The topology of the strong Scott condition

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Abstract: We define a Hausdorff extension of the Scott topology on the domain of ideals of an information system. This topology can be characterized as the topology of the "strong Scott condition", a strong version of the standard Scott condition. To our knowledge this is a new refinement of the Scott topology. We prove its basic properties and we show that in general it properly refines the corresponding Lawson topology, the strong topology associated with the Scott topology and the liminf topology. We also prove some basic results on continuity of functions with respect to this topology and we show how the Alexandrov condition and the Scott condition are related to the introduced notion of a pair of approximation structures for the same set.

1 Information systems and approximation structures

Following the minor modification of the definition of Scott [Scott 1982] due to Larsen and Winskel [Larsen and Winskel 1991], an *information system* is a structure $\mathcal{A} = (A, \mathrm{Con}, \vdash)$, where A is a non-empty set of *tokens* or *data objects*, Con is a non-empty set of finite subsets of A known as *consistent sets* or *formal neighborhoods* and $\vdash$ is a subset of $\mathrm{Con} \times A$, the *entailment* relation, satisfying: (i) if $V \subset U$ and $U \in \mathrm{Con}$, then $V \in \mathrm{Con}$, (ii) if $a \in A$, then $\{a\} \in \mathrm{Con}$, (iii) if $U \vdash a$, then $U \cup \{a\} \in \mathrm{Con}$, (iv) if $U \in \mathrm{Con}$ and $a \in U$, then $U \vdash a$ and (v) if $U, W \in \mathrm{Con}$ and $U \vdash W$, then $W \vdash a \to U \vdash a$, where $U \vdash W$ means that $U \vdash c$, for each $c \in W$.

An *ideal* J of $\mathcal{A}$ is a subset of A satisfying: (i) if U is a finite subset of J, $U \subseteq^{\mathrm{fin}} J$, then $U \in \mathrm{Con}$ and (ii) if $U \subseteq^{\mathrm{fin}} J$ and $U \vdash a$, then $a \in J$. Let $|\mathcal{A}|$ denote the set of ideals of $\mathcal{A}$. If $U \in \mathrm{Con}$, then the deductive closure of U, $\overline{U} = \{a \in A : U \vdash a\}$, is an ideal called *compact*. Let $|\mathcal{A}|_0$ denote the set of all compact ideals J_0 of $\mathcal{A}$. Since $\emptyset \in \mathrm{Con}$, $\perp = \overline{\emptyset}$ is an ideal included in all ideals of $\mathcal{A}$.

A partial ordering $(D, \leq)$ is called a *complete partial ordering* (cpo), if there is a least element $\perp$ and every directed $\mathcal{F} \subseteq D$ has a least upper bound (lub), $\bigvee \mathcal{F}$, in D, where $\mathcal{F} \subseteq D$ is called *directed*, if each finite subset of $\mathcal{F}$ is bounded in $\mathcal{F}$. An element d_0 of a cpo D is called *compact*, if for every directed $\mathcal{F} \subseteq D$

$$d_0 \leq \bigvee \mathcal{F} \to d_0 \leq b \ ,$$

for some $b \in \mathcal{F}$. Let D_0 denote the non-empty set ($\perp \in D_0$) of the compact elements of D. A cpo $(D, \leq)$ is called *algebraic*, if for each $d \in D$ the non-empty set

$$D_0(d) = \{d_0 \in D_0 : d_0 \leq d\}$$

is directed and $d = \bigvee D_0(d)$. In this case we call D_0 the *base* of D. A cpo $(D, \leq)$ is called *consistently complete*, if every bounded (finite) subset of D has a lub in D. A *domain* is a consistently complete, algebraic cpo $(D, \leq, \perp, D_0)$.

It is direct to prove that if $\mathcal{A}$ is an information system, then the structure $(|\mathcal{A}|, \subseteq, \perp, |\mathcal{A}|_0)$ is a domain. Conversely, every domain D can be identified to the domain of ideals of an appropriately defined by D information system $\mathcal{A}_D$ (see [Stoltenberg-Hansen et.al 1994] or [Schwichtenberg and Wainer 2012]).

The fact that each ideal J of an information system $\mathcal{A}$ is the union of all the compact ideals included in it,

$$J = \bigcup_{U \subseteq J} \overline{U},$$

expresses an approximation of the abstract object J by its compact sub-ideals. The concept of approximation can be abstractly defined as follows[1]. If X, P are non-empty sets, a relation $\prec \subseteq P \times X$ is called an *approximation* of X by P and $p \prec x$ is interpreted as "p approximates x". A quasi-ordering $\leq$ (that is a reflexive and a transitive relation) is defined on P by

$$p \leq q \leftrightarrow (\forall x \in X) q \prec x \to p \prec x.$$

If $p \leq q$, then "q is a better approximation than p", since q approximates fewer elements than p. We call an approximation relation *external*, if $P \subseteq \mathcal{P}(X)$ and $p \prec x$ is of the form $x \in p$, in which case $p \leq q \leftrightarrow p \supseteq q$. We call an approximation relation *internal*, if it is not external. The set $P(x) = \{p \in P : p \prec x\}$ contains all P-approximations of x w.r.t. $\prec$.

Suppose that $\prec$ is an approximation of X by P, $\leq$ is the induced quasi-ordering and $\perp \in P$. A structure $\Pi = (X, P, \prec, \leq, \perp)$ is called an *approximation structure for X w.r.t. $\prec$*, if (i) $P(x) = P(y) \to x = y$, (ii) $p, q \prec x \to (\exists r \prec x) p, q \leq r$ and (iii) $(\forall x \in X) \perp \prec x$.

Due to condition (i) each $x \in X$ is uniquely determined by its approximations $P(x)$, while condition (ii) expresses that $P(x)$ is a directed subset of P. The object $\perp$ approximates all elements of X, therefore $\perp \leq p$, for each $p \in P$. We call an approximation structure Π *external*, if its approximation relation is external, and in case $X \in P$ we denote it by $(X, P, \in, \supseteq, X)$. Also, we call an approximation structure Π *strict*, if $(\forall p \in P)(\exists x \in X) p \prec x$, i.e., if P contains no information data irrelevant to X. An approximation structure Π can be

[1] Except the definition of an approximation structure which can be found in [Stoltenberg-Hansen 2001], all the other definitions related to approximations are introduced here.

related to some extra structure of X. For example, if R is a binary relation on X we say that R *is compatible with* Π, if

$$(\forall x, x', p) p \prec x \to R(x, x') \to p \prec x' \ .$$

If $\Pi_1 = (X, P_1, \prec_1, \leq_1, \perp_1)$ and $\Pi_2 = (X, P_2, \prec_2, \leq_2, \perp_2)$ are approximation structures for the same set X satisfying a *compatibility condition* of the form

$$(\forall x)(\forall p_1) p_1 \prec_1 x \to (\exists p_2 \prec_2 x) S(p_2, p_1) \ ,$$

where $S \subseteq P_2 \times P_1$, then we say that Π_1, Π_2 form a *pair of approximation structures* (Π_1, Π_2). The *strong compatibility condition* of a pair (Π_1, Π_2) is the following strong version of its compatibility condition

$$(\forall x)(\forall p_1) p_1 \prec_1 x \to (\exists p_2 \prec_2 x)(\forall q_2) p_2 \leq_2 q_2 \prec_2 x \to S(q_2, p_1) \ .$$

If R is a given binary relation on X and (Π_1, Π_2) is a pair of approximation structures for X such that R is compatible with both Π_1 and Π_2, then we call (Π_1, Π_2) an *Alexandrov pair of approximation structures* for X w.r.t. R. If R is not compatible with Π_1 or Π_2, we call (Π_1, Π_2) a *Scott pair of approximation structures* for X w.r.t. R.

If $(X, \mathcal{T})$ is a T_0 topological space and $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\Pi_1 = (X, \mathcal{T}, \in, \supseteq, X)$ and $\Pi_2 = (X, \mathcal{B} \cup \{X\}, \in, \supseteq, X)$ are two external approximation structures for X. The structure Π_2 is strict, while Π_1 is not. Also, Π_1, Π_2 form a pair because of the trivial compatibility condition $(\forall x)(\forall \mathcal{O}) x \in \mathcal{O} \to (\exists B \ni x) B \subseteq \mathcal{O}$, which can be strengthened to the formula $(\forall x)(\forall \mathcal{O}) x \in \mathcal{O} \to (\exists B \ni x)(\forall B') x \in B' \subseteq B \to B' \subseteq \mathcal{O}$.

