CONSTRUCTIVE AND PREDICATIVE MEASURE THEORY (CPMT) APPLICATION TO THE LMU JUNIOR RESEARCHER FUND

IN PREPARATION FOR THE ERC STARTING GRANT (2022-CALL)

IOSIF PETRAKIS

Mathematisches Institut
Ludwig-Maximilians-Universität München
Theresienstrasse 39, 80333 München

petrakis@math.lmu.de

1. CPMT: Project Summary and Goals

In this section I present a summary of my proposed ERC Starting Grant *Constructive and Predicative Measure Theory* (CPMT). The aim of CPMT is to develop a measure theory within Bishop-style constructive mathematics as a synthesis of the already existing impredicative Bishop-Cheng measure theory and the predicative, but less general, earlier Bishop measure theory.

1.1. **Constructive mathematics.** One of the most important aspects of mathematics is *computation*. A fundamental example of a computation is the association of a number to a mathematical object, like the calculation of the volume of a geometric solid. More generally, a mathematical computation is an algorithm, or a routine that determines a mathematical object from given ones. Such computations, like the Euclidean algorithm of determining the greatest common divisor of two natural numbers, are the backbone of mathematics. The development of modern, abstract mathematics though, obscured a lot the meaning of computation.

E.g., the Dirichlet function Dir on real numbers is defined as follows:

$$\mathtt{Dir}(x) := \begin{cases} 0 & x \text{ is a rational number} \\ 1 & x \text{ is an irrational number.} \end{cases}$$

It is known that there is no algorithm that determines the above case-distinction. Although the Dirichlet function cannot compute its output in a practicable way, it is well-defined in standard, *classical* mathematics, where the computationally problematic case-distinction in its rule is justified by the use of the logical principle of the excluded middle "P or (not P)", where P is any mathematical formula. Mathematics that makes no use of this principle is called *constructive*, and the corresponding logic is called *intuitionistic*. Hence, in constructive mathematics the Dirichlet function cannot be accepted, as its rule does not involve an efficient computation.

If α is a sequence of 0's and 1's, then classically, either all terms α_n of the sequence α are 0, or there is a natural number n such that α_n is 1. In general, what we can practically do is to observe one-by-one each term α_n of the sequence α. If we find a term α_n equal to 1, then the procedure is terminated, but, as there is no guarantee that such a term exists, we cannot decide which one of the two disjuncts of the above disjunction is the case. It turns out that this principle, known as the *limited principle of omniscience* (LPO), is equivalent to many results of classical mathematics, like e.g., the trichotomy law of real numbers (every real number is less than 0, or equal to 0, or larger than 0). As expected, LPO cannot be used in constructive mathematics.

The positive aspect of avoiding in constructive mathematics all computationally inefficient principles of classical mathematics is that the *computational content* of mathematical practice is revealed and preserved. Whatever we do in constructive mathematics expresses an, in principle, efficient computation. The negative aspect of it is that large parts of classical mathematics cannot be accepted as they are. The aim of a constructive mathematician is to find computational alternatives to those classical results that lack, in their classical formulation, a computational meaning.

In the modern era the list of mathematicians that promoted constructive mathematics includes the names of Kronecker, Baire, Borel, Lebesgue, Brouwer, Weyl, Kolmogorov, Markov, and, more recently, of Bishop, Martin-Löf, and of the late Fields medalist Voevodsky. As we explain next, Bishop's system of constructive mathematics, which is the system that we are mainly working in, has a very special relation to classical mathematics.

1.2. **Bishop-style constructive mathematics.** In his seminal book *Foundations of Constructive Analysis* [3] the great analyst Errett Bishop (1928-1983) reconstructed a large part of mathematical analysis using intuitionistic logic, instead of classical logic, a new set theory, very different from the classical Zermelo-Fraenkel set theory, and an innovative approach to the definition of mathematical concepts, in order to be consistent with classical mathematics. Although Brouwer was the first to develop mathematics within intuitionistic logic, to employ a new constructive set theory, the theory of spreads and species, and to find for many classical mathematical concepts the (classically) equivalent formulation that suits best to constructive study, it was Bishop who managed to generate a system of informal constructive mathematics that is known as BISH, which did not contradict the informal system of classical mathematics. Despite the fundamental differences between BISH and classical mathematics, Bishop presented his constructive mathematics in [3] in a way remarkably friendly to classical mathematicians, his main target group.

