ and x are arbitrary, we conclude that $\forall_{x \in [a,b]}((f \circ h_1)(x) = (f \circ h_2)(x))$ i.e., $f \circ h_1 = f \circ h_2$. Since in the above argument f is arbitrarily chosen, we conclude that $\forall_{f \in F}(f(h_1(x)) = f(h_2(x)))$, and by the complete regularity of F we get that $h_1(x) = h_2(x)$, for every $x \in [a, b]$. $\square$

The previous proof works also in the following more general setting. In classical topology it suffices to suppose that Y is a Hausdorff topological space, where the proof of the equality $h_1 = h_2$ is based on the fact that the zero set of a continuous function is a closed subset of its domain (Proposition 5.3.2).

Proposition 5.7.7. *Suppose that X, Y are metric spaces, $D \subseteq Y$ is dense, and F, G are topologies on X, Y, respectively, such that $h \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ implies that h is uniformly continuous. If $h_1, h_2 : Y \to X \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, then, $h_{1|D} = h_{2|D} \to h_1 = h_2$.*

Proposition 5.7.8. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces.*
(i) If $\mathcal{G}$ is isomorphic to the completely regular $\mathcal{F}$, then $\mathcal{G}$ is completely regular.
(ii) If $A \subseteq X$ is inhabited and $\mathcal{F}$ is a completely regular, then $\mathcal{F}_{|A}$ is completely regular.
(iii) $\mathcal{F}$ and $\mathcal{G}$ are completely regular if and only if $\mathcal{F} \times \mathcal{G}$ is completely regular.
(iv) $\mathcal{F} \to \mathcal{G}$ is completely regular if and only if $\mathcal{G}$ is completely regular.
(v) The dual space $\mathcal{F}^*$ of $\mathcal{F}$ is completely regular.

Proof. (i) Suppose that e is an isomorphism between $\mathcal{F}$ and $\mathcal{G}$, and $y_1, y_2 \in Y$. Since e is onto Y and e^* is onto G, we get that $\forall_{g \in G}(g(y_1) = g(y_2)) \leftrightarrow \forall_{g \in G}(g(e(x_1)) = g(e(x_2))) \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2$. Hence, $y_1 = y_2$.
(ii) If $a_1, a_2 \in A$, it suffices to show that $\forall_{f \in F}(f_{|A}(a_1) = f_{|A}(a_2)) \to a_1 = a_2$. The premiss is rewritten as $\forall_{f \in F}(f(a_1) = f(a_2))$, and since F is completely regular, we get that $a_1 = a_2$.
(iii) The hypotheses $\forall_{f \in F}((f \circ \pi_1)(x_1, y_1) = (f \circ \pi_1)(x_2, y_2))$ and $\forall_{g \in G}((g \circ \pi_2)(x_1, y_1) = (g \circ \pi_2)(x_2, y_2))$ imply $\forall_{f \in F}(f(x_1) = f(x_2))$ and $\forall_{g \in G}(g(y_1) = g(y_2))$. For the converse we topologically embed in the obvious way $\mathcal{F}, \mathcal{G}$ into $\mathcal{F} \times \mathcal{G}$ and we use (i) and (ii).
(iv) If $\mathcal{G}$ is completely regular and $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(e_{x,g}(h_1) = e_{x,g}(h_2))$, where by definition this means that $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(g(h_1(x)) = g(h_2(x)))$, then the tightness of $\bowtie_{\mathcal{G}}$ implies that $\forall_{x \in X}(h_1(x) = h_2(x))$ i.e., $h_1 = h_2$. For the converse we suppose that $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(e_{x,g}(h_1) = e_{x,g}(h_2)) \to h_1 = h_2$. If $y_1, y_2 \in G$ and $\forall_{g \in G}(g(y_1) = g(y_2))$, then since $\overline{y_1}, \overline{y_2} \in \mathrm{Const}(X, Y) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, we have that $\forall_{x \in X} \forall_{g \in G}(e_{x,g}(\overline{y_1}) = g(\overline{y_1}(x)) = g(y_1) = g(y_2) = g(\overline{y_2}(x)) = e_{x,g}(\overline{y_2}))$. Hence, $\overline{y_1} = \overline{y_2}$ i.e., $y_1 = y_2$.
(v) Since $\mathcal{F}^* = \mathcal{F} \to \mathcal{R}$ and $\mathcal{R}$ is completely regular, we use (iv). □

The proof of Proposition 5.7.8(iii) works for an arbitrary product of Bishop spaces.

Corollary 5.7.9. *There are epimorphisms in* **crBis** *which are not set-epimorphisms.*

Proof. By Proposition 5.7.8(ii) and Corollary 5.4.4 we get that $\mathcal{I}_{ab}$ and $\mathcal{I}_{ab|D}$, where D is a countable dense subset of $[a, b]$, are completely regular Bishop spaces. If we consider the identity function $\mathrm{id}_D : D \to [a, b]$, it is clear by the $\mathcal{F}$-lifting of morphisms and the fact that $\mathrm{id}_{[a,b]}$ generates the topology on $[a, b]$ that $\mathrm{id}_D \in \mathrm{Mor}(\mathcal{I}_{ab|D}, \mathcal{I}_{ab})$. If $e, j \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{H})$, for some completely regular space $\mathcal{H}$, such that $e \circ \mathrm{id}_D = j \circ \mathrm{id}_D \leftrightarrow e_{|D} = j_{|D}$, by Proposition 5.7.6 we get that $e = j$. Therefore, id_D is an epimorphism in **crBis** which is not a set-epimorphism. □

Proposition 5.7.10. *Suppose that $\mathcal{G} = (Y, G)$ is a completely regular Bishop space and $\theta : X \to Y$ is an injection. The weak Bishop space $\mathcal{F}(\theta)$ induced by θ is completely regular.*

Proof. Using Corollary 5.1.6 it suffices to show for fixed $x_1, x_2 \in X$ that $\forall_{g \in G}((g \circ \theta)(x_1) = (g \circ \theta)(x_1)) \to x_1 = x_2$. The premiss of the previous implication is rewritten as $\forall_{g \in G}(g(\theta(x_1)) = g(\theta(x_1)))$ and the complete regularity of $\mathcal{G}$ implies that $\theta(x_1) = \theta(x_2)$, while the injectivity of θ implies that $x_1 = x_2$. □

Proposition 5.7.11. *If $\mathcal{F} = (X, F)$ is a completely regular Bishop space and $\phi : X \to Y$ is a surjection and an open morphism from $\mathcal{F}$ to $\mathcal{G}_\phi$, then (Y, G_ϕ) is completely regular.*

Proof. If $y_1, y_2 \in Y$, we have that $\forall_{g \in G_\phi}(g(y_1) = g(y_2)) \leftrightarrow \forall_{g \in G_\phi}(g(\phi(x_1)) = g(\phi(x_2)))$. Since ϕ is open, this is equivalent to $\forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2 \to y_1 = y_2$. $\square$

As in classical topology, one can show that the quotient of a completely regular Bishop space need not be completely regular. If F, G_ϕ are completely regular, then it is not necessary that ϕ is open, since in order to have that $f = g \circ \phi$, for some fixed $f \in F$ it must be the case that $\phi(x_1) = \phi(x_2) \to f(x_1) = f(x_2)$, which is not generally the case. In the following example, which also shows that the quotient of a Hausdorff Bishop space need not be Hausdorff and which is the translation of a folklore classical example, one can use any countable dense subset of $\mathbb{R}$ instead of $\mathbb{Q}$.

Example 5.7.12. *Let $\sim$ be the least equivalence relation on $\mathbb{R}$ generated by $x \sim x + q$, where $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. The quotient topology G_π on $\mathbb{R}/{\sim}$ equals* $\mathrm{Const}(\mathbb{R}/{\sim})$*, which is not completely regular.*

Proof. First we show that $\mathbb{R}/{\sim}$ contains at least two points, hence, if $G_\pi = \mathrm{Const}(\mathbb{R}/{\sim})$, G_π cannot be completely regular. It suffices to show that $\sqrt{2} \nsim 0$; one can show trivially inductively (following the inductive definition of $\sim$) that $\forall_{x,y \in \mathbb{R}}(x \sim y \to \exists_{q \in \mathbb{Q}}(y = x + q))$. To show $G_\pi = \mathrm{Const}(\mathbb{R}/{\sim})$ it suffices to show that $g \circ \pi \in \mathrm{Bic}(\mathbb{R}) \to g$ is constant, for every $g : \mathbb{R}/{\sim} \to \mathbb{R}$. For that we fix $x, y \in \mathbb{R}$ and we consider a bounded set B such that $x \in B$. By the uniform continuity of $g \circ \pi$ on B we get that

$$\forall_{y' \in B}(|x - y'| \leq \omega_{g \circ \pi, B}(\epsilon) \to |g([x]_\sim) - g([y']_\sim)| \leq \epsilon).$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every z one can find some $z' \sim z$ such that $z' \in (a, b)$, for every $a, b \in \mathbb{R}$ such that $a < b$; there is some $q \in (a - z, b - z)$, hence $z + q \in (a, b)$. Consequently, there is some $y' \sim y$ such that $y' \in (x - \omega_{g\circ\pi,B}(\epsilon), x + \omega_{g\circ\pi,B}(\epsilon))$, hence $|g([x]_\sim) - g([y']_\sim)| = |g([x]_\sim) - g([y]_\sim)| \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $g([x]_\sim) = g([y]_\sim)$. $\square$

Proposition 5.7.13. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space and $\mathcal{G} = (Y, G)$ is a completely regular Bishop space. If $e : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ which is also a surjection, then the set $\mathrm{Gr}(e) = \{(x, e(x)) \mid x \in X\}$ is closed in the induced neighborhood structure of the product topology $F \times G$.*

Proof. $\mathrm{Gr}(e)$ is closed in the induced neighborhood structure of $F \times G$ if and only if

$$\forall_{(x,y) \in X \times Y}(\forall_{h \in F \times G}(h(x, y) > 0 \to \exists_{(x_0, e(x_0)) \in \mathrm{Gr}(e)}(h(x_0, e(x_0)) > 0)) \to (x, y) \in \mathrm{Gr}(e)).$$

We fix a pair $(x, y) \in X \times Y$ and we suppose that

$$\forall_{h \in F \times G}(h(x, y) > 0 \to \exists_{(x_0, e(x_0)) \in \mathrm{Gr}(e)}(h(x_0, e(x_0)) > 0)).$$

Since $\mathcal{G}$ is completely regular, it suffices to show that $\forall_{g \in G}(g(e(x)) = g(y))$. We fix some $g \in G$ and we suppose that $g(e(x)) \bowtie_{\mathbb{R}} g(y)$. Without loss of generality we assume that $g(e(x)) > g(y)$. If we consider the function

$$h := (g \circ e) \circ \pi_1 - g \circ \pi_2 \in F \times G,$$

we have that $h(x, y) = g(e(x)) - g(y) > 0$. By our hypothesis there exists some $x_0 \in X$ such that $g(e(x_0)) - g(e(x_0)) = 0 > 0$, which is a contradiction. By the tightness of $\bowtie_{\mathbb{R}}$ we conclude that $g(e(x)) = g(y)$. $\square$

5.8 The Tychonoff embedding theorem

In this section we present some "deeper" properties of the completely regular Bishop spaces which show their similarity to the completely regular topological spaces. Most of its content is included in [74].

The following version of the Stone-Čech theorem for Bishop spaces expresses the corresponding "sufficiency" of the completely regular Bishop spaces within **Bis**. Its proof is a translation of the classical Stone-Čech theorem for topological spaces, since the quotient Bishop spaces behave like the quotient topological spaces.

Theorem 5.8.1 (Stone-Čech theorem for Bishop spaces)**.** *For every Bishop space $\mathcal{F} = (X, F)$ there exists a completely regular Bishop space $\rho\mathcal{F} = (\rho X, \rho F)$ and a mapping $\tau : X \to \rho X \in \mathrm{Mor}(\mathcal{F}, \rho\mathcal{F})$ such that the induced mapping τ^* is a ring isomorphism between ρF and F i.e., the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[dr, "f"'] & \rho X \arrow[d, "\rho(f)"] \\ & \mathbb{R}. \end{tikzcd}$$

Proof. We use the equivalence relation $x_1 \sim x_2 \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2))$, for every $x_1, x_2 \in X$, and if $\tau = \pi : X \to X/\sim$, where $x \mapsto [x]_\sim$, we consider the quotient Bishop space $\rho\mathcal{F} = \mathcal{F}/\sim = (X/\sim, G_\pi) = (\rho X, \rho F)$. By Proposition 4.6.4 we have that π is a morphism from $\mathcal{F}$ to $\mathcal{F}/\sim$ and $\rho : F \to G_\pi$ is a ring homomorphism onto G_π. We also know that $\pi^* : G_\pi \to F$ is a ring homomorphism. Since $\rho(g \circ \pi)([x]_\sim) = (g \circ \pi)(x) = g([x]_\sim)$, for every $[x]_\sim \in X/\sim$, we get that $\rho \circ \pi^* = \mathrm{id}_{G_\pi}$. Since $\pi^*(\rho(f)) = \rho(f) \circ \pi = f$, for every $f \in F$, we get that $\pi^* \circ \rho = \mathrm{id}_F$. Hence, π^* is a bijection (see [15], p.17). Finally, $\rho\mathcal{F}$ is completely regular; if $\forall_{g \in \rho F}(g([x_1]_\sim) = g([x_2]_\sim))$, then $\forall_{f \in F}(f(x_1) = f(x_2))$, since $\rho(f) \circ \pi = f$ and $\rho(f) \in \rho F$, therefore $x_1 \sim x_2$ i.e., $[x_1]_\sim = [x_2]_\sim$. $\square$

Definition 5.8.2. *If $\mathcal{F} = (X, F)$ and $\mathcal{G}_i = (Y_i, G_i)$ are Bishop spaces, for every $i \in I$, the family $(h_i)_{i \in I}$, where $h_i : X \to Y_i$, for every $i \in I$, separates the points of X, if*

$$\forall_{x,y \in X}(\forall_{i \in I}(h_i(x) = h_i(y)) \to x = y).$$

Definition 5.8.3. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, a function $e : X \to Y$ is a topological embedding of $\mathcal{F}$ into $\mathcal{G}$, if e is an isomorphism between $\mathcal{F}$ and $\mathcal{G}_{|e(X)}$.*

Note that since X is by hypothesis inhabited, $e(X)$ is an inhabited subset of Y.

Theorem 5.8.4 (Embedding lemma for Bishop spaces)**.** *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G}_i = (Y_i, G_i)$ are Bishop spaces and $h_i : X \to Y_i$, for every i in some index set I. If*
(i) the family of functions $(h_i)_{i\in I}$ separates the points of X,
(ii) $h_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_i)$, for every $i \in I$, and
(iii) $\forall_{f\in F}\exists_{i\in I}\exists_{g\in G_i}(f = g \circ h_i)$,
then the evaluation map

$$e : X \to Y = \prod_{i\in I} Y_i,$$

$$x \mapsto (h_i(x))_{i\in I},$$

is a topological embedding of $\mathcal{F}$ into $\mathcal{G} = \prod_{i\in I} \mathcal{G}_i$.

Proof. First we show that e is 1−1; if $x_1, x_2 \in X$, then $e(x_1) = e(x_2) \leftrightarrow (h_i(x_1))_{i\in I} = (h_i(x_2))_{i\in I} \leftrightarrow \forall_{i\in I}(h_i(x_1) = h_i(x_2)) \to x_1 = x_2$. By the $\mathcal{F}$-lifting of morphisms we have that

$$\begin{aligned} e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) &\leftrightarrow \forall_{g\in G}(g \circ e \in F) \\ &\leftrightarrow \forall_{i\in I}\forall_{g\in G_i}((g \circ \pi_i) \circ e = g \circ (\pi_i \circ e) \in F) \\ &\leftrightarrow \forall_{i\in I}\forall_{g\in G_i}(g \circ h_i \in F) \\ &\leftrightarrow \forall_{i\in I}(h_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_i)). \end{aligned}$$

Next we show that e is open i.e., $\forall_{f\in F}\exists_{g\in G}(f = g \circ e)$. If $f \in F$, by hypothesis (iii) there is some $i \in I$ and some $g \in G_i$ such that $f = g \circ h_i$. Since $g \circ \pi_i \in G = \prod_{i\in I} G_i$ we have that $(g \circ \pi_i)(e(x)) = (g \circ \pi_i)((h_i(x))_{i\in I}) = g(h_i(x)) = f(x)$, for every $x \in X$ i.e., $f = (g \circ \pi_i) \circ e$. Hence e is open as a morphism from $\mathcal{F}$ to $\mathcal{G}$, and also an open morphism from $\mathcal{F}$ to $\mathcal{G}_{|e(X)}$. □

According to the classical Tychonoff embedding theorem, the completely regular topological spaces are precisely those which can be embedded in a product of the closed unit interval $\mathcal{I}$. In the following characterization of the tightness of the canonical apartness relation it is $\mathcal{R}$ which has the role of $\mathcal{I}$.

Theorem 5.8.5 (Tychonoff embedding theorem for Bishop spaces)**.** *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space. Then, $\mathcal{F}$ is completely regular if and only if $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^F$.*

Proof. If $\mathcal{F}$ is completely regular, using the embedding lemma we show that the mapping

$$e : X \to \mathbb{R}^F,$$

$$x \mapsto (f(x))_{f\in F}$$

is a topological embedding of $\mathcal{F}$ into $\mathcal{R}^F$. The topology F is a family of functions of type $X \to \mathbb{R}$ that separates the points of X, since the separation condition is exactly the tightness of $\bowtie_F$. That every $f \in F$ is in $\mathrm{Mor}(\mathcal{F}, \mathcal{R})$ is already explained. If we fix some $f \in F$, then $f = \mathrm{id}_{\mathbb{R}} \circ f$, and since $\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, the condition (iii) of the embedding lemma is satisfied. If $\mathcal{F}$ is topologically embedded into $\mathcal{R}^F$, then $\mathcal{F}$ is completely regular, since by Proposition 5.7.8 a Euclidean Bishop space is completely regular and $\mathcal{F}$ is isomorphic to a subspace of a completely regular space. □

Theorem 5.8.6 (General Tychonoff embedding theorem)**.** *Suppose that $\mathcal{F} = (X, \mathcal{F}(F_0))$ is a Bishop space. Then, $\mathcal{F}$ is completely regular if and only if $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^{F_0}$.*

Proof. If $\mathcal{F}$ is completely regular, we show directly that the mapping

$$e : X \to \mathbb{R}^{F_0},$$

$$x \mapsto (f_0(x))_{f_0 \in F_0}$$

is a topological embedding of $\mathcal{F}$ into $\mathcal{R}^{F_0}$. Since the tightness of $\bowtie_{\mathcal{F}(F_0)}$ is equivalent to $\forall_{x_1, x_2 \in X}(\forall_{f_0 \in F_0}(f_0(x_1) = f_0(x_2)) \to x_1 = x_2))$, we get that e is $1-1$. Using our remark on the relative topology given with a subbase, we get that, since $\mathrm{Bic}(\mathbb{R})^{F_0} = \bigvee_{f_0 \in F_0} \pi_{f_0}$, its restriction to $e(X)$ is

$$(\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)} = (\bigvee_{f_0 \in F_0} \pi_{f_0})_{|e(X)} = \bigvee_{f_0 \in F_0} (\pi_{f_0})_{|e(X)}.$$

By the $\mathcal{F}$-lifting of morphisms we have that

$$e \in \mathrm{Mor}(\mathcal{F}, (\mathcal{R}^{F_0})_{|e(X)}) \leftrightarrow \forall_{f_0 \in F_0}((\pi_{f_0})_{|e(X)} \circ e = f_0 \in \mathcal{F}(F_0)),$$

which holds automatically. In order to prove that e is open it suffices by the $\mathcal{F}$-lifting of openness on the set-epimorphism $e : X \to e(X)$ to show that

$$\forall_{f_0 \in F_0} \exists_{g \in (\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)}} (f_0 = g \circ e).$$

Since $f_0 = (\pi_{f_0})_{|e(X)} \circ e$ and $(\pi_{f_0})_{|e(X)} \in (\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)}$, for every $f_0 \in F_0$, we are done. The converse is proved as in the proof of Theorem 5.8.5. □

If we consider $F_0 = F$, then $\mathcal{F}(F_0) = F$ and the general Tychonoff embedding theorem implies Theorem 5.8.5. If $X = \mathbb{R}^n$, then $\mathrm{Bic}(\mathbb{R})^n = \mathcal{F}(\pi_1, \ldots, \pi_n)$ and the above embedding e is $\mathrm{id}_{\mathbb{R}^n}$, since

$$\vec{x} \mapsto (\pi_1(\vec{x}), \ldots, \pi_n(\vec{x})) = \vec{x}.$$

Following [99], pp.6-7, we show that a diagram like the one in Proposition 5.8.1 commutes, where $\mathbb{R}$ is replaced by the carrier set Y of any completely regular Bishop space $\mathcal{G} = (Y, G)$.

Proposition 5.8.7. *If $\mathcal{F} = (X, F)$ is a Bishop space, $\mathcal{G} = (Y, G)$ is a completely regular Bishop space and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then there exists a mapping $\rho(h) : \rho X \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[dr, "h"'] & \rho X \arrow[d, "\rho(h)"] \\ & Y. \end{tikzcd}$$

Proof. If $\mathbb{R}_g = \mathbb{R}$, for every $g \in G$, by Theorem 5.8.5 we have that the function

$$e : Y \to \prod_{g \in G} \mathbb{R}_g, \quad y \mapsto (g(y))_{g \in G},$$

is a topological embedding of $\mathcal{G}$ into

$$\mathcal{G}' = (\prod_{g \in G} \mathbb{R}_g, \prod_{g \in G} \mathrm{Bic}(\mathbb{R}_g) = \bigvee_{g \in G} \pi_g)).$$

Next we apply Theorem 5.8.1 on $g \circ h$ so that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "\tau"] \arrow[dr, "h"'] & & \rho X \arrow[dd, "\rho(g \circ h)"] \\ & Y \arrow[dr, "g"'] & \\ & & \mathbb{R}. \end{tikzcd}$$

We define the mapping $\Theta : \rho X \to \prod_{g \in G} \mathbb{R}_g$, by $z \mapsto [\rho(g \circ f)(z)]_{g \in G}$. By the $\mathcal{F}$-lifting of morphisms $\Theta \in \mathrm{Mor}(\rho\mathcal{F}, \mathcal{G}') \leftrightarrow \forall_{g \in G}(\pi_g \circ \Theta \in \rho F)$, which holds, since $\pi_g \circ \Theta = \rho(g \circ h) \in \rho F$, for every $g \in G$. Next we show that $\Theta(\rho X) \subseteq e(Y)$ i.e., $\forall_{z \in \rho X}(\Theta(z) \in e(Y))$. If we fix some $g \in G$, we have that there is some $x \in X$ such that $\rho(g \circ h)(z) = \rho(g \circ h)(\pi(x)) = (g \circ h)(x) = g(h(x)) = g(y)$ i.e., the g-component of $h(z)$ is the g-component of $e(y)$, where $y = h(x)$. Because of this inclusion the mapping $\rho(h)$ defined by $\rho(h) := e^{-1} \circ \Theta$

$$\begin{tikzcd} \Theta(\rho X) \arrow[r, "e^{-1}"] & Y \\ \rho X \arrow[u, "\Theta"] \arrow[ur, "\rho(h)"'] & \end{tikzcd}$$

is in $\mathrm{Mor}(\rho\mathcal{F}, \mathcal{G})$ as a composition of morphisms, and the initial diagram commutes, since

$$\begin{aligned} \rho(h)(\tau(x)) &= \rho(h)(z) \\ &= (e^{-1} \circ \Theta)(z) \\ &= e^{-1}[(\rho(g \circ h)(z))_{g \in G}] \\ &= e^{-1}[(g(h(x)))_{g \in G}] \\ &= h(x). \end{aligned}$$

□

Corollary 5.8.8. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then there exists a mapping $P(h) : \rho X \to \rho Y \in \mathrm{Mor}(\rho\mathcal{F}, \rho\mathcal{G})$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\tau"'] & Y \arrow[d, "\tau'"] \\ \rho X \arrow[r, "P(h)"'] & \rho Y, \end{tikzcd}$$

where τ, τ' are the morphisms determined by Theorem 5.8.1 for $\mathcal{F}$ and $\mathcal{G}$, respectively

Proof. By Proposition 5.8.7 the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[dr, "h"'] & \rho X \arrow[dd, "\rho(\tau' \circ h) = P(h)"] \\ & Y \arrow[d, "\tau'"'] \\ & \rho Y, \end{tikzcd}$$

therefore the initial diagram commutes, since $\tau' \circ h = \rho(\tau' \circ h) \circ \tau$. □

Proposition 5.8.9. *Suppose that $\mathcal{F} = (X, F)$ is a completely regular Bishop space, $\mathcal{G} = (Y, G)$ is Bishop space, and $\tau : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Then, τ^* is onto F if and only if τ is a topological embedding of $\mathcal{F}$ into $\mathcal{G}$ such that $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$.*

Proof. We suppose that τ^* is onto F and first we show that τ is $1-1$; suppose that $\tau(x_1) = \tau(x_2)$, for some $x_1, x_2 \in X$. We have that $\forall_{f \in F}(f(x_1) = f(x_2))$, since by the onto hypothesis of τ^*, if $f \in F$, there is some $g \in G$ such that $f(x_1) = (g \circ \tau)(x_1) = g(\tau(x_1)) = g(\tau(x_2)) = (g \circ \tau)(x_2) = f(x_2)$. By the complete regularity of $\mathcal{F}$ we conclude that $x_1 = x_2$. By the $\mathcal{F}$-lifting of morphisms we get directly that if $\tau \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $\tau \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_{|\tau(X)})$. The onto hypothesis of τ^* i.e., $\forall_f \exists_{g \in G}(f = g \circ \tau)$, implies that $\forall_f \exists_{g' \in G_{|\tau(X)}}(f = g' \circ \tau)$, where $g' = g_{|\tau(X)}$. Hence, $\tau : X \to \tau(X)$ is an isomorphism between $\mathcal{F}$ and $\mathcal{G}_{|\tau(X)}$. Next we show that $\{g_{|\tau(X)} \mid g \in G\}$ is a topology on $\tau(X)$, therefore by the definition of the relative topology we get that $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$. Clearly, $\overline{a}_{|\tau(X)} = \overline{a}$, $(g_1 + g_2)_{|\tau(X)} = g_{1|\tau(X)} + g_{2|\tau(X)}$ and $(\phi \circ g)_{|\tau(X)} = \phi \circ g_{|\tau(X)}$, where $\phi \in \mathrm{Bic}(\mathbb{R})$. Suppose that $h : \tau(X) \to \mathbb{R}$, $\epsilon > 0$ and $g \in G$ such that $\forall_{y \in \tau(X)}(|g(y) - h(y)| \leq \epsilon) \leftrightarrow \forall_{x \in X}(|g(\tau(x)) - h(\tau(x))| \leq \epsilon)$. Since $g \circ \tau \in F$ and ϵ is arbitrary, we conclude by the condition BS_4 that $h \circ \tau \in F$, hence, by our onto hypothesis of τ^*, there is some $g \in G$ such that $g \circ \tau = h \circ \tau$ i.e., $g_{|\tau(X)} = h$. For the converse we fix some $f \in F$ and we find $g \in G$ such that $f = \tau^*(g)$. Since $\tau : X \to \tau(X)$ is open, there exists some $g' \in G_{|\tau(X)}$ such that $f = g' \circ \tau$, and since $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$, there is some $g \in G$ such that $g' = g_{|\tau(X)}$, hence $f = g_{|\tau(X)} \circ \tau = g \circ \tau = \tau^*(g)$. □

As in [44], p.155, for $C(X)$, Proposition 5.8.9 implies the Tychonoff embedding theorem; if F is completely regular and $e : X \to \mathbb{R}^F$ is defined by $x \mapsto (f(x))_{f \in F}$, then $e^* : \bigvee_{f \in F} \pi_f \to F$ is onto F, since $e^*(\pi_f) = \pi_f \circ e = f$, therefore e is an embedding of $\mathcal{F}$ into $\mathcal{R}^F$ such that

$$(\bigvee_{f \in F} \pi_f)_{|e(X)} = \{g_{|e(X)} \mid g \in \bigvee_{f \in F} \pi_f\}.$$

Proposition 5.8.10. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, then:*
(i) If $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $\hat{x} \circ e^* = \widehat{e(x)}$, for every $x \in X$, and $e^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*)$.
(ii) The mapping $\hat{}: X \to F^*$, defined by $x \mapsto \hat{x}$, is in $\mathrm{Mor}(\mathcal{F}, \mathcal{F}^{**})$ and it is 1–1 if and only if $\mathcal{F}$ is completely regular.
(iii) If $\mathcal{G}$ is completely regular, then $\forall_{e_1, e_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})}(e_1^* = e_2^* \to e_1 = e_2)$.

