A direct constructive proof of a Stone-Weierstrass theorem for metric spaces

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Abstract. We present a constructive proof of a Stone-Weierstrass theorem for totally bounded metric spaces (**SWtbms**) which implies Bishop's Stone-Weierstrass theorem for compact metric spaces (**BSWcms**) found in [3]. Our proof has a clear computational content, in contrast to Bishop's highly technical proof of **BSWcms** and his hard to motivate concept of a (Bishop-)separating set of uniformly continuous functions. All corollaries of **BSWcms** in [3] are proved directly by **SWtbms**. We work within Bishop's informal system of constructive mathematics BISH.

1 Introduction

According to the classical Stone-Weierstrass theorem (**SWchts**), if X is a compact Hausdorff topological space and Φ is a separating subalgebra of the continuous real-valued functions $C(X)$ on X that contains a non-zero constant function, then the uniform closure of Φ is $C(X)$ (see [10], p.282). Recall that Φ is *separating*, if $\forall_{x,y\in X}(x \neq y \to \exists_{f\in\Phi}(f(x) \neq f(y)))$.
There are some constructive versions of this theorem depending on the notion of space under constructive study. In [1] Banaschewski and Mulvey considered a compact, completely regular locale instead of a compact Hausdorff topological space. In [7] Coquand gave a simple, constructive localic proof of it, replacing the ring structure of $C(X)$ by its lattice structure, while in [8] he studied the usual formulation of the Stone-Weierstrass theorem in this point-free topological framework.
For reasons which we discuss in [15], Bishop did not pursue a constructive reconstruction of abstract topology. Although he introduced two constructive alternatives to the notion of topological space, the notion of *neighborhood space*, see [11], [13], and the notion of *function space*, or *Bishop space*, see [4], [12] and [15]-[17], he never elaborated them, restricting his studies to metric spaces. In [2] and [3] Bishop formulated a theorem of Stone-Weierstrass type for *compact* metric spaces (i.e., complete and totally bounded metric spaces) using the notion of a Bishop-separating set of uniformly continuous functions[1]. Since Bishop's results, as well as ours, hold for totally bounded metric spaces, we formulate all related concepts and results for them without restricting to compact metric

[1] Bishop's original term is that of a separating set, which we avoid here in the presence of the standard classical notion of a separating subset of $C(X)$.

spaces. Recall that a metric space (X, d) is *totally bounded*, if for every $\epsilon > 0$ there exists a finite ϵ-approximation of X, and a set A is *finite* if there exists a one-one mapping of $\{1, \ldots, n\}$ onto A, for some $n > 0$ (see [6], p.29). Hence, a totally bounded metric space is always inhabited.

Throughout this paper (X, d) is a totally bounded metric space, $C_u(X)$ denotes the uniformly continuous real-valued functions on X, and $\varPhi \subseteq C_u(X)$.

Definition 1. *$\varPhi$ is called Bishop-separating, if there is $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

(Bsep_1) *For all $\epsilon > 0$ and $x_0, y_0 \in X$, if $d(x_0, y_0) \geq \epsilon$, there exists $g_{\epsilon,x_0,y_0} \in \varPhi$ such that*

$$\forall_{z \in X}(d_{x_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0,y_0}(z)| \leq \epsilon) \text{ and}$$
$$\forall_{z \in X}(d_{y_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0,y_0}(z) - 1| \leq \epsilon).$$

(Bsep_2) *For all $\epsilon > 0$ and $x_0 \in X$ there exists $g_{\epsilon,x_0} \in \varPhi$ such that*

$$\forall_{z \in X}(d_{x_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0}(z) - 1| \leq \epsilon).$$

Note that in Definition 1 g_{ϵ,x_0,y_0} and g_{ϵ,x_0} are just notations that do not involve the use of some choice principle. Recall also that for every $x_0 \in X$ the map $d_{x_0} : X \to \mathbb{R}$, defined by $x \mapsto d(x_0, x)$, is in $C_u(X)$ with modulus of continuity $\omega_{d_{x_0}} = \mathrm{id}_{\mathbb{R}^+}$. If $a \in \mathbb{R}$, we denote by $\overline{a}$ the constant map on X with value a, and their set by $\mathrm{Const}(X)$. We define

$$U_0(X) := \{d_{x_0} \mid x_0 \in X\}.$$
$$U_0^*(X) := U_0(X) \cup \{\overline{1}\}.$$

We call $\varPhi$ *positively separating*, if $\forall_{x,y \in X}(x \bowtie_d y \to \exists_{g \in \varPhi}(g(x) \bowtie_{\mathbb{R}} g(y)))$, where $x \bowtie_d y \leftrightarrow d(x, y) > 0$, for every $x, y \in X$, and $a \bowtie_{\mathbb{R}} b \leftrightarrow |a - b| > 0 \leftrightarrow a < b \vee b < a$, for every $a, b \in \mathbb{R}$, are the canonical point-point apartness relations on X and $\mathbb{R}$, respectively. The notion of a positively separating set $\varPhi$ is the positive version of the classical notion of a separating subset of $C(X)$ for metric spaces. Clearly, $U_0(X)$ is positively separating.