Large parts of mathematics have been developed within BISH the last fifty years. E.g, in [3], and later with Bridges in [6], Bishop developed the constructive theory of metric spaces, of normed and Banach spaces, of Hilbert spaces, of locally compact Abelian groups and of commutative Banach algebras. In [35] and [31] general algebra and commutative algebra, respectively, are elaborated. In the up-coming[1] *Handbook of Bishop Constructive Mathematics* [8] the whole spectrum of the contemporary research-activity around BISH is presented.

1.3. **Predicativity and Bishop-style constructive mathematics.** The terms *predicative* and *non-predicative* were introduced by Russell in [59], an early attempt to deal with set-theoretic paradoxes within his logicism, and developed further by Poincaré in [57], mainly philosophically, and by Weyl in [70], mainly mathematically. Later the term *impredicative* is also used for non-predicative. Roughly speaking, a definition of a mathematical object O in a totality X is impredicative, or *circular*, if it refers essentially to X. In [22], p. 29, Feferman answers the question "What is predicativity?" by saying that "it is a concept applicable to different foundational stances given by the rejection of the actual infinite for various domains, coupled with its possible limited acceptance for others".

A totality X in BISH is defined through a membership-condition $\mathcal{M}_X(x)$, which reflects the computation to be carried out in order to show that $x \in X$. If the condition $\mathcal{M}_X(x)$ involves quantification over the universe $\mathbb{V}_0$ of (predicative) sets, then the definition of X is impredicative, and X is said to be a *class*. Typical classes in BISH, other than $\mathbb{V}_0$ itself, are the powerset of a set and the totality of all partial functions between two sets. A predicative standpoint within BISH would imply the avoidance of quantification over a class in the definition of a mathematical object. Bishop's position regarding impredicativity is ambivalent. There are instances in Bishop's work indicating that the powerset of a set is treated as a (predicative) set, while other instances clearly indicate that the powerset should be treated as a class. This phenomenon influenced crucially the development of constructive measure theory within BISH.

1.4. **Bishop-style constructive measure theory.** As Bishop and Bridges acknowledge in [6], pp. 215–6, "any constructive approach to mathematics will find a crucial test in its ability to assimilate the intricate body of mathematical thought called measure theory, or integration theory". The standard approach to measure theory (see e.g., [67], [24]) is to take measure as a primitive notion, and to define integration with respect to a given measure. An important alternative, and, as argued by Segal in [63] and [64], a more natural approach to measure theory, is to take the integral on a certain set of functions as a primitive notion, extend its definition to an appropriate, larger set of functions, and then define measure at a later stage. This is the idea of the Daniell integral, defined by Daniell in [18], which was taken further by Weil, Kolmogoroff, and Carathéodory (see [69], [29], and [9], respectively).

In [3] Bishop developed measure theory using the predicative, or non-circular notion of a family of complemented subsets in the definition of measure space. The general measure function is an abstraction of the measure function $A \mapsto \mu(A)$, where A is a member of a family of complemented

[1]The edition of this Handbook is the result of our initiative. Moreover, the Mathematics Institute of LMU, through its Logic group, is one of the most important centers of constructive mathematics worldwide.

subsets of a locally compact metric space X. The use of complemented subsets in order to overcome the difficulties generated in measure theory by the use of negation and negatively defined concepts is one of Bishop's great conceptual achievements, while the use of the concept of a family of complemented subsets is crucial to the predicative character of this notion of measure space. The indexing required behind this first notion of measure space is evident in [3], and sufficiently stressed in [4].

Based on the classical Daniell integral [18], Bishop and Cheng defined in [5] first the notion of an integrable function through the notion of an integration space, and afterwords the measure of an integrable set. The *Bishop-Cheng Measure Theory* (BCMT), which was enriched in [6], was more powerful than *Bishop's Measure Theory* (BMT) in [3]. Nevertheless, it used impredicative, or circular, concepts such as the "set" of all partial functions between two sets, a fact which hindered the extraction of its computational content and its formalisation in a suitable formal language. Based on BCMT, Chan developed in [10]-[15] constructive probability theory and the constructive theory of stochastic processes.