Proof. (i) By the $\mathcal{F}$-lifting of morphisms we have that $e^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*) \leftrightarrow \forall_{x \in X}(\hat{x} \circ e^* \in G^*)$. But $\hat{x} \circ e^* = \widehat{e(x)} \in G^*$, since $(\hat{x} \circ e^*)(g) = \hat{x}(e^*(g)) = \hat{x}(g \circ e) = (g \circ e)(x) = g(e(x)) = \widehat{e(x)}(g)$, for every $g \in G$.
(ii) Since $\mathcal{F}^{**} = (F^*, F^{**})$, where $F^{**} = \mathcal{F}(\{\hat{f} \mid f \in F\})$, $\hat{f} : F^* \to \mathbb{R}$, and $\hat{f}(\theta) = \theta(f)$, for every $\theta \in F^*$, by the $\mathcal{F}$-lifting of morphisms we have that $\hat{} \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}^{**}) \leftrightarrow \forall_{f \in F}(\hat{f} \circ \hat{} \in F)$. But $\hat{f} \circ \hat{} = f$, since $(\hat{f} \circ \hat{})(x) = \hat{f}(\hat{x}) = \hat{x}(f) = f(x)$, for every $x \in X$. Since $\hat{x} = \hat{y} \leftrightarrow \forall_{f \in F}(\hat{x}(f) = \hat{y}(f)) \leftrightarrow \forall_{f \in F}(f(x) = f(y))$, the injectivity of $\hat{}$ implies the tightness of $\bowtie_F$ and vice versa.
(iii) By (i) we have that $\hat{x} \circ e_1^* = \widehat{e_1(x)}$ and $\hat{x} \circ e_2^* = \widehat{e_2(x)}$, for every $x \in X$. Since $e_1^* = e_2^*$, we get that $\widehat{e_1(x)} = \widehat{e_2(x)}$, for every $x \in X$. By (ii) and the complete regularity of $\mathcal{G}$ we get that $e_1(x) = e_2(x)$, for every $x \in X$. $\square$

Definition 5.8.11. *We call an isomorphism $\mathcal{E} : G \to F$ between the Bishop spaces $\mathcal{G}^*$ and $\mathcal{F}^*$ smooth, if there exists an isomorphism $e : X \to Y$ between $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{E} = e^*$ i.e., for every $g \in G$ we have that $\mathcal{E}(g) = e^*(g) = g \circ e$.*

By Proposition 5.8.10(iii) we get that if $\mathcal{E}$ is smooth and $\mathcal{G}$ is completely regular, there exists a unique e such that $\mathcal{E} = e^*$. Note that an abstract isomorphism $\mathcal{E}$ satisfies $\forall_{\phi \in F^*}(\phi \circ \mathcal{E} \in G^*)$ and $\forall_{\theta \in G^*} \exists_{\phi \in F^*}(\theta = \phi \circ \mathcal{E})$, hence $\hat{x} \circ \mathcal{E} = \theta \in G^*$, for every $x \in X$ and for every $y \in Y$ there is some $\phi \in F^*$ such that $\hat{y} = \phi \circ \mathcal{E}$. The smoothness of $\mathcal{E}$ is equivalent to the simplest form of θ and ϕ.

Proposition 5.8.12. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are completely regular Bishop spaces and $\mathcal{E} : G \to F$ is an isomorphism between $\mathcal{G}^*$ and $\mathcal{F}^*$. Then, $\mathcal{E}$ is smooth if and only if*

$$\forall_{x \in X} \exists_{y \in Y}(\hat{x} \circ \mathcal{E} = \hat{y}) \ \ \textit{and} \ \ \forall_{y \in Y} \exists_{x \in X}(\hat{y} = \hat{x} \circ \mathcal{E}).$$

Proof. The necessity follows by applying Proposition 5.8.10(i) on e and e^{-1}. For the converse we suppose that $\forall_{x \in X} \exists_{y \in Y}(\hat{x} \circ \mathcal{E} = \hat{y})$ and we show that $\forall_{x \in X} \exists !_{y \in Y}(\hat{x} \circ \mathcal{E} = \hat{y})$; if $\hat{x} \circ \mathcal{E} = \hat{y_1} = \hat{y_2}$, then by the complete regularity of $\mathcal{G}$ we get that $y_1 = y_2$. We define $e : X \to Y$ by $x \mapsto y$, where y is the unique element of Y such that $\hat{x} \circ \mathcal{E} = \hat{y}$.

Similarly, we suppose that $\forall_{y \in Y} \exists_{x \in X}(\hat{y} = \hat{x} \circ \mathcal{E})$ and we show that $\forall_{y \in Y} \exists!_{x \in X}(\hat{y} = \hat{x} \circ \mathcal{E})$; if $\hat{y} = \hat{x_1} \circ \mathcal{E} = \hat{x_2} \circ \mathcal{E}$, then $\hat{y} \circ \mathcal{E}^{-1} = \hat{x_1} = \hat{x_2}$ and by the complete regularity of $\mathcal{F}$ we conclude that $x_1 = x_2$. We define $j : Y \to X$ by $y \mapsto x$, where x is the unique element of X such that $\hat{x} \circ \mathcal{E} = \hat{y}$. Next we show that $j = e^{-1}$, or equivalently that $e \circ j = \mathrm{id}_Y$ and $j \circ e = \mathrm{id}_X$; for the first equality we have that $\hat{y} = \widehat{j(y)} \circ \mathcal{E}$ and also that $\widehat{j(y)} \circ \mathcal{E} = \widehat{e(j(y))}$, which implies that $\hat{y} = \widehat{e(j(y))}$. By the complete regularity of $\mathcal{G}$ we get that $y = e(j(y))$. For the second equality we have that $\hat{x} \circ \mathcal{E} = \widehat{e(x)}$ and $\widehat{e(x)} = \widehat{j(e(x))} \circ \mathcal{E}$, which implies that $\hat{x} \circ \mathcal{E} = \widehat{j(e(x))} \circ \mathcal{E}$, and consequently $\hat{x} = \widehat{j(e(x))}$. By the complete regularity of $\mathcal{F}$ we get that $x = j(e(x))$. Hence, e is a bijection. Next we show that $\mathcal{E}(g) = g \circ e$, for every $g \in G$; since the first part of our hypothesis can be written as $\forall_{x \in X}(\hat{x} \circ \mathcal{E} = \widehat{e(x)})$, we get that

$$(\hat{x} \circ \mathcal{E})(g) = \widehat{e(x)}(g) \leftrightarrow \mathcal{E}(g)(x) = g(e(x)) \leftrightarrow \mathcal{E}(g)(x) = (g \circ e)(x),$$

for every $g \in G$ and $x \in X$. Since $g \circ e = \mathcal{E}(g) \in F$, for every $g \in G$, we conclude that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, while if $f \in F$, since $\mathcal{E}$ is onto F, there exists some $g \in G$ such that $\mathcal{E}(g) = g \circ e = f$ i.e., e is open. $\square$

Note that a formalization of the previous proof requires Myhill's axiom of nonchoice. If we consider the not completely regular Bishop space $(X, \mathrm{Const}(X))$, the isomorphism $\mathrm{Id} : \mathrm{Const}(X) \to \mathrm{Const}(X)$ satisfies the condition of the previous proposition, except the uniqueness conditions though, which are based on the complete regularity of the involved spaces, and clearly, $\mathrm{Id} = \mathrm{id}_X^*$.

5.9 F-ideals

In this section we study the ideals of a Bishop topology and we translate the Gelfand-Kolmogoroff theorem for compact Hausdorff spaces into TBS.

Definition 5.9.1. *If $\mathcal{F} = (X, F)$ is a Bishop space, an F-ideal, or simply an ideal of F, is an inhabited subset I of F such that $i + j \in I$, if $i, j \in I$ and $f \cdot i \in I$, if $f \in F$ and $i \in I$. The least F-ideal $\mathcal{I}(I_0)$ including an inhabited set $I_0 \subseteq F$ is defined by the inductive rules*

$$\frac{i_0 \in I_0}{i_0 \in \mathcal{I}(I_0)}\,, \quad \frac{i, j \in \mathcal{I}(I_0)}{i + j \in \mathcal{I}(I_0)}\,, \quad \frac{f \in F, i \in \mathcal{I}(I_0)}{f \cdot i \in \mathcal{I}(I_0)}\,.$$

The above inductive rules induce the following induction principle $\mathrm{Ind}_{\mathcal{I}}$ on $\mathcal{I}(I_0)$

$$\begin{aligned}
&\forall_{i_0 \in I0}(P(i_0)) \to \\
&\forall_{i,j \in \mathcal{I}(I_0)}(P(i) \to P(j) \to P(i + j)) \to \\
&\forall_{i \in \mathcal{I}(I_0)} \forall_{f \in F}(P(i) \to P(f \cdot i)) \to \\
&\forall_{i \in \mathcal{I}(I_0)}(P(i)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. If we define the set

$$\sum_n F \cdot I_0 := \{f_1 i_{01} + \ldots f_n i_{0n} \mid f_i \in F, i_{0i} \in I_0, 1 \leq i \leq n, n \in \mathbb{N}\},$$

it is immediate to show inductively that $\mathcal{I}(I_0) \subseteq \sum_n F \cdot I_0$, and since every element of $\sum_n F \cdot I_0$ is trivially in $\mathcal{I}(I_0)$, we get that $\mathcal{I}(I_0) = \sum_n F \cdot I_0$. We denote by $I(F)$ the collection of all F-ideals. If $I, J \in I(F)$ and $i, i_{01}, \ldots, i_{0n} \in F$ we have that

$$\mathcal{I}(i_{01}, \ldots, i_{0n}) = F i_{01} + \ldots + F i_{0n},$$

$$I \vee J := \mathcal{I}(I \cup J) = \sum_n F \cdot (I \cup J) = I + J,$$

$$(I, I_0) := I \vee \mathcal{I}(I_0),$$

$$(I, i) = I + F \cdot i.$$

Definition 5.9.2. *An F-ideal I is called p-closed, if $f \in I$, whenever $f_n \xrightarrow{p} f$ and $f_n \in I$, for every n. If $I_0 \subseteq F$, its zero sets is the collection $Z(I_0) = \{\zeta(i_0) \mid i_0 \in I_0\}$, and I_0 is called fixed, if $\bigcap Z(I_0)$ is inhabited i.e., if there exists some $x \in X$ such that $i_0(x) = 0$, for every $i_0 \in I_0$. If x inhabits $\bigcap Z(I_0)$, we call x a fixing point, or we say that x fixes I_0. We call I_0 free, if*

$$\bigcap Z(F_0) = \emptyset,$$

while we call I_0 2-fixed, if

$$\forall_{i_0, j_0 \in I_0} \exists_{x \in X} (i_0(x) = j_0(x) = 0),$$

and in this case x is called a fixing point of i_0, j_0.

It is clear that if $x \in \zeta(i_0)$, or $x \in \zeta(i_0) \cap \zeta(j_0)$, then $\mathcal{I}(i_0)$ and $\mathcal{I}(i_0, j_0)$ are also fixed, while if $\{i_{01}, \ldots, i_{0n}\}$, where $n > 2$, is 2-fixed, it is not generally the case that $\{i_{01}, \ldots, i_{0n}\}$, and consequently $\mathcal{I}(i_{01}, \ldots, i_{0n})$, is fixed; e.g., the polynomials

$$p_1(x) = (x-1)(x-2), \ p_2(x) = (x-2)(x-3), \ p_3(x) = (x-1)(x-3) \in \mathrm{Bic}(\mathbb{R})$$

form a 2-fixed set which is not fixed. Clearly, $\mathcal{I}(I_0)$ is fixed if and only if I_0 is fixed.

Definition 5.9.3. *An ideal I is called proper, if $\overline{1} \notin I$.*

Note that the notion of a proper ideal is negatively defined. The next proposition shows the "distance" between the classical existence of some $x \in X$ such that $f(x) = 0$, if $f \in I$ and I is an ideal of $C(X)$, and the constructive existence of such a root for some $i \in I$, and $I \in I(F)$.

Proposition 5.9.4. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space and $I \in I(F)$ is proper.*
(i) $\forall_{c>0} \forall_{i \in I} \neg\neg\exists_{x \in X} (|i(x)| < c)$.
(ii) I is 2-fixed if and only if $\zeta(i)$ is inhabited, for every $i \in I$.

Proof. (i) We fix $c > 0$, $i \in I$, and we suppose that $\neg\exists_{x\in X}(|i(x)| < c)$. This implies that $\forall_x(\neg(|i(x)| < c))$, and from that we get that $\forall_{x\in X}(|i(x)| \geq c)$ i.e., $|i| \geq \overline{c} \rightarrow |i|^2 \geq \overline{c^2} > 0$. Since $|i|^2 = i^2 \in I$, by Theorem 5.4.8 we have that $\frac{1}{i^2} \in F$, therefore $\frac{1}{i^2}i^2 = \overline{1} \in I$.
(ii) If I is 2-fixed and $f \in I$, then there exists $x \in X$ such that $i(x)(= i(x)) = 0$. Conversely, if $i, j \in I$, and x inhabits $\zeta(i^2 + j^2)$, then x inhabits $\zeta(i) \cap \zeta(j)$, since $i^2 + j^2 \in I$. □

If F is a given topology on some X, one can define subsets of F using a comprehension principle restricted to appropriate formulas which depend on F, that we call F-*formulas*. The systematic study of F-formulas and the subsets of F defined through them used in the informal theory TBS can lead to a formal treatment of the set theory of the subsets of X and F defined by F-formulas. This is a "local" approach to the problem of what a set or a type is constructively. Its development is not included in this Thesis, but it is postponed after a serious development of TBS (see also section 8.3). The main point of this approach is the formalization not of an abstract and general notion of set, but of the notion set as it appears in an informal mathematical theory such as TBS. Subsequently, we use the notion of an F-formula informally.

Definition 5.9.5. *(i) Suppose that a subset I of $\mathbb{F}(X)$ is defined through an F-formula ϕ. We call I positively proper, if there is some formula θ which contains no negation such that $\theta(f, \vec{x}) \rightarrow \neg\phi(f, \vec{x})$, for every $f \in F$ and $\theta(\overline{1}, \vec{x})$, where $\vec{x}$ is a list of parameters in X.*
(ii) An F-ideal I is called positively maximal, if for every $f \in F$ such that $\theta(f)$ we have that $\overline{1} \in (I, f)$, or equivalently that $(I, f) = F$. If $x_0 \in X$, the subset M_{x_0} of F is defined through the formula $\phi(f, x_0) := f(x_0) = 0$ i.e.,

$$M_{x_0} := \{f \in F \mid f(x_0) = 0\}.$$

We denote by by $M_+(F)$ the collection of all positively maximal F-ideals and by $M_{0,+}(F)$ the collection of all positively maximal fixed F-ideals.

A basic property of M_{x_0} is that

$$\forall_{f\in F}(f - \overline{f(x_0)} \in M_{x_0}).$$

Proposition 5.9.6. *Suppose that $\mathcal{F} = (X, F)$ is a completely regular Bishop space and $x_0 \in X$.*
(i) M_{x_0} is a positively proper, p-closed, fixed ideal of F, and x_0 is its unique fixing point.
(ii) If $I \in I(F)$ such that $M_{x_0} \subseteq I$ and I is fixed, then $M_{x_0} = I$.
(iii) M_{x_0} is a positively maximal ideal of F.

Proof. (i) Clearly, $\overline{0}$ inhabits M_{x_0} which is trivially shown to be an ideal. If

$$\phi(f, x_0, 0) := f(x_0) = 0, \text{ then}$$

$$\theta(f, x_0, 0) := f(x_0) \bowtie_{\mathbb{R}} 0,$$

and since $\overline{1}(x_0) = 1 > 0$, we get that $\theta(\overline{1}, x_0, 0)$. The fact that M_{x_0} is p-closed is easy to show. Clearly, x_0 is a fixing element of M_{x_0}. If y is a fixing point of M_{x_0}, we show that

$y = x_0$. By the complete regularity of F it suffices to show that $\neg(x_0 \bowtie_F y)$. If $x_0 \bowtie_F y$, there exists some $h \in F$ such that $h(x_0) = 0$ and $h(y) = 1$, hence $h \in M_{x_0}$. Since y fixes M_{x_0}, we get that $h(y) = 0$, which is impossible.
(ii) If y is a fixing point of I, then y is a fixing point of M_{x_0}, therefore by (i) we get that $y = x_0$. Since x_0 is the fixing point of I, we get that $I \subseteq M_{x_0}$.
(iii) Suppose that $h \in F$ such that $h(x_0) > 0$. Then $\overline{h(x_0)} - h \in M_{x_0}$, hence

$$(\overline{h(x_0)} - h) + h = \overline{h(x_0)} \in (I, h).$$

Since $\frac{1}{h(x_0)} \in F$, we conclude that $0 + \frac{1}{h(x_0)}\overline{h(x_0)} = \overline{1} \in (I, h)$. If $h \in F$ such that $h(x_0) < 0$, we work similarly. ☐

Note that we cannot show with the previous argument that M_{x_0} is also a maximal ideal, since that would require to know the inverse of $h(x_0) \neq 0$, while constructively the inverse of some real x presupposes the stronger condition $x \bowtie_{\mathbb{R}} 0$ (see [15], p.24).
Next we translate into TBS the classical fact that every topological space is homeomorphic to the space of the fixed maximal ideals of $C(X)$ equipped with the relative hull-kernel topology on the set of all maximal ideals of $C(X)$. Here we do not need to define a topology on the maximal F-ideals first, since we can use Proposition 3.6.11.

Proposition 5.9.7. *If $\mathcal{F} = (X, F)$ is a completely regular Bishop space, then the mapping*

$$e_F : X \to M_0(F),$$

$$x \mapsto M_x$$

is a bijection. The topology $G_F = \{e_f \mid f \in F\}$ on $M_0(F)$, where $e_f \circ e_F = f$, is the unique topology with respect to which e_F is an isomorphism between $\mathcal{F}$ and $(M_0(F), G_F)$.

Proof. Suppose that $x, y \in X$ and that $M_x = M_y$. Then y is a fixing point of M_x, which by Proposition 5.9.6(i) it is equal to the fixing point x of M_x. The rest is an immediate corollary of Proposition 3.6.11. ☐

Proposition 5.9.8. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are completely regular Bishop spaces and $\mathcal{E} = e^* : G \to F$ is a smooth isomorphism between $\mathcal{G}^*$ and $\mathcal{F}^*$.*
(i) $e^*(J)$ is an ideal of F, if J is an ideal of G.
(ii) $e^*(J)$ is proper (2-fixed, fixed, free), if J is proper (2-fixed, fixed, free).
(iii) $e^*(M_y) = M_{e^{-1}(y)}$, for every $y \in Y$.

Proof. (i) It follows directly from Proposition 3.6.13(ii) and (viii). Clearly, if j inhabits J, $e^*(j)$ inhabits $e^*(J)$.
(ii) If $\overline{1} \in e^*(J)$, there exists some $j \in J$ such that $e^*(j) = e^*(\overline{1}) = \overline{1}$, which implies that $j = \overline{1} \in G$, which is a contradiction. It is also easy to see that if y fixes J, then $e^{-1}(y)$ fixes $e^*(J)$. The rest follows easily.
(iii) If $y \in Y$, then by definition we have that $e^*(M_y) = \{e^*(g) = g \circ e \mid g \in M_y\}$. First we get that $e^*(M_y) \subseteq M_{e^{-1}(y)}$, since $(g \circ e)(e^{-1}(y)) = g(y) = 0$, for every $g \in M_y$. For the converse inclusion we have that if $f \in F$ such that $f(e^{-1}(y)) = 0$, then $(g \circ e)(e^{-1}(y)) = g(y) = 0$, where g is determined by the openness of e. Thus, $g \in M_y$ and $e^*(g) = f$. ☐

Proposition 5.9.9. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are completely regular Bishop spaces and $\mathcal{E} = e^* : G \to F$ is a smooth isomorphism between $\mathcal{G}^*$ and $\mathcal{F}^*$. Then the mapping*

$$i : M_0(G) \to M_0(F),$$

$$i(M_y) = e^*[M_y] = M_{e^{-1}(y)},$$

for every $y \in Y$, is an isomorphism between the Bishop spaces $\mathcal{M}_0(\mathcal{G}) = (M_0(G), H_G)$ and $\mathcal{M}_0(\mathcal{F}) = (M_0(F), H_F)$, and the following diagram commutes

$$\begin{tikzcd}
M_0(G) & M_0(F) \arrow[l, "i^{-1}"'] \\
Y \arrow[u, "e_G"] & X. \arrow[l, "e"] \arrow[u, "e_F"']
\end{tikzcd}$$

Proof. First we show that i is a surjection; if $M_x \in M_0(F)$, then $i(M_{e(x)}) = M_{e^{-1}(e(x))} = M_x$. It is also an injection, since $M_{e^{-1}(y_1)} = M_{e^{-1}(y_2)}$ implies, by the complete regularity of $\mathcal{F}$, that $e^{-1}(y_1) = e^{-1}(y_2)$, and since e^{-1} is an injection we get that $y_1 = y_2$. Recall that

$$H_G = \{e_g \mid g \in G\}, \quad e_g \circ e_G = g, \quad e_G(y) = M_y,$$

$$H_F = \{e_f \mid f \in F\}, \quad e_f \circ e_F = f, \quad e_F(x) = M_x.$$

By definition $i \in \mathrm{Mor}(\mathcal{M}_0(\mathcal{G}), \mathcal{M}_0(\mathcal{F})) \leftrightarrow \forall_{f \in F}(e_f \circ i \in H_G)$. If we fix some $f \in F$, we have that $e_f \circ i = e_{f \circ e^{-1}} \in H_G$, since for every $y \in Y$

$$\begin{aligned}
(e_f \circ i)(M_y) &= e_f(M_{e^{-1}(y)}) \\
&= (e_f \circ e_F)(e^{-1}(y)) \\
&= f(e^{-1}(y)) \\
&= (e_{f \circ e^{-1}} \circ e_G)(y) \\
&= e_{f \circ e^{-1}}(e_G(y)) \\
&= e_{f \circ e^{-1}}(M_y).
\end{aligned}$$

To show that i is open we need to show that $\forall_{g \in G} \exists_{f \in F}(e_g = e_f \circ i)$. If $g \in G$, we have that $e_g = e_{g \circ e} \circ i$, since for every $y \in Y$

$$\begin{aligned}
(e_{g \circ e} \circ i)(M_y) &= e_{g \circ e}(M_{e^{-1}(y)}) \\
&= e_{g \circ e}(e_F(e^{-1}(y))) \\
&= (g \circ e)(e^{-1}(y))) \\
&= g(y) \\
&= (e_g \circ e_G)(y)) \\
&= e_g(M_y).
\end{aligned}$$

The commutativity of the above diagram follows now immediately. $\square$

Next we translate within Bishop spaces the theorem of Gelfand and Kolmogoroff, according to which if X, Y are compact Hausdorff spaces and $C(X), C(Y)$ are isomorphic as rings, then X and Y are homeomorphic. The key fact to this result is that every proper ideal of $C(X)$, where X is a compact Hausdorff space, is fixed.

Definition 5.9.10. *We call a Bishop space (X, F), and its topology F, fixed, if every proper ideal of F is fixed.*

Clearly, $\mathrm{Const}(X)$ is fixed, since the only proper ideal of $\mathrm{Const}(X)$ is the zero ideal $\{\overline{0}\}$. A ring homomorphism $T : G \to F$ is also a homomorphism of the algebras G, F, since $\lambda f = \overline{\lambda} f$. Clearly, $T(\overline{a}) = \overline{a}$, for every $a \in \mathbb{R}$. It is also clear that if T is an epimorphism, then $T(J)$ is an ideal of F, if J is an ideal of G; we need the onto hypothesis on T to show that $fT(g) \in T(J)$, for every $f \in F$. If T is also $1-1$, we get that if J is proper, then $T(J)$ is proper. The formalization of the following proof requires again Myhill's axiom of nonchoice.

Theorem 5.9.11 (Gelfand-Kolmogoroff theorem for fixed Bishop spaces)**.** *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are completely regular, fixed Bishop spaces, and $T : G \to F$ is an isomorphism of rings. There exists $\tau : X \to Y$ such that $T = \tau^*$ and τ is an isomorphism between $\mathcal{F}$ and $\mathcal{G}$.*

Proof. (i) First we show that if $y \in Y$, then $T(M_y)$ is a maximal ideal of F. We know already that $T(M_y)$ is proper. Suppose that $x \in X$ such that x fixes $T(M_y)$. Hence, $T(M_y) \subseteq M_x$. But then $T^{-1}(M_x) \supseteq M_y$ is a proper fixed ideal of G. By Proposition 5.9.6(ii) we get that $M_y = T^{-1}(M_x)$ and consequently $T(M_y) = TT^{-1}(M_x) = M_x$, since $T^{-1}(M_x) = \{g \in G \mid T(g) \in M_x\} \to TT^{-1}(M_x) \subseteq M_x$, and if $f \in M_x$, then for the unique $g \in G$ such that $T(g) = h$ we get that $f \in TT^{-1}(M_x)$, since $g \in T^{-1}(M_x)$. Next we show that

$$\forall_{x \in X} \exists !_{y \in Y} (T(M_y) = M_x).$$

If $T(M_y) = T(M_{y'}) = M_x$, then $T^{-1}(M_x) = M_y = M_{y'}$, hence $y = y'$. Next we consider the function

$$\tau : X \to Y,$$

$$\tau(x) = y :\leftrightarrow T(M_y) = M_x,$$

which is $1-1$, since $\tau(x) = \tau(x') = y \leftrightarrow T(M_y) = M_x = M_{x'}$, hence $x = x'$, and onto Y, since if $y \in Y$, then $T(M_y) = M_x$, for some $x \in X$, and consequently $\tau(x) = y$. Next we show that $T = \tau^*$ i.e., $T(g)(x) = g(\tau(x))$, for every $g \in G$ and $x \in X$. We fix $g \in G, x \in X$ and let $y = \tau(x)$. Since $g - \overline{g(y)} \in M_y$ we get that $T(g - \overline{g(y)}) = T(g) - T(\overline{g(y)}) = T(g) - \overline{g(y)} \in M_x$, therefore

$$0 = [T(g) - \overline{g(y)}](x) = T(g)(x) - g(y) = T(g)(x) - g(\tau(x))$$

i.e., $T(g)(x) = g(\tau(x))$. Since x and g were arbitrarily chosen we get the required equality. Similarly we get that $T^{-1}(f) = f \circ \tau^{-1} \in G$, for every $f \in F$. The fact that τ is an isomorphism between $\mathcal{F}$ and $\mathcal{G}$ follows automatically; $\forall_{g \in G}(g \circ \tau = T(g) \in F)$ and $\forall_{f \in F}(f = (f \circ \tau^{-1}) \circ \tau)$. $\square$

Chapter 6

Compactness

In this chapter we introduce the notion of a 2-compact Bishop space as a constructive function-theoretic notion of compactness suitable to TBS. The function-theoretic character of 2-compactness is based on the function-theoretic notions of a Bishop space and of a Bishop morphism. Another notion of compactness, that of pair-compactness, is also introduced. Although 2-compactness is far more superior to pair-compactness, pair-compactness offers an immediate proof of a Stone-Weierstrass theorem for pair-compact Bishop spaces. In between we study some concrete sets, like the Cantor, the Baire space and the Hilbert cube, as Bishop spaces.