Remark 1. If $\varPhi$ is Bishop-separating, then $\varPhi$ is positively separating.

Proof. By the Archimedean property of $\mathbb{R}$ (see [5], p.57), if $x_0, y_0 \in X$ such that $d(x_0, y_0) > 0$, there is some natural number $N > 2$ such that $d(x_0, y_0) > \frac{1}{N}$. By Bsep_1 we have that $|g_{\frac{1}{N},x_0,y_0}(x_0)| \leq \frac{1}{N}$ and $|g_{\frac{1}{N},x_0,y_0}(y_0) - 1| \leq \frac{1}{N}$, for some $g_{\frac{1}{N},x_0,y_0} \in \varPhi$, therefore $g_{\frac{1}{N},x_0,y_0}(x_0) \bowtie_{\mathbb{R}} g_{\frac{1}{N},x_0,y_0}(y_0)$.

In [3], p.106, Bishop formulated a theorem of Stone-Weierstrass type for compact metric spaces using the notion of a Bishop-separating set as the property that corresponds to the classical notion of a separating set in the formulation of **SWchts**. Bishop's proof of this theorem is non-trivial and does not involve the completeness property of X. Following Bishop, we denote by $\mathcal{A}(\varPhi)$ the least subset of $C_u(X)$ that includes $\varPhi$ and it is closed with respect to addition, multiplication, and multiplication by reals. Bishop didn't define $\mathcal{A}(\varPhi)$ inductively but explicitly, as the set of compositions of strict real polynomials in several variables with vectors of elements of $\varPhi$ (see [3], p.105).

Theorem 1 (Bishop's Stone-Weierstrass theorem for totally bounded metric spaces (BSWtbms)). *If Φ is Bishop-separating, then $\mathcal{A}(\Phi)$ is dense in $C_u(X)$.*

The condition of Φ being Bishop-separating implies that the constant map $\overline{1}$ is in the closure of $\mathcal{A}(\Phi)$ (see [3], p.106). Bishop's formulation of **BSWtbms** represents a non-trivial technical achievement, namely to find a formulation of a theorem of Stone-Weierstrass type in the constructive theory of metric spaces that resembles the formulation of the classical **SWchts**. As Coquand and Spitters mention in [9], pp.339-340, constructive proofs using a concrete presentation of topological notions (e.g., the Gelfand spectrum as a lattice) are "more direct than proofs via an encoding of topology in metric spaces, as is common in Bishop's constructive mathematics".
In the next two sections we present a Stone-Weierstrass theorem for metric spaces which avoids the concept of a Bishop-separating set, it has an informative and direct proof, it implies **BSWtbms**, and it proves directly all corollaries of **BSWtbms**.

2 A Stone-Weierstrass theorem for totally bounded metric spaces

Definition 2. *If $f, g \in C_u(X)$ and $\epsilon > 0$, then $f \wedge g := \min\{f, g\}, f \vee g := \max\{f, g\}$, and the uniform closure $\mathcal{U}(\Phi)$ of Φ is defined by*

$$U(g, f, \epsilon) :\leftrightarrow \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) :\leftrightarrow \forall_{\epsilon > 0} \exists_{g \in \Phi}(U(g, f, \epsilon)),$$

$$\mathcal{U}(\Phi) := \{f \in C_u(X) \mid U(\Phi, f)\}.$$

The following remark is immediate to show.

Remark 2. If Φ is closed under addition, multiplication by reals and multiplication, then $\mathcal{U}(\Phi)$ is closed under addition, multiplication by reals and multiplication. Moreover, if Φ is closed under $|.|$, then $\mathcal{U}(\Phi)$ is closed under $|.|$.

The next two lemmas are proved in [3], pp.105-6 (Lemma 5.11 and Lemma 5.12).

