Limit spaces with approximations

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Abstract

Abstracting from a presentation of the density theorem for the hierarchy $\mathrm{Ct}(\rho)$ of countable functionals over $\mathbb{N}$ given by Normann in [13], we define two subcategories of limit spaces, the limit spaces with approximations, and the limit spaces with general approximations, for both of which a density theorem holds directly. We show that these categories are cartesian closed, and we give examples of such limit spaces and of density theorems for hierarchies of functionals over them. Most of our main proofs are within Bishop's informal system of constructive mathematics BISH. In a limit space with (general) approximations the approximation functions are given beforehand as an internal part of the structure under study. In this way limit spaces with (general) approximations form a constructive approach to abstract limit spaces, reflecting at the same time the central idea of Normann's Program of Internal Computability.

1 Introduction

In this paper we generalize Normann's notion of the nth approximation of a functional in the typed hierarchy $\mathrm{Ct}(\rho)$ over $\mathbb{N}$, defining two subcategories of the category of limit spaces **Lim**, the category **Appr** of limit spaces with approximations, and the category **Gappr** of limit spaces with general approximations. These limit spaces, which are studied here constructively, realize in a direct way Normann's notion of internal computability.

Normann formulated the distinction between internal and external computability over a mathematical structure already in [12], and initiated, what we call, a *Program of Internal Computability* in [13]-[16].

According to [12], "the *internal* concepts [of computability] must grow out of the structure at hand, while *external* concepts may be inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways". Normann's motivation behind an internal approach in general computability is technical (see his results in [15]), conceptual (an associate-free description of $\mathrm{Ct}(\rho)$) and practical, since within a weaker concept of computability (if an object is internally computable, then it is also externally computable, but not necessarily the converse) "the weaker tools we use to obtain a result, the more extra knowledge can be obtained from the process of obtaining the result" ([14], p.305). Normann found suitable for his study of internal computability over a mathematical structure the framework of limit spaces. As he mentions in [13], p.474, he finds it "useful to see how far we can get towards constructing an effective infrastructure on such spaces without introducing superstructures and imposing external notions of computability on the given structures ... One way to create a useful part of an infrastructure will be to isolate a dense subset that in some way is effectively dense.".

As we show in this paper, such dense sets are very direct to find in limit spaces with (general) approximations[1]. Although Normann is working in a classical framework, here we study these limit spaces within Bishop's informal system of constructive mathematics BISH (see [1]-[3]). When a proposition is proved with the use of non-constructive methods we write that it is in CLASS, the classical extension of BISH. In order to fix our notation and be self-contained we include some necessary definitions and facts.

2 Basic definitions and facts

A *limit space* is a structure $\mathbb{L} = (X, \lim)$, where X is an inhabited set, and $\lim \subseteq X \times X^{\mathbb{N}}$ is a relation satisfying the following conditions: (i) if $x \in X$ and (x) denotes the constant sequence x, then $\lim(x, (x))$, (ii) if $\mathcal{S}$ denotes the set of all elements of the Baire space $\mathcal{N}$ which are strictly monotone, then[2] $\forall_{\alpha \in \mathcal{S}}(\lim(x, x_n) \rightarrow \lim(x, x_{\alpha(n)}))$, and (iii) if $x \in X$ and $x_n \in X^{\mathbb{N}}$, then $\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}}(\lim(x, x_{\alpha(\beta(n))})) \rightarrow \lim(x, x_n)$. In the literature condition (iii) is usually written as (iii)′ $\neg(\lim(x, x_n)) \rightarrow \exists_{\alpha \in \mathcal{S}} \forall_{\beta \in \mathcal{S}}(\neg\lim(x, x_{\alpha(\beta(n))}))$, but we prefer to have the intuitionistically stronger condition (iii) right from the start[3]. If $\forall_{x,y \in X} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \rightarrow \lim(y, x_n) \rightarrow x = y)$, then the limit space has the *uniqueness property*[4]. A limit space has the *weak uniqueness property*, if $\forall_{x,y \in X}(\lim(x, y) \rightarrow x = y)$. One can show classically[5], that there exists a limit space with the weak uniqueness property which does not have the uniqueness property.