6.1 Compactness in constructive mathematics

Classically, the compactness of a topological space X amounts to the Heine-Borel property i.e., the existence of a finite subcover for every open cover of X. Constructively, compactness is a thorny issue, since there are spaces which are classically compact but we cannot show this constructively i.e., within BISH, since they are not compact in an extension of BISH. Kleene's proof of the existence of a primitive recursive infinite $(\forall_n \exists_u (|u| = n \wedge T(u)))$, binary tree T without an infinite path $(\forall_\alpha \exists_n (\overline{\alpha}(n) \notin T))$ expresses the failure of König's lemma within RUSS, and implies the failure of the Heine-Borel property for the classically compact space $2^{\mathbb{N}}$ in RUSS; if $(u_n)_n$ is the sequence of the binary nodes which are not in T but all their predecessors are (see [2], p.68, for a precise formulation of the constructive character of this sequence), then by the hypothesis of the non-existence of an infinite path we get that $(\mathcal{B}_{u_n})_{n \in \mathbb{N}}$ is an open cover of $2^{\mathbb{N}}$, where $\mathcal{B}_u$ denotes the basic clopen set in the standard topology on $2^{\mathbb{N}}$ of all elements of $2^{\mathbb{N}}$ which extend the node u, with no finite subcover, since if $(\mathcal{B}_{u_n})_{n=1}^{N}$ is a finite subcover, there is by the hypothesis of the infinity of T a node $u \in T$ of length larger than the length of all $u_1, \ldots, u_N$, therefore an extension α of u has to be in some $\mathcal{B}_{u_j}$, hence $u_j \prec u$, and the fact that T is a tree implies that $T(u_j)$, which is a contradiction.

According to the fan theorem, every bar of $2^{\mathbb{N}}$ is uniform, where a *bar* of $2^{\mathbb{N}}$ is a set of binary nodes such that $\forall_\alpha \exists_n (\overline{\alpha}(n) \in B)$, and a bar is *uniform*, if there is some $N \in \mathbb{N}$,

the *uniform bound* of B such that $\forall_\alpha \exists_{m \leq N}(\overline{\alpha}(m) \in B)$. The fan theorem is classically equivalent to König's lemma and the fact that the failure of König's lemma in RUSS implies the failure of the compactness of $2^{\mathbb{N}}$ in RUSS is expected from the classical equivalence between the fan theorem and the open-cover compactness of $2^{\mathbb{N}}$: if $B \subseteq 2^*$ is a bar, then $\{\mathcal{B}(u) \mid u \in B\}$ is an open cover of $2^{\mathbb{N}}$, therefore $\{\mathcal{B}(u_{i_k}) \mid u_{i_k} \in B \wedge k = 1, \ldots, M\}$ is a finite subcover, for some $M > 0$. Hence, $N = \max\{|u_{i_1}|, \ldots, |u_{i_M}|\}$ is a uniform bound of B. For the converse, if every bar in $2^{\mathbb{N}}$ is uniform and $\{\mathcal{B}(u_i) \mid i \in I\}$ is an open cover of $2^{\mathbb{N}}$, then $B = \{u_i \mid i \in I\}$ is a bar, and taking the basic open sets over the nodes with length less than the uniform bound of B we get a finite subcover of $2^{\mathbb{N}}$. In [38], pp.41-43, there is proof within BISH of the equivalence between the Heine-Borel property of $[0, 1]$ and the decidable fan theorem, according to which every decidable bar of $2^{\mathbb{N}}$ is uniform.
Sequential compactness is constructively also not useful, since classically sequential compact sets, like 2, are not constructively sequential compact; a *fleeing property* on $\mathbb{N}$ is a formula ϕ such that $\forall_n(\phi(n) \vee \neg\phi(n))$, but we cannot prove neither $\exists_n(\phi(n))$ nor $\forall_n(\neg\phi(n))$ (we can always provide such properties using some unsolved so far mathematical problem regarding the decimal expansion of π). If $\alpha(n) = 1$, when $\phi(m)$, for some $m \leq n$, and $\alpha(n) = 0$, otherwise, then α has no convergent subsequence, since that would imply $\exists_n(\phi(n)) \vee \forall_n(\neg\phi(n))$. A constructive "at most" notion of sequential compactness, and a constructive "almost" notion of sequential compactness are studied in [22] and in [23], respectively.
As we have seen already, for compactness in metric spaces Bishop used Brouwer's notion of a complete and totally bounded space. Actually, in this sense $2^{\mathbb{N}}$ endowed with its standard metric (see also section 6.2) is a compact metric space. Since Bishop didn't develop some form of abstract topology, he didn't address the question of compactness in a more general setting. According to Diener in [38], pp.15-6, an ideal constructive notion of general compactness should exhibit the following properties:

(a) it would be defined for, and in the language of, topological spaces,
(b) it would be classically equivalent to the Heine-Borel property,
(c) within BISH a complete and totally bounded metric/uniform space would satisfy this notion, and
(d) we would be able to prove deep and meaningful theorems assuming that the underlying space satisfies this notion, for example we would be able to prove the existence of the supremum of the image under some sort of continuous function into the reals.

In [38] Diener defined the notion of neat compactness taken with that of neat completeness as a candidate for such notion in a pre-apartness space. His proof in [38], pp.29-31, of the existence in RUSS of a uniform structure on $2^{\mathbb{N}}$ that induces the usual topology but it is not totally bounded, shows that there is no such ideal notion of compactness.
Within TBS we need to define a constructive notion of compactness such that:

(a$'$) it would be defined for, and in the language of, Bishop spaces. The function-theoretic character of TBS forces us to find a function-theoretic characterization of compactness.
(b$'$) Since our objects are Bishop spaces, it is not possible to have an equivalence to the

Heine-Borel property, a property designed for topological spaces. In a way to be explained later though, if a Bishop space is compact within **Bis**, this will mirror a kind of compactness within **Top**.
(c$'$) within BISH a complete and totally bounded metric space would satisfy this notion, in the only way that makes sense within TBS: a metric space X endowed with the Bishop topology of uniformly continuous functions satisfies this notion.
(d$'$) we would be able to prove deep and meaningful theorems assuming that the underlying Bishop space satisfies this notion.

The question of a function-theoretic characterization of compactness is not recent. Mrówka's classical result in [66], according to which a topological space X is compact if and only if for every topological space Y the projection $\pi_Y : X \times Y \to Y$ "parallel" to the compact factor X is a closed map, is almost such a characterization, since the concept of a closed map is set-theoretic. It was this characterization which inspired a categorical treatment of compactness (see e.g., [32], p.410). In the (non-constructive) work of Escardó [41] and (the constructive) work of Taylor [91] compactness of a topological space X is characterized by the continuity (understood in the standard set-theoretic way) of an appropriate functional of type $S^X \to S$, where S is the Sierpinski space. For the, as expected non function-theoretic, treatment of compactness in the theory of apartness spaces and in formal topology see e.g., [27] and [90], respectively.

6.2 The Cantor and the Baire space as Bishop spaces

In this section we show that the natural product Bishop topology on the Cantor and the Baire space captures the expected topological structure on these sets. First we determine the Bishop topology of the basic sets on which we consider the corresponding products.

Proposition 6.2.1. *For the discrete topologies on* $\mathbb{N}$ *and* $2 = \{0, 1\}$ *we have that* $\mathbb{F}(\mathbb{N}) = \mathrm{Bic}(\mathbb{N}) = \mathcal{F}(\mathrm{id}_{\mathbb{N}})$ *and* $\mathbb{F}(2) = \mathrm{Bic}(2) = \mathcal{F}(\mathrm{id}_2)$*, respectively.*

Proof. Working as in the proof of Proposition 5.5.3(iv) we construct a function ϕ^* which is uniformly continuous on a bounded subset B of $\mathbb{Q}_+$ with modulus of continuity $\omega_{\phi^*,B}(\epsilon) = \min\{\omega_i(\epsilon) \mid n \leq i \leq N\}$, for every $\epsilon > 0$. By Lemma 4.7.13 ϕ^* is extended to some $\phi \in \mathrm{Bic}(\mathbb{R})$ which also extends g. If $f : 2 \to \mathbb{R}$, then f is extended to a function $\hat{f} : \mathbb{Q} \to \mathbb{Q}$ such that $\hat{f} \in \mathrm{Bic}(\mathbb{Q})$ by linearly connecting $f(0)$ and $f(1)$, while $\hat{f}$ is constant $f(0)$ on every $q \leq 0$, and constant $f(1)$ on every $q \geq 1$. By Lemma 4.7.11 there is an extension of $\hat{f}$ which is in $\mathrm{Bic}(\mathbb{R})$. $\square$

Definition 6.2.2. *The Cantor space* $\mathcal{C}$ *and the Baire space* $\mathfrak{N}$ *are the following Bishop spaces*

$$\mathcal{C} = (2^{\mathbb{N}}, \mathrm{Bic}(2)^{\mathbb{N}}),$$

$$\mathfrak{N} = (\mathbb{N}^{\mathbb{N}}, \mathrm{Bic}(\mathbb{N})^{\mathbb{N}}),$$

where by the properties of the product of Bishop spaces with a given subbase we get that the Cantor topology and the Baire topology are given by

$$\mathrm{Bic}(2)^{\mathbb{N}} = \mathcal{F}(\{\mathrm{id}_2 \circ \pi_n \mid n \in \mathbb{N}\}) = \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}) = \bigvee_{n\in\mathbb{N}} \pi_n,$$

$$\mathrm{Bic}(\mathbb{N})^{\mathbb{N}} = \mathcal{F}(\{\mathrm{id}_{\mathbb{N}} \circ \varpi_n \mid n \in \mathbb{N}\}) = \mathcal{F}(\{\varpi_n \mid n \in \mathbb{N}\}) = \bigvee_{n\in\mathbb{N}} \varpi_n,$$

where $\pi_n(\alpha) = \alpha(n)$, for every $\alpha \in 2^{\mathbb{N}}$, and $\varpi_n(\beta) = \beta(n)$, for every $\beta \in \mathbb{N}^{\mathbb{N}}$.
The standard metric ρ on the Cantor (Baire) set is given by

$$\rho(\alpha, \beta) := \inf\{2^{-n} \mid \overline{\alpha}(n) = \overline{\beta}(n)\},$$

for every $\alpha, \beta \in 2^{\mathbb{N}}(\mathbb{N}^{\mathbb{N}})$, and it is constructively well-defined [1].

Next we show in an elementary way and independently from Proposition 6.4.5 the equivalence $\alpha \bowtie_{\rho} \beta \leftrightarrow \alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta$, where $\alpha, \beta \in 2^{\mathbb{N}}$ and $\alpha \bowtie_{\rho} \beta$ is the canonical apartness relation induced by ρ. The converse implication is easily provable with the use of Markov's Principle, since constructively $\alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta \to \neg\neg\exists_n(\pi_n(\alpha) \neq \pi_n(\beta))$; if $\neg\exists_n(\pi_n(\alpha) \neq \pi_n(\beta))$, then $\forall_n(\pi_n(\alpha) = \pi_n(\beta))$, therefore $\alpha = \beta$. The pointwise continuity, or the strong continuity of the elements of the Cantor and the Baire topology, are strong enough to assure that this implication is constructively provable.

Proposition 6.2.3. *(i) If $\alpha, \beta \in 2^{\mathbb{N}}$, then $\alpha \bowtie_{\rho} \beta \leftrightarrow \alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta$.*
(ii) If $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, then $\alpha \bowtie_{\rho} \beta \leftrightarrow \alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \varpi_n} \beta$.

Proof. (i) If $\rho(\alpha, \beta) > 0$, let $n \in \mathbb{N}$ such that $2^{-n} < \rho(\alpha, \beta)$, therefore $\alpha(n) \neq \beta(n) \leftrightarrow \pi_n(\alpha) \bowtie_{\mathbb{R}} \pi_n(\beta)$. Suppose next that $\alpha \bowtie_{\phi} \beta$, for some $\phi \in \bigvee_{n\in\mathbb{N}} \pi_n$. By the $\mathcal{F}$-lifting of strong continuity, or the $\mathcal{F}$-lifting of pointwise continuity, and Remark 2.3.12, we get that $\alpha \bowtie_{\rho} \beta$, since π_n is strongly continuous (and pointwise continuous), for every $n \in \mathbb{N}$.
(ii) We work as in case (i). $\square$

Definition 6.2.4. *If F is a topology on X we say that some $x \in X$ is F-isolated, if there exists some $f \in F$ such that $\{x\} = U(f)$. If (X, N) is a neighborhood space a point $x \in X$ is called isolated in N, if $\{x\}$ is open in N.*

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[1] For the sake of completeness we include here the proof of this fact. By the constructive version of the completeness of $\mathbb{R}$, see [15], p.37, an inhabited, bounded below subset A of $\mathbb{R}$ has an infimum if and only if for every $x, y \in \mathbb{R}$ such that $x < y$, either x is a lower bound of A or there exists $a \in A$ such that $a < y$. Moreover, if A is included in some interval I, it suffices to prove the existence of $\inf A$ considering arbitrary $x, y \in I$ such that $x < y$. Since $\overline{\alpha}(0) = \overline{\beta}(0)$, for every $\alpha, \beta \in 2^{\mathbb{N}}(\mathbb{N}^{\mathbb{N}})$, 1 inhabits the set $A = \{2^{-n} \mid \overline{\alpha}(n) = \overline{\beta}(n)\}$, which is bounded below by 0. If $x, y \in (0, 1] \supseteq A$ such that $x < y$, let n_0 be the minimum natural such that $2^{-n_0} < x$. If $\overline{\alpha}(n_0) = \overline{\beta}(n_0)$, then $2^{-n_0} \in A$ and $2^{-n_0} < x < y$. If $\overline{\alpha}(n_0) \neq \overline{\beta}(n_0)$, then $2^{-n_0} \notin A$ and x is a lower bound of A.

If F is a topology on some X we have that

$$\begin{aligned} x \text{ is isolated in } N(F) &\leftrightarrow \exists_{f \in F}(x \in U(f) \wedge U(f) \subseteq \{x\}) \\ &\leftrightarrow \exists_{f \in F}(U(f) = \{x\}) \\ &\leftrightarrow x \text{ is } F\text{-isolated.} \end{aligned}$$

Next we present some well-known facts on the Cantor and the Baire space with their standard topologies that hold for the Cantor and the Baire space with the corresponding Bishop topologies.

Proposition 6.2.5. *(i) There are no $\bigvee_{n \in \mathbb{N}} \pi_n$-isolated points in $2^{\mathbb{N}}$.*
(ii) $(\bigvee_{n \in \mathbb{N}} \varpi_n)_{|2^{\mathbb{N}}} = \bigvee_{n \in \mathbb{N}} \pi_n$.
(iii) There is a retraction r of $\mathbb{N}^{\mathbb{N}}$ onto $2^{\mathbb{N}}$.
(iv) $\mathcal{C}$ is embedded in $\mathfrak{N}$ i.e., $\forall_{\phi \in \bigvee_{n \in \mathbb{N}} \pi_n} \exists_{\phi^* \in \bigvee_{n \in \mathbb{N}} \varpi_n}(\phi^*_{|2^{\mathbb{N}}} = \phi)$.

Proof. (i) Suppose that there are $\alpha \in 2^{\mathbb{N}}$ and $\phi \in \bigvee_{n \in \mathbb{N}} \pi_n$ such that $\phi(\alpha) = \epsilon > 0$. By Proposition 6.4.7 and the subsequent pointwise continuity of ϕ we have that

$$\forall_{\beta \in 2^{\mathbb{N}}}(\rho(\alpha, \beta) < \delta_\phi(\frac{\epsilon}{4}) \rightarrow |\phi(\alpha) - \phi(\beta)| < \frac{\epsilon}{4}).$$

Hence, for any β such that $\rho(\alpha, \beta) < \delta_\phi(\frac{\epsilon}{4})$ we have that $0 < \frac{3\epsilon}{4} = \phi(\alpha) - \frac{\epsilon}{4} < \phi(\beta) < \phi(\alpha) + \frac{\epsilon}{4}$ i.e., $\beta \in U(\phi)$.
(ii) By Proposition 4.7.2 we have that

$$\begin{aligned} (\bigvee_{n \in \mathbb{N}} \varpi_n)_{|2^{\mathbb{N}}} &= \mathcal{F}(\{\varpi_{n|2^{\mathbb{N}}} \mid n \in \mathbb{N}\}) \\ &= \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}) \\ &= \bigvee_{n \in \mathbb{N}} \pi_n. \end{aligned}$$

(iii) We define the function $r : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$, defined by $\alpha \mapsto r(\alpha)$, where

$$r(\alpha)(n) = \begin{cases} 1 & \text{, if } \alpha(n) \neq 0 \\ 0 & \text{, ow.} \end{cases}$$

Clearly, $r(\alpha) = \alpha$, for every $\alpha \in 2^{\mathbb{N}}$. In order to conclude that r is a retraction of $\mathbb{N}^{\mathbb{N}}$ onto $2^{\mathbb{N}}$ it suffices by (ii) to show that $r \in \mathrm{Mor}(\mathfrak{N}, \mathcal{C}) \leftrightarrow \forall_{n \in \mathbb{N}}(\pi_n \circ r \in \bigvee_{n \in \mathbb{N}} \varpi_n)$. If we fix some $n \in \mathbb{N}$, we have that

$$(\pi_n \circ r)(\alpha) = r(\alpha)(n) = \begin{cases} 1 & \text{, if } \alpha(n) \neq 0 \\ 0 & \text{, ow.} \end{cases}$$

for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. We consider the function $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$ as defined in Lemma 6.4.2, and the shifting function $s_n : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$, where $s_n(\alpha)(i) = \alpha(n + i)$, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$. As in the proof of Lemma 6.4.3 for the shifting function on $2^{\mathbb{N}}$, we have that $s_n \in \mathrm{Mor}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}})$. Since by Lemma 6.4.2 $\phi \in \mathbb{N}^{\mathbb{N}}$, by the definition of a Bishop morphism we have that $\phi \circ s_n \in \bigvee_{n\in\mathbb{N}} \varpi_n$. Since

$$\phi(s_n(\alpha)) = \begin{cases} 1 & \text{, if } s_n(\alpha)(0) = \alpha(n) \neq 0 \\ 0 & \text{, ow,} \end{cases}$$

we conclude that $\pi_n \circ r = \phi \circ s_n \in \bigvee_{n\in\mathbb{N}} \varpi_n$.
(iv) This is immediate by (ii) and Proposition 5.5.3(vii).

□

To show the dimensionless character of $\mathfrak{N}$ and $\mathcal{C}$ as it is expressed in Proposition 6.2.7 we need to secure that we can have within BISH a countably infinite partition of $\mathbb{N}$ such that $\mathbb{N} = \bigcup_{i\in\mathbb{N}} N_i$, where $N_i \cap N_j = \emptyset$ and $N_i = \{k_1^{(i)}, k_2^{(i)}, k_3^{(i)}, \ldots\}$, for every $i \in \mathbb{N}$.

Lemma 6.2.6. *There exists a sequence $(e_m)_{m\in\mathbb{N}}$ of bijections $e_m : \mathbb{N} \to \mathrm{rng}(e_m) \subseteq \mathbb{N}$ such that $\mathbb{N} = \bigcup_{m\in\mathbb{N}} \mathrm{rng}(e_m)$ and $\mathrm{rng}(e_m) \cap \mathrm{rng}(e_l) = \emptyset$, for every $m, l \in \mathbb{N}$ such that $m \neq l$.*

Proof. We define the functions $e_m : \mathbb{N} \to \mathrm{rng}(e_m)$, for every $m > 0$, by

$$e_m(i) := 2^m i + e_m(0),$$
$$e_1(0) := 0,$$
$$e_{m+1}(0) := e_m(0) + 2^{m-1}.$$

If $\sigma > 1$, we have that $e_{m+\sigma}(0) = e_m(0) + 2^{m-1} + 2^m + 2^{m+1} + \ldots + 2^{m+\sigma-2}$, hence, if $\sigma \geq 1$

$$e_{m+\sigma}(0) = e_m(0) + \sum_{l=m-1}^{m+\sigma-2} 2^l.$$

We get $\mathrm{rng}(e_m) \cap \mathrm{rng}(e_l) = \emptyset$, for every $m, l \in \mathbb{N}$ such that $m \neq l$, since

$$\begin{aligned} e_m(i) = e_{m+\sigma}(j) &\leftrightarrow 2^m i + e_m(0) = 2^{m+\sigma} j + e_{m+\sigma}(0) \\ &\leftrightarrow 2^m i + e_m(0) = 2^{m+\sigma} j + e_m(0) + \sum_{l=m-1}^{m+\sigma-2} 2^l \\ &\leftrightarrow 2^m(i - 2^\sigma j) = \sum_{l=m-1}^{m+\sigma-2} 2^l \\ &\leftrightarrow i - 2^\sigma j = 2^{-m} \sum_{l=m-1}^{m+\sigma-2} 2^l \\ &\leftrightarrow i - 2^\sigma j = \sum_{l=m-1}^{m+\sigma-2} 2^{l-m} \\ &\leftrightarrow \mathbb{Z} \ni i - 2^\sigma j = 2^{-1} + \sum_{l=m}^{m+\sigma-2} 2^{l-m} \in \mathbb{Q} \setminus \mathbb{Z}. \end{aligned}$$

If $i \in \mathbb{N}$, then the algorithm to find the unique $k > 0$ such that $i \in \mathrm{rng}(e_k)$ is the following: first we find the index m such that $e_1(0) < e_2(0) < \ldots < e_{m-1}(0) \leq i < e_m(0)$ and second we check if $i \in \mathrm{rng}(e_1)$, or if $i \in \mathrm{rng}(e_2)$, ..., or if $i \in \mathrm{rng}(e_{m-1})$, where each such sub-step is a finite procedure "bounded" by i itself. The algorithm terminates because at every stage m all numbers smaller or equal to $e_m(0)$ are already included in some $\mathrm{rng}(e_k)$, for some $k \in \{0, \ldots, m-1\}$. We show that

$$\forall_{m>0} \forall_{n \leq e_m(0)} \exists_{i \leq m} (n \in \mathrm{rng}(e_i)).$$

If $m = 1$, then $e_1(0) = 0$ and $0 \in \mathrm{rng}(e_1)$. Since the case $e_2(0) = 1$, we suppose next that $\forall_{n \leq e_m(0)} \exists_{i \leq m} (n \in \mathrm{rng}(e_i))$ and we show that

$$\forall_{n \leq e_{m+1}(0)} \exists_{i \leq m+1} (n \in \mathrm{rng}(e_i)) \leftrightarrow \forall_{n \leq e_m(0)+2^{m-1}} \exists_{i \leq m} (n \in \mathrm{rng}(e_i)),$$

and that $m > 2$. If $n \leq e_m(0)$, then we use the inductive hypothesis. Hence, we need only to show that $\forall_{e_m(0)<n<e_m(0)+2^{m-1}} \exists_{i \leq m} (n \in \mathrm{rng}(e_i))$, and since the even numbers between $e_m(0)$ and $e_{m+1}(0)$ are in $\mathrm{rng}(e_1)$ we need only to show that

$$\forall_{j \in \{1,2,\ldots,(2^{m-2}-1)\}} \exists_{i \leq m} (e_m(0) + j2 \in \mathrm{rng}(e_i)).$$

We show that there are $i < m, \sigma \geq 1$ and $\lambda \in \mathbb{N}$ such that $m = i + \sigma$ and

$$\begin{aligned}
e_{i+\sigma}(0) + j2 = \lambda 2^i + e_i(0) &\leftrightarrow e_i(0) + \sum_{l=i-1}^{i+\sigma-2} 2^l + j2 = \lambda 2^i + e_i(0) \\
&\leftrightarrow \lambda = 2^{-i}\Big(\sum_{l=i-1}^{i+\sigma-2} 2^l + j2\Big) \\
&\leftrightarrow \lambda = \sum_{l=i-1}^{i+\sigma-2} 2^{l-i} + j2^{1-i} \\
&\leftrightarrow \lambda = 2^{-1} + \sum_{l=i}^{i+\sigma-2} 2^{l-i} + j2^{1-i} \\
&\leftrightarrow 2^{-1} + j2^{1-i} \in \mathbb{N}.
\end{aligned}$$

If $2 \nmid j$ i.e., if j is odd, then for $i = 2$ we get $2^{-1} + j2^{-1} = (1+j)2^{-1} \in \mathbb{N}$, since $1 + j$ is even. Recall that by our assumption at the beginning of the last step of the inductive proof we have that $m > 2$. If $2 \mid j$, let t be the largest power of 2 such that $j = 2^t j'$, for some odd j'. Then

$$2^{-1} + j2^{1-i} = 2^{-1} + 2^t j' 2^{1-i} = 2^{-1} + j' 2^{t+1-i},$$

hence, if $i = t+2$ we get that $2^{-1} + j'2^{t+1-i} = 2^{-1}(1+j') \in \mathbb{N}$, and since $j \in \{1, 2, \ldots, (2^{m-2} - 1)\}$, we have that $t \leq m-3 < m-2 \rightarrow i = t+2 < m$. $\square$

Although there are many elementary proofs of this fact, here we work completely constructively, using the minimum number theory required and fully formalizing the intuitive idea of taking the "half" of an infinite countable set at each step. The first step is to isolate the even numbers, the second is to isolate from the odd numbers the "even-odd" numbers $1, 5, 9, 13, \ldots,$, the third step is to isolate from the "odd-odd" numbers $3, 7, 11, 15, 19\ldots,$ the "even-odd-odd" numbers $3, 11, 19, \ldots,$, and so on.