Lemma 1. *If $\mathrm{Const}(X) \subseteq \Phi$, and Φ is closed under addition and multiplication (or if Φ is closed under addition, multiplication by reals, and multiplication), then $\mathcal{U}(\Phi)$ is closed under $|.|, \vee$ and $\wedge$.*

Lemma 2. *If Φ is closed under addition, multiplication by reals, and multiplication, and $f \in \mathcal{U}(\Phi)$ such that $\forall_{x \in X}(|f(x)| \geq c)$, for some $c > 0$, then $\frac{1}{f} \in \mathcal{U}(\Phi)$.*

Corollary 1. *If $x_0, y_0 \in X$ such that $d(x_0, y_0) > 0$, then $\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X)))$.*

Proof. If $x \in X$, then $0 < d(x_0, y_0) \leq d(x_0, x) + d(x, y_0) = d_{x_0}(x) + d_{y_0}(x)$ i.e., $d(x_0, y_0) \leq d_{x_0} + d_{y_0} \in \mathcal{A}(U_0(X))$. By Lemma 2 we get that $\frac{1}{d_{x_0}+d_{y_0}} \in \mathcal{U}(\mathcal{A}(U_0(X)))$, therefore $\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X)))$.

The existence of $x_0, y_0 \in X$ such that $d(x_0, y_0) > 0$ is equivalent to the positivity of the diameter of (X, d) (see the footnote in the proof of Lemma 3).

Definition 3. *If* $\mathbb{F}(X)$ *denotes the set of real-valued functions on* X*, the set of Lipschitz functions* $\mathrm{Lip}(X)$ *on* (X, d) *is defined by*

$$\mathrm{Lip}(X, k) := \{f \in \mathbb{F}(X) \mid \forall_{x,y \in X}(|f(x) - f(y)| \leq kd(x, y))\},$$

$$\mathrm{Lip}(X) := \bigcup_{k \geq 0} \mathrm{Lip}(X, k).$$

Remark 3. The set $\mathrm{Lip}(X) \subseteq C_u(X)$ includes $U_0(X)$, $\mathrm{Const}(X)$ and it is closed under addition, multiplication by reals, and multiplication.

Proof. If $x_0 \in X$, then $|d(x_0, x) - d(x_0, y)| \leq d(x, y)$, for every $x, y \in X$, therefore $U_0(X) \subseteq \mathrm{Lip}(X, 1)$. Clearly, $\mathrm{Const}(X) \subseteq \mathrm{Lip}(X, k)$, for every $k \geq 0$. Recall that $f \cdot g = \frac{1}{2}((f + g)^2 - f^2 - g^2)$, and if $M_f > 0$ is a bound of f, it is immediate to see that

$$f \in \mathrm{Lip}(X, k_1) \to g \in \mathrm{Lip}(X, k_2) \to f + g \in \mathrm{Lip}(X, k_1 + k_2),$$

$$f \in \mathrm{Lip}(X, k) \to \lambda \in \mathbb{R} \to \lambda f \in \mathrm{Lip}(X, |\lambda| k),$$

$$f \in \mathrm{Lip}(X, k) \to f^2 \in \mathrm{Lip}(X, 2M_f k).$$

Lemma 3. *If* $\Phi = \mathcal{A}(U_0^*(X))$*, then* $\mathrm{Lip}(X) \subseteq \mathcal{U}(\Phi)$*.*

Proof. It suffices to show that $\mathrm{Lip}(X, 1) \subseteq \mathcal{U}(\Phi)$, since if $f \in \mathrm{Lip}(X, k)$, for some $k > 0$, then $\frac{1}{k} f \in \mathrm{Lip}(X, 1)$ and we have, for every $\epsilon > 0$ and $\theta \in \Phi$,

$$U(\theta, \frac{1}{k} f, \frac{\epsilon}{k}) \to U(k\theta, f, \epsilon).$$

Suppose next that $f \in \mathrm{Lip}(X, 1)$ and $\epsilon > 0$. We find $g \in \mathcal{U}(\Phi)$ such that $U(g, f, \epsilon)$, therefore $f \in \mathcal{U}(\mathcal{U}(\Phi)) = \mathcal{U}(\Phi)$. More specifically, if $\{z_1, \ldots, z_m\}$ is an $\frac{\epsilon}{2}$-approximation of X, we find $g \in \mathcal{U}(\Phi)$ such that $g(z_i) = f(z_i)$, for every $i \in \{1, \ldots, m\}$, and $|g(x) - g(z_i)| = |g(x) - f(z_i)| \leq \frac{\epsilon}{2}$, for every $x \in X$ and z_i such that $d(x, z_i) \leq \frac{\epsilon}{2}$. Consequently,

$$\begin{aligned} |g(x) - f(x)| &\leq |g(x) - g(z_i)| + |g(z_i) - f(z_i)| + |f(z_i) - f(x)| \\ &\leq \frac{\epsilon}{2} + 0 + d(z_i, x) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