1.5. **Bishop-style constructive measure theory after Bishop.** Recognising the above problem of BCMT, Coquand, Palmgren and Spitters, in [16], [66] and [17], considered instead the algebraic, and point-free framework of Boolean rings and vector lattices to develop measure theory. In analogy to Segal's notion of a probability algebra, found in [63], [64], the starting notion in [16] is a boolean ring equipped with an apartness relation and a measure function, on which integrable and measurable functions can be defined. One can show that the integrable sets of BCMT form such a ring. Motivated by the work of Spitters, recently Ishihara developed in [26] a topological approach to constructive integration theory. Despite its abstract character, post-Bishop constructive measure theory is considered conceptually and technically simpler than BCMT.

1.6. **`CPMT`: goals and central questions.** My general aim is to formulate and develop the *Constructive and Predicative Measure Theory* (`CPMT`), a synthesis of the Bishop Measure Theory BMT and of the Bishop-Cheng Measure Theory BCMT, in order to combine the predicative character of the former with the richness in results and the generality of the latter. More specifically, I want to address the following central research questions:

(Q1) Which are the main theoretical tools in the formulation and the development of `CPMT`?

(Q2) Is it possible to develop a corresponding to `CPMT` topological measure theory?

(Q3) Is it possible to formalise `CPMT`?

1.7. **Originality and significance of `CPMT`.** In the general framework of computable mathematics outside BISH there are many approaches to measure and probability theory. There is an extended literature in intuitionistic measure theory (see e.g., [25]), in measure theory within the computability framework of Type-2 Theory of Effectivity (see e.g., [19]), in Russian constructivism (especially in the work of Šanin [60] and Demuth [7]), in Type Theory, where the main interest lies in the creation of probabilistic programming (see e.g., [2]), and recently also in Homotopy Type Theory (HoTT) (see [20]), where univalent techniques are applied to probabilistic programming. The revival of BMT, and the development of a constructive measure theory from BMT *and* BCMT is, to our knowledge, a completely original approach to the subject.

The significance of a predicative, Bishop-style, constructive measure theory is twofold. First, it will include important measure-theoretic notions and results from BMT and BCMT, two highly significant approaches to constructive measure theory. The post-Bishop approach to the subject, despite its elegance, is not a direct constructive version of the classical theory, and not as elaborated as BCMT. Second, and in relation to question (Q3), the elimination of the impredicative elements of BCMT will facilitate its formalisation, as well as the interaction between `CPMT` and the aforementioned approaches to computational measure theory outside BISH.

Regarding (Q2): In the classical study of measure and integration, topological considerations arise very often (see e.g., [36]). The construction of Lebesgue measure on $\mathbb{R}$, the notion of compact approximation, the general Radon measure, the probabilistic applications of weak convergence theory, are examples of notions and results that depend on topologically based ideas. The constructive approach to these topics requires a degree of compatibility between the under development measure theory CPMT and constructive topology, before a genuine *constructive topological measure theory* can be elaborated. As we explain in the next section, question (Q2) is particularly related to our research on constructive topology. In a series of papers, [42]-[48] and [50]-[51], we elaborate the theory of Bishop spaces, a function-theoretic approach to constructive topology. The possible development of the measure theory of Bishop spaces will be a significant new enrichment of the theory of Bishop spaces.

Regarding (Q3): The formalisation of CPMT is expected to be carried out through a computer software, a so-called *proof assistant*. A proof assistant allows users to practice mathematical activities on the computer by declaring axioms, defining functions, computing values, stating and proving theorems, and so on. In order to make use of a proof assistant, users have to write down their mathematical ideas following a strict syntax that the proof assistant employs so that the machine can read and interpret them. Such a rigorous writing-down process is called a formalisation. The main advantage of such a formalisation is that the computer guides users to do all related activities correctly, as the correctness of each single reasoning step is mechanically checked. An expected remarkable outcome concerning correctness in CPMT is predicativity, since any inadvertent use of impredicativity will be cleared out through the help of the machine. As it was explained in section 1.1, the lack of an efficient computation concerning the formula of LPO