Proposition 6.2.7. *The finite or countably infinite products of the Bishop spaces $\mathfrak{N}$ and $\mathcal{C}$ are isomorphic to $\mathfrak{N}$ and $\mathcal{C}$, respectively.*

Proof. We prove the countably infinite case for $\mathfrak{N}$. The other cases are shown similarly. If $(e_m)_{m\in\mathbb{N}}$ is a sequence of bijections as in Lemma 6.2.6, the function

$$\mathcal{E} : \mathbb{N}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}},$$

$$\alpha \mapsto \mathcal{E}(\alpha),$$

$$\mathcal{E}(\alpha) := (\mathcal{E}(\alpha)(1), \mathcal{E}(\alpha)(2), \mathcal{E}(\alpha)(3), \ldots),$$

$$\mathcal{E}(\alpha)(m)(n) := \alpha(e_m(n)),$$

for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $m, n \in \mathbb{N}$, is an isomorphism between $\mathfrak{N}$ and $\mathfrak{N}^{\mathbb{N}}$. The function $\mathcal{E}$ is an injection since, for every $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ we have that

$$\begin{aligned}
\mathcal{E}(\alpha) = \mathcal{E}(\beta) &\leftrightarrow \forall_{m\in\mathbb{N}}(\mathcal{E}(\alpha)(m) = \mathcal{E}(\beta)(m)) \\
&\leftrightarrow \forall_{m\in\mathbb{N}}\forall_{n\in\mathbb{N}}(\mathcal{E}(\alpha)(m)(n) = \mathcal{E}(\beta)(m)(n)) \\
&\leftrightarrow \forall_{m\in\mathbb{N}}\forall_{n\in\mathbb{N}}(\alpha(e_m(n)) = \beta(e_m(n))) \\
&\leftrightarrow \forall_{i\in\mathbb{N}}(\alpha(i) = \beta(i)) \\
&\leftrightarrow \alpha = \beta.
\end{aligned}$$

It is also a surjection since, if $A = (A(1), A(2), \ldots,) \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, we define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by $\alpha(i) = A(m)(n)$ and $i = e_m(n)$; this is well-defined, since $i \in \mathbb{N} = \bigcup_{m\in\mathbb{N}} \mathrm{rng}(e_m)$, there exists $m \in \mathbb{N}$ such that $i \in \mathrm{rng}(e_m) \leftrightarrow i = e_m(n)$, for some unique m, since the elements of the partition of $\mathbb{N}$ are pairwise disjoint, and for some unique n, since e_m is an injection, for every $m \in \mathbb{N}$. If we fix $m \in \mathbb{N}$, then $\mathcal{E}(\alpha)(m)(n) = \alpha(e_m(n)) = \alpha(i) = A(m)(n)$, for every $n \in \mathbb{N}$, hence $\mathcal{E}(\alpha)(m) = A(m)$, and since m is arbitrary, we get that $\mathcal{E}(\alpha) = A$. Next we show that

$$\mathcal{E} \in \mathrm{Mor}(\mathfrak{N}, \mathfrak{N}^{\mathbb{N}}) \leftrightarrow \forall_{m,n\in\mathbb{N}}((\varpi_m \circ \pi_n) \circ \mathcal{E} \in \bigvee_{n\in\mathbb{N}} \varpi_n).$$

Since for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ we have that

$$\begin{aligned}
((\varpi_m \circ \pi_n) \circ \mathcal{E})(\alpha) &= (\varpi_m \circ \pi_n)(\mathcal{E}(\alpha)) \\
&= \varpi_m(\mathcal{E}(\alpha)(n)) \\
&= \mathcal{E}(\alpha)(n)(m) \\
&= \alpha(e_n(m)) \\
&= \varpi_{e_n(m)}(\alpha),
\end{aligned}$$

we get that $(\varpi_m \circ \pi_n) \circ \mathcal{E} = \varpi_{e_n(m)} \in \bigvee_{n\in\mathbb{N}} \varpi_n$. By the proof of the surjectivity of $\mathcal{E}$ we have that

$$\mathcal{E}^{-1} : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \quad A \mapsto \mathcal{E}^{-1}(A),$$

$$\mathcal{E}^{-1}(A)(i) := A(m)(n), \quad i = e_m(n),$$

$$\mathcal{E}^{-1} \in \mathrm{Mor}(\mathfrak{N}^{\mathbb{N}}, \mathfrak{N}) \leftrightarrow \forall_{m\in\mathbb{N}}(\varpi_m \circ \mathcal{E}^{-1} \in \bigvee_{n\in\mathbb{N}} \varpi_n).$$

For every $A \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ we have that

$$\begin{aligned}(\varpi_m \circ \mathcal{E}^{-1})(A) &= \varpi_m(\mathcal{E}^{-1}(A)) \\ &= \mathcal{E}^{-1}(A)(m) \\ &\overset{m=e_i(j)}{=} A(i)(j) \\ &= (\varpi_j \circ \pi_i)(A),\end{aligned}$$

therefore, $\varpi_m \circ \mathcal{E}^{-1} \overset{m=e_i(j)}{=} \varpi_j \circ \pi_i \in \bigvee_{m,n\in\mathbb{N}}(\varpi_m \circ \pi_n)$. $\square$

As we have said in section 4.1, if I is an arbitrary set, a Boolean Bishop space is an I-product $\mathit{2}^I = (2^I, \mathrm{Bic}(2)^I = \bigvee_{i\in I} \varpi_i)$ of the Bishop space $\mathit{2} = (2, \mathcal{F}(\mathrm{id}_2))$. Next we show that the Cantor space is isomorphic to the Cantor set endowed with the relative topology of $[0,1]$. If Ca denotes the Cantor set i.e., the reals in $[0,1]$ that do not require the use of the digit 1 in their triadic expansion, the *Cantor set space* is the Bishop space

$$\mathfrak{C} = (\mathrm{Ca}, \mathrm{Bic}[0,1]_{|\mathrm{Ca}} = \mathcal{F}(\mathrm{id}_{[0,1]_{|\mathrm{Ca}}}) = \mathcal{F}(\mathrm{id}_{\mathrm{Ca}})).$$

Proposition 6.2.8. *The Cantor space $\mathcal{C}$ is isomorphic to the Cantor set space $\mathfrak{C}$.*

Proof. By Proposition 4.1.11 it suffices to show that the Bishop space

$$\mathcal{C}' = (\{0,2\}^{\mathbb{N}}, \prod_{n\in\mathbb{N}} \mathrm{Bic}(\{0,2\}) = \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}))$$

is isomorphic to the Cantor set space, since $\mathcal{C}$ is isomorphic to $\mathcal{C}'$. The function

$$\phi : \{0,2\}^{\mathbb{N}} \to \mathrm{Ca},$$

$$\{0,2\}^{\mathbb{N}} \ni i = (i_1, i_2, \ldots,) \mapsto \sum_{k=1}^{\infty} \frac{i_k}{3^k},$$

is clearly a bijection. Moreover, $\phi \in \mathrm{Mor}(\mathcal{C}', \mathfrak{C}) \leftrightarrow \mathrm{id}_{\mathrm{Ca}} \circ \phi = \phi \in \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\})$. By the comparison test and the condition BS4 we get that

$$\mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}) \ni \phi_n = \sum_{k=1}^{n} \frac{1}{3^k}\pi_k \xrightarrow{u} \phi \in \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}),$$

since for every $n \geq n_0$ such that $\frac{1}{3^{n_0}} < \epsilon$ we have that

$$\begin{aligned}
|\phi_n(i) - \phi(i)| &= |\sum_{k=1}^{\infty} \frac{i_k}{3^k} - \sum_{k=1}^{n} \frac{i_k}{3^k}| = |\sum_{k=n+1}^{\infty} \frac{i_k}{3^k}| \\
&\leq \sum_{k=n+1}^{\infty} |\frac{i_k}{3^k}| = \sum_{k=n+1}^{\infty} \frac{i_k}{3^k} \\
&\leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} = 2\sum_{k=n+1}^{\infty} \frac{1}{3^k} = 2\sum_{k=1}^{\infty} \frac{1}{3^{n+k}} \\
&= 2\sum_{k=1}^{\infty} \frac{1}{3^n 3^k} = 2\frac{1}{3^n}\sum_{k=1}^{\infty} \frac{1}{3^k} = 2\frac{1}{3^n}\frac{1}{2} \\
&= \frac{1}{3^n} < \frac{1}{3^{n_0}} < \epsilon.
\end{aligned}$$

To prove that the set-epimorphism ϕ is open it suffices by the $\mathcal{F}$-lifting of openness to show that $\forall_k \exists_{g \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}})}(\pi_k = g \circ \phi)$. Property Tri gives that $\forall_{x \in \mathrm{Ca}}(x > \frac{1}{3} \vee x < \frac{2}{3})$, which entails that $\forall_{x \in \mathrm{Ca}}(x \leq \frac{1}{3} \vee x \geq \frac{2}{3})$, since $\forall_{x \in \mathrm{Ca}}(x < \frac{2}{3} \rightarrow x \leq \frac{1}{3})$ and $\forall_{x \in \mathrm{Ca}}(x > \frac{1}{3} \rightarrow x \geq \frac{2}{3})$. In order to show that

$$g_1 = [(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{2}}) \vee \overline{0}] \cdot \frac{\overline{2}}{|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{2}}|} \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}}),$$

it suffices by Theorem 5.4.8 to show that $|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{2}}| \geq \overline{c}$, for some $c > 0$. If $x \in \mathrm{Ca}$ such that $0 \leq x \leq \frac{1}{3}$, then $\frac{1}{6} \leq \frac{1}{2} - x = |x - \frac{1}{2}| \leq \frac{1}{2}$, while if $x \in \mathrm{Ca}$ such that $\frac{2}{3} \leq x \leq 1$, then $\frac{1}{6} \leq x - \frac{1}{2} = |x - \frac{1}{2}| \leq \frac{1}{2}$ i.e., for every $x \in \mathrm{Ca}$ we have that $|x - \frac{1}{2}| \geq \frac{1}{6} > 0$. If $x \in \mathrm{Ca}$ such that $0 \leq x \leq \frac{1}{3}$, we get that $g_1(x) = 0$, while if $x \in \mathrm{Ca}$ such that $\frac{2}{3} \leq x \leq 1$, we get that $g_1(x) = 2$ i.e.,

$$\pi_1 = g_1 \circ \phi, \quad \text{where } g_1 \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}}).$$

In a similar way we get that if $x \in \mathrm{Ca}$ such that $0 \leq x \leq \frac{1}{3}$, then $x \leq \frac{1}{9} \vee x \geq \frac{2}{9}$, while if $x \in \mathrm{Ca}$ such that $\frac{2}{3} \leq x \leq 1$, then $x \leq \frac{7}{9} \vee x \geq \frac{8}{9}$. In order to show that

$$\begin{aligned}
g_2 = (g_1 \vee \overline{0})((\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{5}{6}}) \vee \overline{0}) \cdot \frac{1}{|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{5}{6}}|} + \\
+ (\overline{2} - g_1)[(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{6}}) \vee \overline{0}] \cdot \frac{1}{|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{6}}|} \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}})
\end{aligned}$$

it suffices by Theorem 5.4.8 to show that $|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{6}}| \geq \overline{c}$, for some $c > 0$. Working as above we find that $|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{6}}| \geq \overline{\frac{1}{18}}$. If $x \in \mathrm{Ca}$ such that $0 \leq x \leq \frac{1}{9}$, we get that $g_2(x) = 0$, since $g_1(x) = 0$ and $x - \frac{1}{6} \leq \frac{1}{9} - \frac{1}{6} = -\frac{1}{18} < 0$. If $x \in \mathrm{Ca}$ such that $\frac{2}{9} \leq x \leq \frac{1}{3}$, we get that $g_2(x) = 2$, since $g_1(x) = 0$ and $x - \frac{1}{6} \geq \frac{2}{9} - \frac{1}{6} = \frac{1}{18} > 0$. If $x \in \mathrm{Ca}$ such that $\frac{2}{3} \leq x \leq \frac{7}{9}$, we get that $g_1(x) = 2$ and the second term of $g_2(x)$ vanishes and consequently $g_2(x) = 0$, since $x - \frac{5}{6} \leq -\frac{1}{18} < 0$ and the first term of $g_2(x)$ vanishes too. If $x \in \mathrm{Ca}$ such that $\frac{8}{9} \leq x \leq 1$, we get again that $g_1(x) = 2$ and the second term of $g_2(x)$ vanishes and consequently $g_2(x) = 2$, since $|x - \frac{5}{6}| = x - \frac{5}{6} \geq \frac{1}{18} > 0$ i.e.,

$$\pi_2 = g_2 \circ \phi, \quad \text{where } g_2 \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}}).$$

Working similarly we get that $\pi_3 = g_3 \circ \phi$, where

$$\begin{aligned} g_3 = {} & (g_1 \vee \overline{0})\llbracket(g_2 \vee \overline{0})[(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{17}{18}}) \vee \overline{0}]\frac{1}{4|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{17}{18}}|} + \\ & + (\overline{2} - g_2)[(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{13}{18}}) \vee \overline{0}]\frac{1}{4|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{13}{18}}|}\rrbracket + \\ & + (\overline{2} - g_1)\llbracket(g_2 \vee \overline{0})[(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{5}{18}}) \vee \overline{0}]\frac{1}{4|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{5}{18}}|} + \\ & + (\overline{2} - g_2)[(\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{18}}) \vee \overline{0}]\frac{1}{4|\mathrm{id}_{\mathrm{Ca}} - \overline{\frac{1}{18}}|}\rrbracket. \end{aligned}$$

One can extend the above definition of $g_k \in \mathcal{F}(\mathrm{id}_{\mathrm{Ca}})$ for arbitrary k so that $\pi_k = g_k \circ \phi$. $\square$

Although the above constructive proof is more complex than the classical proof that ϕ is a homeomorphism between the corresponding topological spaces, see e.g., [40], p.104, it seems to us more informative, since the openness of ϕ forced us to express the corresponding projections as elements of $\mathcal{F}(\mathrm{id}_{\mathrm{Ca}})$ i.e., to find an explicit way to $\mathcal{F}$-generate them by $\mathrm{id}_{\mathrm{Ca}}$. It is evident by the previous argument why the functions g_k on the whole of $[0,1]$ cannot be in $\mathcal{F}(\mathrm{id}_{[0,1]})$ i.e., a uniformly continuous function; for example, for $k = 1$ since the middle third interval is not missing we can have an $x < \frac{1}{3}$ and some $y > \frac{1}{3}$ such that $|x - y| = y - x$ is arbitrarily small while $g_1(x) = 0$ and $g_1(y) = 1$. We can show similarly that the function

$$\psi : \{0,2\}^{\mathbb{N}} \to [0,1],$$

$$\{0,2\}^{\mathbb{N}} \ni i = (i_1, i_2, \ldots,) \mapsto \sum_{k=1}^{\infty} \frac{i_k}{2^{k+1}} \in \mathrm{Mor}(\mathcal{C}', \mathcal{I}),$$

is not an injection, and we need the dyadic expansion of the elements of $[0,1]$ to get that ψ is a surjection, a fact which is equivalent to LLPO, therefore constructively unacceptable.

6.3 The Hilbert cube as a Bishop space

Definition 6.3.1. *If $l^2(\mathbb{N}) := \{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}$, the Hilbert cube space $\mathcal{I}^{\infty}$ is the Bishop space*

$$\mathcal{I}^{\infty} := (I^{\infty}, (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^{\infty}}),$$

$$I^{\infty} := \{(x_n) \in l^2(\mathbb{N}) \mid \forall_{n \in \mathbb{N}}(|x_n| \leq \frac{1}{n})\}.$$

Proposition 6.3.2. *The Hilbert cube space is isomorphic to $\mathcal{I}^{\mathbb{N}}$.*

Proof. Since $\mathcal{I}$ is isomorphic to $\mathcal{I}_{(-1)1}$, by Proposition 4.1.11 it suffices to show that the Hilbert cube space is isomorphic to $\mathcal{I}^{\mathbb{N}}_{(-1)1}$. We show that the bijection

$$\phi : I^{\infty} \to [-1,1]^{\mathbb{N}}$$

$$(x_1, x_2, x_3, \ldots) \mapsto (x_1, 2x_2, 3x_3, \ldots)$$

is an open morphism. By definition and the $\mathcal{F}$-lifting of morphisms we have that

$$\begin{aligned}(\mathrm{Bic}[-1,1])^{\mathbb{N}} &= \mathcal{F}(\{f \circ \pi_i \mid f \in \mathrm{Bic}[-1,1], i \in \mathbb{N}\}) \\ &= \mathcal{F}(\{\mathrm{id}_{|[-1,1]} \circ \pi_i \mid i \in \mathbb{N}\}) \\ &= \mathcal{F}(\{\pi_i \mid i \in \mathbb{N}\})\end{aligned}$$

$$\begin{aligned}{(\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}}_{|I^{\infty}} &= \mathcal{F}(G_{0,I^{\infty}}) \\ G_{0,I^{\infty}} &= \{\pi_{i|I^{\infty}} \mid i \in \mathbb{N}\}.\end{aligned}$$

$$\phi \in \mathrm{Mor}(\mathcal{I}^{\infty}, \mathcal{I}^{\mathbb{N}}_{(-1)1}) \leftrightarrow \forall_{i\in\mathbb{N}}((\mathrm{id}_{|[-1,1]} \circ \pi_i) \circ \phi \in {(\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}}_{|I^{\infty}}).$$

For every $(x_n)_n \in I^{\infty}$ we have that

$$(\mathrm{id}_{|[-1,1]} \circ \pi_i) \circ \phi = \bar{i} \cdot \pi_{i|I^{\infty}} \in (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}_{|I^{\infty}},$$

since a topology is closed with respect to the product of functions and

$$\begin{aligned}[(\mathrm{id}_{|[-1,1]} \circ \pi_i) \circ \phi]((x_n)_n) &= (\mathrm{id}_{|[-1,1]} \circ \pi_i)(\phi((x_n)_n)) \\ &= \pi_i(x_1, 2x_2, 3x_3, \ldots) \\ &= ix_i \\ &= (\bar{i} \cdot \pi_{i|I^{\infty}})((x_n)_n),\end{aligned}$$

for every $(x_n)_n \in I^{\infty}$. We also have that

$$\pi_{i|I^{\infty}} = \overline{\frac{1}{i}} \cdot [(\mathrm{id}_{|[-1,1]} \circ \pi_i) \circ \phi] = [(\overline{\frac{1}{i}} \cdot \mathrm{id}_{|[-1,1]}) \circ \pi_i] \circ \phi,$$

since

$$(\overline{\frac{1}{i}} \cdot \mathrm{id}_{|[-1,1]})(ix_i) = \overline{\frac{1}{i}}(ix_i) \cdot \mathrm{id}_{|[-1,1]}(ix_i) = \frac{1}{i}(ix_i) = x_i.$$

Also, since $\overline{\frac{1}{i}} \cdot \mathrm{id}_{|[-1,1]} \in \mathrm{Bic}[-1,1]$, we get that

$$(\overline{\frac{1}{i}} \cdot \mathrm{id}_{|[-1,1]}) \circ \pi_i \in (\mathrm{Bic}[-1,1])^{\mathbb{N}}$$

and by the $\mathcal{F}$-lifting of openness we conclude that the set-epimorphism ϕ is open. $\square$

Note that in the previous proof we avoided to metrize the product $\mathcal{I}^{\mathbb{N}}_{(-1)1}$, something which is done in the classical proof (see [40], p.193). I.e., the above isomorphism of Bishop spaces is independent from the corresponding metrics.

6.4 $\mathfrak{2}$-compactness generalizes metric compactness

Definition 6.4.1. *A Bishop space (X, F) is called $\mathfrak{2}$-compact, if there exists some set I and a function $e : 2^I \to X$ such that $e \in \mathrm{setEpi}(\mathfrak{2}^I, \mathcal{F})$.*

Note that classically a metric space X is totally bounded if and only if there exists some uniformly continuous bijection $e : B \to X$, where B is a subset of the Cantor set (see [70], p.153). Since $\mathrm{id}_{2^I} \in \mathrm{Mor}(\mathfrak{2}^I, \mathfrak{2}^I)$, a Boolean Bishop space is $\mathfrak{2}$-compact. In this section we show that every compact metric space endowed with its uniform topology is $\mathfrak{2}$-compact (Theorem 6.4.6), therefore the notion of $\mathfrak{2}$-compactness includes two paradigmatic cases of compactness. Without loss of mathematical essence all compact metric spaces considered here are inhabited and have a positive diameter, so that we can use a corollary of the constructive Stone-Weierstrass theorem requiring these conditions (see the proof of Theorem 6.4.6). Note that if (X, d) is a totally bounded space, then its *diameter* exists[2], where

$$\mathrm{diam}(X) := \sup\{d(x, y) \mid x, y \in X\}.$$

Lemma 6.4.2. *The function $\phi : 2^{\mathbb{N}} \to \mathbb{R}$, defined, for every $\alpha \in 2^{\mathbb{N}}$, by*

$$\phi(\alpha) = \begin{cases} 1 & , \text{ if } \alpha(0) \neq 0 \\ 0 & , \text{ ow} \end{cases}$$

belongs to the Cantor topology $\bigvee_{n \in \mathbb{N}} \pi_n$. The similarly defined function on $\mathbb{N}^{\mathbb{N}}$ belongs to the Baire topology $\bigvee_{n \in \mathbb{N}} \varpi_n$.

Proof. We present the proof for the case of the Cantor topology, but we write it so that it includes the case of the Baire topology too. First we note that ϕ is well-defined, since $\alpha(0) \in 2$. By BS_4 it suffices to show that $U(\bigvee_{n \in \mathbb{N}} \pi_n, \phi)$. For that we show that there is some $g \in \bigvee_{n \in \mathbb{N}} \pi_n$ such that $U(g, \phi, \epsilon)$, for every $0 < \epsilon < 1$; if $\epsilon' > 0$, there exists some $n \in \mathbb{N}$ such that $n > 0$ and $\epsilon' > \frac{1}{n}$ (see [25], p.27). Since $\frac{1}{n} < 1$, if we have that $U(g, \phi, \frac{1}{n})$, we get trivially that $U(g, \phi, \epsilon')$. If we fix some $\epsilon \in (0, 1)$, we consider any real σ such that $0 < \sigma \leq \frac{\epsilon}{1-\epsilon}$. In this case we get that

$$|\frac{1}{1+\sigma} - 1| = 1 - \frac{1}{1+\sigma} = \frac{\sigma}{1+\sigma} \leq \epsilon.$$

We also have that

$$\forall_{n \geq 1}(\frac{1}{1+\sigma} \leq \frac{n}{n+\sigma} < 1).$$

We define the function

$$g := \frac{\pi_0}{\pi_0 + \overline{\sigma}} \in \bigvee_{n \in \mathbb{N}} \pi_n,$$

[2] In [15], p.94, this proof is omitted as trivial. For the sake of completeness we outline here the proof: since d_{x_0} is uniformly continuous, for every $x_0 \in X$, the function $\delta : X \to \mathbb{R}$, defined by $x_0 \mapsto \delta(x_0)$, where $\delta(x_0) = \sup\{d(x_0, x) \mid x \in X\}$, is well-defined by Corollary 4.3 in [15], p.94. It is easy to see that δ is uniformly continuous with modulus of continuity $\omega_\delta(\epsilon) = \epsilon$, and $\mathrm{diam}(X) = \sup\{\delta(x_0) \mid x_0 \in X\}$.

since $\pi_0 + \overline{\sigma} \geq \overline{\sigma} \in \bigvee_{n\in\mathbb{N}} \pi_n$, therefore by Theorem 5.4.8 its inverse $\frac{1}{\pi_0+\overline{\sigma}}$ is in $\bigvee_{n\in\mathbb{N}} \pi_n$ too. Next we show that

$$U(g,\phi,\epsilon) :=\forall_{\alpha\in 2^{\mathbb{N}}}(|g(\alpha)-\phi(\alpha)| = |\frac{\alpha(0)}{\alpha(0)+\sigma} - \phi(\alpha)| \leq \epsilon).$$

If $\alpha(0) = 0$, then $\phi(\alpha) = g(\alpha) = 0$, and we are done. If $\alpha(0) \neq 0 \leftrightarrow \alpha(0) = n \geq 1$, then[3]

$$|\frac{\alpha(0)}{\alpha(0)+\sigma} - \phi(\alpha)| = |\frac{n}{n+\sigma} - 1| = 1 - \frac{n}{n+\sigma} \leq 1 - \frac{1}{1+\sigma} = \frac{\sigma}{1+\sigma} \leq \epsilon.$$

□

The following lemmas are formulated only for the Cantor space but their proofs are automatically applicable to the Baire space. If $\alpha \in 2^{\mathbb{N}}$, or $\alpha \in \mathbb{N}^{\mathbb{N}}$, and $n \in \mathbb{N}$, then $\overline{\alpha}(n) = (\alpha(0),\ldots,\alpha(n-1))$ is the n-th initial segment of α ($\overline{\alpha}(0)$ is the empty sequence).

Lemma 6.4.3. *If $\alpha \in 2^{\mathbb{N}}$ and $i \geq 1$, the function $\theta_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, defined by*

$$\theta_{\alpha,i}(\beta) = \begin{cases} 1 & , \text{ if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 0 & , \text{ ow} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$, belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$.

Proof. We show by induction on i that $\forall_{i\geq 1}(\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n))$. First we show that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,1} \in \bigvee_{n\in\mathbb{N}} \pi_n)$. On the set 2 we define the operation $\dot{-}$ by the rules $0 \dot{-} 1 = 1 \dot{-} 0 = 1$ and $1 \dot{-} 1 = 0 \dot{-} 0 = 0$ i.e., $j \dot{-} k = |j-k|$, for every $j,k \in 2$. We fix some $\alpha \in 2^{\mathbb{N}}$ and we show that if ϕ is the element of $\bigvee_{n\in\mathbb{N}} \pi_n$ from Lemma 6.4.2, and if $S_\alpha : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $S_\alpha(\beta) = \alpha \dot{-} \beta$, where $(\alpha \dot{-} \beta) = \alpha(n) \dot{-} \beta(n)$, for every $n \in \mathbb{N}$, we have that

$$\theta_{\alpha,1} = (\overline{1} - \phi) \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since $\overline{1} - \phi \in \bigvee_{n\in\mathbb{N}} \pi_n$ and $S_\alpha \in \mathrm{Mor}(\mathcal{C},\mathcal{C})$, therefore by the definition of the Bishop morphism we get that $(\overline{1}-\phi) \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$. To show that $S_\alpha \in \mathrm{Mor}(\mathcal{C},\mathcal{C})$ it suffices by the $\mathcal{F}$-lifting of morphisms to show that $\pi_n \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$, which is true, since $\pi_n \circ S_\alpha = |\overline{\pi_n(\alpha)} - \pi_n| \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$. It is straightforward to check that $\theta_{\alpha,1}(\beta) = (\overline{1} - \phi)(S_\alpha(\beta)$; recall only that $\overline{\alpha}(1) = \overline{\beta}(1) \leftrightarrow \alpha(0) = \beta(0)$. Next we suppose that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n)$ and we show that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i+1} \in \bigvee_{n\in\mathbb{N}} \pi_n)$. For that we consider the shifting function $s_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, defined by $s_i(\beta)(n) = \beta(n+i)$, for every $n \in \mathbb{N}$. Again we have that $s_i \in \mathrm{Mor}(\mathcal{C},\mathcal{C})$, since $\pi_n \circ s_i = \pi_{n+i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$. Moreover,

$$\theta_{\alpha,i+1} = \theta_{\alpha,i} \cdot (\theta_{s_i(\alpha),1} \circ s_i) \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since $\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$ by the inductive hypothesis on α, while $\theta_{s_i(\alpha),1} \in \bigvee_{n\in\mathbb{N}} \pi_n$, by the case $i = 1$ on the sequence $s_i(\alpha)$, and $\theta_{s_i(\alpha),1} \circ s_i \in \bigvee_{n\in\mathbb{N}} \pi_n$ by the definition of a Bishop morphism. Since $\theta_{s_i(\alpha),1}(s_i(\beta)) = 1$, if $\alpha(i) = \beta(i)$ and $\theta_{s_i(\alpha),1}(s_i(\beta)) = 0$, otherwise, it is immediate to see that $\theta_{\alpha,i}(\beta)\theta_{s_i(\alpha),1}(s_i(\beta)) = \theta_{\alpha,i+1}(\beta)$. □

[3]Note that the last equivalence works simultaneously for both the Cantor and the Baire space.

Lemma 6.4.4. *If $\alpha \in 2^{\mathbb{N}}$ and $i \geq 1$, the function $\eta_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, defined by*

$$\eta_{\alpha,i}(\beta) = \begin{cases} 2^{-i} & , \textit{ if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 3 & , \textit{ ow} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$, belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$.

Proof. First we define the functions $\theta^*_{\alpha,i}, \theta^{**}_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, where

$$\theta^*_{\alpha,i}(\beta) = \begin{cases} 1 & , \text{ if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 3 & , \text{ ow} \end{cases}$$

$$\theta^{**}_{\alpha,i}(\beta) = \begin{cases} 1 & , \text{ if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 2 & , \text{ ow} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$. If we show that $\theta^{**}_{\alpha,i}, \theta^*_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, then we have that

$$\eta_{\alpha,i} = (2^{-i}\theta^*_{\alpha,i})(\theta^{**}_{\alpha,i})^i \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since a topology is closed under products and powers. For the above equality we have that $(2^{-i}\theta^*_{\alpha,i})(\beta)(\theta^{**}_{\alpha,i})^i(\beta) = 2^{-i}1^i = 2^{-i}$, if $\overline{\alpha}(i) = \overline{\beta}(i)$, and $(2^{-i}\theta^*_{\alpha,i})(\beta)(\theta^{**}_{\alpha,i})^i(\beta) = (2^{-i}3)2^i = 3$, otherwise. If $\theta_{\alpha,i}$ is the function defined in Lemma 6.4.3, we get that

$$\theta^{**}_{\alpha,i} = \overline{2} - \theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

$$\theta^*_{\alpha,i} = \theta^{**}_{\alpha,i} + (\overline{1} - \theta_{\alpha,i}) \in \bigvee_{n\in\mathbb{N}} \pi_n.$$

□

The previous lemmas prepare the proof of the next proposition which is necessary to our proof of Theorem 6.4.6. If $\alpha \in 2^{\mathbb{N}}$, we denote by ρ_α the uniformly continuous function $\rho_\alpha : 2^{\mathbb{N}} \to \mathbb{R}$, where $\beta \mapsto \rho(\alpha, \beta)$, for every $\beta \in 2^{\mathbb{N}}$. We use the same notation ρ_α, if $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Proposition 6.4.5. *The Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ includes the set $\{\rho_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$.*

Proof. If $\alpha \in 2^{\mathbb{N}}$ is fixed and $i \geq 1$, we show first that the function $\sigma_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\sigma_{\alpha,i}(\beta) = \begin{cases} 2^{-i} & , \text{ if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 2^{-m} & , \overline{\alpha}(m) = \overline{\beta}(m) \text{ and } \alpha(m) \neq \beta(m), \end{cases}$$

belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$. Clearly, $\sigma_{\alpha,i}(\beta)$ is well-defined, since if $\overline{\alpha}(i) \neq \overline{\beta}(i)$, there is a unique $m < i$ such that $\overline{\alpha}(m) = \overline{\beta}(m)$ and $\alpha(m) \neq \beta(m)$. If $\eta_{\alpha,i}$ is the function defined in Lemma 6.4.4, then we have that

$$\sigma_{\alpha,i} = \bigwedge_{j=1}^{i} \eta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since, if $\overline{\alpha}(i) = \overline{\beta}(i)$, then $\overline{\alpha}(j) = \overline{\beta}(j)$, for every $j \leq i$, while if $\overline{\alpha}(i) \neq \overline{\beta}(i)$, then

$$\eta_{\alpha,i}(\beta) = \ldots = \eta_{\alpha,m+1}(\beta) = 3 \;\text{ and }\; \eta_{\alpha,m}(\beta) = 2^{-m}, \ldots, \eta_{\alpha,1}(\beta) = 1,$$

therefore $\bigwedge_{j=1}^{i} \eta_{\alpha,i}(\beta) = 2^{-m}$. Clearly, $\sigma_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, since a topology is a $(\wedge, \vee)$-lattice. Next we fix some $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \epsilon$. We show that

$$U(\sigma_{\alpha,n_0}, \rho_\alpha, \epsilon) := \forall_{\beta\in 2^{\mathbb{N}}}(|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| \leq \epsilon).$$

If $\overline{\alpha}(n_0) = \overline{\beta}(n_0)$, then $\rho_\alpha(\beta) \leq 2^{-n_0}$, and

$$|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| = |2^{-n_0} - \rho_\alpha(\beta)| = 2^{-n_0} - \rho_\alpha(\beta) \leq 2^{-n_0} < \epsilon.$$

If $\overline{\alpha}(n_0) \neq \overline{\beta}(n_0)$, then $\rho_\alpha(\beta) = \sigma_{\alpha,n_0}(\beta)$ and we get that $|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| = 0 \leq \epsilon$. Since by Lemma 6.4.4 we have that $\sigma_{\alpha,n_0} \in \bigvee_{n\in\mathbb{N}} \pi_n$, and $\epsilon > 0$ is arbitrarily chosen, we get that $U(\bigvee_{n\in\mathbb{N}} \pi_n, \rho_\alpha)$, therefore by the condition BS_4 we conclude that $\rho_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$. □

Suppose that $\alpha \bowtie_\rho \beta$, hence $\rho(\alpha, \beta) = \rho_\alpha(\beta) > 0$. Since $\rho_\alpha(\alpha) = 0$ and by Proposition 6.4.5 $\rho_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$, we get, in a much less elementary way than by Proposition 6.2.3 that $\alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta$.