A limit space induces a natural topology $\mathcal{T}_{\lim}$, the Birkhoff-Baer topology, according to which a set $\mathcal{O} \subseteq X$ is lim-*open*, if $\forall_{x \in \mathcal{O}} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \rightarrow \mathrm{ev}(x_n, \mathcal{O}))$, where, if $A \subseteq X$, we define $\mathrm{ev}(x_n, A) :\leftrightarrow \exists_{n_0} \forall_{n \geq n_0}(x_n \in A)$. A topological space $(X, \mathcal{T})$ induces a limit space $(X, \lim_{\mathcal{T}})$, where $\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x$, and $x_n \xrightarrow{\mathcal{T}} x$ denotes the convergence of x_n to x w.r.t. the topology $\mathcal{T}$. If $\mathbb{L}$ is a limit space, it is direct to see that $\lim(x, x_n) \rightarrow (x_n \xrightarrow{\mathcal{T}_{\lim}} x)$ i.e.,

[1]Limit spaces, as special case of weak limit spaces, have also been studied within Type-2 Theory of Effectivity (see the work of Schröder [21] and [22]), but from a non-constructive and an external computability point of view.

[2]If $(x_n)_n \in X^{\mathbb{N}}$ we write for simplicity $\lim(x, x_n)$ instead of $\lim(x, (x_n)_n)$, and $\lim(x, x)$ instead of $\lim(x, (x))$. If it is necessary, we write $\lim_n(x, x_n)$ to specify the convergence w.r.t. n. Usually one finds the notation $\lim_n x_n = x$ instead of $\lim(x, x_n)$.

[3]Menni and Simpson in [10] also use (iii) instead of (iii)′ in the definition of a limit space.

[4]A limit space with the uniqueness property is what Kuratowski calls in [9] an $\mathcal{L}^*$-space.

[5]All proofs not included here can be found in [18].

$\lim \subseteq \lim_{\mathcal{T}_{\lim}}$. A limit space is called *topological*, if $\lim = \lim_{\mathcal{T}_{\lim}}$. It is also direct that $\mathcal{T} \subseteq \mathcal{T}_{\lim_{\mathcal{T}}}$. A topological space is called *sequential*, if $\mathcal{T} = \mathcal{T}_{\lim_{\mathcal{T}}}$.

A set $F \subseteq X$ is called lim-*closed*, if it is the complement of a lim-open set, and in CLASS we have that F is lim-closed iff $\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}}(x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F)$. One can show that if $\mathbb{L}$ is topological and the induced topological space is 1st countable, the previous characterization of a lim-closed set can be carried out in BISH. A similar remark holds for other classical results within limit spaces.

A set $D \subseteq X$ is called lim-*dense*, if $\forall_{x \in X} \exists_{d_n \in D^{\mathbb{N}}}(\lim(x, d_n))$, while a limit space is called lim-*separable*, if there is a countable lim-dense subset of it. It is direct to show in BISH that if D is a lim-dense set, then D is dense in $(X, \mathcal{T}_{\lim})$, while one can show in CLASS that if D is dense in $(X, \mathcal{T})$, then it is not generally the case that D is $\lim_{\mathcal{T}}$-dense[6].

It is also trivial within BISH that if $(X, \lim)$ and $(Y, \lim)$ are limit spaces[7], and $\lim((x, y), (x_n, y_n)) :\leftrightarrow \lim(x, x_n) \wedge \lim(y, y_n)$, then $(X \times Y, \lim)$ is a limit space. Moreover, if $(X, \lim)$ and $(Y, \lim)$ are topological limit spaces (with the uniqueness property), then $(X \times Y, \lim)$ is a topological limit space (with the uniqueness property).

If $(X, \lim)$ and $(Y, \lim)$ are limit spaces, a function $f : X \rightarrow Y$ is called lim-*continuous*, if $\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \rightarrow \lim(f(x), f(x_n)))$. Clearly, a constant function and the identity function are lim-continuous. Of course, lim-continuity is closed under composition of functions. If $Y \subseteq X$, then $(Y, \lim_Y)$ is the *relative* limit space, where $\forall_{y \in Y} \forall_{y_n \in Y^{\mathbb{N}}}(\lim_Y(y, y_n) :\leftrightarrow \lim(y, y_n))$. The topology $\mathcal{T}_{\lim_Y}$ induced by the relative lim-relation includes the restriction $\mathcal{T}_{\lim|Y}$ of the topology induced by the initial lim on Y, and if $f : (X, \lim) \rightarrow (Y, \lim)$ is lim-continuous, then $f : (X, \lim) \rightarrow (f(X), \lim_{f(X)})$ is lim-continuous too.