$$\forall_{n\in\mathbb{N}}(\alpha_n = 0) \text{ or } \exists_{n\in\mathbb{N}}(\alpha_n = 1),$$

suggests that a formula in constructive mathematics poses (or is) a computational problem; this is exactly the idea of Kolmogorov in his interpretation of intuitionistic logic, given in [28]. As already suggested, a constructive proof contains a computational interpretation, or a "solution" to the problem which the proved formula specifies. *Program extraction* is a procedure of translating a proof into an executable program code, which moreover is provably a solution of the corresponding formula. Obviously, program extraction is hard to manually practice for pen and paper proofs, but it is fully automatised by some proof assistant. It opens broad possibilities of applications of constructive mathematics in fields other than logic and mathematics. To answer (Q3) means to practice program extraction in CPMT. In order to accomplish this task, it is crucial to have a constructive and predicative measure theory formalised in a proof assistant, something which has not been done so far.

2. Development of CPMT

Next I describe my research work related to questions (Q1)-(Q2).

2.1. **My work regarding (Q1).** The main theoretical tool in the formulation and the development of CPMT is going to be the theory of set-indexed families of sets. The notion of a set-indexed family of sets avoids the use of the powerset as a set, and it has already used by Bishop in BMT, in some specific cases, in exactly this way. As Bishop acknowledges though in [4], p. 67, the use of appropriate set-indexed families of sets, where the index-set is also equipped with operations that are lifted to the sets themselves, explains how "all of the material in [3], appropriately modified, can be comfortably formalised".

• In my recent work [49], and in great detail in my submitted Habilitation Thesis *Families of Sets in Bishop Set Theory* [50], I elaborate the various notions of set-indexed families of sets that are found in Bishop's set theory and are used in BISH. For that, I develop *Bishop Set Theory* (BST), an informal, constructive theory of totalities and assignment routines that serves as a "completion" of Bishop's theory of sets found in [3], chapter 3. Its first aim is to fill in the "gaps" in Bishop's account of the powerset, or highlight the fundamental notions, like dependent assignment routines, that were somehow suppressed by Bishop. Its second aim is to serve as an intermediate step between Bishop's theory of sets and an adequate and faithful, in Fefereman's sense [21], formalisation of BISH. To assure faithfulness, we use concepts or principles that appear, explicitly or implicitly, in BISH.

The features of BST that "complete" Bishop's account of set theory in [3] are:

(1) The explicit use of a universe of (predicative) sets (Bishop used such a universe only implicitly).
(2) A clear distinction between sets and classes.
(3) The explicit use of dependent operations.

• In my submitted work [52] I use BST in order to make explicit the algorithmic content of several constructive proofs by defining a Brouwer-Heyting-Kolmogorov-interpretation of certain formulas of BISH *within* BST. Through the notion of a set with a proof-relevant equality the first level of the identity type of intensional Martin-Löf Type Theory is translated into BISH. Moreover, several notions and facts from Homotopy Type Theory are interpreted in BISH.

• In my submitted work [48] I apply the notion of a family of sets indexed by a directed set and its corresponding dependent sums and products to the theory of Bishop spaces. The successful treatment of spectra of Bishop spaces and their limits in the framework of indexed families of sets paved the way for a similar application of set-indexed families of sets in measure theory.

• In his Master Thesis [71], and working under my supervision, Zeuner translated results from BCMT avoiding the use of impredicative concepts[2]. The Bishop-Cheng definition of the "set" L^1 (or L^p, where $p \geq 1$) of *integrable functions* is also impredicative, as it rests on the use of the totality of strongly extensional, real-valued partial functions on a set X (Definition (2.1) in [6], p. 222). In [71], pp. 49–60, the pre-integration space L^1 of *canonically* integrable functions is studied instead within BST, as the completion of an integration space. The set L^1 is predicatively defined in [3], p. 190, as an integrable function is an appropriate measurable function, which is defined using quantification over the set-indexed family of integrable sets in a Bishop measure space.