Theorem 6.4.6. *Suppose that X is a compact metric space and $\mathcal{U}(X) = (X, C_u(X))$ is the corresponding uniform Bishop space. Then the following hold:*
(i) If $e : 2^{\mathbb{N}} \to X$ is uniformly continuous, then $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X))$.
(ii) $\mathcal{U}(X)$ is a $\mathtt{2}$-compact Bishop space.

Proof. (i) By Corollary 3.8.4 we have that if X is a compact metric space with positive diameter, then $\mathcal{F}(U_0(X)) = C_u(X)$. Since the metric space $(2^{\mathbb{N}}, \rho)$ is complete and totally bounded i.e., it is a compact metric space in Bishop's sense, and since its diameter is $1 > 0$, we get that $\mathcal{F}(U_0(2^{\mathbb{N}})) = C_u(2^{\mathbb{N}})$. Because of Proposition 6.4.5 and the fact that $\mathcal{F}(F_0)$ is the least topology including F_0, the Cantor topology includes the uniform topology i.e.,

$$C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n.$$

By the $\mathcal{F}$-lifting of morphisms we have that $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X)) \leftrightarrow \forall_{x_0\in X}(d_{x_0} \circ e \in \bigvee_{n\in\mathbb{N}} \pi_n)$. Since the composition of uniformly continuous functions is a uniformly continuous function, we get that $d_{x_0} \circ e \in C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n$, for every $x_0 \in X$.
(ii) Since X is an inhabited compact metric space, there exists a uniformly continuous function e from $2^{\mathbb{N}}$ onto X (see [20], p.106 for a proof of this fact in BISH). By (i) we get that $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X))$, hence $e \in \mathrm{setEpi}(\mathcal{C}, \mathcal{U}(X))$ i.e., $\mathcal{U}(X)$ is $\mathtt{2}$-compact. □

Proposition 6.4.7. *All the elements of the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ are uniformly continuous functions, and $\bigvee_{n\in\mathbb{N}} \pi_n = C_u(2^{\mathbb{N}})$.*

Proof. First we show that $\{\pi_n \mid n \in \mathbb{N}\} \subseteq C_u(2^{\mathbb{N}})$. If we fix some $n \in \mathbb{N}$ and some $0 < \epsilon < 1$, we define $\omega_{\pi_n}(\epsilon) = 2^{-n}$. If $\alpha, \beta \in 2^{\mathbb{N}}$ such that $\rho(\alpha, \beta) < 2^{-n} \leftrightarrow \rho(\alpha, \beta) \leq 2^{-(n+1)}$, then $\overline{\alpha}(n+1) = \overline{\beta}(n+1)$, hence $\alpha(n) = \beta(n)$ and $|\pi_n(\alpha) - \pi_n(\beta)| = |\alpha(n) - \beta(n)| = 0 < \epsilon$. Since every function π_n is bounded, by Proposition 3.4.9 we get that $\bigvee_{n\in\mathbb{N}} \pi_n \subseteq C_u(2^{\mathbb{N}})$. Since in the proof of Theorem 6.4.6 we showed that $C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n$, we get the required equality. □

Theorem 6.4.8 (Fan theorem for the Cantor topology)**.** *If $2^{\mathbb{N}}$ is equipped with the Cantor metric and $\mathbb{N}$ with the discrete metric, then*

$$\mathrm{Mor}(\mathcal{C}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))) = C_u(2^{\mathbb{N}}, \mathbb{N}).$$

Proof. If $\phi : 2^{\mathbb{N}} \to \mathbb{N}$, then by Proposition 6.2.1 and the $\mathcal{F}$-lifting of morphisms we get

$$\begin{aligned}
\phi \in \mathrm{Mor}(\mathcal{C}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))) &\leftrightarrow \mathrm{id}_{\mathbb{N}} \circ \phi \in \bigvee_{n\in\mathbb{N}} \pi_n \\
&\leftrightarrow \phi \in \bigvee_{n\in\mathbb{N}} \pi_n \\
&\leftrightarrow \phi \in C_u(2^{\mathbb{N}}) = C_u(2^{\mathbb{N}}, \mathbb{R}) \\
&\leftrightarrow \phi \in C_u(2^{\mathbb{N}}, \mathbb{N})
\end{aligned}$$

For the last equivalence we need to show the equivalence between the following

$$\text{(I)} \quad \rho(\alpha, \beta) \leq \omega_\phi(\epsilon) \to |\phi(\alpha) - \phi(\beta)| \leq \epsilon$$

$$\text{(II)} \quad \rho(\alpha, \beta) \leq \omega^*_\phi(\epsilon) \to d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) \leq \epsilon,$$

$$d_{\mathbb{N}}(n, m) = \begin{cases} 1 & \text{, if } n \neq m \\ 0 & \text{, ow.} \end{cases}$$

Suppose (I) and let $\epsilon > 0$. We define $\omega^*_\phi(\epsilon) = \omega_\phi(\epsilon \wedge \frac{1}{2})$. By Tri we have that $\frac{1}{2} < \epsilon \vee \epsilon < 1$. In both cases we get that $\epsilon \wedge \frac{1}{2} < 1$, hence $\rho(\alpha, \beta) \leq \omega_\phi(\epsilon \wedge \frac{1}{2}) \to |\phi(\alpha) - \phi(\beta)| \leq \epsilon \wedge \frac{1}{2} < 1 \to \phi(\alpha) = \phi(\beta) \to d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) = 0 \leq \epsilon$. Next we suppose (II) and let $\epsilon > 0$. We define $\omega_\phi(\epsilon) = \omega^*_\phi(\epsilon \wedge \frac{1}{2})$. If $\rho(\alpha, \beta) \leq \omega^*_\phi(\epsilon \wedge \frac{1}{2})$, then $d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) \leq \epsilon \wedge \frac{1}{2} < 1 \to \phi(\alpha) = \phi(\beta)$, hence $|\phi(\alpha) - \phi(\beta)| \leq \epsilon$. □

By Proposition 6.4.7 the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ i.e., the set $\mathrm{Mor}(\mathcal{C}, \mathcal{R})$, "captures" exactly the set of uniformly continuous functions on $2^{\mathbb{N}}$ without a compactness assumption, while by Theorem 6.4.8 the Bishop morphisms between $\mathcal{C}$ and $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ "capture" the uniformly continuous functions with respect to the corresponding metrics. The following corollary of Proposition 6.4.7 and the Backward uniform continuity theorem is another such "capture" of uniform continuity by the notion of Bishop morphism. The discrete Bishop space $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ in Theorem 6.4.8 is replaced by a compact metric space endowed with the uniform topology.

Corollary 6.4.9. *If X is a compact metric space, then $C_u(2^{\mathbb{N}}, X) = \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X))$.*

Proof. Since $C_u(2^{\mathbb{N}}) = \bigvee_{n\in\mathbb{N}} \pi_n$ by Corollary 3.8.9 of BUCT we get that $h \in \mathrm{Mor}(\mathcal{C},\mathcal{U}(X))$ if and only if $h \in \mathrm{Mor}(\mathcal{U}(2^{\mathbb{N}}),\mathcal{U}(X))$ if and only if h is uniformly continuous. □

We consider the above results as one more indication of the usefulness of considering Bishop continuity in order to solve Bishop's problem of constructivising topology.
Since Lemma 6.4.3 holds also for the Baire space, we formulate its next corollary for it, although it holds for the Cantor space too. Recall that

$$\mathcal{B}(u) = \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid \overline{\alpha}(|u|) = u\}$$

is a basic open set in the standard topology of the Baire space, where $|u|$ denotes the length of a finite sequence $u \in \mathbb{N}^*$, while a closed subset of the Baire space in the standard topology, or a *spread*, M, is characterized by the property

$$\forall_{\alpha\in\mathbb{N}^{\mathbb{N}}}(\forall_{k\in\mathbb{N}}\exists_{\beta\in M}(\overline{\alpha}(k) = \overline{\beta}(k)) \to \alpha \in M).$$

Clearly, $\mathcal{B}(u)$ is a spread.

Proposition 6.4.10. *A spread M of the Baire space is a closed set in the induced by the Baire topology neighborhood structure $N(\bigvee_{n\in\mathbb{N}} \varpi_n)$, and the standard topology on the Baire space shares the same open sets with $\mathcal{T}_{N(\bigvee_{n\in\mathbb{N}} \varpi_n)}$.*

Proof. According to the definition of a closed set in the induced by some topology neighborhood structure given in section 3.7, we need to show that

$$\forall_{\alpha\in\mathbb{N}^{\mathbb{N}}}(\forall_{\phi\in\bigvee_{n\in\mathbb{N}}\pi_n}(\phi(\alpha) > 0 \to \exists_{\beta\in M}(\phi(\beta) > 0)) \to \alpha \in M).$$

We fix some $\alpha \in \mathbb{N}^{\mathbb{N}}$, we suppose that $\forall_{\phi\in\bigvee_{n\in\mathbb{N}}\pi_n}(\phi(\alpha) > 0 \to \exists_{\beta\in M}(\phi(\beta) > 0))$ and we show that $\alpha \in M$. By the aforementioned characterization of a spread it suffices to show $\forall_{k\in\mathbb{N}}\exists_{\beta\in M}(\overline{\alpha}(k) = \overline{\beta}(k))$. We consider the function $\theta_{\alpha,k} \in \bigvee_{n\in\mathbb{N}} \pi_n$ by Lemma 6.4.3. Since $\theta_{\alpha,k}(\alpha) = 1$, we get by our hypothesis that there exists some $\beta \in M$ such that $\theta_{\alpha,k}(\beta) > 0 \leftrightarrow \theta_{\alpha,k}(\beta) = 1 \leftrightarrow \overline{\alpha}(k) = \overline{\beta}(k)$. Since $\mathcal{B}(u) = U(\theta_{\alpha,|u|})$, for every $u \in \mathbb{N}^*$, we get that the standard topology on the Baire space shares the same open sets with the neighborhood structure induced by the Baire topology. □

Hence, $\mathcal{B}(u)$ is clopen in the induced by the Baire topology neighborhood structure too.

6.5 Properties of 2-compact spaces

Proposition 6.5.1. *If $\mathcal{F} = (X, F)$ is a 2-compact Bishop space, $\mathcal{G} = (Y, G)$ is a Bishop space and $h : X \to Y \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$, then $\mathcal{G}$ is 2-compact.*

Proof. If $e : 2^I \to X \in \mathrm{setEpi}(2^I, \mathcal{F})$, then $h \circ e : 2^I \to Y \in \mathrm{setEpi}(2^I, \mathcal{G})$. □

By Proposition 6.5.1 if $\mathcal{F}$ is 2-compact and $e : X \to Y$ is onto Y, then the quotient Bishop space $\mathcal{G}_e$ is 2-compact. For the next expected fact recall that a set Y is *finite*, if there is some $n \in \mathbb{N}$ and a bijection $j : n \to Y$, where $n = \{0, 1, \ldots, n-1\}$ (see [15], p.18).

Proposition 6.5.2. *If Y is an inhabited finite set, then $(Y, \mathbb{F}(Y))$ is $\mathfrak{2}$-compact.*

Proof. Suppose that there is some $n > 0$ and a bijection $j : n \to Y$, hence $Y = \{j_0, \ldots, j_{n-1}\}$. It is trivial to see that j is an isomorphism between $(Y, \mathbb{F}(Y))$ and $\mathfrak{n} = (n, \mathbb{F}(n))$, where $\mathbb{F}(n) = \mathcal{F}(\mathrm{id}_n)$. Therefore, it suffices to show that $\mathfrak{n}$ is $\mathfrak{2}$-compact. We show that there exists a function $e : 2^n \to n \in \mathrm{setEpi}(\mathfrak{2}^n, \mathfrak{n})$ i.e., e is onto n and $\mathrm{id}_n \circ e = e \in \bigvee_{l \in n} \varpi_l$. If $i \in n$, let $\pi_i \in 2^n$ defined as $\pi_i(l) = 1$, if $l = i$, and $\pi_i(l) = 0$, otherwise. Then the function $e := \sum_{l=0}^{n-1} \bar{l} \varpi_l \in \bigvee_{l \in n} \varpi_l$, and $e(\pi_i) = i$, for every $i \in n$. $\square$

Proposition 6.5.3 (Countable Tychonoff theorem)**.** *If for every $n \in \mathbb{N}$ the Bishop space $\mathcal{F}_n = (X_n, F_n)$ is $\mathfrak{2}$-compact, then the product $\prod_{n \in \mathbb{N}} \mathcal{F}_n$ is $\mathfrak{2}$-compact.*

Proof. By the definition of $\mathfrak{2}$-compactness there exist some I_n and some $e_n : 2^{I_n} \to X_n$ such that $e_n \in \mathrm{Mor}(\mathfrak{2}^{I_n}, \mathcal{F}_n)$, for every $n \in \mathbb{N}$. Without loss of generality we assume that the sets $(I_n)_n$ are pairwise disjoint, since it is straightforward to see that there is an isomorphism between the Bishop spaces 2^{I_n} and $2^{I_n \times \{n\}}$. If $I = \bigcup_{n \in \mathbb{N}} I_n$ and $X = \prod_{n \in \mathbb{N}} X_n$, we define

$$E : 2^I \to X,$$

$$E(\alpha) := (e_n(\alpha_{|I_n}))_{n \in \mathbb{N}},$$

$$\alpha_{|I_n} : I_n \to 2,$$

$$\alpha_{|I_n}(i) = \alpha(i),$$

for every $i \in I_n$. In order to show that E is onto X we fix some $(x_n)_{n \in \mathbb{N}} \in X$, and since there exists some $\beta_n \in 2^{I_n}$ such that $e_n(\beta_n) = x_n$, for every $n \in \mathbb{N}$, we define $\alpha \in 2^I$ by $\alpha(i) = \beta_n(i)$, where n is the unique index n for which $i \in I_n$, for every $i \in I$. In other words, $\alpha_{|I_n} = \beta_n$, for every $n \in \mathbb{N}$. Hence, $E(\alpha) = (e_n(\alpha_{|I_n}))_{n \in \mathbb{N}} = (e_n(\beta_n))_{n \in \mathbb{N}} = (x_n)_{n \in \mathbb{N}}$. By the $\mathcal{F}$-lifting of morphisms we have that

$$E \in \mathrm{Mor}(\mathfrak{2}^I, \prod_{n \in \mathbb{N}} \mathcal{F}_n) \leftrightarrow \forall_{n \in \mathbb{N}} \forall_{f \in F_n} ((f \circ \pi_n) \circ E \in \bigvee_{i \in I} \varpi_i).$$

In order to show that we define the function

$$\mathrm{cut}_n : 2^I \to 2^{I_n},$$

$$\alpha \mapsto \alpha_{|I_n},$$

for every $\alpha \in 2^I$, and we show that $\mathrm{cut}_n \in \mathrm{Mor}(\mathfrak{2}^I, \mathfrak{2}^{I_n}) \leftrightarrow \forall_{j \in I_n} (\pi_j \circ \mathrm{cut}_n \in \bigvee_{i \in I} \varpi_i)$, for every $n \in \mathbb{N}$. Since $(\pi_j \circ \mathrm{cut}_n)(\alpha) = \mathrm{cut}_n(\alpha)(j) = \alpha_{|I_n}(j) = \alpha(j) = \varpi_j(\alpha)$, for every $\alpha \in 2^I$, we get that $\pi_j \circ \mathrm{cut}_n = \varpi_j \in \bigvee_{i \in I} \varpi_i$. If we fix some $n \in \mathbb{N}$ and some $f \in F_n$, then

$$[(f \circ \pi_n) \circ E](\alpha) = (f \circ \pi_n)((e_n(\alpha_{|I_n}))_{n \in \mathbb{N}}) = (f \circ e_n)(\alpha_{|I_n}) = [(f \circ e_n) \circ \mathrm{cut}_n](\alpha),$$

for every $\alpha \in 2^I$. Hence, $(f \circ \pi_n) \circ E = (f \circ e_n) \circ \mathrm{cut}_n = f \circ (e_n \circ \mathrm{cut}_n) \in \bigvee_{i \in I} \varpi_i$, since $e_n \circ \mathrm{cut}_n : 2^I \to X_n \in \mathrm{Mor}(\mathfrak{2}^I, \mathcal{F}_n)$ as a composition of morphisms, and consequently $f \circ (e_n \circ \mathrm{cut}_n) \in \bigvee_{i \in I} \varpi_i$, by the definition of a morphism and the fact that $f \in F_n$. $\square$

Corollary 6.5.4. *The Hilbert cube $\mathcal{I}^\infty$ is a 2-compact Bishop space.*

Proof. Since the topology on $[-1,1]$ is the uniform one, by Theorem 6.4.6 we get that $\mathcal{I}_{(-1)1}$ is 2-compact, while by Proposition 6.5.3 we have that $\mathcal{I}_{(-1)1}^{\mathbb{N}}$ is 2-compact. We show that $\mathcal{I}^\infty$ is isomorphic to $\mathcal{I}_{(-1)1}^{\mathbb{N}}$, therefore by Proposition 6.5.1 we get that $\mathcal{I}^\infty$ is 2-compact. It suffices to show that the bijection

$$e : I^\infty \to [-1,1]^{\mathbb{N}},$$
$$(x_1, x_2, x_3, \ldots) \mapsto (x_1, 2x_2, 3x_3, \ldots)$$

is an open morphism. By the properties of the product and relative Bishop topology, and the $\mathcal{F}$-lifting of morphisms we have that

$$(\mathrm{Bic}[-1,1])^{\mathbb{N}} = \mathcal{F}(\{\mathrm{id}_{|[-1,1]} \circ \pi_n \mid n \in \mathbb{N}\}) = \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}),$$
$$(\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^\infty} = \mathcal{F}(\{\pi_{n|I^\infty} \mid n \in \mathbb{N}\}),$$
$$e \in \mathrm{Mor}(\mathcal{I}^\infty, \mathcal{I}_{(-1)1}^{\mathbb{N}}) \leftrightarrow \forall_{n\in\mathbb{N}}(\pi_n \circ e \in (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^\infty}).$$

Since

$$(\pi_n \circ e)((x_m)_m) = \pi_n(x_1, 2x_2, 3x_3, \ldots) = nx_n = (\overline{n} \cdot \pi_{n|I^\infty})((x_m)_m),$$

for every $(x_m)_m \in I^\infty$, we get that

$$\pi_n \circ e = \overline{n} \cdot \pi_{n|I^\infty} \in (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}_{|I^\infty},$$
$$\pi_{n|I^\infty} = \overline{\frac{1}{n}} \cdot (\pi_n \circ e) = (\overline{\frac{1}{n}} \cdot \pi_n) \circ e,$$

and since $\overline{\frac{1}{n}} \cdot \pi_n \in (\mathrm{Bic}[-1,1])^{\mathbb{N}}$, for every $n \in \mathbb{N}$, we conclude by Proposition 3.6.10 that the set-epimorphism e is open, hence e is an isomorphism between $\mathcal{I}^\infty$ and $\mathcal{I}_{(-1)1}^{\mathbb{N}}$. □

Although e is the bijection used in the classical proof too, here we avoided to use the metric on the product $\mathcal{I}_{(-1)1}^{\mathbb{N}}$ (see [40], p.193), showing that the above isomorphism of Bishop spaces is independent from the corresponding metrics.

Proposition 6.5.5. *If $\mathcal{F} = (X, F)$ is 2-compact, then $\mathcal{F}$ is pseudo-compact.*

Proof. If $e : 2^I \to X \in \mathrm{setEpi}(2^I, \mathcal{F})$ and $f \in F$, then $f \circ e \in \bigvee_{i\in I} \varpi_i$. Since ϖ_i is bounded, for every $i \in I$, by the $\mathcal{F}$-lifting of boundedness we get that every element of $\bigvee_{i\in I} \varpi_i$ is bounded. Hence, $f(X) = (f \circ e)(2^I)$ is a bounded subset of $\mathbb{R}$. □

There are properties of 2-compactness though, which deviate from the classical notion of compactness. Hannes Diener suggested to me an example of a 2-compact space (X, F) for which it is not possible to accept constructively that $f(X)$ has a supremum, for some $f \in F$. Also, in the case of metric spaces 2-compactness does not imply metric compactness. The reason behind such phenomena is the generality of the index set I in the definition of 2-compactness. There are many issues requiring further study regarding 2-compactness. The characterization of the 2-compact subspaces of a 2-compact space, the isomorphism of two 2-compact spaces whenever their topologies are isomorphic as rings, and the transfer of properties of a Boolean space to a 2-compact Bishop space are only some of them.

6.6 Pair-compact Bishop spaces

The concept of a pair-compact Bishop space is the "least" necessary notion of compactness on Bishop spaces for the proof of a Stone-Weierstrass theorem within TBS. Its definition requires quantification over all subsets of a Bishop topology, therefore it cannot be understood as completely constructive notion. If we understood the notion of a subset of a Bishop topology as a concept defined through some specific F-formula, we may avoid the full power set axiom and restrict to the *well-constructed* subsets of F (see also the relevant comments in section 8.3 and the discussion on impredicativity within the theory of apartness spaces in [27], pp.61-63). We also include here the classical proofs of some versions of the Stone-Weierstrass theorem for pair-compact Bishop spaces because they guided the constructive version of the Stone-Weierstrass theorem for pair-compact Bishop spaces[4]. This section is a translation of the section 16.2 of the book of Gillman and Jerison [44] into TBS. The definition of a pair-compact Bishop space is motivated by the fact that if X is a compact topological space and A is a sublattice of $C(X)$, then $\overline{A}$, the closure of A in the metric topology $d(f,g)_\infty$, contains every function $f \in C(X)$ that can be approximated at each pair of points in X by a function from A (see [44], p.242).

Definition 6.6.1. *A topology F on some inhabited set X is called pair-compact, if*

$$\forall_{f\in F}(\forall_{\mathrm{Const}(X)\subseteq\Phi\subseteq F}\forall_{x\bowtie_F y}\forall_{\epsilon>0}\exists_{\phi\in\Phi}(|\phi(x)-f(x)|<\epsilon \ \wedge \ |\phi(y)-f(y)|<\epsilon)\to f\in\mathcal{U}(\Phi)).$$

In other words, F is pair-compact, if the uniform closure $\mathcal{U}(\Phi)$ contains every function in F that can be approximated as close as we want at each pair of points in X by some function from Φ, for every $\mathrm{Const}(X)\subseteq\Phi\subseteq F$. The last condition is considered in order to get trivially that $\mathrm{Const}(X)$ is a pair-compact topology on X.

Proposition 6.6.2. *If $\mathcal{F}=(X,F)$ is a pair-compact Bishop space, $\mathcal{G}=(Y,G)$ is a Bishop space and $h\in\mathrm{setEpi}(\mathcal{F},\mathcal{G})$, then $\mathcal{G}$ is pair-compact.*

Proof. We fix $g\in G$, $\Theta\subseteq G$, $\epsilon>0$, $y_1,y_2\in Y$ such that $y_1\bowtie_{g'}y_2$, for some $g'\in G$ and we suppose that

$$\exists_{\theta\in\Theta}(|\theta(y_1)-g(y_1)|<\epsilon \ \wedge \ |\theta(y_2)-g(y_2)|<\epsilon).$$

We show that $g\in\mathcal{U}(\Theta)$. Since $h\in\mathrm{Mor}(\mathcal{F},\mathcal{G})$ we have that $f=g\circ h\in F$. We consider the set $\Phi=\Theta\circ h=\{\theta\circ h \mid \theta\in\Theta\}$, which trivially includes $\mathrm{Const}(X)$. If $x_1,x_2\in X$ such that $h(x_1)=y_1$ and $h(x_2)=y_2$, then $y_1\bowtie_{g'}y_2\leftrightarrow h(x_1)\bowtie_{g'}h(x_2)\leftrightarrow x_1\bowtie_{g'\circ h}x_2$. The above inequalities are written as

$$|(\theta\circ h)(x_1)-g(h(x_1))|<\epsilon \ \wedge \ |(\theta\circ h)(x_2)-g(h(x_2))|<\epsilon.$$

By the pair-compactness of $\mathcal{F}$ we conclude that $f=g\circ h\in\mathcal{U}(\Phi)$. Since the condition $\forall_{\phi\in\Theta\circ h}\exists_{\theta\in\Theta}(\phi=\theta\circ h)$ is satisfied in a trivial way, by the $\mathcal{U}$-lifting of openness we get that $\forall_{\phi\in\mathcal{U}(\Theta\circ h)}\exists_{\theta\in\mathcal{U}(\Theta)}(\phi=\theta\circ h)$. This implies that for the element $g\circ h$ of $\mathcal{U}(\Theta\circ h)$, there is some $\theta\in\mathcal{U}(\Theta)$ such that $g\circ h=\theta\circ h$, hence $g=\theta$, therefore $g\in\mathcal{U}(\Theta)$. $\square$

[4]Note that there is a constructive version of the Stone-Weierstrass theorem in the theory of frames and locales by Coquand in [35].

By Proposition 4.7.3(ii) we get that the image of a pair-compact Bishop space under a morphism endowed with the relative Bishop topology is a pair-compact Bishop space.

Theorem 6.6.3 (Stone-Weierstass theorem for pair-compact Bishop spaces I)**.** *Suppose that (X, F) is a pair-compact Bishop space and $\Phi \subseteq F$ such that*
(i) $\mathrm{Const}(X) \subseteq \Phi$*,*
(ii) Φ *is closed under addition and multiplication,*
(iii) $x \bowtie_F y \to x \bowtie_\Phi y$*, for every* $x, y \in X$*.*
Then Φ *is a base of* F*.*

Proof. We fix some $f \in F$ and we show that $f \in \mathcal{U}(\Phi)$. Since $\mathrm{Const}(X) \subseteq \Phi$ by the pair-compactness of F it suffices to show that

$$\forall_{x \bowtie_F y} \forall_{\epsilon > 0} \exists_{\phi \in \Phi} (|\phi(x) - f(x)| < \epsilon \ \wedge \ |\phi(y) - f(y)| < \epsilon).$$

We fix $x, y \in X$ such that $x \bowtie_F y$. By condition (iii) there is some $\theta \in \Phi$ such that $x \bowtie_\theta y$. The function

$$\phi(z) := f(x) \frac{\theta(z) - \theta(y)}{\theta(x) - \theta(y)} + g(y) \frac{\theta(z) - \theta(x)}{\theta(y) - \theta(x)}$$

belongs to Φ by condition (i) and (ii). Moreover $\phi(x) = f(x)$ and $\phi(y) = f(y)$, therefore the above conjunction is satisfied in a trivial way and uniformly for every $\epsilon > 0$. $\square$

Note that condition (iii) above is always the case, if Φ is a base of F (Proposition 5.1.9(i)). The proof of the subsequent second version of the Stone-Weierstrass theorem for pair-compact Bishop spaces is classical and shows how strong condition (iii) is.