If $\mathbb{R}$ is the set of reals and $\mathbb{Q}$ is the set of rationals, then the structure $\Pi_1 = (\mathbb{R}, \{[p,q] : p, q \in \mathbb{Q}, p \leq q\} \cup \{[-\infty, +\infty]\}, \in, \supseteq, [-\infty, +\infty])$ is a strict and external approximation structure for $\mathbb{R}$. If $\mathbb{Q}^{\mathbb{N}}$ is the set of sequences of rationals, then the relation $(q_n) \prec x \leftrightarrow (q_n) \twoheadrightarrow x$ is an approximation of $\mathbb{R}$ by $\mathbb{Q}^{\mathbb{N}}$, where $(q_n) \twoheadrightarrow x$ denotes that x is an accumulation point of (q_n) within the standard topology of $\mathbb{R}$. Also, $(p_n) \leq (q_n) \leftrightarrow L(p_n) \supseteq L(q_n)$, where $L(p_n)$ denotes the set of limit points of the sequence (p_n). It is direct to see that $\Pi_2 = (\mathbb{R}, \mathbb{Q}^{\mathbb{N}}, \twoheadrightarrow, \leq, \mathbb{Q})$ is an internal approximation structure for $\mathbb{R}$, which is not strict since the sequence $q_n = n$, for each n, has no accumulation points. Structures Π_1, Π_2 form a Scott pair of approximations for $\mathbb{R}$ w.r.t. $<$, the standard ordering of reals, since they satisfy the compatibility condition $(\forall x)(\forall [p,q]) x \in [p,q] \to (\exists (q_n) \twoheadrightarrow x)(q_n) \subseteq [p,q]$, but $<$ is not compatible neither with Π_1 nor with Π_2; if $x < x'$ and $x \in [p,q]$ or $(q_n) \twoheadrightarrow x$, then we cannot infer neither $x' \in [p,q]$ nor $(q_n) \twoheadrightarrow x'$.

The *Scott topology* $\mathcal{S}$ on the set of ideals $|\mathcal{A}|$ of an information system $\mathcal{A}$ has as basis all sets

$$O_U = \{J \in |\mathcal{A}| : U \subseteq^{\text{fin}} J\} \ .$$

Since $(|\mathcal{A}|, \mathcal{S})$ is a T_0-space, following our first example of an approximation structure we get that $\Pi_{\mathcal{S}} = (|\mathcal{A}|, \mathcal{S}, \in, \supseteq, |\mathcal{A}|)$ is an external approximation structure for $|\mathcal{A}|$. It is also standard (see [Stoltenberg-Hansen et.al 1994] or [Schwichtenberg and Wainer 2012]) that $\mathcal{O} \in \mathcal{S}$ if and only if $\mathcal{O}$ satisfies the *Alexandrov condition*

$$(\forall J, J')(\forall \mathcal{O}) J \in \mathcal{O} \to J \subseteq J' \to J' \in \mathcal{O}$$

and the *Scott condition*

$$(\forall J)(\forall \mathcal{O}) J \in \mathcal{O} \to (\exists \overline{U} \subseteq J) \overline{U} \in \mathcal{O} \ .$$

Within our terminology the Alexandrov condition expresses that the ordering $\subseteq$ of ideals is compatible with the approximation structure $\Pi_{\mathcal{S}}$ and is justified through a "forcing" interpretation of $J \in \mathcal{O}$ given by Smyth [Smyth 1988]. According to it, an open set $\mathcal{O}$ is thought of as an "observable property" and elementhood $J \in \mathcal{O}$ is interpreted as J "forces" property $\mathcal{O}$ to hold, or J contains enough information for property $\mathcal{O}$ to hold. To interpret the Scott condition in our language we need to connect $\Pi_{\mathcal{S}}$ to another approximation structure for $|\mathcal{A}|$.

If $\mathcal{A}$ is an information system then $\Pi_0 = (|\mathcal{A}|, |\mathcal{A}|_0, \subseteq, \dashv, \bot)$ is a strict and internal approximation structure[2] for $|\mathcal{A}|$, where $\overline{U} \leq \overline{V} \leftrightarrow \overline{U} \subseteq \overline{V} \leftrightarrow U \dashv V$. It is trivial that the ordering of ideals $\subseteq$ is compatible with Π_0 too. The Scott condition is a compatibility condition between the structures $\Pi_{\mathcal{S}}$ and Π_0, which form an Alexandrov pair of approximation structures $(\Pi_{\mathcal{S}}, \Pi_0)$ for $|\mathcal{A}|$ w.r.t. $\subseteq$.

The Alexandrov condition together with the Scott condition imply the following strong version of the Scott condition

$$(\forall J)(\forall \mathcal{O}) J \in \mathcal{O} \to (\exists \overline{U} \subseteq J)(\forall \overline{V})[\overline{U} \subseteq \overline{V} \subseteq J \to \overline{V} \in \mathcal{O}] \ ,$$

which we call the *strong Scott condition*. Clearly, the strong Scott condition is the strong compatibility condition of the pair $(\Pi_{\mathcal{S}}, \Pi_0)$ corresponding to its compatibility condition given by the Scott condition. Although the strong Scott condition trivially implies the Scott condition, the strong Scott condition does not imply the Alexandrov condition and Scott pairs of approximation structures for $|\mathcal{A}|$ w.r.t. $\subseteq$ can be found.

[2] In the same way, an approximation structure $(D, D_0, \preccurlyeq, \leq, \bot)$ corresponds to any domain $(D, \preccurlyeq, \bot, D_0)$. One can also show (see [Stoltenberg-Hansen 2001]) that if $(X, P, \prec, \leq, \bot)$ is an approximation structure for X such that $(P, \leq)$ is a *cusl*, i.e., a partial ordering with a least element such that if p, q are bounded elements of P, then $p \vee q \in P$, an information system $\mathcal{A}_P$ is naturally defined by it. Moreover, the order ideals of $(P, \leq)$, i.e., its directed and downward closed subsets, are the ideals of $\mathcal{A}_P$.

2 Basic properties of the topology of the strong Scott condition

In order to be self-contained we include the following standard definitions[3]. A *directed set* $(\Lambda, \leq)$ is a quasi-ordering, every pair of which has an upper bound. If X is a non-empty set, a *net* in X is a function $x : \Lambda \to X$, where $(\Lambda, \leq)$ is a directed set, denoted by $(x_\lambda)_{\lambda\in\Lambda}$, or simply by (x_λ). If $(M, \preceq)$ $(\Lambda, \leq)$ are directed sets, a function $\varphi : M \to \Lambda$ is called a *directed map*, if $(\forall\lambda \in \Lambda)(\exists\mu(\lambda) \in M)(\forall\mu \succeq \mu(\lambda))\varphi(\mu) \geq \lambda$. If $(x_\lambda)_{\lambda\in\Lambda}$, $(y_\mu)_{\mu\in M}$ are nets in X, then $(y_\mu)_{\mu\in M}$ is a *subnet* of $(x_\lambda)_{\lambda\in\Lambda}$ if there is a directed map $\varphi : M \to \Lambda$ such that $y_\mu = x_{\varphi(\mu)}$, for each μ. We denote a subnet $(y_\mu)_{\mu\in M}$ of $(x_\lambda)_{\lambda\in\Lambda}$ by $(x_{\varphi(\mu)})_{\mu\in M}$, or simply by (x_{λ_μ}). A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is a *constant net*, if there is some $x \in X$ such that $x_\lambda = x$, for each $\lambda \in \Lambda$. We denote such a constant net by (x). If $(X, \mathcal{T})$ is a topological space and $(x_\lambda)_\lambda \subseteq X$, the net $(x_\lambda)_\lambda$ *converges* to $x \in X$, $(x_\lambda)_\lambda \stackrel{\mathcal{T}}{\to} x$ or simply $(x_\lambda)_\lambda \to x$, if $(\forall\mathcal{O} \ni x)(\exists\lambda_0)(\forall\lambda \geq \lambda_0)x_\lambda \in \mathcal{O}$. Of course, it suffices for $\mathcal{O}$ to be a basic open set. In the case of the Scott topology the above convergence takes the form

$$(I_\lambda)_\lambda \stackrel{\mathcal{S}}{\to} J \leftrightarrow (\forall U \subseteq^{\text{fin}} J)(\exists\lambda_0)(\forall\lambda \geq \lambda_0)\overline{U} \subseteq I_\lambda \ .$$