We define

$$g := \bigwedge_{k=1}^{m} (\overline{f(z_k)} + d_{z_k}).$$

Since $\overline{f(z_k)} + d_{z_k} \in \Phi$ and since by Lemma 1 $\mathcal{U}(\Phi)$ is closed under $\wedge$ we get $g \in \mathcal{U}(\Phi)$. Moreover,

$$g(z_i) = \bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(z_i)) \leq f(z_i) + d_{z_i}(z_i) = f(z_i).$$

For the converse inequality we suppose that $g(z_i) < f(z_i)$ and reach a contradiction (here we use the fact that $\neg(a < b) \to a \geq b$, for every $a, b \in \mathbb{R}$ (see [3], p.26)). If $a, b, c \in \mathbb{R}$, then one shows[2] that $a \wedge b < c \to a < c \vee b < c$. Hence

$$\bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(z_i)) < f(z_i) \to \exists_{j \in \{1,\ldots,m\}} (f(z_j) + d(z_j, z_i) < f(z_i))$$

$$\to d(z_j, z_i) < f(z_i) - f(z_j) \leq |f(z_i) - f(z_j)| \leq d(z_j, z_i),$$

which is a contradiction. Using the equality $g(z_i) = f(z_i)$ we have that

$$g(x) = \bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(x)) \leq f(z_i) + d_{z_i}(x) \to$$

$$g(x) - g(z_i) \leq f(z_i) + d_{z_i}(x) - g(z_i) = f(z_i) + d_{z_i}(x) - f(z_i) = d(x, z_i) \leq \frac{\epsilon}{2}.$$

If $k \in \{1, \ldots, m\}$, then $f(z_i) - f(z_k) \leq |f(z_i) - f(z_k)| \leq d(z_i, z_k) \leq d(z_i, x) + d(x, z_k)$, therefore

$$\begin{aligned}
&\forall_{k \in \{1,\ldots,m\}} (f(z_i) - d(z_i, x) \leq f(z_k) + d(z_k, x)) \to \\
&f(z_i) - d(z_i, x) \leq \bigwedge_{k=1}^{m} (f(z_k) + d(z_k, x)) \leftrightarrow \\
&f(z_i) - \bigwedge_{k=1}^{m} (f(z_k) + d(z_k, x)) \leq d(z_i, x) \to \\
&g(z_i) - g(x) \leq d(z_i, x) \to \\
&g(z_i) - g(x) \leq \frac{\epsilon}{2}.
\end{aligned}$$

From $g(x) - g(z_i) \leq \frac{\epsilon}{2}$ and $g(z_i) - g(x) \leq \frac{\epsilon}{2}$ we get $|g(x) - g(z_i)| \leq \frac{\epsilon}{2}$.

[2] The proof goes as follows. By the constructive trichotomy property (see [3], p.26) either $a < c$ or $a \wedge b < a$. In the first case we get immediately what we want to show. In the second case we get that $b \leq a$, since if $b > a$, we have that $a = a \wedge b < a$, which is a contradiction. Thus $a \wedge b = b$ and the hypothesis $a \wedge b < c$ becomes $b < c$.

Lemma 4. *If $f \in C_u(X)$ and $\epsilon > 0$, there exist $\sigma > 0$ and $g, g^* \in \mathrm{Lip}(X, \sigma)$ such that*
(i) $\forall_{x \in X}(f(x) - \epsilon \leq g(x) \leq f(x) \leq g^*(x) \leq f(x) + \epsilon)$.
(ii) For every $e \in \mathrm{Lip}(X, \sigma)$, if $e \leq f$, then $e \leq g$.
(iii) For every $e^* \in \mathrm{Lip}(X, \sigma)$, if $f \leq e^*$, then $g^* \leq e^*$.

Proof. (i) Let ω_f be a modulus of continuity of f and $M_f > 0$ a bound of f. We define the functions $h_x : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ by

$$h_x := f + \sigma d_x,$$

$$\sigma := \frac{2M_f}{\omega_f(\epsilon)} > 0,$$

$$g(x) := \inf\{h_x(y) \mid y \in X\} = \inf\{f(y) + \sigma d(x, y) \mid y \in X\},$$