Definition 6.6.4. *If* $\mathcal{F} = (X, F)$ *is a Bishop space and* Φ *is an inhabited subset of* F *the set* $\mathrm{St}(\Phi)$ *of the stationary sets of* Φ *and the set* $\mathrm{St}_n(\Phi)$ *of the* n*-stationary sets of* Φ *are defined by*

$$A \in \mathrm{St}(\Phi) :\leftrightarrow \forall_{f \in \Phi} (f_{|A} \textit{ is constant}),$$
$$A \in \mathrm{St}_n(\Phi) :\leftrightarrow A \in \mathrm{St}(\Phi) \wedge |A| = n,$$

where $|A| = n$ *denotes that* $A = \{x_1, \ldots, x_n\}$*, for some* $x_1, \ldots, x_n \in X$ *such that* $x_i \neq x_j$*, for every* $i \neq j$*, if* $n > 1$*.*

Note that since the proofs of the subsequent versions of the Stone-Weierstrass theorem are classical, we do not consider some apartness relation on X. We denote by $\mathrm{Singl}(X)$ the set of all singletons of X. Clearly, $\mathrm{Singl}(X) = \mathrm{St}_1(\Phi)$, for every inhabited set Φ. The trivial proof of the next proposition is omitted.

Proposition 6.6.5. *Suppose that* $\mathcal{F} = (X, F)$ *is a Bishop space,* $\bowtie$ *is a point-point apartness relation on* X*, and* $\Phi \subseteq F$ *is inhabited.*
(i) $A \in \mathrm{St}(\Phi) \to B \subseteq A \to B \in \mathrm{St}(\Phi)$*.*
(ii) $\mathrm{Singl}(X) \subseteq \mathrm{St}(\Phi) \subseteq \mathcal{P}(X)$*.*
(iii) $\mathrm{St}(\Phi) = \mathrm{St}(\Phi \cup \mathrm{Const}(X))$*.*
(iv) $\mathrm{St}(\mathrm{Const}(X)) = \mathcal{P}(X)$*.*
(v) If Φ *is a* $\bowtie$*-separating family i.e.,* $\bowtie \subseteq \bowtie_\Phi$*, then* $\mathrm{St}(\Phi) = \mathrm{Singl}(X)$*.*
(vi) If F *is a completely regular topology, then* $\mathrm{St}(F) = \mathrm{Singl}(X)$*.*

Proposition 6.6.6. *If $\mathcal{F} = (X, F)$ is a Bishop space and Φ is an inhabited subset of F, then $\mathrm{St}(\mathcal{F}(\Phi)) = \mathrm{St}(\Phi)$ and $\mathrm{St}_n(\mathcal{F}(\Phi)) = \mathrm{St}_n(\Phi)$, for every $n \in \mathbb{N}$.*

Proof. It is immediate to see that $\mathrm{St}(\mathcal{F}(\Phi)) \subseteq \mathrm{St}(\Phi)$, since if every function in $\mathcal{F}(\Phi)$ is constant on some $A \subseteq X$, then every function in Φ is constant on A. The converse inclusion is exactly Proposition 3.4.8. For the n-stationary sets the proof is the same. $\square$

Following [44], p.242, the version of the classical Stone-Weierstrass theorem that we will translate first into TBS is the following: if X is a compact topological space and A is a subring of $C(X)$ that contains all the constant functions, then $\overline{A}$ is the family of all functions in $C(X)$ that are constant on every stationary set of A.

Proposition 6.6.7. *If F is a topology on X and $\Phi \subseteq F$, then the sets*

$$G = \{f \in F \mid \forall_{A \in \mathrm{St}(\Phi)}(f_{|A} \textit{ is constant})\},$$

$$G_2 = \{f \in F \mid \forall_{A \in \mathrm{St}_2(\Phi)}(f_{|A} \textit{ is constant})\}$$

are topologies on X such that

$$\Phi \subseteq G \subseteq G_2 \subseteq F,$$

$$\mathrm{St}(\Phi) = \mathrm{St}(G),$$

$$\mathrm{St}_2(F_0) = \mathrm{St}_2(G_2).$$

Proof. It is immediate from the definition of $\mathrm{St}(\Phi)$ that $\Phi \subseteq G$. We show next that G is a Bishop topology. The fact that $\mathrm{Const}(X) \subseteq G$ is trivial. Next we fix $f_1, f_2 \in F$ and $A \in \mathrm{St}(\Phi)$ such that $f_{1|A} = \overline{c_1}_{|A}$ and $f_{2|A} = \overline{c_2}_{|A}$, for some $c_1, c_2 \in \mathbb{R}$. Since $(f_1 + f_2)_{|A} = f_{1|A} + f_{2|A} = \overline{c_1}_{|A} + \overline{c_2}_{|A} = (\overline{c_1 + c_2})_{|A}$, we conclude that $f_1 + f_2 \in G$. If $\phi \in \mathrm{Bic}(\mathbb{R})$, then since $(\phi \circ f_1)_{|A} = \phi \circ f_{1|A} = \overline{\phi(c_1)}$, we conclude that $\phi \circ f_1 \in G$. If $f_n \subseteq G$ such that $f_n \stackrel{u}{\to} f$, for some $f \in F$, then $f_{n|A} \stackrel{u}{\to} f_{|A}$ and clearly, $f_{n|A} \stackrel{p}{\to} f_{|A}$. Let $x, y \in A$. Since $f_n(x) \to f(x)$, $f_n(y) \to f(y)$ and $f_n(x) = f_n(y)$, for every $n \in \mathbb{N}$, we conclude that $f(x) = f(y)$, and since x, y are arbitrary elements of A, we get that $f \in F$.
The inclusion $\mathrm{St}(\Phi) \subseteq \mathrm{St}(G)$ follows automatically by the definition of G, while for the inverse inclusion, if $B \in \mathrm{St}(G)$ and $f \in \Phi$, then $\Phi \in G$ and $f_{|B}$ is constant, therefore $B \in \mathrm{St}(\Phi)$. By Proposition 6.6.6 we get that $\mathrm{St}(\mathcal{F}(\Phi)) = \mathrm{St}(G)$. For the last equality we work similarly. $\square$

Theorem 6.6.8 (Stone-Weierstass theorem for pair-compact Bishop spaces II, CLASS)**.** *If $\mathcal{F} = (X, F)$ is a pair-compact Bishop space and $\Phi \subseteq F$ is closed with respect to addition and multiplication of functions and it includes the constant functions, then*

$$G = \{f \in F \mid \forall_{A \in \mathrm{St}(\Phi)}(f_{|A} \textit{ is constant})\} \subseteq \mathcal{U}(\Phi).$$

Proof. We fix $f \in G$ and $x \neq y \in X$. By the double negation shift (DNS), which implies that $\forall_{x,y \in \mathbb{R}}(x = 0 \vee x \neq 0)$ one gets[5] that either $f(x) = f(y)$ or $f(x) \neq f(y)$. In the first case we have that $\overline{c}_{|\{x,y\}} = f_{|\{x,y\}}$, where $f(x) = f(y) = c$, and $\overline{c} \in F_0$, since F_0 includes the constant functions. If $f(x) = a$ and $f(y) = b$, where $a \neq b$, then there exists classically some $g \in F_0$ such that $g(x) \neq g(y)$; suppose that $\forall_{g \in F_0}(g(x) = g(y))$. Then $\{x, y\} \in \mathrm{St}(F_0)$, and since $f \in G$ we conclude that $f(x) = f(y)$, which is a contradiction. Consider next a function $g \in F_0$ such that $g(x) \neq g(y)$. Then the function

$$h(z) := a\frac{g(z) - g(y)}{g(x) - g(y)} + b\frac{g(z) - g(x)}{g(y) - g(x)}$$

is in F_0, since F_0 is closed with respect to addition and multiplication, while $h(x) = a$ and $h(y) = b$. In both cases we have found an element of F_0 which is trivially arbitrary close to f on $\{x, y\}$. Since F is pair-compact, we get that $f \in \mathcal{F}(F_0)$. $\square$

According to the next corollary, it suffices to formulate the conclusion of the Stone-Weierstrass theorem restricting to the 2-stationary sets of Φ.

Corollary 6.6.9 (Stone-Weierstrass theorem for pair-compact Bishop spaces III, CLASS)**.** *If $\mathcal{F} = (X, F)$ is a pair-compact Bishop space and $\Phi \subseteq F$ is closed with respect to addition and multiplication of functions and it includes the constant functions, then*

$$G_2 = \{f \in F \mid \forall_{A \in \mathrm{St}_2(\Phi)}(f_{|A} \textit{ is constant})\} \subseteq \mathcal{U}(\Phi).$$

Proof. Working as in the previous proof, if $f \in G_2$, $x \neq y \in X$ and we consider the case $f(x) = a$ and $f(y) = b$, where $a \neq b$, then there exists some $g \in \Phi$ such that $g(x) \neq g(y)$, since otherwise the condition $\forall_{g \in \Phi}(g(x) = g(y))$ implies that $\{x, y\} \in \mathrm{St}_2(\Phi)$, and since $f \in G_2$ we conclude that $f(x) = f(y)$, which is a contradiction. $\square$

The equality between the stationary sets of two Bishop spaces does not imply the equality of the spaces. For example, if F, G are two different Hausdorff topologies on some X, then $\mathrm{St}(F) = \mathrm{St}(G) = \mathrm{Singl}(X)$. Note that if we add the hypothesis of separation on Φ the classical character of the previous proof is restricted to the use of DNS; if $f \in G$ and $x \neq y \in X$ such that $f(x) = a$, $f(y) = b$ and $a \neq b$ we get easily that the above defined h is in Φ, where g is any element of Φ which separates x and y. Thus, the proof of the translation within Bishop spaces of the following more well-known version of the Stone-Weierstrass theorem is within BISH + DNS: if X is a compact topological space and A is a closed subalgebra that separates points and includes the constant functions, then $A = C(X)$.

Theorem 6.6.10 (Stone-Weierstass theorem for pair-compact Bishop spaces IV, DNS)**.** *If $\mathcal{F} = (X, F)$ is a pair-compact Bishop space and $\Phi \subseteq F$ is a $\neq$-separating family that includes the constant functions and is closed with respect to addition and multiplication of functions, then Φ is a base of F.*

[5]It suffices to show that $\neg\neg x = 0 \to x = 0$, which is an immediate consequence of DNS.

Proof. By the separating hypothesis on Φ and Proposition 6.6.5(v) we get that $\mathrm{St}(\Phi) = \mathrm{Singl}(X)$. By Theorem 6.6.8 we have that

$$F = \{f \in F \mid \forall_{\{x\} \in \mathrm{Singl}(X)}(f_{|\{x\}} \text{ is constant})\} \subseteq \mathcal{U}(\Phi).$$

$\square$

Chapter 7

Basic homotopy theory of Bishop spaces

> To be able to make certain quotient and glueing constructions it is necessary to have a constructive theory of more general topological spaces than metric spaces.
>
> Erik Palmgren, 2009

To paraphrase a comment of Bridges and Vîţă on locale theory and general topology, however, remarkable though the development of HoTT has been, it does not, in our view, provide the straightforward elementary constructive counterpart of classical homotopy theory. In this section we present a constructive reconstruction of basic homotopy theory within TBS. A similar study within formal topology was initiated by Palmgren in [71]. Since TBS is a function-theoretic approach to constructive topology, and since classical homotopy theory contains many function-theoretic concepts, it seems natural to try to develop such a reconstruction within TBS.

If (X, F) is a Bishop space, an F-path is a morphism from $[0,1]$, endowed with the topology of the uniformly continuous functions, to (X, F). In contrast to the "logical" character of paths in HoTT, not every Bishop space has the path-joining property (PJP). We study the rich class of codense Bishop spaces, which generalizes in TBS the class of complete metric spaces, and we show that every codense Bishop space has the PJP. Also, we study Bishop spaces with the homotopy-joining or the loop homotopy-joining property. With such concepts as starting point we can start translating some basic facts of the classical theory of the homotopy type into TBS.

7.1 F-paths

If $(X, \mathcal{T})$ is a topological space, a path in X is a continuous functions $\gamma : [a, b] \to X$. In [15], pp.148-9, paths in $\mathbb{C}$ were defined with the additional property of piecewise differentiability. Since continuity is expressed in TBS through the notion of a Bishop morphism, we use the following natural definition of a path in a Bishop space.

Definition 7.1.1. *If $\mathcal{F} = (X, F)$ is a Bishop space, an F-path, or simply a path, is a mapping $\gamma : [a, b] \to X$, where $a, b \in \mathbb{R}$ and $a < b$, such that $\gamma \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F})$.*

We denote by $\Gamma(F)$ the set of all F-paths of a Bishop space $\mathcal{F}$. As in [15], the interval $[a, b]$ is called the *parameter interval* of γ, while $\gamma(a)$ and $\gamma(b)$ are the *left* and *right end points* of γ, respectively. In case these end points are equal, the path γ is said to be *closed*, or a *loop*. If $x \in X$, the constant path $\overline{x}$ on $[a, b]$ is in $\mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F})$ and it is called the *stationary path* at x. A *linear map* $\lambda_Y : Y \to \mathbb{R}$, where $Y \subseteq \mathbb{R}$ has the form $At + B$, for some $A, B \in \mathbb{R}$, and for every $t \in Y$. For the sake of completeness we include in the following lemma some elementary facts on paths.

Lemma 7.1.2. *Suppose that $a, b, c, d, k, l \in \mathbb{R}$ such that $a < b$, $c < d$ and $k < l$.*
(i) There is a unique linear map $\lambda_{ab,cd} : [a, b] \to [c, d]$ with $\lambda_{ab,cd}(a) = c$ and $\lambda_{ab,cd}(b) = d$.
(ii) There is a unique linear map $\lambda_{ab,ba} : [a, b] \to [a, b]$ with $\lambda_{ab,ba}(a) = b$ and $\lambda_{ab,ba}(b) = a$.
(iii) $\lambda_{cd,kl} \circ \lambda_{ab,cd} = \lambda_{ab,kl}$.
(iv) $\lambda_{cd,ab} \circ \lambda_{ab,cd} = \mathrm{id}_{[a,b]}$.
(v) $\lambda_{ab,cd}$ is an isomorphism between $\mathcal{I}_{ab}$ and $\mathcal{I}_{cd}$, and $\lambda_{ab,ba}$ is an automorphism of $\mathcal{I}_{ab}$.

Proof. (i) and (ii) It is easy to see that the system $\lambda_{ab,cd}(t) = B + At \wedge \lambda_{ab,cd}(a) = c \wedge \lambda_{ab,cd}(b) = d$ has as solution the function $\lambda_{ab,cd} = \frac{bc-ad}{b-a} + \frac{d-c}{b-a}t$, while the system $\lambda_{ab,ba}(t) = B + At \wedge \lambda_{ab,ba}(a) = b \wedge \lambda_{ab,ba}(b) = a$ has as solution the function $\lambda_{ab,ba} = a + b - t$.
(iii) Given the expressions for $\lambda_{ab,cd}$ and $\lambda_{cd,kl}$ determined in (i) the required form of their composition follows from a straightforward calculation.
(iv) By (iii) we have that $(\lambda_{cd,ab} \circ \lambda_{ab,cd})(t) = \lambda_{ab,ab}(t) = \frac{ba-ab}{b-a} + \frac{b-a}{b-a}t = t$, for every $t \in [a, b]$.
(v) It suffices to show that $\lambda_{ab,cd} \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{I}_{cd})$, since by (iv) its inverse (it is trivial to see that $\lambda_{ab,cd}$ is $1{-}1$ and onto $[c, d]$) is shown similarly that it in $\mathrm{Mor}(\mathcal{I}_{cd}, \mathcal{I}_{ab})$. By the $\mathcal{F}$-lifting of morphisms it suffices to show that $\mathrm{id}_{[c,d]} \circ \lambda_{ab,cd}$ is uniformly continuous on $[a, b]$. Since $|\lambda_{ab,cd}(t_1) - \lambda_{ab,cd}(t_2)| = |\frac{bc-ad}{b-a} + \frac{d-c}{b-a}t_1 - (\frac{bc-ad}{b-a} + \frac{d-c}{b-a}t_2)| = |\frac{d-c}{b-a}||t_1 - t_2|$, we get that the modulus of continuity of $\lambda_{ab,cd}$ is $\omega_{\lambda_{ab,cd}}(\epsilon) = |\frac{b-a}{d-c}|\epsilon$. $\square$

Note that the notation $\lambda_{ab,ba}$, where $a < b$ implies that $\neg(b < a)$, is justified by the fact that $\lambda_{ab,ba}(t) = \frac{b^2-a^2}{b-a} + \frac{a-b}{b-a}t = a + b - t$, therefore we can forget in this case our convention $c < d$ for $\lambda_{ab,cd}$. If we add a positive constant to the denominator of $\omega_{\lambda_{ab,cd}}(\epsilon)$, we do not need to know that $c \bowtie_{\mathbb{R}} d$ and we get the following necessary fact to the proof of the path-connectedness of $\mathcal{R}$ (see section 7.2).

Remark 7.1.3. *If $c, d \in \mathbb{R}$ the map $\lambda : [0, 1] \to \mathbb{R}$, where $\lambda(t) = (d - c)t + c$, for every $t \in [0, 1]$ is in $\mathrm{Mor}(\mathcal{I}, \mathcal{R})$.*

Proof. If $\sigma > 0$, then $\omega_\lambda(\epsilon) := \frac{\epsilon}{|d-c|+\sigma}$ is a modulus of continuity of λ, since $|\lambda(t_1) - \lambda(t_2)| = |d-c||t_1 - t_2| < |d-c|\frac{\epsilon}{|d-c|+\sigma} = \frac{|d-c|}{|d-c|+\sigma}\epsilon < \epsilon$. $\square$

The relation $\gamma \sim \delta$, γ is *equivalent* to δ, on $\Gamma(F)$ is defined by $\gamma \sim \delta :\leftrightarrow \gamma = \delta \circ \lambda_{ab,cd}$

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\lambda_{ab,cd}"] \arrow[dr, "\gamma"'] & {[c,d]} \arrow[d, "\delta"] \\
& \mathbb{R}.
\end{tikzcd}$$

By Lemma 7.1.2 $\gamma \sim \delta$ is an equivalence relation. If $\gamma : [a,b] \to X$, the *negative* of γ is the path $-\gamma = \gamma \circ \lambda_{ab,ba}$. Clearly, $(-\gamma)(a) = \gamma(b)$ and $(-\gamma)(b) = \gamma(a)$. We call two F-paths $\gamma : [a,b] \to X$ and $\delta : [c,d] \to X$ *summable*, $\gamma S \delta$, if $\gamma(b) = \delta(c)$. Clearly, $\gamma S(-\gamma)$. Combining the notions found in [15], pp.134-5, and in [71], p.238, we give the following definition.

Definition 7.1.4. *A Bishop space* (X, F) *has the path-joining property* (PJP)*, in symbols* PJP($\mathcal{F}$)*, if for every* $\gamma, \delta \in \Gamma(F)$ *such that* $\gamma : [a,b] \to X$ *and* $\delta : [c,d] \to X$ *are summable, there exists an* F*-path* $\zeta : [a, b+d-c] \to X$ *satisfying*

$$\zeta(s) = \begin{cases} \gamma(s) & , \textit{ if } s \in [a,b] \\ \delta(\lambda_{b(b+d-c),cd}(s)) & , \textit{ if } s \in [b, b+d-c]. \end{cases}$$

Since constructively it is not acceptable that if $a < b < c$, then $s \in [a,c] \to s \in [a,b] \vee s \in [b,c]$, we cannot expect that every Bishop space has the PJP i.e., that such a path ζ can be found for every pair of summable paths (see Proposition 7.3.2). Note that in HoTT the composition of paths is guaranteed by the transitivity of equality.
If (X,F) is completely regular, then the uniqueness of the aforementioned path ζ is shown as follows: if there was some path ζ' with the same values on $[a,b]$ and $[b, b+d-c]$, then $\zeta_{|D_{ab}} = \zeta'_{|D_{ab}}$ and $\zeta_{|D_{b(b+d-c)}} = \zeta'_{|D_{b(b+d-c)}}$, where D_{ab} is dense in $[a,b]$ and $D_{b(b+d-c)}$ is dense in $[b, b+d-c]$. By Lemma 2.2.8 $D = D_{ab} \cup D_{b(b+d-c)}$ is dense in $[a, b+d-c]$ we get that $\zeta_{|D} = \zeta'_{|D}$, therefore by Proposition 5.7.6 we get that $\zeta = \zeta'$. In this case we write $\zeta = \gamma + \delta$ and we call ζ the *addition* of γ and δ. In the rest of this chapter all Bishop spaces considered are completely regular. Since

$$\begin{aligned}
\lambda_{b(b+d-c),cd}(s) &= \frac{(b+d-c)c - bd}{d-c} + \frac{d-c}{d-c}s \\
&= \frac{bc + dc - c^2 - bd}{d-c} + s \\
&= \frac{bc - bd}{d-c} + \frac{(d-c)c}{d-c} + s \\
&= c - b + s,
\end{aligned}$$

if $b = c$, we get, the expected by the general definition, equality $\lambda_{cd,cd} = \mathrm{id}_{[c,d]}$ and in this case the addition of γ and δ is reduced to

$$(\gamma + \delta)(s) = \begin{cases} \gamma(s) & \text{, if } s \in [a, b] \\ \delta(s) & \text{, if } s \in [b, c]. \end{cases}$$

Definition 7.1.5. *If F is a topology on X and $x, y \in X$, then x, y are path-connected in F, $x \overset{\Gamma(F)}{\sim} y$, if $\exists_{\gamma \in \Gamma(F)}(\gamma(a) = x \wedge \gamma(b) = y)$, where a, b are the end points of the domain of γ.*

It is clear from Lemma 7.1.2 that we can choose any parameter interval for the F-path connecting a pair of elements of X, although usually we take $[0, 1]$ as the parameter interval of an F-path. If $x \overset{\Gamma(F)}{\sim} y$ and $\gamma \in \Gamma(F)$ such that γ connects x and y, we write $x \overset{\gamma}{\sim} y$.

Remark 7.1.6. *If (X, F) has the path-joining property, then $\overset{\Gamma(F)}{\sim}$ is an equivalence relation.*

Proof. If $x, y, z \in X$, then $x \overset{\overline{x}}{\sim} x$, and if $x \overset{\gamma}{\sim} y$, then $y \overset{-\gamma}{\sim} x$. If $\gamma : [0, \frac{1}{2}] \to X$ and $\delta : [\frac{1}{2}, 1] \to X$ such that $x \overset{\gamma}{\sim} y$ and $y \overset{\delta}{\sim} z$, respectively, then $x \overset{\gamma+\delta}{\sim} z$. $\square$

If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\gamma \in \Gamma(F)$, then $h \circ \gamma \in \Gamma(G)$ i.e., $\mathrm{Mor}(\mathcal{F}, \mathcal{G}) \circ \Gamma(F) \subseteq \Gamma(G)$.

Definition 7.1.7. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, the path-exponential topology $F \overset{\gamma}{\to} G$ on $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is defined by*

$$\begin{aligned} F \overset{\gamma}{\to} G &:= \mathcal{F}(\{e_{\gamma^*,g} \mid \gamma \in \Gamma(F), g \in G\}), \\ e_{\gamma^*,g} &: \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R} \\ e_{\gamma^*,g}(h) &= \sup g(h(\gamma^*)) \\ &= \sup\{g(h(\gamma(t))) \mid t \in [\gamma(a), \gamma(b)]\} \\ &= ||g \circ h \circ \gamma||_\infty. \end{aligned}$$

We denote the *path-exponential Bishop space* by $\mathcal{F} \overset{\gamma}{\to} \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \overset{\gamma}{\to} G)$. Note that $e_{\gamma^*,g}(h)$ is well-defined, since $g \circ h \circ \gamma : [\gamma(a), \gamma(b)] \to \mathbb{R} \in \mathrm{Mor}(\mathcal{I}_{\gamma(a)\gamma(b)}, \mathcal{F})$, hence $g \circ h \circ \gamma$ is uniformly continuous. We may fix $[0, 1]$ as the parameter interval of the paths above.

Remark 7.1.8. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces.*
(i) The map e defined by $h \mapsto \inf g(h(\gamma^*)) = \inf\{g(h(\gamma(t))) \mid t \in [\gamma(a), \gamma(b)]\}$ is in $F \overset{\gamma}{\to} G$.
(ii) $F \to G \subseteq F \overset{\gamma}{\to} G$.
(iii) The map $j : Y \to \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, $y \mapsto \overline{y}$, is in $\mathrm{Mor}(\mathcal{G}, \mathcal{F} \overset{\gamma}{\to} \mathcal{G})$.

Proof. (i) It is immediate to see that $e = -e_{\gamma^*,-g} \in F \overset{\gamma}{\to} G$.
(ii) If $x \in X$ and we consider the stationary path $\overline{x}$ at x, we have that $e_{\overline{x}^*,g}(h) = ||g \circ h \circ \overline{x}||_\infty = g(h(x)) = e_{x,g}(h)$, for every $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$.
(iii) By the $\mathcal{F}$-lifting of morphisms $j \in \mathrm{Mor}(\mathcal{G}, \mathcal{F} \overset{\gamma}{\to} \mathcal{G}) \leftrightarrow \forall_{\gamma \in \Gamma(F)} \forall_{g \in G}(e_{\gamma^*,g} \circ j \in G)$. But $e_{\gamma^*,g} \circ j = g$, since, for every $y \in Y$, we have that $(e_{\gamma^*,g} \circ j)(y) = e_{\gamma^*,g}(j(y)) = e_{\gamma^*,g}\overline{y}) = \sup g(\overline{y}(\gamma^*)) = \sup g(y) = g(y)$. $\square$

It is expected that $F \overset{\gamma}{\to} G$ is larger than $F \to G$, and it is not clear if $F \overset{\gamma}{\to} G \subseteq F \Rightarrow G$.

7.2 Path-connectedness and 2-connectedness

In [15], p.79, Bishop defined a function space $\mathcal{F} = (X, F)$ to be *connected*, if $\forall_{f \in F}(\overline{f(X)}$ is a convex subset of $\mathbb{R})$, where a set $C \subseteq \mathbb{R}$ is called *convex*, if $\forall_{a,b \in C}\forall_{t \in [0,1]}(tx + (1-t)y \in C)$, and $\overline{f(X)}$ is the closure of $f(X)$ in the neighborhood space $\mathbb{R}$. The use of the closure of $f(X)$ in this definition is due to the fact that e.g., in the case of $\mathrm{Bic}([a, b])$ which we want to consider as connected, $f([a, b])$ is not generally convex, since that would imply the validity of the intermediate value theorem IVT (see [15], p.40). By the constructive version of IVT, plus the principle of (probably not decidable) countable choice we get a sequence of $f(X)$ converging to $y \in [f(a), f(b)]$, hence, the closure of $f(X)$ is convex. Classically, a convex subset of $\mathbb{R}$ is an interval, and conversely each interval is convex. A trivial example of a connected topology is $\mathrm{Const}(X)$. Also, $\mathcal{R}$ is expected to be connected. In [15], p.79, Bishop included as an exercise the fact that a Bishop space (X, F) is connected if and only if

$$\forall_{x,y \in X}\forall_{f_1,\ldots,f_n \in F_b}\forall_{\epsilon > 0}\exists_{x_1,\ldots,x_m \in X}(x = x_1 \ \wedge \ x_m = y$$

$$\wedge \ \forall_{1 \leq i \leq n}\forall_{1 \leq j \leq m-1}(|f_i(x_j) - f_i(x_{j+1})| \leq \epsilon)).$$

In our opinion Bishop's notion of connectedness seems too complex to be a workable one. In this section we introduce the notion of the path-connected Bishop space and the more general notion of a 2-connected Bishop space. The latter, also found in the literature of categorical topology, seems to us a simple and workable notion of connectedness for TBS.