Consider Λ to be a fixed index set for the ideals $|\mathcal{A}|$ of an information system $\mathcal{A}$, and Λ_0 the subset of Λ which indexes $|\mathcal{A}|_0$. That is, $|\mathcal{A}| = \{J_\lambda : \lambda \in \Lambda\}$ and $|\mathcal{A}|_0 = \{J_\lambda : \lambda \in \Lambda_0\}$. The sets Λ, Λ_0 are ordered according to the condition $\lambda \leq \lambda^{'} \leftrightarrow J_\lambda \subseteq J_{\lambda^{'}}$. If J is a fixed ideal of $\mathcal{A}$, the sets

$$\Lambda(J) = \{\lambda \in \Lambda : J_\lambda \subseteq J\} \ \text{ and } \ \Lambda_0(J) = \{\lambda \in \Lambda_0 : J_\lambda \subseteq J\}$$

are directed subsets of Λ and we call the corresponding nets in $|\mathcal{A}|$

$$(J_\lambda)_{\lambda\in\Lambda(J)} \ \text{ and } \ (J_\lambda)_{\lambda\in\Lambda_0(J)} \ ,$$

the *ideal net* and the *compact net* of J, respectively. Obviously, $J = \bigcup_{\lambda\in\Lambda_0(J)} J_\lambda$ and each $(\Lambda(J), \leq)$ or $(\Lambda_0(J), \leq)$ has the same least element $\lambda_\perp$, for which $J_{\lambda_\perp} = \perp$.

Note that if $J \in |\mathcal{A}|_0$, then $\Lambda_0(J)$ is not necessarily a finite set. To show this we construct an information system $\mathcal{A}$, a compact ideal of which has infinite compact sub-ideals. Let $A = \{a_0, a_1, a_2, ...\}$, Con be the set $\mathcal{P}^{\text{fin}}(A)$ of all finite subsets of A and the entailment relation be defined as follows: $a_0 \vdash a_n$, for each n, and each finite subset of $A \setminus \{a_0\}$ entails only itself. It is easy to see that $(A, \text{Con}, \vdash)$ is an information system such that $\overline{\{a_0\}} = A$, while its sub-ideals $\overline{\{a_{i_1}, ..., a_{i_n}\}} = \{a_{i_1}, ..., a_{i_n}\}$, where each $a_{i_k} \neq a_0$, are infinite. Moreover, the set $A \setminus \{a_0\}$ is a non-compact ideal of $\mathcal{A}$ which is included to a compact one $(A \setminus \{a_0\} \subseteq A = \overline{\{a_0\}})$.

[3] For all the topological notions not defined here see [Dugungji 1990].

Also, if $J \notin |\mathcal{A}|_0$, then $\Lambda_0(J)$ is necessarily infinite. If $J \notin |\mathcal{A}|_0$ and

$$\Lambda_0(J) = \{\lambda_1, ..., \lambda_n\},$$

for some n, i.e.,

$$J = \bigcup_{i=1}^{n} J_{\lambda_k} = \overline{\{a_1^{(1)}, ..., a_{m_1}^{(1)}\}} \cup ... \cup \overline{\{a_1^{(n)}, ..., a_{m_n}^{(n)}\}},$$

then the ideal

$$J_0 = \overline{\{a_1^{(1)}, ..., a_{m_1}^{(1)}, ..., a_1^{(n)}, ..., a_{m_n}^{(n)}\}}$$

belongs to $|\mathcal{A}|_0$, since the finite subset $\{a_1^{(1)}, ..., a_{m_1}^{(1)}, ..., a_1^{(n)}, ..., a_{m_n}^{(n)}\}$ of J belongs to Con. By the deductive closure of J, $J_0 \subseteq J$. Also, if $b \in J$, then $b \in J_{\lambda_k}$, for some k, and since $J_{\lambda_k} \subseteq J_0$, $b \in J_0$ too. The conclusion $J = J_0$ though, contradicts our hypothesis $J \notin |\mathcal{A}|_0$.

It is also trivial to show that if $J \in |\mathcal{A}|$, then

$$(J_\lambda)_{\lambda \in \Lambda(J)} \stackrel{\mathcal{S}}{\to} J \ \text{ and } \ (J_\lambda)_{\lambda \in \Lambda_0(J)} \stackrel{\mathcal{S}}{\to} J \ .$$

There are nets though, converging to J w.r.t. the Scott topology that are not contained in the compact net of J. If $I \supseteq J$, then the constant net (I) converges to J w.r.t. $\mathcal{S}$ and it is not contained in the compact net of J (actually $\overline{\{I\}}$, the $\mathcal{S}$-closure of $\{I\}$, is equal to $\downarrow I = \{J' \in |\mathcal{A}| : J' \subseteq I\}$). Thus, a net $(I_\mu)_\mu$ may converge to some $J \in |\mathcal{A}|$ w.r.t. to $\mathcal{S}$, though keeping "information distance" from J. In other words, $\mathcal{S}$-convergence permits information irrelevance, exactly because of the Alexandrov condition.

We define a Hausdorff extension σ of the Scott topology on $|\mathcal{A}|$ (and in a similar way on an abstract domain) as the induced topology of a convergence space in net form (see [Heinze et.al 2001]). If X is a non-empty set, the pair $(X, (\mathcal{N}_x)_{x \in X})$ is called a *convergence space in net form*, if for each x, $\mathcal{N}_x$ is a family of nets in X such that (i) if (x) is a constant net in X, then $(x) \in \mathcal{N}_x$ and (ii) if (x_λ) is a net in X such that: $(x_\lambda) \in \mathcal{N}_x$, and (x_{λ_μ}) is a subnet of (x_λ), then $(x_{\lambda_\mu}) \in \mathcal{N}_x$. An $\mathcal{O} \subseteq X$ is called $\mathcal{N}$-open if

$$(\forall x \in \mathcal{O})(\forall (x_\lambda)_{\lambda \in \Lambda} \in \mathcal{N}_x)(\exists \lambda_0 \in \Lambda)(\forall \lambda \geq \lambda_0) x_\lambda \in \mathcal{O}$$

i.e., if for each point x of $\mathcal{O}$ all nets in $\mathcal{N}_x$ lie eventually in $\mathcal{O}$. The collection of all $\mathcal{N}$-open sets satisfies the properties of a topology $\mathcal{N}$, which is called the *induced topology* of a convergence space in net form. Clearly, if $(X, \mathcal{T})$ is a topological space and (x_λ) is a net in X such that $(x_\lambda) \in \mathcal{N}_x \leftrightarrow (x_\lambda) \stackrel{\mathcal{T}}{\to} x$, then $(X, (\mathcal{N}_x)_{x \in X})$ is a convergence space in net form and its induced topology is identical to $\mathcal{T}$. Also, if $\mathcal{N}_x$ contains only the constant nets, then $\mathcal{N} = \mathcal{P}(X)$.