for every $x \in X$. Note that $g(x)$ is well-defined, since $h_x \in C_u(X)$ and the infimum of h_x exists (see [3], p.94 and p.38). First we show that $g \in \mathrm{Lip}(X, \sigma)$. If $x_1, x_2, y \in X$ the inequality $d(x_1, y) \leq d(x_2, y) + d(x_1, x_2)$ implies that $f(y) + \sigma d(x_1, y) \leq (f(y) + \sigma d(x_2, y)) + \sigma d(x_1, x_2)$, hence $g(x_1) \leq (f(y) + \sigma d(x_2, y)) + \sigma d(x_1, x_2)$, therefore $g(x_1) \leq g(x_2) + \sigma d(x_1, x_2)$, or $g(x_1) - g(x_2) \leq \sigma d(x_1, x_2)$. Starting with the inequality $d(x_2, y) \leq d(x_1, y) + d(x_1, x_2)$ and working similarly we get that $g(x_2) - g(x_1) \leq \sigma d(x_1, x_2)$, therefore $|g(x_1) - g(x_2)| \leq \sigma d(x_1, x_2)$. Next we show that

$$\forall_{x \in X}(f(x) - \epsilon \leq g(x) \leq f(x)).$$

Since $f(x) = f(x) + \sigma d(x, x) = h_x(x) \geq \inf\{h_x(y) \mid y \in X\} = g(x)$, for every $x \in X$, we have that $g \leq f$. Next we show that $\forall_{x \in X}(f(x) - \epsilon \leq g(x))$. For that we fix $x \in X$ and we show that $\neg(f(x) - \epsilon > g(x))$. Note that if $A \subseteq \mathbb{R}, b \in \mathbb{R}$, then[3] $b > \inf A \to \exists_{a \in A}(a < b)$. Therefore,

$$\begin{aligned}
&f(x) - \epsilon > g(x) \leftrightarrow \\
&f(x) - \epsilon > \inf\{f(y) + \sigma d(x, y) \mid y \in X\} \to \\
&\exists_{y \in X}(f(x) - \epsilon > f(y) + \sigma d(x, y)) \leftrightarrow \\
&\exists_{y \in X}(f(x) - f(y) > \epsilon + \sigma d(x, y)).
\end{aligned}$$

For this y we show that $d(x, y) \leq \omega_f(\epsilon)$. If $d(x, y) > \omega_f(\epsilon)$, we have that

$$2M_f \geq f(x) + M_f \geq f(x) - f(y) > \epsilon + 2M_f \frac{d(x, y)}{\omega_f(\epsilon)} > \epsilon + 2M_f > 2M_f,$$

which is a contradiction. Hence, by the uniform continuity of f we get that $|f(x) - f(y)| \leq \epsilon$, therefore the contradiction $\epsilon > \epsilon$ is reached, since

$$\epsilon \geq |f(x) - f(y)| \geq f(x) - f(y) > \epsilon + \sigma d(x, y) \geq \epsilon.$$

[3] By the definition of $\inf A$ in [3], p.37, we have that $\forall_{\epsilon > 0} \exists_{a \in A}(a < \inf A + \epsilon)$, therefore if $b > \inf A$ and $\epsilon = b - \inf A > 0$ we get that $\exists_{a \in A}(a < \inf A + (b - \inf A) = b)$.

Next we define the functions $h_x^*: X \to \mathbb{R}$ and $g^*: X \to \mathbb{R}$ by

$$h_x^* := f - \sigma d_x,$$
$$g^*(x) := \sup\{h_x^*(y) \mid y \in X\} = \sup\{f(y) - \sigma d(x,y) \mid y \in X\},$$

for every $x \in X$, and $\sigma = \frac{2M_f}{\omega_f(\epsilon)}$. Similarly[4] to g we get that $g^* \in \mathrm{Lip}(X, \sigma)$ and

$$\forall_{x \in X}(f(x) \leq g^*(x) \leq f(x) + \epsilon).$$

(ii) Let $e \in \mathrm{Lip}(X, \sigma)$ such that $e \leq f$. If we fix some $x \in X$, then for every $y \in X$ we have that $e(x) - e(y) \leq |e(x) - e(y)| \leq \sigma d(x,y)$, hence $e(x) \leq e(y) + \sigma d(x,y) \leq f(y) + \sigma d(x,y)$, therefore $e(x) \leq g(x)$.
(iii) Let $e^* \in \mathrm{Lip}(X, \sigma)$ such that $f \leq e^*$. If we fix some $x \in X$, then for every $y \in X$ we have that $e^*(y) - e^*(x) \leq |e^*(y) - e^*(x)| \leq \sigma d(x,y)$, hence $f(y) - \sigma d(x,y) \leq e^*(y) - \sigma d(x,y) \leq e^*(x)$, therefore $g^*(x) \leq e^*(x)$.