Definition 7.2.1. *A Bishop space (X, F) is path-connected, if $x \overset{\Gamma(F)}{\sim} y$, for every $x, y \in X$.*

By Lemma 7.1.2 and Remark 7.1.3 we get that $\mathcal{I}_{ab}$ and $\mathcal{R}$ are path-connected.

Proposition 7.2.2. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces.*
(i) If $h \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F}$ is path-connected, then $\mathcal{G}$ is path-connected.
(ii) $\mathcal{F} \times \mathcal{G}$ is path-connected if and only if $\mathcal{F}$ and $\mathcal{G}$ are path-connected.
(iii) If $\mathcal{F}$ is path-connected and $\tau : X \to 2 \in \mathrm{Mor}(\mathcal{F}, \mathbf{2})$, then τ is constant.

Proof. (i) If $y_1, y_2 \in Y$, there are $x_1, x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. If $x_1 \overset{\gamma}{\sim} x_2$, then $y_1 \overset{h \circ \gamma}{\sim} y_2$.
(ii) If $\mathcal{F}$ and $\mathcal{G}$ are path-connected and $x_1 \overset{\gamma}{\sim} x_2$ and $y_1 \overset{\delta}{\sim} y_2$, then $(x_1, y_1) \overset{\varepsilon}{\sim} (x_2, y_2)$, where $\varepsilon(t) = (\gamma(t), \delta(t)) \in \mathrm{Mor}(\mathcal{I}, \mathcal{F} \times \mathcal{G})$ by Corollary 4.1.7. If $\mathcal{F} \times \mathcal{G}$ is path-connected, $x_1, x_2 \in X$, and if $(x_1, y) \overset{\gamma}{\sim} (x_1, y)$, where y inhabits Y, then $x_1 \overset{\pi_X \circ \gamma}{\sim} x_2$ and $\pi_X \circ \gamma \in \mathrm{Mor}(\mathcal{I}, \mathcal{F})$ as a composition of morphisms.
(iii) Let $\tau \in \mathrm{Mor}(\mathcal{F}, \mathbf{2}) \leftrightarrow \mathrm{id}_2 \circ \tau = \tau \in F$, $x, y \in X$, and $\gamma : [0, 1] \to X \in \mathrm{Mor}(\mathcal{I}, \mathcal{F})$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since $\tau \circ \gamma \in \mathrm{Mor}(\mathcal{I}, \mathbf{2})$, we have that $\tau \circ \gamma$ is uniformly continuous, therefore by Proposition 2.3.3 it is constant. Hence $\tau(\gamma(0)) = \tau(\gamma(1)) \leftrightarrow \tau(x) = \tau(y)$. $\square$

Definition 7.2.3. *A Bishop space (X, F) is 2-connected, if $\forall_{\tau \in \mathrm{Mor}(\mathcal{F}, \mathbf{2})}(\tau$ is constant).*

Proposition 7.2.2(iii) says that a path-connected Bishop space is 2-connected. It is clear that the Bishop spaces $\mathcal{I}$, $\mathcal{R}$ and $((a,b), \mathrm{Bic}(a,b))$ are 2-connected (use Remark 7.1.3).

Proposition 7.2.4. *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces.*
(i) If $h \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F}$ is 2-connected, then $\mathcal{G}$ is 2-connected.
(ii) $\mathcal{F} \times \mathcal{G}$ is 2-connected if and only if $\mathcal{F}$ and $\mathcal{G}$ are 2-connected.

Proof. The proof is immediate.

□

7.3 Codense Bishop spaces

In this section we translate into TBS the result of Palmgren in [71] that a complete metric space has the PJP property. For that we define the notion of a codense Bishop space which can be seen as a kind of generalization of a complete metric space. First we adopt Palmgren's proof, found in [71], p.239, of the fact that there is a metric space such that if X satisfies PJP, then the non-acceptable principle LLPO follows, in the proof of Proposition 7.3.2.

Lemma 7.3.1. *Suppose that $a, b, c, d \in \mathbb{R}$ such that $a < c < d < b$.*
(i) $\forall_{x \in \mathbb{R}}(x \in [a,b] \vee x \leq c \vee x \geq d)$.
(ii) The inclusion $[a,b] \subseteq [a,c] \cup [c,b]$ implies $\forall_{x \in \mathbb{R}}(x \leq c \vee x \geq c)$.

Proof. (i) If $x \in \mathbb{R}$, then by Tri we have that $x > a \vee x < c$, hence $x \geq a \vee x \leq c$. If $x \geq a$, then by Tri again we have that $x \geq d \vee x \leq b$, hence if $x \leq b$, we get that $x \in [a,b]$.
(ii) Suppose that $[a,b] \subseteq [a,c] \cup [c,b]$ and $x \in \mathbb{R}$. By (i) we consider the following cases: if $x \in [a,b]$, then $x \in [a,c] \cup [c,b]$, hence $x \leq c \vee x \geq c$. If $x \leq c$, we are done, while if $x \geq d$, then $x \geq c$. □

Proposition 7.3.2 (Palmgren)**.** *There is a Bishop space $\mathcal{F}$ such that* $\mathrm{PJP}(\mathcal{F}) \to \mathrm{LLPO}$.

Proof. If $X = [-1,0] \cup [0,1]$ and $\mathcal{F} = (X, \mathrm{Bic}(\mathbb{R})_{|X} = \mathcal{F}(\mathrm{id}_{|X}))$, then it is straightforward to show that the functions $i : [-1,0] \to X$, defined by $s \mapsto s$, and $j : [0,1] \to X$, defined by $t \mapsto t$, are in $\mathrm{Mor}(\mathcal{I}_{(-1)0}, \mathcal{F})$ and in $\mathrm{Mor}(\mathcal{I}_{01}, \mathcal{F})$, respectively, therefore $i, j \in \Gamma(F)$. Since i, j are also summable, by PJP there is some $\zeta : [-1,1] \to X$ such that $\zeta(x) = x$, for every $x \in [-1,0] \cup [0,1]$. Next we show that $\zeta(x) = x$, for every $x \in [-1,1]$; If $k : X \to [-1,1]$, it is also immediate to see that $k \in \mathrm{Mor}(\mathcal{F}, \mathcal{I}_{(-1)1})$, and also that the morphisms $k \circ \zeta$ and $\mathrm{id}_{[-1,1]}$ are equal on $[-1,0] \cup [0,1]$, hence equal on a dense subset D of $[-1,1]$. Since the topology $C_u([-1,1]) = \mathrm{Bic}(\mathbb{R})_{|[-1,1]}$, and $\mathrm{Bic}(\mathbb{R})$ is completely regular, we have by Proposition5.7.8(ii) that $\mathcal{I}_{(-1)1}$ is completely regular, therefore by Proposition 5.7.6 we conclude that $k \circ \zeta = \mathrm{id}_{[-1,1]}$. By this we get that $\zeta = \mathrm{id}_{[-1,1]}$ too, since $k(\zeta(x)) = x \leftrightarrow \zeta(x) = x$, for every $x \in [-1,1]$. Consequently $[-1,1] \subseteq [-1,0] \cup [0,1]$, which by Lemma 7.3.1 gives that $\forall_{x \in \mathbb{R}}(x \leq c \vee x \geq c)$, which implies LLPO (see [25], p.10). □

Definition 7.3.3. *If $\mathcal{F} = (X, F)$ is a completely regular Bishop, $\mathcal{F}$, or F, is called codense, if for every $a, b \in \mathbb{R}$ such that $a < b$, for every $D \subseteq [a, b]$ dense in $[a, b]$ and for every $e : D \to X$ such that $e \in \mathrm{Mor}(\mathcal{I}_{ab|D}, \mathcal{F})$, there exists a map $h : [a, b] \to X$ such that $h \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F})$ and $h(d) = e(d)$, for every $d \in D$.*

Since a codense space is by definition completely regular, by Proposition 5.7.6 we have that the extension h of e is unique. It is also easy to see that we get the codensity of F, if the above property is satisfied for a single interval $[a, b]$. At first sight it seems that the above formulation of codensity requires quantification over the subsets of $[a, b]$, since it requires a certain property to hold for every dense subset D of $[a, b]$. Since a dense set here can be a countable set, it suffices to work with functions $d : \mathbb{N} \to [a, b]$ such that $d(\mathbb{N}) = D$ is dense in $[a, b]$, hence codensity can be formulated in function-theoretic terms only.

Remark 7.3.4. *$\mathcal{R}$ is a codense Bishop space.*

Proof. If $e : D \to \mathbb{R}$, where D is a dense subset of $[a, b]$, such that $e \in \mathrm{Mor}(\mathcal{I}_{ab|D}, \mathcal{R})$, then by the $\mathcal{F}$-lifting of morpphisms and Corollary 4.7.12 we get $e = f_{|D}$, where $f \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{R})$. $\square$

Since a locally compact metric space is complete (see [15], p.110), the next proposition is a generalization of the previous remark and because of Theorem 7.3.6 it corresponds to Palmgren's result that a complete metric space has the PJP (see [71], p.238). Note that, except from $\mathcal{F}(U_0(X))$, within TBS we defined no special Bishop topology on a complete metric space, and we work mainly with locally compact metric spaces.

Proposition 7.3.5. *If (X, d) is a locally compact metric space, then the Bishop space $\mathcal{X} = (X, \mathrm{Bic}(X))$ is codense.*

Proof. Suppose that $e : D \to X \in \mathrm{Mor}(\mathcal{I}_{ab|D}, \mathcal{X}) \leftrightarrow \forall_{\mathrm{Bic}(X)}(f \circ e \in C_u(D))$. First we show that e is uniformly continuous. For that we show that $e(D)$ is a bounded subset of X. Let $d_0 \in D$ and $h : [a, b] \to \mathbb{R}$ is the unique uniform continuous extension of the uniformly continuous function by our hypothesis $d_{e(d_0)} \circ e : D \to \mathbb{R}$, according to Lemma 4.7.11. If $M > 0$ is a bound of $h([a, b])$, then M is also a bound of $(d_{e(d_0)} \circ e)(D) = d_{e(d_0)}(e(D))$, since $(d_{e(d_0)})(D) = h(D) \subseteq h([a, b])$. Working as in the proof of Proposition 3.8.10(i) we have that for every $d_1, d_2 \in D$

$$\begin{aligned} d(e(d_1), e(d_2)) &\leq d(e(d_1), e(d_0)) + d(e(d_0), e(d_2)) \\ &= d_{e(d_0)}(e(d_1)) + d_{e(d_0)}(e(d_2)) \\ &\leq M + M \\ &= 2M. \end{aligned}$$

Since X is locally compact, $e(D)$ is included in a compact subset K of X, and consequently $e : D \to K$. Next we show that $e \in \mathrm{Mor}(\mathcal{I}_{ab|D}, C_u(K)) \leftrightarrow \forall_{g \in C_u(K)}(g \circ e \in C_u(D))$; if $g \in C_u(K)$, then by Corollary 5.4.6 there is some $f \in \mathrm{Bic}(X)$ such that $g = f_{|K}$, therefore $g \circ e = f_{|K} \circ e = f \circ e \in C_u(D)$, by our initial hypothesis. By the Backward uniform continuity theorem we conclude that e is uniformly continuous.

Since X is also complete, by Lemma 4.7.11 again we get that there is a unique uniformly continuous function $\theta : [a,b] \to X$ which extends e. We conclude our proof by showing that $\theta \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{X}) \leftrightarrow \forall_{f \in \mathrm{Bic}(X)}(f \circ \theta \in C_u([a,b]))$. If $f \in \mathrm{Bic}(X)$, and since $\theta([a,b])$ is bounded, we get that $f \circ \theta = f_{|\theta([a,b])} \circ \theta \in C_u(D)$ as a composition of two uniformly continuous functions. □

Theorem 7.3.6. *A codense Bishop space $\mathcal{F}$ has the path-joining property.*

Proof. We fix two F-paths $\gamma : [a,b] \to X$ and $\delta : [c,d] \to X$ such that $\gamma S \delta$, and we show that there exists an F-path $\gamma + \delta : [a, b+d-c] \to X$ satisfying

$$(\gamma+\delta)(s) = \begin{cases} \gamma(s) & \text{, if } s \in [a,b] \\ \delta(\lambda_{b(b+d-c),cd}(s)) & \text{, if } s \in [b, b+d-c]. \end{cases}$$

If D_{ab} and $D_{b(b+d-c)}$ are dense subsets of $[a,b]$ and $[b,b+d-c]$, respectively, and $D_{ab} \cap D_{b(b+d-c)} = \{b\}$, then by Lemma 2.2.8 the set $D = D_{ab} \cup D_{b(b+d-c)}$ is dense in $[a, b+d-c]$. We define $e : D \to X$ by

$$e(d) = \begin{cases} \gamma(d) & \text{, if } d \in D_{ab} \\ \delta(\lambda_{b(b+d-c),cd}(d)) & \text{, if } d \in D_{b(b+d-c)}, \end{cases}$$

which is well defined, since by the hypothesis $\gamma S \delta$ we get $e(b) = \gamma(b) = \delta(\lambda_{b(b+d-c),cd}(b)) = \delta(c)$. We show first that $e \in \mathrm{Mor}(\mathcal{I}_{a(b+d-c)_{|D}}, \mathcal{F})$ i.e., $f \circ e \in \mathrm{Bic}([a,b+d-c]_{|D})$, for every $f \in F$. We fix some $f \in F$, and by Corollary 4.7.12 and the fact that $\mathrm{Bic}([a,b]) = C_u([a,b])$ we get that $\mathrm{Bic}([a,b])_{|D} = C_u([a,b])_{|D} = C_u(D)$ it suffices to show that $f \circ e : D \to \mathbb{R}$ is uniformly continuous. Since $\gamma \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{F})$ and $\delta \in \mathrm{Mor}(\mathcal{I}_{cd}, \mathcal{F})$ we have that $f \circ \gamma \in \mathrm{Bic}([a,b])$ and $f \circ \delta \in \mathrm{Bic}([c,d])$ i.e., $f \circ \gamma$ and $f \circ \delta$ are uniformly continuous functions on $[a,b]$ and $[c,d]$ with modulus of continuity $\omega_{f\circ\gamma}$ and $\omega_{f\circ\delta}$, respectively. We define $\omega_{f\circ e}(\epsilon) := \min\{\omega_{f\circ\gamma}(\frac{\epsilon}{2}), \omega_{f\circ\delta}(\frac{\epsilon}{2})\}$. Then, if $d_1 \in D_{ab}$ and $d_2 \in D_{b(b+d-c)}$ and $|d_1 - d_2| \leq \omega_{f\circ e}(\epsilon)$, we have that

$$\begin{aligned} |f(e(d_1)) - f(e(d_2))| &\leq |f(e(d_1)) - f(e(b))| + |f(e(b)) - f(e(d_2))| \\ &= |f(\gamma(d_1)) - f(\gamma(b))| + |f(\delta(c)) - f(\delta(d_2))| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

since in this case $|d_1 - b| \leq |d_1 - d_2| \leq \omega_{f\circ e}(\epsilon) \leq \omega_{f\circ\gamma}(\frac{\epsilon}{2})$ and $|d_2 - c| \leq |d_1 - d_2| \leq \omega_{f\circ e}(\epsilon) \leq \omega_{f\circ\delta}(\frac{\epsilon}{2})$. The cases $d_1, d_2 \in D_{ab}$ and $d_1, d_2 \in D_{b(b+d-c)}$ are trivial. Applying the codensity of $\mathcal{F}$ on $[a, b+d-c]$ and e we get the existence of a map $h : [a, b+d-c] \to X$ which is in $\mathrm{Mor}(\mathcal{I}_{a(b+d-c)}, \mathcal{F})$ and extends e i.e.,

$$h_{|D_{ab}} = e_{|D_{ab}} = \gamma_{|D_{ab}},$$

$$h_{|D_{b(b+d-c)}} = e_{|D_{b(b+d-c)}} = \delta_{|\lambda_{b(b+d-c),cd}(D_{b(b+d-c)})} = (\delta \circ \lambda_{b(b+d-c),cd})_{|D_{b(b+d-c)}}.$$

Applying the codensity of $\mathcal{F}$ on the pairs $[a,b]$, $\gamma_{|D_{ab}}$ and $[b, b+d-c]$, $(\delta \circ \lambda_{b(b+d-c),cd})_{|D_{b(b+d-c)}}$ we get that there is a morphism-extension of $\gamma_{|D_{ab}}$ on $[a,b]$ and of $(\delta \circ \lambda_{b(b+d-c),cd})_{|D_{b(b+d-c)}}$ on $[b, b+d-c]$. Hence, $h_{|[a,b]} = \gamma$, and $h_{|[b,b+d-c]} = \delta \circ \lambda_{b(b+d-c),cd}$. □

Note that we used here repeatedly the fact that the restriction of a morphism is also a morphism, Proposition 4.7.3(iii), and Corollary 5.4.3 in order to assure that, for example, $h_{|[a,b]} \in \mathrm{Bic}[a,b]$.

7.4 Homotopic morphisms

Definition 7.4.1. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, and $a, b \in \mathbb{R}$ such that $a < b$, we say that h_1 is homotopic to h_2 with respect to $[a, b]$, in symbols $h_1 \simeq_{ab} h_2$, if*

$$\exists_{\Phi \in \mathrm{Mor}(\mathcal{I}_{ab} \times \mathcal{F}, \mathcal{G})} \forall_{x \in X} (\Phi(a, x) = h_1(x) \wedge \Phi(b, x) = h_2(x)).$$

Following the presentation of the standard homotopy relation between continuous functions in [40], Chapter XV, we write $\Phi : h_1 \simeq_{ab} h_2$ when the morphism Φ establishes a homotopy relation between h_1 and h_2. If h_2 is a constant morphism we say that h_1 is *nullhomotopic* and we write $h_1 \simeq_{ab} 0$. The choice of the interval in the previous definition is irrelevant.

Remark 7.4.2. *If $a, b, cd \in \mathbb{R}$ such that $a < b$ and $c < d$, then $h_1 \simeq_{ab} h_2 \rightarrow h_1 \simeq_{cd} h_2$.*

Proof. Suppose that $\Phi_{ab} : h_1 \simeq_{ab} h_2$. We define the function $\Phi_{cd} : [c, d] \times X \rightarrow Y$ by $\Phi_{cd}(s, x) = \Phi_{cd}(\lambda_{ab,cd}(t), x) := \Phi_{ab}(t, x)$. Since $\lambda_{ab,cd}$ is an isomorphism between $\mathcal{I}_{ab}$ and $\mathcal{I}_{cd}$, by Proposition 4.1.10 we have that $\Phi_{cd} \in \mathrm{Mor}(\mathcal{I}_{cd} \times \mathcal{F}, \mathcal{G})$ and for each $x \in X$ we get that $\Phi_{cd}(c, x) = \Phi_{cd}(\lambda_{ab,cd}(a), x) = \Phi_{ab}(a, x) = h_1(x)$, and $\Phi_{cd}(d, x) = \Phi_{cd}(\lambda_{ab,cd}(b), x) = \Phi_{ab}(b, x) = h_2(x)$. □

Because of the previous fact, generally we will not specify the ends of the interval in the notation of the homotopy relation.

Proposition 7.4.3. *The relation $h_1 \simeq h_2$ is reflexive and symmetric.*

Proof. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, we define $\Phi : [a, b] \times X \rightarrow Y$ by $\Phi(t, x) := h(x)$ i.e., $\Phi = h \circ \pi_1$, and since, for each $g \in G$, $g \circ \Phi = g \circ (h \circ \pi_1) = (g \circ h) \circ \pi_1 = f \circ \pi_1 \in F \times \mathrm{Bic}([a, b])$, for some $f \in F$. Therefore, $\Phi : h \simeq h$. Suppose next that $\Phi : h_1 \simeq h_2$. We define the function $\Phi' : [a, b] \times X \rightarrow Y$ by $\Phi'(s, x) = \Phi'(\lambda_{ab,ba}(t), x) := \Phi(t, x)$. Since $\lambda_{ab,ba}$ is an automorphism of $\mathcal{I}_{ab}$, by Proposition 4.1.10 we have that $\Phi' \in \mathrm{Mor}(\mathcal{I}_{ab} \times \mathcal{F}, \mathcal{G})$ and for every $x \in X$ we get that $\Phi'(a, x) = \Phi_{cd}(\lambda_{ab,ba}(b), x) = \Phi(b, x) = h_2(x)$, $\Phi'(b, x) = \Phi'(\lambda_{ab,ba}(a), x) = \Phi(a, x) = h_1(x)$. □

As in the case of the path-joining property we cannot show that the relation $h_1 \simeq h_2$ is transitive for arbitrary Bishop spaces $\mathcal{F}$ and $\mathcal{G}$. Simplifying the notation used in the definition of PJP we give the following definition.

Definition 7.4.4. *A pair of Bishop spaces $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ has the homotopy-joining property* (HJP)*, in symbols* HJP$(\mathcal{F}, \mathcal{G})$*, if for every $\Phi : [a, b] \times X \rightarrow$*

$Y \in \mathrm{Mor}(\mathcal{I}_{ab} \times \mathcal{F}, \mathcal{G})$ and $\Theta : [b, c] \times X \to Y \in \mathrm{Mor}(\mathcal{I}_{bc} \times \mathcal{F}, \mathcal{G})$, such that $\Phi_b = \Theta_b$, there exists a function $\Psi : [a, c] \times X \to Y \in \mathrm{Mor}(\mathcal{I}_{ac} \times \mathcal{F}, \mathcal{G})$ satisfying

$$\Psi(s, x) = \begin{cases} \Phi(s, x) & , \textit{if } s \in [a, b] \\ \Theta(s, x) & , \textit{if } s \in [b, c]. \end{cases}$$

Note that in the definition of Palmgren of HJP for metric spaces, in [71], pp.239-40, the functions that correspond to the Bishop morphisms Φ and Θ are considered uniformly continuous, and not just continuous as in the classical setting, in order to avoid the uniform continuity theorem. Since the Bishop morphism generally captures uniform continuity, the above definition for Bishop spaces corresponds exactly to the one given by Palmgren for metric spaces.

If $\mathcal{O} = (\{x_0\}, C_u(\{x_0\})$, then $\mathrm{HJP}(\mathcal{O}, \mathcal{G})$ is reduced to $\mathrm{PJP}(\mathcal{G})$, hence by Proposition 7.3.2 HJP cannot hold for arbitrary pairs of Bishop spaces. Working as in the proof of Theorem 7.3.6 we show for Bishop spaces the following special case of a result of Palmgren for metric spaces (Theorem 4, in [71], p.240).

Proposition 7.4.5. *If (X, d) is a compact metric space, (Y, ρ) is a locally compact metric space and $\mathcal{U}(X) = (X, C_u(X))$, $\mathcal{Y} = (Y, \mathrm{Bic}(Y))$, then $\mathrm{HJP}(\mathcal{X}, \mathcal{Y})$.*

Proof. Suppose that $\Phi : [a, b] \times X \to Y \in \mathrm{Mor}(\mathcal{I}_{ab} \times \mathcal{U}(X), \mathcal{Y}) \leftrightarrow \forall_{g \in \mathrm{Bic}(Y)}(g \circ \Phi \in C_u([a, b]) \times C_u(X))$, and $\Theta : [b, c] \times X \to Y \in \mathrm{Mor}(\mathcal{I}_{bc} \times \mathcal{U}(X), \mathcal{Y}) \leftrightarrow \forall_{g \in \mathrm{Bic}(Y)}(g \circ \Phi \in C_u([b, c]) \times C_u(X))$ such that $\Phi_b = \Theta_b$. If $D_{ac} = D_{ab} \cup D_{bc}$ where D_{ab} is dense in $[a, b]$ and D_{bc} is dense in $[b, c]$ such that $D_{ab} \cap D_{bc} = \{b\}$, the function $\Psi : D_{ac} \times X \to Y$, where

$$\Psi(d, x) = \begin{cases} \Phi(d, x) & , \text{if } d \in [a, b] \\ \Theta(d, x) & , \text{if } d \in [b, c], \end{cases}$$

is well-defined. Next we show that Ψ is uniformly continuous i.e.,

$$|d_1 - d_2| + d(x_1, x_2) \leq \omega_\Psi(\epsilon) \to \rho(\Psi(d_1, x_1), \Psi(d_2, x_2)) \leq \epsilon,$$

for every $d_1, d_2 \in D_{ac}, x_1, x_2 \in X$, and for some modulus ω_Ψ. We consider the less trivial case $d_1 \in D_{ab}$ and $d_2 \in D_{bc}$, and we have that

$$\begin{aligned} \rho(\Psi(d_1, x_1), \Psi(d_2, x_2)) &\leq \rho(\Psi(d_1, x_1), \Psi(b, x_1)) + \rho(\Psi(b, x_1), \Psi(b, x_2)) + \rho(\Psi(b, x_2), \Psi(d_2, x_2)) \\ &= \rho(\Phi(d_1, x_1), \Phi(b, x_1)) + \rho(\Phi(b, x_1), \Phi(b, x_2)) + \rho(\Theta(b, x_2), \Theta(d_2, x_2)) \\ &= \rho(\Phi_{x_1}(d_1), \Phi_{x_1}(b)) + \rho(\Phi_b(x_1), \Phi_b(x_2)) + \rho(\Theta_{x_2}(b), \Theta_{x_2}(d_2)). \end{aligned}$$

By Proposition 4.1.12(iii) we get that $\Phi_{x_1} : [a, b] \to Y \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{Y}), \Phi_b : X \to Y \in \mathrm{Mor}(\mathcal{U}(X), \mathcal{Y})$ and $\Theta_{x_2} : [b, c] \to Y \in \mathrm{Mor}(\mathcal{I}_{bc}, \mathcal{Y})$. Since Y is locally compact and $[a, b], [b, c], X$ are compact, by Proposition 3.8.10(iii) we conclude that Φ_{x_1}, Φ_b and Θ_{x_2} are uniformly continuous. Since $|d_1 - b|, |d_2 - b| \leq |d_1 - d_2|$, for the following modulus we get

$$\omega_\Psi(\epsilon) \leq \min\{\omega_{\Phi_b}(\frac{\epsilon}{3}), \omega_{\Phi_{x_1}}(\frac{\epsilon}{3}), \omega_{\Theta_{x_2}}(\frac{\epsilon}{3})\},$$

$$|d_1 - b| \leq \omega_{\Phi_{x_1}}(\frac{\epsilon}{3}) \to \rho(\Phi_{x_1}(d_1), \Phi_{x_1}(b)) \leq \frac{\epsilon}{3},$$

$$d(x_1, x_2) \leq \omega_{\Phi_b}(\frac{\epsilon}{3}) \to \rho(\Phi_b(x_1), \Phi_b(x_2)) \leq \frac{\epsilon}{3},$$

$$|d_2 - b| \leq \omega_{\Theta_{x_2}}(\frac{\epsilon}{3}) \to \rho(\Theta_{x_2}(b), \Theta_{x_2}(d_2)) \leq \frac{\epsilon}{3},$$

$$\rho(\Psi(d_1, x_1), \Psi(d_2, x_2)) \leq \epsilon.$$

By Lemma 4.7.11 there is a uniform continuous extension Ψ^* of Ψ on $[a, c] \times X$. We show that $\Psi^* \in \mathrm{Mor}(\mathcal{I}_{ac} \times \mathcal{U}(X), \mathcal{Y}) \leftrightarrow \forall_{g \in \mathrm{Bic}(Y)}(g \circ \Psi^* \in C_u([a, b]) \times C_u(X)$; let $g \in \mathrm{Bic}(Y)$ and since $\Psi^*([a, c] \times X)$ is a bounded subset of Y, $g \circ \Psi^* = g_{|\Psi^*([a,c] \times X)} \circ \Psi^*$ is uniformly continuous on $[a, c] \times X$ as the composition of uniformly continuous functions. Since $C_u([a, b]) \times C_u(X) = C_u([a, c] \times X)$ we reach our conclusion. The fact that Ψ^* is the required extension of Φ and Θ follows from Proposition 5.7.7; by Proposition 3.8.10(iii) we have that the corresponding morphisms are always uniformly continuous functions and since we have $\Psi^*_{|D_{ab} \times X} = \Phi_{|D_{ab} \times X}$, $\Psi^*_{|D_{bc} \times X} = \Theta_{|D_{bc} \times X}$, and the fact that all related topologies on metric spaces are completely regular, the hypotheses of Proposition 5.7.7 are satisfied. □

The following generalization of the previous result relies on Bridges's forward uniform continuity theorem, hence we rely on the antithesis of Specker's theorem.