To each ideal J of $|\mathcal{A}|$ we assign a family of nets σ_J containing all subnets of the compact net $(J_\lambda)_{\lambda \in \Lambda_0(J)}$ of J and all the constant nets with value J. Then

$$(|\mathcal{A}|, (\sigma_J)_{J \in |\mathcal{A}|})$$

is a convergence space in net form, since a subnet of a constant net (J) is also a constant net with value J, and a subnet of a subnet of the compact net of J is again a subnet of the compact net of J. A σ-open set $\mathcal{O}$ is defined by the condition

$$(\forall J \in \mathcal{O})(\forall (I_\lambda)_{\lambda \in \Lambda} \in \sigma_J)(\exists \lambda_0 \in \Lambda)(\forall \lambda \geq \lambda_0) I_\lambda \in \mathcal{O},$$

which is reduced to

$$(\forall J \in \mathcal{O})(\exists \lambda_0 \in \Lambda(J))(\forall \lambda \in \Lambda_0(J)) \lambda \geq \lambda_0 \to J_\lambda \in \mathcal{O},$$

since if $J \in \mathcal{O}$, all constant nets (J) are automatically included in $\mathcal{O}$, and if the whole compact net of J satisfies the initial openness condition, then each subnet of it satisfies the openness condition too. Since the condition which characterizes a σ-open set is the strong Scott condition we call the induced topology σ of the convergence space in net form $(|\mathcal{A}|, (\sigma_J)_{J \in |\mathcal{A}|})$ the *topology of the strong Scott condition.*

Theorem 1 *The family of sets of ideals* $(J^*_{\lambda_0})_{\lambda_0 \in \Lambda_0(J), J \in |\mathcal{A}|}$, *where*

$$J^*_{\lambda_0} = \{J_\lambda | \lambda \in \Lambda_0(J) \wedge \lambda \geq \lambda_0\} \cup \{J\},$$

is a basis for the topology σ *on* $|\mathcal{A}|$. *Moreover,* $(|\mathcal{A}|, \sigma)$ *is a totally disconnected Hausdorff space which is not zero-dimensional.*

Proof. Each set $J^*_{\lambda_0}$ is σ-open since it satisfies automatically the strong Scott condition. Also, if $J \in \mathcal{O} \in \sigma$, then the strong Scott condition determines some $\lambda_0 \in \Lambda_0(J)$ such that $J \in J^*_{\lambda_0} \subseteq \mathcal{O}$.

To show that σ is Hausdorff we suppose that J_1, J_2 are ideals such that there is $a \in J_1 \setminus J_2$ (the case $a \in J_2 \setminus J_1$ is treated in a similar way). Hence, $\{a\} = J_{1_{\lambda_0}}$, for some $\lambda_0 \in \Lambda_0(J_1)$. Obviously, $J_1 \in J^*_{1_{\lambda_0}}$ and $J_2 \in J^*_{2_{\lambda_\perp}}$, while $J^*_{1_{\lambda_0}} \cap J^*_{2_{\lambda_\perp}} = \emptyset$, since otherwise a would belong to J_2.

In order to show that $(|\mathcal{A}|, \sigma)$ is totally disconnected we remark first that if $\overline{U} \in |\mathcal{A}|_0$, then there is some $\lambda_0 \in \Lambda_0(\overline{U})$ such that $\overline{U} = \overline{U}_{\lambda_0}$. Since $\{\overline{U}\} = \overline{U}^*_{\lambda_0}$, each $\{\overline{U}\}$ belongs to the above basis. Suppose that B is a set of ideals such that $B \supsetneq \{J\}$, for some $J \in |\mathcal{A}|$. If B contains a compact ideal $\overline{U}$, then B is not connected, since it can be written as the union $B = \{\overline{U}\} \cup B \setminus \{\overline{U}\}$, that is, the union of disjoint open sets in B. If B contains no compact ideal, then $J^*_{\lambda_\perp} \cap B = \{J\}$, therefore $\{J\}$ is open in B. Again B is not connected since it is written as $B = \{J\} \cup B \setminus \{J\}$. Thus, the connected components of $(|\mathcal{A}|, \sigma)$ are the singletons.

To prove that the space $(|\mathcal{A}|, \sigma)$ is not zero-dimensional, it suffices to show that in general we cannot find a clopen set K such that $J \in K \subseteq J^*_{\lambda_\perp}$. If there was such an open K contained in $J^*_{\lambda_\perp}$, it would contain a basic open set $J^*_{\lambda_0}$, for some $\lambda_0 \in \Lambda_0(J)$. The closure of such a set though, may contain a non-compact ideal $I \subsetneq J$, so K cannot be included to $J^*_{\lambda_\perp}$. Let $A = \{a_0, a_1, a_2, ...\}$ and $(A, \mathcal{P}^{\mathrm{fin}}(A), \vdash)$ be the information system we defined previously in order to show that a compact ideal may contain an infinite number of compact sub-ideals. If B is an infinite subset of $A \setminus \{a_0\}$, then B is a non-compact ideal. Suppose that there is a σ-clopen set K such that $B \in K \subseteq B^*_{\lambda_\perp}$. Since K is σ-open there is some $\lambda_0 \in \Lambda_0(B)$ such that $B^*_{\lambda_0} \subseteq K$. If $a_l \in B \setminus B_{\lambda_0}$, then $B \setminus \{a_l\}$ is a non-compact ideal other than B. We show that $B \setminus \{a_l\} \in \overline{K} = K$, where $\overline{K}$ is the σ-closure of K, therefore K cannot be contained in $B^*_{\lambda_\perp}$. It suffices to define a net in K that σ-converges to $B \setminus \{a_l\}$. If $(M, \preceq)$ is a directed set and $(I_\mu)_{\mu \in M}$ is a net of $|\mathcal{A}|$, then its σ-convergence is characterized by

$$(I_\mu)_\mu \xrightarrow{\sigma} J \;\leftrightarrow\; (\forall \lambda_0 \in \Lambda(J))(\exists \mu_0)(\forall \mu \succeq \mu_0) I_\mu \in J^*_{\lambda_0}.$$

The set $\Gamma = \{U : B_{\lambda_0} \subseteq U \subseteq^{\mathrm{fin}} B \setminus \{a_l\}\}$ is a net in K such that $B \setminus \{a_l\} = \bigcup \Gamma$, since if $a_k \in B \setminus \{a_l\}$, then $B_{\lambda_0} \cup \{a_k\} \in \Gamma$. If $V \subseteq^{\mathrm{fin}} B \setminus \{a_l\}$, then $V \cup B_{\lambda_0} \in \Gamma$ and all the ideals of Γ that contain $V \cup B_{\lambda_0}$ also contain V, therefore $\Gamma \xrightarrow{\sigma} B \setminus \{a_l\}$.

While the last argument of the previous proof shows that the σ-open set $J^*_{\lambda_\perp}$ is not generally σ-closed, the corresponding net ideal $\downarrow J$ is σ-clopen. If $I \in \downarrow J$, then each I_λ, where $\lambda \in \Lambda(I)$, is included in I, therefore the whole compact net of I is included in $\downarrow J$. The set $\downarrow J$ is also σ-closed because its complement is σ-open. If $I \notin \downarrow J$, then there is some $a \in I \setminus J$, therefore $I^*_{\mu_0}$ is included in the complement of $\downarrow J$, for some $\mu_0 \in \Lambda_0(I)$ such that $\overline{\{a\}} = I_{\mu_0}$.

Note that if $J \notin |\mathcal{A}|_0$, no $J^*_{\lambda_0}$ is contained in $\{J\}$ and consequently $\{J\} \notin \sigma$. Hence, σ is not generally equal to the power set $\mathcal{P}(|\mathcal{A}|)$. Since $\{\overline{U}\} \in \sigma$, the whole $|\mathcal{A}|_0$ is σ-open, as the union of σ-open sets. Also, $\{\overline{U}\}$ is clopen, since $(|\mathcal{A}|, \sigma)$ is T_1. Moreover, the set $\lambda_0(J) = \{J_\lambda : \lambda \in \Lambda_0(J) \wedge \lambda \geq \lambda_0\}$, where $\lambda_0 \in \Lambda_0(J)$, is σ-open, since $\lambda_0(J) = (|\mathcal{A}| \setminus \{J\}) \cap J^*_{\lambda_0}$. For each $J_\lambda \in \lambda_0(J)$, the basic open sets containing J_λ and included in $\lambda_0(J)$ are the singletons $\{J_\lambda\}$.

The fact that $\{\overline{U}\} \in \sigma$ shows how "thin" a σ-open set can be, a fact in complete contrast to the S-open sets that satisfy the Alexandrov condition. Thus, a σ-open set satisfies the strong Scott condition but not necessarily the Alexandrov condition. Within the terminology of the previous section the approximation structures $\Pi_\sigma = (|\mathcal{A}|, \sigma, \in, \supseteq, |\mathcal{A}|)$ and $\Pi_0 = (|\mathcal{A}|, |\mathcal{A}|_0, \subseteq, \dashv, \perp)$ for $|\mathcal{A}|$ form a Scott pair of approximation structures (Π_σ, Π_0) for $|\mathcal{A}|$ w.r.t. $\subseteq$.