Next proposition, the version of which for Hausdorff spaces is standard (see e.g., [5], p.140), is within BISH, if we restrict to topological limit spaces inducing a 1st countable topology, while it is within CLASS, if we pose no restrictions on them. The importance of proving for limit spaces with the uniqueness property facts that hold on Hausdorff spaces is due to the following simple fact within BISH: if $(X, \mathcal{T}_{\lim})$ is a T_2-space, then $(X, \lim)$ has the uniqueness property, while the converse doesn't hold in general (see [4] p.485).

Proposition 1. *Suppose that* $(X, \lim)$ *is a limit space,* $(Y, \lim)$ *is a limit space with the uniqueness property,* D *is a* lim-*dense subset of* X*, and* $f, g : X \rightarrow Y$ *are* lim-*continuous functions. Then the following hold:*

(i) If $f_{|D} = g_{|D}$*, then* $f = g$.
(ii) If $f : (D, \lim_D) \rightarrow (Y, \lim)$ *is* lim-*continuous, then it has at most one* lim-*continuous extension to* X.
(iii) The set $Z(f, g) = \{x \in X \mid f(x) = g(x)\}$ *is* lim-*closed.*
(iv) The graph $\mathbb{G}_f$ *of* f *is* lim-*closed in* $(X \times Y, \lim)$.
(v) If f *is* 1-1*, then* $(X, \lim)$ *has the uniqueness property.*

[6] E.g., the set of irrational numbers $\mathbb{I}$ is dense in $(\mathbb{R}, \mathcal{T}_{\text{coc}})$, but it is not lim-dense in $(\mathbb{R}, \lim_{\mathcal{T}_{\text{coc}}})$, where $\mathcal{T}_{\text{coc}}$ is the cocountable topology on $\mathbb{R}$.

[7] For the sake of simplicity we use the same symbol for the limit relations on X and Y.

Proof. Straightforward. $\square$

If $(X, \lim)$ is a limit space, $A \subseteq X$ is called a lim-*retract* of X, if there is a lim-continuous function $r : X \to A$ such that $r(a) = a$, for each $a \in A$. Again we get within limit spaces the standard extension property of topological retracts.

Proposition 2 (BISH)**.** *If* $(X, \lim), (Y, \lim)$ *are limit spaces,* A *is a* lim*-retract of* X *and* $f : (A, \lim_A) \to (Y, \lim)$ *is* lim*-continuous, then* f *has a* lim*-continuous extension* $F : (X, \lim) \to (Y, \lim)$.

Proof. If we define $F = f \circ r$, then F is lim-continuous as a composition of lim-continuous functions, and $F(a) = f(r(a)) = f(a)$, for each $a \in A$. $\square$

Proposition 3 (BISH)**.** *(i) If* $(X, \lim)$ *and* $(Y, \lim)$ *are limit spaces and* $f : X \to Y$ *is* lim*-continuous, then* $f : (X, \mathcal{T}_{\lim}) \to (Y, \mathcal{T}_{\lim})$ *is continuous.*
(ii) If $(X, \lim)$ *is a limit space and* $(Y, \lim)$ *is a topological limit space, then the converse to (i) holds.*

Proof. (i) Suppose that $\mathcal{O}_Y$ is open in Y, $x \in f^{-1}(\mathcal{O}_Y)$ and x_n is a sequence in X such that $\lim(x, x_n)$. Since f is lim-continuous we get $\lim(f(x), f(x_n))$. Since $f(x) \in \mathcal{O}_Y$, $\mathrm{ev}(f(x_n), \mathcal{O}_Y)$, therefore $\mathrm{ev}(x_n, f^{-1}(\mathcal{O}_Y))$ i.e., $f^{-1}(\mathcal{O}_Y)$ is open.
(ii) Suppose that $f : (X, \mathcal{T}_{\lim}) \to (Y, \mathcal{T}_{\lim})$ is continuous, and that $x \in X, x_n \in X^{\mathbb{N}}$ such that $\lim(x, x_n)$. We show that $\lim(f(x), f(x_n))$, which, since $\lim = \lim_{\mathcal{T}_{\lim}}$ in Y, amounts to $f(x_n) \overset{\mathcal{T}_{\lim}}{\to} f(x)$. Suppose that $\mathcal{O}_Y$ is open in Y and that $f(x) \in \mathcal{O}_Y$; we show that $\mathrm{ev}(f(x_n), \mathcal{O}_Y)$. Since $x \in f^{-1}(\mathcal{O}_Y) \in \mathcal{T}_{\lim}$, the hypothesis $\lim(x, x_n)$ implies $\mathrm{ev}(x_n, f^{-1}(\mathcal{O}_Y))$ i.e., $\mathrm{ev}(f(x_n), \mathcal{O}_Y)$.
$\square$

One can curry within BISH the proof of the following basic theorem[8].