Hence g is the largest function in $\mathrm{Lip}(X, \sigma)$ which is smaller than f, and g^* is the smallest function in $\mathrm{Lip}(X, \sigma)$ which is larger than f. So, if there is some $e' \in \mathrm{Lip}(X)$ such that $e' \leq f$ and $g(x) < e'(x)$, for some $x \in X$, then $e' \in \mathrm{Lip}(X, \sigma')$, for some $\sigma' > \sigma$. It is interesting that Lemma 4 is in complete analogy to the McShane-Kirszbraun theorem. To make this clear we include a constructive version of this theorem (for a classical presentation see [18], p.6). Recall that $A \subseteq X$ is *located*, if the distance $d(x, A) := \inf\{d(x,y) \mid y \in Y\}$ exists for every $x \in X$, and that a located subset of a totally bounded metric space is totally bounded (see [3], p.95).

Proposition 1 (McShane-Kirszbraun theorem for totally bounded metric spaces). *If $\sigma > 0$, $A \subseteq X$ is located, and $f: A \to \mathbb{R} \in \mathrm{Lip}(A, \sigma)$, then there exist $g, g^* \in \mathrm{Lip}(X, \sigma)$ such that $g_{|A} = g^*_{|A} = f$ and for every $e \in \mathrm{Lip}(X, \sigma)$ such that $e_{|A} = f$ we have that $g^* \leq e \leq g$.*

Proof. The functions g, g^* defined by $g(x) := \inf\{f(a) + \sigma d(x,a) \mid a \in A\}$, and $g^*(x) := \sup\{f(a) - \sigma d(x,a) \mid a \in A\}$, for every $x \in X$, are well-defined and satisfy the required properties.

Corollary 2. *$\mathcal{U}(\mathrm{Lip}(X)) = C_u(X)$.*

Proof. If $\epsilon > 0$, then the functions $g, g^* \in \mathrm{Lip}(X, \sigma)$ of Lemma 4 satisfy $U(g, f, \epsilon)$, $U(g^*, f, \epsilon)$, respectively.

Next follows our Stone-Weierstrass theorem for totally bounded metric spaces.

Theorem 2 (Stone-Weierstrass theorem for totally bounded metric spaces (SWtbms)). *If $\Phi = \mathcal{A}(U_0^*(X))$, then $C_u(X) = \mathcal{U}(\Phi)$.*

Proof. First we show that $C_u(X) \subseteq \mathcal{U}(\Phi)$. If $f \in C_u(X)$ and $\epsilon > 0$, then by Corollary 2 there exists $h \in \mathrm{Lip}(X)$ such that $U(h, f, \frac{\epsilon}{2})$, while by Lemma 3 there exists $g \in \Phi$ such that $U(g, h, \frac{\epsilon}{2})$. Consequently, $U(g, f, \epsilon)$. The converse inclusion follows from the immediate fact that all elements of $\mathcal{U}(\Phi)$ are in $C_u(X)$.

[4] To show that $\neg(g^*(x) > f(x) + \epsilon)$ we just use the fact that if $A \subseteq \mathbb{R}, b \in \mathbb{R}$, then $\sup A > b \to \exists_{a \in A}(a > b)$. The function g^* is mentioned in [19], where non-constructive properties of the classical $(\mathbb{R}, <)$ are used.

3 Corollaries of SWtbms

Proposition 2. **SWtbms** *implies* **BSWtbms**

Proof. The proof follows immediately by inspection of the proof of **BSWtbms** in [3], pp.106-8. Bishop shows there that if Φ is Bishop-separating, then $\overline{1} \in \mathcal{U}(\mathcal{A}(\Phi))$, and by his Lemma 5.14.1 one shows that $U_0(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi))$ - this is a slight simplification of the final part of Bishop's proof that $C_u(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi))$. Since $U_0^*(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi))$, then by Remark 2 $\mathcal{A}(U_0^*(X)) \subseteq \mathcal{U}(\mathcal{A}(\Phi))$, therefore $C_u(X) = \mathcal{U}(\mathcal{A}(U_0^*(X))) \subseteq \mathcal{U}(\mathcal{U}(\mathcal{A}(\Phi))) = \mathcal{U}(\mathcal{A}(\Phi))$.

In the proof of Corollary 5.16 in [3], pp.108-9, it is shown that if (X, d) has positive diameter, then $\mathcal{A}(U_0(X))$ is a Bishop-separating set, therefore by **BSWtbms** we get that $\mathcal{U}(\mathcal{A}(U_0(X))) = C_u(X)$. Hence **SWtbms** is only "slightly" stronger than **BSWtbms**. If we use **SWtbms**, we get immediately the same result.

Corollary 3. *If* (X, d) *has positive diameter, then* $\mathcal{U}(\mathcal{A}(U_0(X))) = C_u(X)$.