The special feature of a σ-basic open set $J^*_{\lambda_0}$ is that it represents a degree of approximation of J with no irrelevancies, since each element of $J^*_{\lambda_0}$ "contains information" which is already in J. In contrast to the Scott topology, the topology of the strong Scott condition permits no irrelevancies. An interpretation of $J \in \mathcal{O}$ which justifies the strong Scott condition but not the Alexandrov condition is that of $\mathcal{O}$ as a "degree of nearness" or as a "degree of approximation" of J. The Scott condition is just a compatibility condition; if $\mathcal{O}$ is near J, then it must be near to some internal approximation J_{λ_0} of J. The strong Scott condition is then automatically satisfied; for each $\lambda \geq \lambda_0$, if $\mathcal{O}$ is near J_{λ_0}, then it is already near J_λ. Regarding the Alexandrov condition though, if $\mathcal{O}$ is near J it cannot be near to all ideals $I \supset J$, without distorting the notion of nearness.

Next we compare the topology of the strong Scott condition with some well known refinements of the Scott topology. The basic open sets for the *Lawson topology* $\mathcal{L}$ on $|\mathcal{A}|$ (see [Gierz et.al 2003]) are of the form $G \setminus \uparrow\{J_1, ..., J_n\}$, where G is a Scott-open set, $\uparrow J = \{I \in |\mathcal{A}| : I \supseteq J\}$ and $\uparrow\{J_1, ..., J_n\} = \bigcup_{i=1}^n \uparrow J_i$. Also, the basic open sets for the *strong topology* Σ on $|\mathcal{A}|$ associated with $\mathcal{S}$ (see [Gierz et.al 2003, p. 428]) are of the form $G \cap F$, where G, F are $\mathcal{S}$-open and $\mathcal{S}$-closed sets, respectively.

Theorem 2 *If $\mathcal{S}, \mathcal{L}, \Sigma, \sigma$ are the Scott, Lawson, strong and the topology of the strong Scott condition on $|\mathcal{A}|$, respectively, then*

$$\mathcal{S} \subsetneq \mathcal{L} \subsetneq \Sigma \subsetneq \sigma,$$

i.e., all these are in general different refinements of the Scott topology included in the topology of the strong Scott condition. Moreover, the basic $\mathcal{S}$-open sets $\mathcal{O}_U$ are σ-clopen.

Proof. First we show that $\mathcal{S} \subseteq \sigma$. It suffices to show that $\mathcal{O}_U \in \sigma$, for each $\mathcal{S}$-basic open set $\mathcal{O}_U$. If $J \in \mathcal{O}_U$, then there is some $\lambda_0 \in \Lambda(J)$ such that $\overline{U} = J_{\lambda_0}$. Since for each $\lambda \geq \lambda_0$ we have that $J_\lambda \supseteq \overline{U}$, we conclude that $J_\lambda \in \mathcal{O}_U$. If $U = \{a_1, ..., a_n\}$, for some n, then $\mathcal{O}_U$ is also σ-closed. Since $\mathcal{O}_U = \bigcap_{i=1}^n \mathcal{O}_{\{a_i\}}$, it suffices to show that any set of the form $\mathcal{O}_{\{a\}}$ is σ-closed. But if $I \in |\mathcal{A}| \setminus \mathcal{O}_{\{a\}}$, then all the elements I_λ of its compact net are in $|\mathcal{A}| \setminus \mathcal{O}_{\{a\}}$.

It is trivial by the definition of $\mathcal{L}$ that $\mathcal{S} \subseteq \mathcal{L}$ and $\mathcal{S} \subseteq \Sigma$. If $J \neq \perp$, then the set $|\mathcal{A}| \setminus \uparrow J = \{I \in |\mathcal{A}| : I \not\supseteq J\}$ is $\mathcal{L}$-open but not $\mathcal{S}$-open ($\perp \not\supseteq J$ and the Alexandrov condition would imply that $J \not\supseteq J$ too), therefore $\mathcal{S} \subsetneq \mathcal{L}$.

To prove that $\mathcal{L} \subseteq \Sigma$ it suffices to show that for each J in an $\mathcal{L}$-basic open set $G \setminus \uparrow\{J_1, ..., J_n\}$ there is a Σ-basic open set $G' \cap F$ such that $J \in G' \cap F \subseteq G \setminus \uparrow\{J_1, ..., J_n\}$. Since $J \in G$, there is some $U \subseteq^{\text{fin}} J$ such that $\mathcal{O}_U \subseteq G$. Since the set $\downarrow J$ is $\mathcal{S}$-closed, the intersection $\mathcal{O}_U \cap \downarrow J$ contains J and is included in $G \setminus \uparrow\{J_1, ..., J_n\}$; if I is any ideal for which $U \subseteq I \subseteq J$, then I cannot be contained in $\uparrow\{J_1, ..., J_n\}$ because if $I \supseteq J_l$, for some $l \in \{1, ..., n\}$, then $J \supseteq J_l$ too.

Next we show that in general $\mathcal{L} \subsetneq \Sigma$ by proving the existence of a Σ-open set which is not $\mathcal{L}$-open. We remark that a singleton $\{\overline{U}\} = \mathcal{O}_U \cap \downarrow\overline{U}$ is Σ-open (and of course, not $\mathcal{S}$-open). If we consider the information system $(A, \mathcal{P}^{\text{fin}}(A), \vdash)$, where $A = \{a_0, a_1, a_2, ...\}$, then by the previous remark the set $\{\{a_1\}\}$ is a Σ-open set. In order to be $\mathcal{L}$-open too it must be written in the form $G\setminus\uparrow\{J_1, ..., J_n\}$, for some $G \in \mathcal{S}$ and $J_1, ..., J_n \in |\mathcal{A}|$. Since $\{a_1\} \in G \setminus \uparrow\{J_1, ..., J_n\}$ each $\{a_1, a_k\}$, for $k > 1$, belongs to G and must contain some J_l, where $l \in \{1, ..., n\}$, in order that $\{\{a_1\}\} = G \setminus \uparrow\{J_1, ..., J_n\}$. But if $\{a_1, a_k\} \supseteq J_l$, either $J_l = \{a_k\}$ or $J_l = \{a_1, a_k\}$. Since there are only n ideals $J_1, ..., J_n$, this condition cannot hold for each $k > 1$, therefore $\{\{a_1\}\}$ is not $\mathcal{L}$-open.

In order to show that $\Sigma \subseteq \sigma$ we prove that each Σ-basic open set is also σ-open. If $J \in G \cap F$, where G, F are $\mathcal{S}$-open and $\mathcal{S}$-closed, respectively, then $I^*_{\lambda_0} \subseteq G$, for some $\lambda_0 \in \Lambda(J)$. Since a $\mathcal{S}$-closed set is a lower set, $\downarrow J \subseteq F$, therefore $I^*_{\lambda_0} \subseteq F$ too.

To show that $\Sigma \subsetneq \sigma$ we prove that there are σ-basic open sets which cannot be Σ-open. Suppose J is a non-compact ideal of an information system $\mathcal{A}$ such that if $J \in G$, where $G \in \mathcal{S}$, then for each $J_{\lambda_0} \in G$ there exists a non-compact ideal I such that $J_{\lambda_0} \subsetneq I \subsetneq J$. For example, if we consider again the information system $(A, \mathcal{P}^{\text{fin}}(A), \vdash)$, where $A = \{a_0, a_1, a_2, ...\}$, the ideal $A \setminus \{a_0\}$ is such an ideal; if some $\{a_{i_1}, ..., a_{i_n}\}$ is contained in an $\mathcal{S}$-open set G containing $A \setminus \{a_0\}$, then $\{a_{i_1}, ..., a_{i_n}\} \subsetneq A \setminus \{a_0, a_l\} \subsetneq A \setminus \{a_0\}$, for each $a_l \notin \{a_{i_1}, ..., a_{i_n}\}$. In this case no set of the form $G \cap L$, where $G \in \mathcal{S}$ and L is a lower set, that contains J is included to $J^*_{\lambda_0}$, i.e., $J^*_{\lambda_0}$ is not Σ-open. Since $J_{\lambda_0} \in G$ and G is $\mathcal{S}$-open, $I \in G$, while since $J \in L$ and L is a lower set, $I \in L$, i.e., $I \in G \cap L$, therefore $G \cap L$ cannot be included to a set like $J^*_{\lambda_0}$ in which the only non-compact ideal is J.