Theorem 4. *If* $(X, \lim)$ *and* $(Y, \lim)$ *are limit spaces, then the following hold:*
(i) If $X \to Y$ *is the set of all* lim*-continuous functions* $f : X \to Y$ *and*

$$\lim(f, f_n) :\leftrightarrow \forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(f(x), f_n(x_n)))$$

then $(X \to Y, \lim)$ *is a limit space.*
(ii) The space $(X \to Y, \lim)$ *has the weak uniqueness property iff* $(Y, \lim)$ *has it.*
(iii) The space $(X \to Y, \lim)$ *has the uniqueness property iff* $(Y, \lim)$ *has it.*

Clearly,the evaluation map $\omega : (X \to Y) \times X \to Y$ defined by $(f, x) \mapsto f(x)$ is lim-continuous.

Proposition 5 (CLASS)**.** *If* $(X, \lim)$ *is a limit space with the uniqueness property, and* X *is a finite set, then* $\lim(x, x_n) \leftrightarrow \mathrm{ev}(x_n, \{x\})$*. Moreover, if* $(Y, \lim)$ *is a limit space, then any function* $f : X \to Y$ *is* lim*-continuous.*

[8] The proof of a necessary lemma for case (i), which is Theorem 3 in [9] p.188, rests on the principle of dependent choices on $\mathbb{N}$, which is accepted in BISH (see [2] p. 75).

Proof. Suppose that $\lim(x, x_n)$ and x_n is not eventually the constant sequence x. Hence, there is some $\alpha \in \mathcal{S}$ such that $x_{\alpha(n)} \neq x$, for each n, and $\lim(x, x_{\alpha(n)})$. Using the infinite pigeonhole principle[9] we get an element $x' \in (X \setminus \{x\})$ appearing infinitely many times, or, in other words, there is some $\beta \in \mathcal{S}$ such that $x_{\alpha(\beta(n))} = x'$, for each n. Since $\lim(x, x_{\alpha(\beta(n))})$ and $\lim(x', x_{\alpha(\beta(n))})$, the uniqueness property of $(X, \lim)$ implies that $x' = x$, which is absurd. Moreover, if $f : X \to Y$ is any function, and $\lim(x, x_n)$ i.e., x_n is eventually x, then $f(x_n)$ is eventually $f(x)$, and we get $\lim(f(x), f(x_n))$. $\square$

3 Limit spaces with Approximations

Scarpellini introduced limit spaces in computability at higher types in [19], while Hyland in [6] showed that Scarpellini's hierarchy is identical to Kleene's hierarchy of countable functionals over $\mathbb{N}$.

In [13], p.470, Normann presented this hierarchy using limit spaces and the corresponding density theorem using the notion of the nth approximation of a functional, for each $n \in \mathbb{N}$. Here we generalize Normann's presentation by defining two new subcategories of limit spaces, the limit spaces with approximations, and the limit spaces with general approximations. As Scott's information systems have the approximation objects (formal neighborhoods) as primitive notions, forming a constructive counterpart to abstract algebraic domains, the approximation functions in a limit space with approximations are given beforehand too.

A *limit space with approximations* is a structure $\mathbb{A} = (X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ such that $(X, \lim)$ is a limit space, and, for each $n \in \mathbb{N}$ the approximation functions $\mathrm{Appr}_n : X \to X$ satisfy the following properties:

(i) Appr_n is lim-continuous.
(ii) $\mathrm{Appr}_n(\mathrm{Appr}_m(x)) = \mathrm{Appr}_{\min(n,m)}(x)$, for each $x \in X$.
(iii) $D_n = \mathrm{Appr}_n(X) = \{\mathrm{Appr}_n(x) \mid x \in X\}$ is an inhabited finite set.
(iv) $\lim(x, x_n) \to \lim(x, \mathrm{Appr}_n(x_n))$, for each $x \in X$ and $x_n \in X^{\mathbb{N}}$.