• In [54] and in [55], the notions of a *pre-measure space* and of a *pre-integration space* are introduced. Roughly speaking, a pre-measure space is a structure $(I, \wedge, \vee, \sim, \mu)$, where $\wedge, \vee, \sim$ are operations on I and $\mu : I \to [0, +\infty)$ such that certain axioms are satisfied, which guarantee that a measure function can be defined on a family of complemented[3] subsets of a set X, if it is indexed on I. The definition of a pre-integration space follows the same pattern. By considering set-indexed families of partial functions and certain operations on the index-set[4], Bishop's use of the class of all partial functions between two sets is avoided.

• In our joint work in preparation [56], Zeuner and I are elaborating Zeuner's predicative definition of the set of integrable functions L^1 that was introduced in [71].

2.2. **My work regarding (Q2).** Working constructively has a dramatic effect on general topology. For example, the complement of a closed set, like the singleton $\{0\}$ of $\mathbb{R}$, is not open, as that would imply that a non-zero real number is "apart" from 0 i.e., either larger than 0 or smaller than 0, something that cannot be captured computationally. Hence, the fundamental notion of a topological space is not suitable to constructive, or computational, study.

• In my PhD Thesis and in a series of papers [42]-[48] and [50]-[51] I develop the theory of Bishop spaces, a constructive, function-theoretic approach to general topology. This notion was introduced by Bishop in his seminal book [3], but it was never really developed. A Bishop topology on a set X is a set of functions F of type $X \to \mathbb{R}$ such that F includes the constant functions and it is closed under addition, composition with the Bishop-continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and uniform limits. The pair $\mathcal{F} := (X, F)$ is called then a *Bishop space*. This notion of topology is in analogy to Spanier's quasi-topology and to Grothendieck's topology on a category. The fundamental idea behind all these notions is that continuity comes first and the notion of space is defined later. Every Bishop space generates a (completely regular) topological space.

The main advantage of working with this notion of space is that it is function-theoretic, in contrast to other constructive, set-theoretic approaches to the notion of topological space. Experience has shown that function-theoretic objects suit better to computational study and to implementation in a programming language. As in the case of categorical topology, the main target in the development of point-function topology of Bishop spaces is the reconstruction of the standard set-based topological notions in a function-theoretic framework that will facilitate the extraction of algorithms from proofs. A source of concepts and ideas for the success of this program was the classical theory of the rings of continuous functions (see e.g., the classic book [23]), where properties of the rings $C(X)$ and $C^*(X)$ that determine the topological space X are studied).

[2]These first steps towards CPMT were announced at the conference "Mathematical Logic and Constructivity", at the Department of Mathematics of Stockholm University, 20-23.08.2019, where I was invited speaker, and Zeuner gave a contributed talk. Zeuner is now a PhD student at the the Department of Mathematics of Stockholm University.

[3]The notion of a complemented subset, defined in [3], p. 66, is one of the most important positive notions introduced by Bishop to overcome the difficulties that negatively defined concepts generate in constructive mathematics.

[4]The use of appropriate operations on the index-set help us also to avoid the principle of countable choice in Bishop's proofs. For the significance of practicing constructive mathematics without countable choice see [58] and [62]).

• In my recent work [46] I study the Borel sets and the Baire sets generated by a Bishop topology F on a set X. These are inductively defined sets of F-complemented subsets of X. Because of the constructive definition of the Borel sets of F, and in contrast to classical topology, I showed that the two families, the Borel and the Baire sets of F, are equal.

• In my recent work [51] I extended [46] with the study of Baire one functions over a Bishop topology.

• In my work in progress [53], which is in the synthetic spirit of `CPMT`, I define the notion of an *integration algebra* in a Bishop topology F, in order to generalise concepts and results from the integration theory of locally compact metric spaces to the integration theory of general Bishop spaces. An integration algebra Φ in a Bishop topology F is a subalgebra and a sublattice of F, while an *integral* on Φ is a certain linear map $\mu : \Phi \to \mathbb{R}$. Abstracting from the properties of the functions with compact support on a locally compact metric space, I define the notion of an *algebra of test functions* in F. These notions and the results related to them corroborate my conjecture that BMT and BCMT can eventually be synthesised.