Proof. The hypothesis of positive diameter implies the hypothesis of Corollary 1, therefore $\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X))) \subseteq C_u(X)$. Hence $U_0^*(X) \subseteq \mathcal{U}(\mathcal{A}(U_0(X)))$, and by Remark 2 we get that $\mathcal{A}(U_0^*(X)) \subseteq \mathcal{A}(\mathcal{U}(\mathcal{A}(U_0(X)))) = \mathcal{U}(\mathcal{A}(U_0(X)))$. Thus $C_u(X) = \mathcal{U}(\mathcal{A}(U_0^*(X))) \subseteq \mathcal{U}(\mathcal{U}(\mathcal{A}(U_0(X)))) = \mathcal{U}(\mathcal{A}(U_0(X)))$.

Next we prove Corollary 5.15 in [3], p.108 and its finite version using **SWtbms**. If $(X, d), (Y, \rho)$ are totally bounded, then $(X \times Y, \sigma)$ is totally bounded, where $\sigma((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2)$, for every $x_1, x_2 \in X$ and $y_1, y_2 \in Y$; if $A = \{x_1, \ldots, x_n\}$ is an $\frac{\epsilon}{2}$-approximation of X and $B = \{y_1, \ldots, y_m\}$ is an $\frac{\epsilon}{2}$-approximation of Y, then $A \times B$ is an ϵ-approximation of $X \times Y$. We denote by π_1 the projection of $X \times Y$ onto X and by π_2 its projection onto Y.

Corollary 4. *If* $(X, d), (Y, \rho)$ *are totally bounded metric spaces and*

$$\Phi := \{\sum_{i=1}^{n} (f_i \circ \pi_1)(g_i \circ \pi_2) \mid f_i \in C_u(X), g_i \in C_u(Y), 1 \leq i \leq n, n \in \mathbb{N}\},$$

then $\mathcal{U}(\Phi) = C_u(X \times Y)$.

Proof. Clearly, $\Phi \subseteq C_u(X \times Y)$, Φ is an algebra (actually, $\Phi = \mathcal{A}((C_u(X) \circ \pi_1) \cup (C_u(Y) \circ \pi_2))$, where $C_u(X) \circ \pi_1 = \{f \circ \pi_1 \mid f \in C_u(X)\}$ and $C_u(Y) \circ \pi_2 = \{g \circ \pi_2 \mid g \in C_u(Y)\}$), and $\mathcal{U}(\Phi) \subseteq C_u(X \times Y)$. The constant $\overline{1}$ on $X \times Y$ is equal to $(\overline{1} \circ \pi_1)(\overline{1} \circ \pi_2)$. If $x_0, x \in X$ and $y_0, y \in Y$, then $\sigma_{(x_0, y_0)}((x, y)) = \sigma((x_0, y_0), (x, y)) = d(x_0, x) + \rho(y_0, y) = d_{x_0}(x) + \rho_{y_0}(y) = (d_{x_0} \circ \pi_1)((x, y)) + (\rho_{y_0} \circ \pi_2)((x, y))$, therefore $\sigma_{(x_0, y_0)} = (d_{x_0} \circ \pi_1) + (\rho_{y_0} \circ \pi_2) = (d_{x_0} \circ \pi_1)(\overline{1} \circ \pi_2) + (\overline{1} \circ \pi_1)(\rho_{y_0} \circ \pi_2) \in \Phi$. Since $U_0^*(X \times Y) \subseteq \mathcal{U}(\Phi)$, by **SWtbms** we get that $C_u(X \times Y) \subseteq \mathcal{U}(\Phi)$.

If (X_n, d_n) is totally bounded, where without loss of generality $d_n \leq \overline{1}$, for every $n \in \mathbb{N}$, then (X, σ_∞), where $X = \prod_{n=1}^{\infty} X_n$ and $\sigma_\infty((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$, is totally bounded; if $A(X_n, \epsilon)$ is an ϵ-approximation of X_n and $x_{0,n}$ inhabits X_n, then $A(X, \epsilon) = \prod_{k=1}^{n_0} A(X_k, \frac{2^{k-1}\epsilon}{n_0}) \times \prod_{k=n_0+1}^{\infty} \{x_{0,k}\}$ is an ϵ-approximation of X, where $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0+1}^{\infty} \frac{1}{2^k} \leq \frac{\epsilon}{2}$.