Condition (ii) can also be written as the conjunction of the following two clauses:

$$\text{(iia)} \quad n \leq m \to \mathrm{Appr}_n(\mathrm{Appr}_m(x)) = \mathrm{Appr}_n(x),$$

$$\text{(iib)} \quad m < n \to \mathrm{Appr}_n(\mathrm{Appr}_m(x)) = \mathrm{Appr}_m(x),$$

which express a natural compatibility between the approximations; the approximation of a bigger approximation is the initial one (iia), while the bigger approximation of a smaller one cannot add new information to it (iib). Of course, condition (ii) implies

$$\text{(iic)} \quad \mathrm{Appr}_n(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(x).$$

[9] A proof that the infinite pigeonhole principle is equivalent to the limited principle of omniscience over Veldman's very weak system BIM can be found in [17].

A direct consequence of condition (ii) is the following:

$$\text{(iid)} \quad n < m \to D_n \subseteq D_m,$$

since, if $y \in \mathrm{Appr}_n(X)$, there exists some $x \in X$ such that $y = \mathrm{Appr}_n(x)$, therefore $y = \mathrm{Appr}_m(\mathrm{Appr}_n(x))$ i.e., $y \in \mathrm{Appr}_m(X)$. It is direct from (iid) that the set $\mathcal{B} = \{D_n \mid n \in \mathbb{N}\}$ is a countable filter base on X, since by their definition D_n are inhabited[10], and $D_n \cap D_m = D_{\min(n,m)}$.

If a structure $\mathbb{A}$ satisfies condition (iic) (iii) and (iv), but not necessarily conditions (i) and (ii), then we call it a *limit space with general approximations*. A significant property of limit spaces with (general) approximations, and a central reason for their study, is that a density theorem holds for them in a direct way.

Proposition 6 (density theorem, BISH)**.** *If $\mathbb{A}$ is a limit space with (general) approximations and $x \in X$, then $\lim(x, \mathrm{Appr}_n(x))$. Moreover, the set*

$$D = \bigcup_{n \in \mathbb{N}} D_n$$

is an enumerable dense subset of $(X, \mathcal{T}_{\lim})$.

Proof. By condition (iv), considering the constant sequence (x), we get directly $\lim(x, x) \to \lim(x, \mathrm{Appr}_n(x))$ i.e., D is a lim-dense subset of X. Therefore, by a remark of section 2 we have that D is a dense subset of $(X, \mathcal{T}_{\lim})$. Of course, D is enumerable as a countable union of finite sets. □

Note that in the previous proof we only used conditions (iii) and (iv) of limit spaces with approximations. A limit space which is not lim-separable, like $(X, \lim_{\mathrm{di}})$, where X is an uncountable set, and $\lim_{\mathrm{di}}$ is the limit relation generating the discrete topology on X, cannot be a limit space with approximations.

A special extension theorem holds also directly for the limit spaces with approximations; for each function f defined on D there is a sequence of lim-continuous functions which extend uniformly arbitrary big "parts" of f.

Proposition 7 (extension theorem, BISH)**.** *If $\mathbb{A}$ is a limit space with approximations, then each set D_n is a* lim*-retract of X. If $(Y, \lim)$ is a limit space, any* lim*-continuous function $f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a* lim*-continuous extension $F_n : (X, \lim) \to (Y, \lim)$. If $f : (D, \lim_D) \to (Y, \lim)$ is* lim*-continuous, then there is a sequence $(F_n)_n$ of* lim*-continuous functions $F_n : X \to Y$ such that $F_{n|D_n} = f_{|D_n}$ and $F_{n+1|D_n} = F_{n|D_n}$, for each n.*

Proof. Since each function $\mathrm{Appr}_n : (X, \lim) \to (X, \lim)$ is lim-continuous, each function $\mathrm{Appr}_n : (X, \lim) \to (D_n, \lim_{D_n})$ is lim-continuous too. Since any $a \in \mathrm{Appr}_n(X)$ has the form $\mathrm{Appr}_n(x)$, for some $x \in X$, we get that $\mathrm{Appr}_n(a) = \mathrm{Appr}_n(\mathrm{Appr}_n(x)) = a$. By Proposition 2 we get that a lim-continuous function

[10] Although in [1], p.124, the elements of a filter are defined to be non-void, we prefer to consider them as inhabited sets.

$f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a lim-continuous extension F. By (i) $f_n = f_{|D_n}$ is extended to a lim-continuous function $F_n : X \to Y$, and since by (iid) $D_n \subseteq D_{n+1}$ we get that $F_{n+1|D_n} = F_{n|D_n}$, for each n. $\square$

Note that in the previous proof we used not only condition (iic), but also condition (i) of our definition and condition (iid), therefore $\mathbb{A}$ cannot be a limit space with general approximations. Since D_n is finite we know classically (Proposition 5) that any function $f : (D_n, \lim_{D_n}) \to (Y, \lim)$ is lim-continuous, therefore it is extendable to X. The extension theorem, though, is independent from the cardinalities of D_n.