Next I describe my planned research work related to question (Q3).

2.3. **My planned work regarding (Q3).** I plan to work on the formal aspects of `CPMT` by translating some of my basic notions and results into Martin-Löf Type Theory (MLTT) (see [32], [33], [34]), through the translation of a Bishop set as a setoid in MLTT and the translation of a set-indexed family of sets as a setoid-indexed family of setoids (see [41]). The type-theoretic translation of Bishop's set theory in the theory of setoids (see the work of Palmgren [37]-[41]) has become nowadays the standard way to understand Bishop sets. The identity type of MLTT expresses though, in a proof-relevant way, the existence of the least reflexive relation on a type, a fact that has no counterpart in Bishop's set theory. As a consequence, the free setoid on a type is definable (see [39], p. 90), and the presentation axiom for setoids can be shown. Because of these unexpected, from the Bishop set theory point of view, consequences of the identity type in intensional MLTT, an extensional version of MLTT seems more suitable as a formal framework of `CPMT`. At present, the use of univalent techniques of [68] in the formalisation of `CPMT` seems less clear. I need to explore first the possibility of employing univalent concepts in the theory of setoids, and then to examine their application to the translation of `CPMT` into the theory of setoids. The use of univalent techniques is though more expected, if the theory of setoids that is going to be adopted is intentional rather than extensional.

The role of Dr. Kenji Miyamoto, a member of the Logic group of the Mathematics Institute of LMU, regarding (Q3), is going to be crucial. His research task will be to formalise several theoretical results of `CPMT` in a suitable proof assistant, so that the formalised proofs are mechanically checked and certified to be logically correct and also predicative. It may be required to improve the proof assistant itself, so that both proof checking and program extraction are carried out for `CPMT`.

References

[1] European Research Council: Transition of the `ERC` to Horizon Europe–Operational Considerations, available at `https://erc.europa.eu/news/transition-ERC-to-Horizon-Europe-Operational-considerations`.

[2] S. Bhat, A. Agarwal, R. Vuduc: A type theory for probability density functions, POPL'12, ACM, 2012, 545–556.

[3] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[4] E. Bishop: Mathematics as a Numerical Language, in [27], 1970, 53–71.

[5] E. Bishop and H. Cheng: *Constructive Measure Theory*, Mem. Amer. Math. Soc. 116, 1972.

[6] E. Bishop and D. S. Bridges: *Constructive Analysis*, Grundlehren der math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[7] D. Bridges, O. Demuth: On the Lebesgue measurability of continuous functions in constructive analysis, Bulletin of the American Mathematical Society, Vol. 24, No. 2, 1991, 259–276.

[8] D. S. Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg (Eds.): *Handbook of Bishop Constructive Mathematics*, in preparation, expected year of publication 2021.

[9] C. Carathéodory: *Mass und Integral und ihre algebraisierung*, Springer Basel AG, 1956.

[10] Y. K. Chan: A constructive approach to the theory of stochastic processes, Transactions of the American Mathematical Society, Vol. 65, 1972, 37–44.

[11] Y. K. Chan: A constructive study of measure theory, Pacific Journal of Mathematics, Vol. 41, No. 1, 1972, 63–79.

[12] Y. K. Chan: A constructive approach to the theory of stochastic processes, Transactions of the American Mathematical Society, Vol. 165, 1972, 37–44.

[13] Y. K. Chan: Notes on constructive probability theory, The Annals of Probability, Vol. 2, No. 1, 1974, 51–75.

[14] Y. K. Chan: A short proof of an existence theorem in constructive measure theory, Proceedings of the American Mathematical Society, Vol. 48, No. 2, 1975, 435–436.