Since $\mathcal{L} \subset \Sigma \subset \sigma$ the approximation structures $\Pi_{\mathcal{L}} = (|\mathcal{A}|, \mathcal{L}, \in, \supseteq, |\mathcal{A}|)$ and $\Pi_{\Sigma} = (|\mathcal{A}|, \Sigma, \in, \supseteq, |\mathcal{A}|)$ for $|\mathcal{A}|$ form with Π_0 the Scott pairs of approximation structures $(\Pi_{\mathcal{L}}, \Pi_0)$ and (Π_{Σ}, Π_0) for $|\mathcal{A}|$ w.r.t. $\subseteq$, respectively. With respect, though, to the "degree of approximation"-interpretation of $J \in \mathcal{O}$ the Σ-open sets are closer to the σ-open sets while the $\mathcal{L}$-open sets are between the $\mathcal{S}$-open and the σ-open sets. If $U \subseteq^{\text{fin}} J$, then J belongs to the Σ-basic open set $\mathcal{O}_U \cap \downarrow J = \{I \in |\mathcal{A}| : U \subseteq I \subseteq J\}$ which expresses a degree of nearness of J with no irrelevancies. The difference with the σ-approximation is that non-compact sub-ideals of J other than J can be contained in $\mathcal{O}_U \cap \downarrow J$. On the other hand, a Lawson open set $\mathcal{O}_U$ that contains an ideal J may contain information irrelevant to J (for example, consider any $\mathcal{L}$-open set containing $\{a_1\}$ in the case of the previous proof of $\mathcal{L} \subsetneq \Sigma$). But still the $\mathcal{L}$-basic open set $G\setminus\uparrow\{J_1, ..., J_n\}$ contains less irrelevance w.r.t. an element J of it than G itself.

A refinement of the Lawson topology on a cpo $(D, \leq)$ can be defined in the following way (see [Gierz et.al 2003]). If $(d_j)_j$ is a net in D and $e \in D$, then e is an *eventual lower bound* of $(d_j)_j$, if $(\exists j_0)(\forall j \succeq j_0)e \leq d_j$. The element d of D is the *liminf* of a net $(d_j)_j$, $d = \underline{\lim}(d_j)$, if (i) d is the supremum of all eventual lower bounds of $(d_j)_j$, and (ii) there is some directed set $\mathcal{F}$ of eventual lower bounds of $(d_j)_j$ such that $d = \bigvee \mathcal{F}$. Obviously, if (d) is a constant net, then $d = \underline{\lim}(d)$, and if $(d_j)_j$ is a monotone net (i.e., $(\forall j, j')j \preceq j' \to d_j \leq d_{j'}$), then $\underline{\lim}(d_j) = \sup(d_j)$. In [Gierz et.al 2003, p. 231], it is proved that d is the liminf of all subnets of $(d_j)_j$ if and only if $d = \underline{\lim}(d_j)$ and $(\forall c)[(\forall j)(\exists j' \succeq j)c \leq d_{j'}] \to c \leq d$. In case one of these two equivalent conditions holds for a pair $(d, (d_j)_j)$ we say that the pair $(d, (d_j)_j)$ satisfies the *ξ-condition*. A set $\mathcal{O} \subseteq D$ is called ξ-open or open w.r.t. the *liminf topology* ξ on D, if for each $d \in \mathcal{O}$ and for each pair $(d, (d_j)_j)$ satisfying the ξ-condition, the net $(d_j)_j$ is eventually in $\mathcal{O}$.

Theorem 3 *If ξ is the liminf topology on the set of ideals $|\mathcal{A}|$ of an information system $\mathcal{A}$, then the topology of the strong Scott condition σ is generally a proper refinement of ξ, in symbols, $\xi \subsetneq \sigma$.*

Proof. If $J \in |\mathcal{A}|$, then the pair $(J, (J_\lambda)_{\lambda \in \Lambda_0(J)})$ satisfies the ξ-condition, since the net $\Lambda_0(J)$ is monotone and J is the supremum of $(J_\lambda)_{\lambda \in \Lambda_0(J)}$ and of all its subnets. If $\mathcal{O}$ is a ξ-open set, the net $(J_\lambda)_{\lambda \in \Lambda_0(J)}$ is eventually in $\mathcal{O}$, therefore $\mathcal{O}$ is also σ-open.

To show that $\xi \subsetneq \sigma$ we consider the information system $\mathcal{A} = (A, \mathcal{P}^{\text{fin}}(A), \vdash)$, where A is an uncountable infinite set and each finite subset of A entails itself. Obviously, each subset of A is an ideal. The set Γ of all countable subsets of A is a directed set, which we also write as a net $(\Gamma_\mu)_\mu$, where $\mu \preceq \mu' \leftrightarrow \Gamma_\mu \subseteq \Gamma_{\mu'}$ w.r.t. a fixed indexing of the ideals of $\mathcal{A}$. Since $(\Gamma_\mu)_\mu$ is a monotone net, we get that $\underline{\lim}(\Gamma_\mu) = \sup(\Gamma_\mu) = A$, since an upper bound of Γ includes each singleton in A, therefore it is equal to A. Also, the condition $(\forall B)[(\forall \mu)(\exists \mu' \succeq \mu)B \subseteq \Gamma_{\mu'}] \to B \subseteq A$ is trivially satisfied. Thus, the pair $(A, (\Gamma_\mu)_\mu)$ satisfies the ξ-condition, $A \in A^*_{\lambda_\perp}$, but the net $(\Gamma_\mu)_\mu$ cannot be eventually in the σ-open set $A^*_{\lambda_\perp}$. The set $A^*_{\lambda_\perp}$ contains A and all the finite subsets of A, while for each μ there is some $\mu' \succeq \mu$ such that $\Gamma_{\mu'}$ is an infinite, proper subset of A. Hence, $A^*_{\lambda_\perp}$ is not a ξ-open set.

We define the *extended compact net* $(J_\lambda)_{\lambda \in \Lambda_0^+(J)}$ of J as the set of all ideals J_λ, where $\Lambda_0^+(J) = \Lambda_0(J) \cup \{\lambda_J\}$ and $J_{\lambda_J} = J$, w.r.t. the fixed indexing of the ideals of $\mathcal{A}$. Obviously, $(\Lambda_0^+(J), \leq)$ is a directed set with a minimum, $\lambda_\perp$, and a maximum, λ_J, element.

Theorem 4 *Suppose (J) is a constant net with value J, (J_{λ_ν}) is a subnet of the compact net of J, and $(I_\mu)_{\mu \in M}$ is a net of $|\mathcal{A}|$ converging to J with respect to σ. Then (J) and (J_{λ_ν}) converge to J with respect to σ, $(I_\mu)_{\mu \in M}$ is eventually a constant net with value J, if $J \in |\mathcal{A}|_0$, and $(I_\mu)_{\mu \in M}$ is eventually a subnet of the extended compact net of J, if $J \notin |\mathcal{A}|_0$.*

Proof. The constant net (J) trivially σ-converges to J. If (J_{λ_ν}) is a subnet of the compact net of J, φ is the corresponding directed map and $\lambda_0 \in \Lambda_0(J)$ is fixed, then there is some ν_0 such that, for each $\nu \geq \nu_0$, $\varphi(\nu) \geq \lambda_0$ and $J_{\lambda_\nu} = J_{\varphi(\nu)} \supseteq J_{\lambda_0}$. Hence, the subnet (J_{λ_ν}) σ-converges to J.

If $J \in |\mathcal{A}|_0$ and $(I_\mu)_{\mu \in M} \overset{\sigma}{\to} J$, applying the definition of σ-convergence for the σ-open set $\{J\}$ of J we get that $(I_\mu)_{\mu \in M}$ is eventually a constant net with value J.

If $J \notin |\mathcal{A}|_0$ and $(I_\mu)_{\mu \in M} \overset{\sigma}{\to} J$, applying the definition of σ-convergence for the σ-open set $J^*_{\lambda_\perp} = \{J_\lambda | \lambda \in \Lambda_0^+(J)\}$, we get some μ_0, such that for each $\mu \geq \mu_0$, $I_\mu \in J^*_{\lambda_\perp}$, i.e., all these elements of $(I_\mu)_{\mu \in M}$ belong to the extended compact net of J. If $M_0 = \{\mu \in M : \mu \succeq \mu_0\}$, then $(M_0, \preceq)$ is also a directed set. The condition of σ-convergence says exactly that the map $\varphi : (M_0, \preceq) \to (\Lambda_0^+(J), \leq)$ defined by $\varphi(\mu) = \lambda_0$, if $I_\mu = J_{\lambda_0}$, while $\varphi(\mu) = \lambda_J$, if $I_\mu = J$, is a directed map. By its definition though, $I_\mu = J_{\varphi(\mu)}$, for each $\mu \succeq \mu_0$, therefore $(I_\mu)_{\mu \in M_0}$ is a subnet of the extended compact net of J.