Proposition 7.4.6 (AS)**.** *If (X, d) is a compact metric space, (Y, ρ) is a complete metric space and $\mathcal{U}(X) = (X, C_u(X))$, $\mathcal{Y} = (Y, \mathcal{F}(U_0(Y)))$, then* $\mathrm{HJP}(\mathcal{X}, \mathcal{Y})$.

Proof. The proof is similar to the proof of Proposition 7.4.5. We only need the FUCT to establish that the functions $\Phi_{x_1} : [a, b] \to Y \in \mathrm{Mor}(\mathcal{I}_{ab}, \mathcal{Y})$, $\Phi_b : X \to Y \in \mathrm{Mor}(\mathcal{U}(X), \mathcal{Y})$ and $\Theta_{x_2} : [b, c] \to Y \in \mathrm{Mor}(\mathcal{I}_{bc}, \mathcal{Y})$ are uniformly continuous functions, and that the hypothesis of Proposition 5.7.7 that the corresponding morphisms are always uniformly continuous functions is satisfied. □

Proposition 7.4.7. *If $\mathcal{F}, \mathcal{G}$ are Bishop spaces such that* $\mathrm{HJP}(\mathcal{F}, \mathcal{G})$*, then the corresponding relation $h_1 \simeq h_2$ is transitive.*

Proof. Suppose that $h_1, h_2, h_3 : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that $h_1 \simeq h_2$ and $h_2 \simeq h_3$ i.e., there exists $\Phi : [0, 1] \times X \to Y \in \mathrm{Mor}(\mathcal{I} \times \mathcal{F}, \mathcal{G})$ and $\Theta : [1, 2] \times X \to Y \in \mathrm{Mor}(\mathcal{I}_{12} \times \mathcal{F}, \mathcal{G})$ such that $\Phi(0, x) = h_1(x), \Phi(1, x) = h_2(x), \Theta(1, x) = h_2(x)$ and $\Theta(2, x) = h_3(x)$, for every $x \in X$. Since $\mathrm{HJP}(\mathcal{F}, \mathcal{G})$, there exists $\Psi : [0, 2] \times X \to Y \in \mathrm{Mor}(\mathcal{I} \times \mathcal{F}, \mathcal{G})$ such that

$$\Psi(s, x) = \begin{cases} \Phi(s, x) & \text{, if } s \in [0, 1] \\ \Theta(s, x) & \text{, if } s \in [1, 2], \end{cases}$$

hence $\Psi(0, x) = \Phi(0, x) = h_1(x)$ and $\Psi(2, x) = \Theta(2, x) = h_3(x)$, for every $x \in X$. □

One could argue, as Beeson in [2], pp.26-27, that

> The classical results of algebraic topology do not require the general concept of a topological space. If we content ourselves to treat metric spaces, then the standard treatments of the homotopy and homology groups are quite straightforwardly constructive ... It is quite essential to deal with uniformly continuous functions, and not just with continuous functions.

Apart from the argument of Palmgren used as the epigraph of this chapter, it is natural to have a unique notion of continuity in a homotopy theory of metric spaces. In [71] Palmgren formulated PJP for metric spaces using continuity, while he formulated HJP for metric spaces using uniform continuity, in order to avoid the use of the uniform continuity theorem. The Bishop morphism is a common notion of continuity that expresses the homotopic notions such that different, ad hoc definitions are avoided. Since the constructive theory of metric spaces of Bishop includes such definitions, the notion of Bishop morphism remedies this problem, with the cost of having to struggle in order to achieve the same generality of results.

Definition 7.4.8. *If $\mathcal{F} = (X, F)$ is a Bishop space and $x_0 \in X$ we define the sets*

$$\Omega(X, F) := \{\gamma \in \Gamma(F) \mid \gamma \textit{ is a loop}\},$$

$$\Omega(X, x_0, F) := \{\gamma \in \Omega(X, F) \mid \gamma \textit{ is a loop at } x_0\}.$$

If $\gamma, \delta \in \Omega(X, x_0, F)$, we say that γ is loop-homotopic to δ, in symbols $\gamma \simeq \delta$, if

$$\exists_{H \in \mathrm{Mor}(\mathcal{I}^2, \mathcal{F})} \forall_{s,t \in [0,1]} (H(0, t) = \gamma(t) \wedge H(1, t) = \delta(t) \wedge H_s \in \Omega(X, x_0, F)).$$

$\mathcal{F}$ has the loop homotopy-joining property, in symbols LHJP$(\mathcal{F})$*, if* HJP$(\mathcal{I}, \mathcal{F})$ *for the elements of $\Omega(X, x_0, F)$ and for every $x_0 \in X$.*

By Proposition 7.4.5 we get that a locally compact metric space has the LHJP, and with the use of AS by Proposition 7.4.6 a complete metric space has the LHJP. Since the relation of loop homotopy on $\Omega(X, x_0, F)$ is an equivalence relation for a Bishop space (X, F) with the LHJP, we can start the constructive study the homotopy group of (X, F) i.e., the equivalence classes $\pi_1(X, x_0, F)$ with their standard product, as in the classical homotopy theory. As Bishop notes in [7], p.56, "elementary algebraic topology should be constructive", but such a reconstruction has not been carried yet. This chapter includes only the very first steps to such a reconstruction.

Chapter 8

Concluding Comments

8.1 Overview of this Thesis

In this Thesis we elaborated the theory of Bishop spaces (TBS), a constructive function-theoretic approach to topology with points. Based on Bishop's definition of a function space, here called a Bishop space, given in [6], and Bridges's definition of a morphism between function spaces, here called a Bishop morphism, given in [19], we presented a translation of many concepts and results from the classical theory of the rings of continuous functions and of the theory of topological spaces into TBS. Our main tool was the inductively defined concept $\mathcal{F}(F_0)$ of the least Bishop topology including a given set F_0 of real-valued functions on some X. We established connections between the category of Bishop spaces **Bis** with the category of the neighborhood spaces **Nbh**, the category of limit spaces **Lim** and the category of apartness spaces **Apart**

$$\begin{tikzcd}
\mathrm{Lim} \arrow[dr, shift left] & & \mathrm{Nbh} \arrow[dl, shift left] \\
& \mathrm{Bis} \arrow[ul, shift left] \arrow[ur, shift left] \arrow[dl, shift left] & \\
\mathrm{Apart} \arrow[ur, shift left] & &
\end{tikzcd}$$

and, as we explain in section 8.3 it is possible to connect Bishop spaces in a natural way with uniform spaces and Bishop's ecclesiastical spaces defined in [8].
Constructive topology, as standard topology, is both a language and a tool of study of concrete spaces. As we showed in Chapter 4, the language of TBS contains most of the important notions of topologies of functions that correspond to the various notions of topologies of opens. What distinguishes TBS from standard topology, and maybe what distinguishes constructive analysis from classical analysis, is the notion of apartness, providing information that is "lost" in the classical setting. The canonical point-point and set-set apartness relations induced by a Bishop topology and studied in Chapter 5 are central to a direct, constructive understanding of facts within the classical theory of $C(X)$.
In Chapter 6 we showed that TBS can be used as a tool in the study of concrete spaces

seen as Bishop spaces,and we introduced the notion of a 2-compact Bishop topology, a constructive function-theoretic notion of compactness that emerged from the study of the Cantor space as a Bishop space. The function-theoretic character of the concepts of the classical homotopy theory met TBS in Chapter 7 of this Thesis, where a development of the basic homotopy theory of Bishop spaces is initiated.

8.2 Open questions

We outline here some open questions arising from our current development of TBS.

1. Is it possible to generalize TBS in a fruitful way relying on a completely inductive notion of a Bishop space by replacing BS_3 with the closure of a Bishop topology under composition with a notion of an inductively defined continuous real function?
In this way the ad hoc character of Bishop-continuity $\mathrm{Bic}(\mathbb{R})(\phi)$ would be avoided, although such an enterprise requires a redevelopment of Bishop's basic constructive analysis in order to prove non-trivial theorems like the Tietze theorem or the Stone-Weierstrass theorem along these lines.

2. Is a notion of a Stone-Čech compactification of a Bishop space possible within TBS? One has to check if Coquand's constructive notion of an ultrafilter found in [34] could be helpful.

3. Can we establish a connection between TBS and formal topology? This is also related to the question whether a point-free version of TBS exists.

4. Is there a notion of exponential topology that corresponds to the standard compact-open topology?

5. It is immediate to see that

$$\phi \in C_u(2^{\mathbb{N}}) \leftrightarrow \exists_{N>0}\forall_{\alpha,\beta}(\overline{\alpha}(N) = \overline{\beta}(N) \to \phi(\alpha) = \phi(\beta)).$$

Since for every $\phi \in C_u(2^{\mathbb{N}})$ there is some $N > 0$ with the above property, the corresponding choice principle guarantees the existence of a FAN functional

$$\mathrm{FAN} : \mathrm{Mor}(2^{\mathbb{N}}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))) \to \mathbb{N},$$

$$\phi \mapsto \mathrm{FAN}(\phi)$$

$$\forall_{\alpha,\beta}(\overline{\alpha}(\mathrm{FAN}(\phi)) = \overline{\beta}(\mathrm{FAN}(\phi)) \to \phi(\alpha) = \phi(\beta)).$$

The "continuity" of FAN depends on the Bishop topology defined on the exponential space $\mathrm{Mor}(2^{\mathbb{N}}, (\mathbb{N}, \mathbb{F}(\mathbb{N})))$. For example, if we use the p-exponential topology we have that

$$\mathrm{FAN} \in \mathrm{Mor}((\mathrm{Mor}(2^{\mathbb{N}}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))), \bigvee_{n\in\mathbb{N}} \pi_n \to \mathcal{F}(\mathrm{id}_{\mathbb{N}})), (\mathbb{N}, \mathbb{F}(\mathbb{N}))) \leftrightarrow$$

$$\leftrightarrow \mathrm{id}_{\mathbb{N}} \circ \mathrm{FAN} \in \bigvee_{n \in \mathbb{N}} \pi_n \to \mathcal{F}(\mathrm{id}_{\mathbb{N}})$$
$$\leftrightarrow \mathrm{FAN} \in \bigvee_{n \in \mathbb{N}} \pi_n \to \mathcal{F}(\mathrm{id}_{\mathbb{N}})$$
$$\leftrightarrow \mathrm{FAN} \in \bigvee_{\alpha \in 2^{\mathbb{N}}} e_{\alpha, \mathrm{id}_{\mathbb{N}}}$$
$$\leftrightarrow \mathrm{FAN} \in \bigvee_{\alpha \in 2^{\mathbb{N}}} \hat{\alpha},$$

since, for every $\alpha \in 2^{\mathbb{N}}$ and $h \in \mathrm{Mor}(2^{\mathbb{N}}, (\mathbb{N}, \mathbb{F}(\mathbb{N})))$ we have that

$$e_{\alpha, \mathrm{id}_{\mathbb{N}}}(h) = \mathrm{id}_{\mathbb{N}}(h(\alpha)) = h(\alpha) = \hat{\alpha}(h).$$

It is open if we can approximate FAN through the set $\{\hat{\alpha} \mid \alpha \in 2^{\mathbb{N}}\}$ and the properties of a Bishop topology. It is obvious that the continuity of FAN is related to the previous open question.

6. Is there a notion of local-compactness within TBS?

7. Is there a notion of metrizability within TBS? Note that classically a compact space is metrizable if and only if there exists a countable separating family of (continuous) functions and a compact Hausdorff space X is metrizable if and only if there exists a continuous $f : X \times X \to \mathbb{R}$ such that $\zeta(f) = \Delta = \{(x, x) \mid x \in X\}$.

7. A topological space is called *functional*, if its topology is generated by some Bishop topology. A functional space is completely regular. Is the converse also true?

8.3 Future Work

The development of TBS is far from complete. There are are many issues to be addressed that can, hopefully, reinforce our thesis that TBS is a fruitful approach to constructive topology. We outline here some basic directions in the future development of TBS.

1. The systematic treatment of Bishop sets and functions.

a. As we saw there is no general elementhood, no general equality and no general inequality for Bishop sets, but all these notions are given beforehand. If we define though, an equality on X as an equivalence relation on X, then we use the notion of a relation i.e., of a certain subset of $X \times X$ (see [15], p.23). In this case we need to define the notion of subset (and product) first which clearly rests on the notion of equality. To avoid a circle like that it seems that we need to define the equality on X by some formula $\mathrm{Eq}(x, y)$ satisfying the obvious conditions, and then we can show that the set

$$= := \{(x, y) \in X \times X \mid \mathrm{Eq}(x, y)\}$$

is a binary relation on X. For similar reasons we may define an apartness relation on X via a formula $\mathrm{Ap}(x,y)$ satisfying the obvious conditions, and then we can show that

$$\bowtie := \{(x,y) \in X \times X \mid \mathrm{Ap}(x,y)\}$$

is a binary relation on X.

b. The only basic Bishop set is $\mathbb{N}$. One could study extensions of Bishop's theory of sets where other basic sets, like a geometric continuum, are considered.

c. An essential ingredient of the notion of a Bishop set is the *canonical* construction k_x of some abstract $x \in X$. This is a notion vaguely understood within BISH. For example, Bridges and Reeves in [21] consider no construction attached to some $n \in \mathbb{N}$, although one could argue that there is a canonical construction k_n attached to every n. The issue of the canonical construction is more involved in the case of a non-basic set.

2. The systematic study of the subsets of X or of F defined through some F-formula i.e., a formula which depends only on X and F in the informal theory TBS. This study can lead to a formal treatment of the set theory of the subsets of X and of F appearing within TBS, forming a "local" approach to the problem of what a set or a type is constructively.

3. The formal study of the inductive definitions used in TBS.

4. The study of Bishop's ecclesiastical spaces based on his unpublished manuscript [8] and their relation to TBS.

5. The study of Bishop spaces with approximations in the style of [76]. The translation of the limit spaces with approximations into TBS is connected to the use of the appropriate exponential topology for Bishop spaces, and it could offer an alternative form of space in the semantics of computability at higher types.

6. The study of the relation between completely regular Bishop spaces and uniform spaces in order to translate into TBS the existing interaction between uniform spaces and $C(X)$, established, for example, in [44], Chapter 15.

Definition 8.3.1. *If X is an inhabited set, d, e are pseudometrics on X and Δ is a set of pseudometrics on X we define:*

$$\mathbb{D}(X) := \{d : X \to [0, +\infty) \mid d \textit{ is a pseudometric}\},$$

$$U(d, e, \delta, \epsilon) := \forall_{x,y \in X}(d(x,y) \leq \delta \to e(x,y) \leq \epsilon).$$

$$U(\Delta, e) := \forall_{\epsilon > 0} \exists_{\delta > 0} \exists_{d \in \Delta}(U(d, e, \delta, \epsilon)).$$

Definition 8.3.2. *A uniform space is a structure $\mathcal{D} = (X, D)$, where X is an inhabited set and D is a uniformity i.e., an inhabited subset of $\mathbb{D}(X)$ satisfying the following conditions:*

(D_1) $d_1, d_2 \in D \to d_1 \vee d_2 \in D$.
(D_2) $e \in \mathbb{D}(X) \to U(D, e) \to e \in D$.

Clearly, we get that

(D_3) $a \geq 0 \to d \in D \to \overline{a}d \in D$, and

(D_4) $e \leq d \to d \in D \to e \in D$.

Definition 8.3.3. *The least uniformity $\mathcal{D}(D_0)$ including the subbase D_0, where $D_0 \subseteq \mathbb{D}(X)$, is defined as follows:*

$$\frac{d_0 \in D_0}{d_0 \in \mathcal{D}(D_0)} \quad \frac{d_1, d_2 \in \mathcal{D}(D_0)}{d_1 \vee d_2 \in \mathcal{D}(D_0)},$$

$$\frac{(d \in \mathcal{D}(D_0),\ U(d, e, \delta, \epsilon))_{\epsilon > 0}}{e \in \mathcal{D}(D_0)}.$$

If P is any property on $\mathbb{D}(X)$, the following induction principle $\mathrm{Ind}_{\mathcal{D}(D_0)}$ on $\mathcal{D}(D_0)$ is induced:

$$\begin{aligned}
&\forall_{d_0 \in D_0}(P(d_0)) \to \\
&\forall_{d_1, d_2 \in \mathcal{D}(D_0)}(P(d_1) \to P(d_2) \to P(d_1 \vee d_2)) \to \\
&\forall_{e \in \mathcal{D}(D_0)}(\forall_{\epsilon > 0} \exists_{\delta > 0} \exists_{d \in \mathcal{D}(D_0)}(P(d) \ \wedge \ U(d, e, \delta, \epsilon)) \to P(e)) \to \\
&\forall_{d \in \mathcal{D}(D_0)}(P(d)).
\end{aligned}$$

Definition 8.3.4. *If $\Delta_0 \subseteq \mathbb{D}(X)$, then Δ_0 is a base of a uniformity D, if*

$$D = \mathcal{U}(\Delta_0) = \{d \in \mathbb{D}(X) \mid U(\Delta_0, d)\}.$$

In complete analogy to our development of Bishop spaces we define the following apartness relation on X induced by some uniformity D on X.

Definition 8.3.5. *If (X, D) is a uniform space, its canonical apartness relation $\bowtie_D$ induced by D is defined, for every $x, y \in X$ by*

$$x \bowtie_D y :\leftrightarrow \exists_{d \in D}(d(x, y) > 0).$$

In analogy to the case of Bishop spaces it is easy to see that

$$\bowtie_D \ \text{ is tight } \ \leftrightarrow \forall_{x, y \in X}(\forall_{d \in D}(d(x, y) = 0) \to x = y),$$

$$\bowtie_{\mathcal{D}(D_0)} \ \text{ is tight } \ \leftrightarrow \forall_{x, y \in X}(\forall_{d_0 \in D_0}(d_0(x, y) = 0) \to x = y).$$

Definition 8.3.6. *If $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$ are uniform spaces, a function $h : X \to Y$ is a morphism between $\mathcal{D}$ and $\mathcal{E}$, if and only if*

$$\forall_{e \in E}(e \odot h \in D),$$

$$(e \odot h)(x, y) := e(h(x), h(y))$$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "D \ni e \odot h"'] & Y \arrow[d, "e \in E"] \\
& {[0, \infty).}
\end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{D}, \mathcal{E})$ the morphisms between $\mathcal{D}$ and $\mathcal{E}$.

These morphisms are the arrows in the category of uniform spaces **Uni**. The structural similarity between the function-theoretic notions of Bishop morphism and the morphism between uniform spaces is obvious. As in the case of Bishop spaces one can show inductively the corresponding *$\mathcal{F}$-lifting of morphisms*: if $\mathcal{E} = (Y, \mathcal{D}(E_0))$, then

$$h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{E}) \leftrightarrow \forall_{e_0 \in E_0}(e_0 \odot h \in D),$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "D \ni e_0 \odot h"'] & Y \arrow[d, "e_0 \in E_0"] \\ & {[0, \infty).} \end{tikzcd}$$

The generation of a base out of a subbase is much simpler in the case of uniform spaces; if D_0 is a subbase of D, then

$$\mathcal{D}(D_0) = \{\bigvee_{i=1}^{n} d_{0i} \mid n \in \mathbb{N}, d_{0i} \in D_0, 1 \leq i \leq n\}$$

is a base of D. If (X, σ) is a metric space, then $(X, U(\sigma))$ is a *metric* uniform space, and $U(\sigma) = \mathcal{D}(\sigma)$. The uniform space $(\mathbb{R}, U(d_{\mathbb{R}}))$ is the *uniform space of reals*. Note that the standard name for some $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ is *uniformly continuous* function, and by the $\mathcal{F}$-lifting of morphisms we get immediately that $h \in \mathrm{Mor}((X, U(\sigma)), (Y, U(\rho)))$ if and only if h is uniformly continuous. Hence, for a notion of space close to Bishop space, the notion of morphism, which has a function-theoretic nature very similar to that of a Bishop morphism, is traditionally considered as a generalization of uniform continuity!
The connection between uniform spaces and Bishop spaces can be established by the following results.

Proposition 8.3.7. *The mapping τ*

$$(X, F) \overset{\tau}{\mapsto} (X, D(F))$$

$$D(F) := \mathcal{D}(D_0(F))$$

$$D_0(F) := \{d_f \mid f \in F\}$$

$$d_f(x, y) := |f(x) - f(y)|$$

$$\mathrm{Mor}(\mathcal{F}, \mathcal{G}) \ni h \overset{\tau}{\mapsto} h \in \mathrm{Mor}(\tau(\mathcal{F}), \tau(\mathcal{G}))$$

is a covariant functor between **Bis** *and* **Uni**. *Moreover, one can show that*

$$\bowtie_F \ \textit{is tight} \ \leftrightarrow \ \bowtie_{D(F)} \ \textit{is tight.}$$

If $\mathcal{D} = (X, D)$ is a uniform space and $h : X \to \mathbb{R}$, then

$$h \in \mathrm{Mor}(\mathcal{D}, \mathcal{R}) \leftrightarrow d_{\mathbb{R}} \odot h \in D \leftrightarrow d_h \in D.$$

Proposition 8.3.8. *If $\mathcal{D} = (X, D)$ is a uniform space, then*

$$\mathrm{Mor}(\mathcal{D}, \mathcal{R})^* = \mathrm{Mor}(\mathcal{D}, \mathcal{R}) \cap \mathbb{F}_b(X)$$

is a Bishop topology on X.

Proof. If $a \in \mathbb{R}$ we have that $\overline{a} \in \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*$ if and only if $d_{\overline{a}} = \overline{0} \in D$. If $h_1, h_2 \in \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*$, then $d_{h_1+h_2} \leq \overline{2}(d_{h_1} \vee d_{h_2})$, and we use the inductive hypotheses on h_1, h_2 and the property D_4. Next we suppose that $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*$ and if $\phi \in \mathrm{Bic}(\mathbb{R})$ we get that $\phi \circ h \in \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*$ using D_2 and the uniform continuity of ϕ on the bounded subset $h(X)$ of $\mathbb{R}$, which takes the following form:

$$\forall_{x,y \in X}(d_h(x, y) \leq \omega_{\phi,h(X)}(\epsilon) \rightarrow d_{\phi \circ h}(x, y) \leq \epsilon).$$

If $j \in \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*$ such that $U(j, h, \frac{\epsilon}{3})$, then with a standard $\frac{\epsilon}{3}$-argument we have that

$$\forall_{x,y \in X}(d_j(x, y) \leq \frac{\epsilon}{3} \rightarrow d_h(x, y) \leq \epsilon).$$

$\square$

Proposition 8.3.9. *One can show that the mapping ν*

$$(X, F) \stackrel{\nu}{\mapsto} (X, \mathrm{Mor}(\mathcal{D}, \mathcal{R})^*)$$

$$\mathrm{Mor}(\mathcal{D}, \mathcal{E}) \ni h \stackrel{\nu}{\mapsto} h \in \mathrm{Mor}(\nu(\mathcal{D}), \nu(\mathcal{E}))$$

is a covariant functor between **Uni** *and* **Bis**. *Moreover,*

$$\bowtie_{\mathrm{Mor}(\mathcal{D},\mathcal{R})^*} \ \textit{is tight} \ \rightarrow \bowtie_D \ \textit{is tight.}$$

It is clear that there are many connections between Bishop spaces and uniform spaces still to be explored.

7. The study of the relations between the various notion of compactness within TBS found in this Thesis: 2-compact, paircompact and fixed Bishop spaces.

8. The constructive study of the Stone-Weierstrass theorem for the various notions of compactness within TBS. This is related of course, to the necessary further investigation of compactness within TBS.
We note here that one could transfer the property of pair-compactness from the whole topology to the base. More concretely, if $\Phi \subseteq F$, where F is a topology on some X, we define

$$\Phi^{[2]} := \{f \in F \mid \forall_{x,y \in X}(x \bowtie_F y \rightarrow \forall_{\epsilon>0}\exists_{\phi \in \Phi}(|\phi(x) - f(x)| < \epsilon \wedge |\phi(y) - f(y)|)\}.$$

One can show the following theorem of Stone-Weierstrass type for pseudo-compact Bishop spaces (without a given subbase) working as in the proof of Theorem 6.6.8.

Theorem 8.3.10. *Suppose that (X, F) is a pseudo-compact Bishop space and $\Phi \subseteq F$ such that:*
(i) $\mathrm{Const}(X) \subseteq \Phi$*,*
(ii) Φ is closed under addition and multiplication,
(iii) $x \bowtie_F y \to x \bowtie_\Phi y$, for every $x, y \in X$,
(iv) $\Phi^{[2]} \subseteq \Phi$.
Then Φ is a base of F.

Since Bishop's proof of the Stone-Weierstrass theorem for compact metric spaces relies on the too special and too technical notion of separating set, one would like to have a proof of the following corollary of the Stone-Weierstrass theorem through the above version of the Stone-Weierstrass theorem for pseudo-compact Bishop spaces.

Theorem 8.3.11. *Suppose that (X, d) is a compact metric space and $\Phi \subseteq C_u(X)$ such that:*

(i) $\mathrm{Const}(X) \subseteq \Phi$*,*
(ii) Φ is closed under addition and multiplication,
(iii) $U_0(X) = \{d_x \mid x \in X\} \subseteq \Phi$.

Then Φ is a base of $C_u(X)$.

Although it is immediate to show that the condition $x \bowtie_F y \to x \bowtie_\Phi y$, for every $x, y \in X$, is satisfied, it is still open if one can show constructively that $\Phi^{[2]} \subseteq \Phi$, for some Φ satisfying the conditions of the previous theorem.

9. The search for a constructive proof of Bridges's forward uniform continuity theorem. So far FUCT rests on the antithesis of Specker's theorem AS which lies beyond BISH. We expect that it can be proved within BISH in case the codomain is a complete metric space, but even this is still open.

10. The development of the homotopy theory of Bishop spaces. It is expected then to prove within TBS that $\mathcal{R}^n$ is not Bishop-isomorphic to $\mathcal{R}^m$, if $n \neq m$. Such a negative result, but not exclusively within TBS, can be reached as follows: The topology on $\mathbb{R}$ induced by the the Bishop topology $\mathrm{Bic}(\mathbb{R})$ is the standard topology of opens on $\mathbb{R}$ and the same holds for the finite products of $\mathbb{R}$. Since isomorphic Bishop spaces induce homeomorphic topological spaces, if the Bishop spaces $\mathcal{R}^n$ and $\mathcal{R}^m$ were isomorphic, then the induced topological spaces would be homeomorphic, which is impossible. Clearly, an "internal" proof of such a result is related to a deeper study the topological objects of TBS up to homotopy equivalence. Moreover, it is desirable to develop the constructive theory of homology and homotopy groups in the computational framework of TBS. As it is noted by Beeson in [2], p.27, one has to overcome the difficulty pointed by Bishop in [7], p.56, in the definition of the singular cohomology groups.

11. The study of the notion of dimension of a Bishop space. It seems natural to define $\dim(F) = n$, if n is the least cardinality of a subbase F_0 of F.

12. The connection of TBS to constructive measure theory as it is developed in [15] and [89].

13. The interpretation of TBS into Type Theory and the implementation of this interpretation in an appropriate proof assistant. The formalization of TBS and its translation to Type Theory would serve the *extraction of its computational content*, as this is understood in Theoretical Computer Science. There are three main characteristics of TBS which facilitate its translation to Type Theory, where the primitive notion of type replaces the notion of set and corresponds directly to the notion of a data-type in programming languages (see [52]).

a. The use of intuitionistic logic. The intuitionistic character of Type Theory is responsible for its intrinsic computational meaning, which is of interest to computer scientists (see [52], p.58).

b. Its inductive character. One of the advantages of Type Theory is its simple and effective techniques for working with inductively defined structures.

c. Its function-theoretic character. In Type Theory given types A and B we can construct the type $A \to B$ of functions with domain A and codomain B.

The last step of this part of our future work is the implementation of the formalized parts of TBS in an appropriate proof assistant[1]. The translation of TBS to Type Theory is necessary, as Type Theory has already been implemented in existing proof assistants. We believe that this aspect of TBS distinguishes it from most of the other approaches to constructive topology and reveals its interdisciplinary character.

[1]Like Coq (see [33]), or even Minlog (see [65]), since the presence of partiality in Minlog does not seem to affect TBS, only its implementation.

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