[15] Y. K. Chan: *Foundations of Constructive Probability Theory*, arXiv:1906.01803v2, 2019.
[16] T. Coquand, E. Palmgren: Metric Boolean algebras and constructive measure theory, Arch. Math. Logic 41, 2002, 687–704.
[17] T. Coquand, B. Spitters: Integrals and valuations, Journal of Logic & Analysis, 1:3, 2009, 1–22.
[18] P. J. Daniell: A general form of integral, Annals of mathematics, Second Series, 19 (4), 1918, 279-294.
[19] A. Edalat: A computable approach to measure and integration theory, Information and Computation 207, 2009, 642–659.
[20] F. Faissole, B. Spitters: Synthetic topology in Homotopy Type Theory for probabilistic programming, preprint, 2016.
[21] S. Feferman: Constructive theories of functions and classes, in Boffa et. al. (Eds.) *Logic Colloquium 78*, North-Holland, 1979, 159–224.
[22] S. Feferman: Predicativity, in S. Shapiro (Ed.) *Handbook of the Philosophy of Mathematics and Logic*, Oxford University Press, 2005.
[23] L. Gillman and M. Jerison: *Rings of Continuous Functions*, Van Nostrand, 1960.
[24] P. R. Halmos: *Measure theory*, Springer-Verlag New York Inc. 1974.
[25] A. Heyting: *Intuitionism, An introduction*, North-Holland, 1956.
[26] H. Ishihara: A constructive integration theory: a topological approach, preprint, 2020.
[27] A. Kino, J. Myhill and R. E. Vesley (Eds.): *Intuitionism and Proof Theory*, North-Holland, 1970.
[28] A. N. Kolmogoroff: Zur Deutung der intuitionistischen Logik, Mathematische Zeitschrift 35, 1932, 58–65.
[29] A. N. Kolmogoroff: Algèbres de Boole métrique complètes, VI Zjazd Mathematyków Polskich, Warwaw, 1948, 21–30.
[30] N. Kraus, T. Altenkirch: Free Higher Groups in Homotopy Type Theory, LICS'18.
[31] H. Lombardi, C. Quitté: *Commutative Algebra: Constructive Methods*, Springer, 2015.
[32] P. Martin-Löf: An intuitionistic theory of types: predicative part, in H. E. Rose and J. C. Shepherdson (Eds.) *Logic Colloquium'73, Proceedings of the Logic Colloquium*, volume 80 of Studies in Logic and the Foundations of Mathematics, pp.73-118, North-Holland, 1975.
[33] P. Martin-Löf: *Intuitionistic type theory: Notes by Giovanni Sambin on a series of lectures given in Padua, June 1980*, Napoli: Bibliopolis, 1984.
[34] P. Martin-Löf: An intuitionistic theory of types, in G. Sambin, J. M. Smith (Eds.) *Twenty-five years of constructive type theory* (Venice, 1995), volume 36 of Oxford Logic Guides, Oxford University Press, 1998, 127–172.
[35] R. Mines, F. Richman, W. Ruitenburg: *A course in constructive algebra*, Springer Science+Business Media New York, 1988.
[36] J. C. Oxtoby: *Measure and Category*, Springer-Verlag, second edition, 1980.
[37] E. Palmgren: Bishop's set theory, Slides from TYPES Summer School 2005, Gothenburg, in `http://staff.math.su.se/palmgren/`.
[38] E. Palmgren: Bishop-style constructive mathematics in type theory - A tutorial, Slides, 2013, in `http://staff.math.su.se/palmgren/`.
[39] E. Palmgren: *Lecture Notes on Type Theory*, manuscript, 2014.
[40] E. Palmgren, O. Wilander: Constructing categories and setoids of setoids in type theory, Logical Methods in Computer Science. 10 (2014), Issue 3, paper 25.
[41] E. Palmgren: Constructions of categories of setoids from proof-irrelevant families, Archive for Mathematical Logic 56, 2017, 51–66.
[42] I. Petrakis: *Constructive Topology of Bishop Spaces*, PhD Thesis, Ludwig-Maximilians-Universität, München, 2015.
[43] I. Petrakis: Completely Regular Bishop Spaces, in A. Beckmann, V. Mitrana and M. Soskova (Eds.) Evolving Computability, CiE 2015, Springer LNCS 9136, 2015, 302–312.
[44] I. Petrakis: The Urysohn Extension Theorem for Bishop Spaces, in S. Artemov and A. Nerode (Eds.) *Symposium on Logical Foundations of Computer Science 2016*, LNCS 9537, Springer, 2016, 299–316.
[45] I. Petrakis: A constructive function-theoretic approach to topological compactness, Proceedings of the 31st Annual ACM-IEEEE Symposium on Logic in Computer Science (LICS 2016), July 5-8, 2016, NYC, USA, 605–614.
[46] I. Petrakis: Borel and Baire sets in Bishop Spaces, in F. Manea et. al. (Eds): Computing with Foresight and Industry, CiE 2019, LNCS 11558, Springer, 2019, 240–252.
[47] I. Petrakis: Constructive uniformities of pseudometrics and Bishop topologies, Journal of Logic and Analysis 11:FT2, 2019, 1–44.
[48] I. Petrakis: Direct spectra of Bishop spaces and their limits, submitted to Logical Methods in Computer Science, 2019, 51 pages, available at `https://arxiv.org/abs/1907.03273`.
[49] I. Petrakis: Dependent Sums and Dependent Products in Bishop's Set Theory, in P. Dybjer et. al. (Eds) TYPES 2018, LIPIcs, Vol. 130, Article No. 3, 2019.
[50] I. Petrakis: Embeddings of Bishop spaces, Journal of Logic and Computation, exaa015, 2020, `https://doi.org/10.1093/logcom/exaa015`.
[51] I. Petrakis: Functions of Baire-class one over a Bishop topology, in M. Anselmo et. al. (Eds.) *Beyond the Horizon of Computability*, in M. Anselmo et al. (Eds.) *Beyond the Horizon of Computability*, CiE 2020, Springer, LNCS 12098, 215–227,
[52] I. Petrakis: Proof-relevance in Bishop-style constructive mathematics, submitted to Mathematical Structures in Computer Science, 2020, 37 pages.
[53] I. Petrakis: *Integrals in Bishop spaces*, in preparation, 2020.
[54] I. Petrakis: *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU, Munich, submitted, 2020, available at `http://www.mathematik.uni-muenchen.de/~petrakis/content/Theses.php`