3 σ-Continuity

If $\mathcal{A}, \mathcal{B}$ are information systems and $f : |\mathcal{A}| \to |\mathcal{B}|$, then (see [Stoltenberg-Hansen et.al 1994] or [Schwichtenberg and Wainer 2012]) f is continuous with respect to $\mathcal{S}$, if and only if f is monotone and commutes with direct unions, i.e., for every directed set $\mathcal{F} \subseteq |\mathcal{A}|$

$$f(\bigcup_{J \in \mathcal{F}} J) = \bigcup_{J \in \mathcal{F}} f(J) \ ,$$

if and only if f is monotone and satisfies the *principle of finite support*

$$b \in f(J) \to (\exists U \subseteq^{\text{fin}} J) b \in f(\overline{U}) \ .$$

If f is monotone and satisfies the principle of finite support, then f satisfies what we call the *strong principle of finite support*

$$b \in f(J) \to (\exists U \subseteq^{\text{fin}} J)(\forall \overline{V})[\overline{U} \subseteq \overline{V} \subseteq J \to b \in f(\overline{V})] \ .$$

As the σ-open sets satisfy the strong Scott condition but not in general the Alexandrov condition, the σ-continuous functions satisfy the strong principle of finite support without being, in general, monotone. If $f : (|\mathcal{A}|, \sigma_{\mathcal{A}}) \to (|\mathcal{B}|, \sigma_{\mathcal{B}})$, where $\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}$ are the topologies of the strong Scott condition on $|\mathcal{A}|, |\mathcal{B}|$ respectively, then the condition of continuity of f through nets is $(I_\mu)_\mu \overset{\sigma_{\mathcal{A}}}{\to} J \to (f(I_\mu))_\mu \overset{\sigma_{\mathcal{B}}}{\to} f(J)$, where $(I_\mu)_\mu$ is any net in $|\mathcal{A}|$. If we consider the compact net of J as a net σ-converging to J, then we get $(J_\lambda)_\lambda \overset{\sigma_{\mathcal{A}}}{\to} J \to (f(I_\lambda))_\lambda \overset{\sigma_{\mathcal{B}}}{\to} f(J)$. If $b \in f(J)$, then $\overline{\{b\}} = f(J)_{\mu_0}$, for some $\mu_0 \in \Lambda_0(f(J))$. From the convergence $(f(J_\lambda))_\lambda \to f(J)$ though, $(\exists \lambda_0 \in \Lambda_0(J))(\forall \lambda \geq \lambda_0) f(J_\lambda) \in f(J)^*_{\mu_0}$. Since, for each such λ, $f(J_\lambda) \supseteq \overline{\{b\}}$, this means that f satisfies the strong principle of finite support

$$b \in f(J) \to (\exists \lambda_0 \in \Lambda_0(J))(\forall \lambda \geq \lambda_0) b \in f(J_\lambda) \ ,$$

without being necessarily monotone.

Next we give a simple example of a σ-continuous function which is not $\mathcal{S}$-continuous. If $\mathbb{B}^\perp = \{\mathsf{tt}, \mathsf{ff}, \perp\}$ is the flat boolean set ordered by the relation $\leq = \{(\mathsf{tt}, \mathsf{tt}), (\mathsf{ff}, \mathsf{ff}), (\perp, \mathsf{tt}), (\perp, \mathsf{ff})\}$, then $\mathcal{B} = ((\mathbb{B}^\perp, \leq)$ is a domain with $\mathcal{B}_0 = \mathbb{B}^\perp$. The $\mathcal{S}$-topology on $\mathbb{B}^\perp$ is $\{\emptyset, \{\mathsf{tt}\}, \{\mathsf{ff}\}, \{\mathsf{tt}, \mathsf{ff}\}, \mathbb{B}^\perp\}$, i.e., $\mathcal{S}$ is T_0 but not T_1. Any T_1 topology on $\mathbb{B}^\perp$ though, like the σ-topology, turns $\mathbb{B}^\perp$ to a discrete space (the singletons $\{\mathsf{tt}\}, \{\mathsf{ff}\}$ are closed and so is their union $\{\mathsf{tt}, \mathsf{ff}\}$, therefore $\{\perp\}$ is σ-open). Hence, any function $f : \mathbb{B}^\perp \to \mathbb{B}^\perp$ is σ-continuous, while it cannot be $\mathcal{S}$-continuous, if it is not monotone (consider for example, the function $f = \{(\mathsf{ff}, \mathsf{ff}), (\mathsf{tt}, \perp), (\perp, \mathsf{tt})\}$).

Theorem 5 *Suppose $f : (|\mathcal{A}|, \mathcal{T}_{\mathcal{A}}) \to (|\mathcal{B}|, \mathcal{T}_{\mathcal{B}})$ is a continuous function, where $\mathcal{T}$ is any refinement of the Scott topology such that each $\mathcal{T}$-open set satisfies the Scott condition. Then f is continuous with respect to the corresponding Scott topologies on $|\mathcal{A}|, |\mathcal{B}|$ if and only if f is monotone.*

Proof. The monotonicity of an $\mathcal{S}$-continuous function is directly implied by the characterization of $\mathcal{S}$-continuity. For the converse it suffices to show that if f is monotone and $\mathcal{T}$-continuous, then $f^{-1}(\mathcal{O}_U)$ is $\mathcal{S}_{\mathcal{A}}$-open, for each basic open set $\mathcal{O}_U$ in $\mathcal{S}_{\mathcal{B}}$. Suppose that $\mathcal{O}_U$ is such a fixed set. Since $\mathcal{S} \subset \mathcal{T}$, $\mathcal{O}_U$ is $\mathcal{T}$-open too. By the $\mathcal{T}$-continuity of f, $f^{-1}(\mathcal{O}_U)$ is $\mathcal{T}_{\mathcal{A}}$-open, therefore $f^{-1}(\mathcal{O}_U)$ satisfies the Scott condition. It remains to show that $f^{-1}(\mathcal{O}_U)$ satisfies the Alexandrov condition. If $J \in f^{-1}(\mathcal{O}_U)$, that is, if $f(J) \supseteq U$, and if $I \supseteq J$, then by the monotonicity of f we get that $f(I) \supseteq f(J) \supseteq U$. Hence, $I \in f^{-1}(\mathcal{O}_U)$ too.

If $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ and $C_\sigma(|\mathcal{A}|, |\mathcal{B}|)$ are the sets of $\mathcal{S}$ and σ-continuous functions between the sets of ideals of two information systems $\mathcal{A}$ and $\mathcal{B}$ respectively, the next proposition shows that $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|) \subset C_\sigma(|\mathcal{A}|, |\mathcal{B}|)$.

Theorem 6 *If $f : (|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}) \to (|\mathcal{B}|, \mathcal{S}_{\mathcal{B}})$ is continuous, then f is also continuous with respect to the corresponding σ-topologies on $|\mathcal{A}|, |\mathcal{B}|$.*

Proof. Suppose that $f : (|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}) \to (|\mathcal{B}|, \mathcal{S}_{\mathcal{B}})$ is continuous. In order to show that f is also σ-continuous it suffices to show that $f^{-1}(J^*_{\lambda_0}) \in \sigma_{\mathcal{A}}$, for each $\sigma_{\mathcal{B}}$-basic open set $J^*_{\lambda_0}$.