Corollary 5. *If (X, σ_∞) is the product of a sequence $(X_n, d_n)_{n=1}^\infty$ of totally bounded metric spaces, then $\mathcal{U}(\Phi) = C_u(X)$, where*

$$\Phi_0 := \{\prod_{i=1}^n (f_i \circ \pi_i) \mid f_i \in C_u(X_i), 1 \leq i \leq n, n \in \mathbb{N}\},$$

$$\Phi := \{\sum_{k=1}^n h_k \mid h_k \in \Phi_0, 1 \leq k \leq n, n \in \mathbb{N}\}.$$

Proof. Without loss of generality let $d_n \leq \overline{1}$, for every $n \in \mathbb{N}$. The only difference with the proof of Corollary 4 is treated as follows. If $(x_n^0)_{n=1}^\infty \in X$ and $\epsilon > 0$, let

$$g := \sum_{k=1}^{n_0} \frac{d_{k,x_k^0} \circ \pi_k}{2^k} = \sum_{k=1}^{n_0} (\frac{d_{k,x_k^0}}{2^k}) \circ \pi_k \in \Phi,$$

where $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0+1}^\infty \frac{1}{2^k} \leq \epsilon$. We get $U(g, \sigma_{\infty,(x_n^0)_{n=1}^\infty}, \epsilon)$, since

$$|g((y_n)_{n=1}^\infty) - \sigma_{\infty,(x_n^0)_{n=1}^\infty}((y_n)_{n=1}^\infty)| = |\sum_{k=n_0+1}^\infty \frac{d_{k,x_k^0}(y_k)}{2^k}| \leq \sum_{k=n_0+1}^\infty |\frac{d_k(x_k^0, y_k)}{2^k}| \leq \epsilon.$$

Recall that a totally bounded metric space is separable (see [3], p.94). The separability of $C_u(X)$ follows by the next corollary.

Corollary 6. *If $Q = \{q_n \mid n \in \mathbb{N}\}$ is dense in (X, d), $U_0(Q) := \{d_{q_n} \mid n \in \mathbb{N}\}$, and $\Phi_0^* = \mathcal{A}(U_0(Q) \cup \{\overline{1}\})$, then $\mathcal{U}(\Phi_0^*) = C_u(X)$.*

Proof. If $(x_n)_{n=1}^\infty \in X^\mathbb{N}$ converges pointwise to x, then $(d_{x_n})_{n=1}^\infty$ converges uniformly to d_x [i.e., if $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}(d(x_n, x) \leq \epsilon)$, then $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_{y\in X}(|d(x_n, y) - d(x, y)| \leq \epsilon)$]. If $\epsilon > 0$ and $n \geq n_0$, then $d(x_n, y) \leq d(x_n, x) + d(x, y) \rightarrow d(x_n, y) - d(x, y) \leq d(x_n, x) \leq \epsilon$, and similarly $d(x, y) - d(x_n, y) \leq d(x_n, x) \leq \epsilon$. By **SWtbms** it suffices to show that $U_0(X) \subseteq \mathcal{U}(\mathcal{A}(U_0(Q)))$. If $d_x \in U_0(X)$, for some $x \in X$, and $(q_{k_n})_{n=1}^\infty$ is a subsequence of Q that converges pointwise to x, then $(d_{q_{k_n}})_{n=1}^\infty$ converges uniformly to d_x, therefore $d_x \in \mathcal{U}(\mathcal{A}(U_0(Q)))$.

4 Concluding comments

We presented a direct constructive proof of **SWtbms** with a clear computational content. Its translation to Type Theory and its implementation to a proof assistant like Coq are expected to be straightforward. Although **SWtbms** does not look like a theorem of Stone-Weierstrass type, as **BSWtbms** does, it has certain advantages over it. Its proof is "natural", in comparison to Bishop's technical proof and his difficult to motivate concept of a Bishop-separating set. As we explained, **SWtbms** implies **BSWtbms**, and all applications of **BSWtbms** in [3] are proved directly by **SWtbms**. We know of no application of **BSWtbms** which cannot be derived by **SWtbms** (in [3] we found only one application of

Corollary 4 and one of the Weierstrass approximation theorem[5]).
An interesting question related to Corollary 2 is if for (X, d) totally bounded and (Y, ρ) complete metric space, the set of Lipschitz functions $\mathrm{Lip}(X, Y)$ between them is a dense subset of the uniformly continuous functions $C_u(X, Y)$ between them. A similar classical result, see [14], requires a Lipschitz extension property, which indicates that the correlation of Lemma 4 to the McShane-Kirszbraun theorem may not be accidental.

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[5] The first, in p.414, is the uniform approximation of a test function $f(x, y)$ on $G \times G$, where G is a locally compact group, by finite sums of the form $\sum_i f_i(x) g_i(y)$, and the second, in p.375, is a density theorem in the theory of Hilbert spaces.