[55] I. Petrakis, M. Zeuner: Pre-measure spaces and pre-integration spaces in predicative Bishop-Cheng measure theory, to be submitted to Archive for Mathematical Logic, 2020.
[56] I. Petrakis, M. Zeuner: Pre-measure spaces and pre-integration spaces in predicative Bishop-Cheng measure theory, part II, to be submitted to Archive for Mathematical Logic, 2020.
[57] H. Poincaré: Les mathematiques et la logique, Revue de métaphysique et de morale 14, 294–317.
[58] F. Richman: Constructive mathematics without choice, in [61], pp.199–205.
[59] B. Russell: On some difficulties in the theory of transfinite numbers and order types,Proc. London Mathematical
[60] N. Šanin: *Constructive Real Numbers and Function Spaces*, Translations of Mathematical Monographs, vol. 21, AMS, Providence Rhode Island, 1968.
[61] P. Schuster, U. Berger, H. Osswald (Eds.): *Reuniting the Antipodes Constructive and Nonstandard Views of the Continuum*, Proc. 1999 Venice Symposium, Dordrecht: Kluwer, 2001.
[62] P. Schuster: Countable choice as a questionable uniformity principle, Phil. Math. (3) 12, 2004, 104-134.
[63] I. Segal: Abstract probability spaces and a theorem of Kolmogoroff, Amer. J. Math. 76, 1954, 721-732.
[64] I. Segal: Algebraic integration theory, Bull. Amer. Math. Soc. 71, 1965, 419-489.
[65] B. Spitters: Constructive and intuitionistic integration theory and functional analysis. PhD Thesis, University of Nijmegen, 2002.
[66] B. Spitters: Constructive algebraic integration theory. Ann. Pure Appl. Logic 137(1-3), 2006, 380-390.
[67] S. J. Taylor: *Introduction to Measure and Integration*, Cambridge University Press, 1973.
[68] The Univalent Foundations Program: *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, Princeton, 2013.
[69] A. Weil: Calcul des probabilités, méthode aximatique, intégration, Revue Sci. (Rev. Rose Illus.) 78, 1940, 201–208.
[70] H. Weyl: *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit, Leipzig, 1918.
[71] M. Zeuner: *Families of Sets in Constructive Measure Theory*, Master Thesis, LMU, 2019.