First we consider the case where the $\sigma_{\mathcal{B}}$-basic open set is of the form $\{\overline{U}\}$, where $U = \{b_1, ..., b_n\} \in \mathrm{Con}_{\mathcal{B}}$. If $I \in f^{-1}(\{\overline{U}\})$, i.e., $f(I) = \overline{U}$, then by the principle of finite support for f there are sets $V_1, ..., V_n \subseteq^{\text{fin}} I$ such that $b_1 \in f(\overline{V_1}), ..., b_n \in f(\overline{V_n})$. But then the following implication holds

$$\overline{\bigcup_{i=1}^{n} V_i} = I_{\mu_0} \to \overline{U} \subseteq f(I_{\mu_0}) \ ,$$

where $\mu_0 \in \Lambda_0(I)$ is the index corresponding to the ideal $\overline{\bigcup_{i=1}^n V_i}$ of I. To show this implication we use the monotonicity of f; $\overline{V_i} \subseteq I_{\mu_0} \to f(\overline{V_i}) \subseteq f(I_{\mu_0})$, for each $i \in \{1, ..., n\}$. Consequently, $\bigcup_{i=1}^n f(\overline{V_i}) \subseteq f(I_{\mu_0})$, and the inclusions $U \subseteq \bigcup_{i=1}^n f(\overline{V_i}) \subseteq f(I_{\mu_0})$ imply that $\overline{U} \subseteq f(I_{\mu_0})$. For each $\mu \geq \mu_0$ the monotonicity of f implies that $\overline{U} \subseteq f(I_{\mu_0}) \subseteq f(I_\mu) \subseteq f(I) = \overline{U}$, which means that $f(I_\mu) = \overline{U}$, for each such μ. This proves though, that the compact net of I is eventually in $f^{-1}(\{\overline{U}\})$, i.e., $f^{-1}(\{\overline{U}\})$ is a σ-open set.

An immediate consequence of the previous case is that if $J \in |\mathcal{B}|$, the inverse image $f^{-1}(\lambda_0(J))$ of the $\sigma_{\mathcal{B}}$-open set $\lambda_0(J)$ is a $\sigma_{\mathcal{A}}$-open set. Since $\lambda_0(J) = \{J_\lambda : \lambda \in \Lambda_0(J) \wedge \lambda \geq \lambda_0\} = \bigcup_{\lambda \geq \lambda_0} \{J_\lambda\} \in \sigma_{\mathcal{B}}$,

$$f^{-1}(\lambda_0(J)) = f^{-1}(\bigcup_{\lambda \geq \lambda_0} \{J_\lambda\}) = \bigcup_{\lambda \geq \lambda_0} f^{-1}(\{J_\lambda\}) \ ,$$

i.e., $f^{-1}(\lambda_0(J))$ is σ-open as the union of the σ-opens sets $f^{-1}(\{J_\lambda\})$.

Next we consider the general case, i.e., the inverse image of a $\sigma_{\mathcal{B}}$-basic open set $J^*_{\lambda_0}$. If $J \in |\mathcal{B}|_0$, then $J^*_{\lambda_0} = \lambda_0(J)$, therefore this case is reduced to the previous one.

If $J \notin |\mathcal{B}|_0$ and since f commutes with direct unions,

$$I \in f^{-1}(J^*_{\lambda_0}) \leftrightarrow f(I) = f(\bigcup_{\mu \in \Lambda_0(I)} I_\mu) = \bigcup_{\mu \in \Lambda_0(I)} f(I_\mu) = J_\lambda \ ,$$

for some $\lambda \geq \lambda_0$ and $\lambda \in \Lambda_0^+(J)$. If $\lambda \neq \lambda_J$, then $f(I)$ is a compact ideal and by the first case the compact net of I is eventually in $f^{-1}(\{J_\lambda\})$, therefore in $f^{-1}(J^*_{\lambda_0})$.

If $\lambda = \lambda_J$, i.e., if $f(I) = J$, then we distinguish again two cases. If there is some $\mu_0 \in \Lambda_0(I)$ such that $f(I_{\mu_0}) = J$, then by the monotonicity of f we get that $f(I_\mu) = J$, for each $\mu \geq \mu_0$, i.e., the compact net of I is eventually in $f^{-1}(J^*_{\lambda_0})$.

Suppose that $f(I_\mu) \subsetneq J$, for each $\mu \in \Lambda_0(I)$. Since $J_{\lambda_0} = \overline{U} = \overline{\{b_1, ..., b_n\}}$, for some $\{b_1, ..., b_n\} \in \mathrm{Con}_{\mathcal{B}}$, and $f(I) = J$, then $\{b_1, ..., b_n\} \subseteq f(I)$. By the principle of finite support for f we consider again a compact sub-ideal $\overline{\bigcup_{i=1}^n V_i} = I_{\mu_0}$ of I, for some $\mu_0 \in \Lambda_0(I)$, for which $f(I_{\mu_0}) \supseteq J_{\lambda_0}$. Hence, by monotonicity of f

$$I_\mu \supseteq I_{\mu_0} \to f(I_\mu) \supseteq f(I_{\mu_0}) \supseteq J_{\lambda_0},$$

for each $\mu \geq \mu_0$. Therefore, the compact net of I is eventually in $f^{-1}(J^*_{\lambda_0})$.

One can show that the set $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ acquires a structure of a domain either by identifying it with the set of ideals of a new information system $\mathcal{A} \to \mathcal{B}$ (see [Stoltenberg-Hansen et.al 1994] or [Schwichtenberg and Wainer 2012]), or by showing that the partial ordering $(C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|), \leq)$, where $f \leq g \leftrightarrow (\forall J) f(J) \subseteq g(J)$, satisfies an appropriate criterion of Eršov (see [Berger 1993]). The Scott topology on the domain $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ is proved to be the topology of the pointwise convergence.

Theorem 7 *If $(f_\mu)_\mu$ is a net in $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ and $f \in C_\sigma(|\mathcal{A}|, |\mathcal{B}|)$, such that $(f_\mu) \overset{p}{\to} f$, i.e., $(f_\mu(J)) \overset{\sigma}{\to} f(J)$, for each $J \in |\mathcal{A}|$, then $f \in C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$.*

Proof. If $(f_\mu) \overset{p}{\to} f$, then in order to show that $f \in C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ it suffices to show by Theorem 5 that f is monotone. Suppose that $J_1, J_2 \in |\mathcal{A}|$ and $J_1 \subseteq J_2$. We show $b \in f(J_1) = I_1 \to b \in f(J_2) = I_2$. Since $b \in I_1$, $\overline{\{b\}} = I_{1_{\lambda_0}}$, for some $\lambda_0 \in \Lambda_0(I_1)$. Since

$$(f_\mu(J_1)) \overset{\sigma}{\to} I_1,$$

there is some μ_0 such that, for each $\mu \geq \mu_0$, $f_\mu(J_1) \in I^*_{1_{\lambda_0}}$, i.e., $f_\mu(J_1) \supseteq \overline{\{b\}}$. By monotonicity of each function f_μ we get that $b \in f_\mu(J_1) \to b \in f_\mu(J_2)$. Since

$$(f_\mu(J_2)) \overset{\sigma}{\to} I_2,$$

there is some μ_0' such that, for each $\mu \geq \mu_0'$, $f_\mu(J_2) \in I^*_{2_{\nu_\perp}}$, where $\nu_\perp$ is the minimum of $\Lambda_0(I_2)$. Hence, for each such μ, $I_2 \supseteq f_\mu(J_2) \supseteq \perp$. If we consider $\mu_0'' \geq \mu_0, \mu_0'$, then, for each $\mu \geq \mu_0''$, we get $I_2 \supseteq f_\mu(J_2) \ni b$, i.e., $b \in I_2 = f(J_2)$.

These are some first basic results on σ-continuity. In subsequent work we intend to study the extendability of a function $f_0 : |\mathcal{A}|_0 \to |\mathcal{B}|$, where $\mathcal{A}, \mathcal{B}$ are given information systems, to a σ-continuous function $\widehat{f_0} : |\mathcal{A}| \to |\mathcal{B}|$. Note that the set of compact ideals $|\mathcal{A}|_0$ is dense w.r.t. all topologies mentioned here. In the case of the Scott topology the criterion for the existence of such an extension is the monotonicity of f_0, while in the case of the other finer topologies mentioned here more general notions of "monotonicity" need to be studied. Also, the characterization of the ordering structure $(C_\sigma(|\mathcal{A}|, |\mathcal{B}|), \leq)$, where $\leq$ is the pointwise ordering of functions in $C_\sigma(|\mathcal{A}|, |\mathcal{B}|)$, and its relation to the corresponding space of continuous functions w.r.t. the other refinements of the Scott topology, are some of the many related open questions.

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