Remark 8 (BISH)**.** *There exists a limit space with approximations which does not have the uniqueness property.*

Proof. If X is a finite set of more than two elements and $\lim = X \times X^{\mathbb{N}}$, we define $\mathrm{Appr}_n(x) = x$, for each $n \in \mathbb{N}$. Trivially, the approximations functions Appr_n are lim-continuous, and satisfy condition (ii). By definition, $\mathrm{Appr}_n(X)$ is a finite set, and condition (iv) also follows trivially. Since any sequence converges to any point of X, this limit space doesn't have the uniqueness property. $\square$

The next two results express that the categories **Appr** and **Gappr** are cartesian closed (the morphisms are the lim-continuous functions, and the terminal object is excluded).

Proposition 9 (BISH)**.** *If $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ and $(Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ are limit spaces with (general) approximations, and if we define on $X \times Y$*

$$\mathrm{Appr}_n(x, y) := (\mathrm{Appr}_n(x), \mathrm{Appr}_n(y)),$$

for each n, then $(X \times Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ is a limit space with (general) approximations, where lim *is the already defined* lim*-relation on $X \times Y$.*

Proof. Continuity of Appr_n on $X \times Y$ is reduced to the continuity of the approximation functions on X and Y. Condition (ii) (or (iic) follows directly by the definition of $\mathrm{Appr}_n(x, y)$ and the corresponding properties on X and Y. The range $\mathrm{Appr}_n(X \times Y)$ is finite as a subset of the finite set $\mathrm{Appr}_n(X) \times \mathrm{Appr}_n(Y)$, while condition (iv) follows directly from the definition of lim on $X \times Y$ and the corresponding properties on X and Y. $\square$

Theorem 10 (BISH)**.** *If $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ and $(Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ are limit spaces with (general) approximations, and if we define, for each n and $f \in X \to Y$,*

$$f \mapsto \mathrm{Appr}_n(f),$$

$$\mathrm{Appr}_n(f)(x) := \mathrm{Appr}_n(f(\mathrm{Appr}_n(x))),$$

for each $x \in X$, then $(X \to Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ is a limit space with (general) approximations, where lim *is the already defined* lim*-relation on $X \to Y$.*

Proof. (i) If $n \in \mathbb{N}$, we need to show that $\lim_k(f, f_k) \to \lim_k(\mathrm{Appr}_n(f), \mathrm{Appr}_n(f_k))$, where, by definition, the hypothesis amounts to

$$\forall_{x\in X}\forall_{x_k\in X^{\mathbb{N}}}(\lim_k(x, x_k) \to \lim_k(f(x), f_k(x_k))),$$

and the conclusion to

$$\forall_{x\in X}\forall_{x_k\in X^{\mathbb{N}}}(\lim_k(x, x_k) \to \lim_k(\mathrm{Appr}_n(f)(x), \mathrm{Appr}_n(f_k)(x_k))).$$

We fix $x \in X$ and $x_k \in X^{\mathbb{N}}$ such that $\lim(x, x_k)$. Since the corresponding approximation function Appr_n on X is lim-continuous, we get

$$\lim_k(x, x_k) \to \lim_k(\mathrm{Appr}_n(x), \mathrm{Appr}_n(x_k)).$$

If we apply the definition of $\lim(f, f_k)$ on $x' = \mathrm{Appr}_n(x)$ and ${x'}_k = \mathrm{Appr}_n(x_k)$, we get

$$\lim_k(f(\mathrm{Appr}_n(x)), f_k(\mathrm{Appr}_n(x_k))).$$

Since the corresponding approximation function Appr_n on Y is continuous, we get

$$\lim_k(\mathrm{Appr}_n(f(\mathrm{Appr}_n(x))), \mathrm{Appr}_n(f_k(\mathrm{Appr}_n(x_k)))) \leftrightarrow$$
$$\leftrightarrow \lim_k(\mathrm{Appr}_n(f)(x), \mathrm{Appr}_n(f_k(x_k))).$$

(ii) It suffices to show that

$$\forall_{x\in X}(\mathrm{Appr}_n(\mathrm{Appr}_m(f))(x) = \mathrm{Appr}_{\min(n,m)}(f)(x)).$$

If we fix some $x \in X$, then

$$\begin{aligned}
\mathrm{Appr}_n(\mathrm{Appr}_m(f))(x) &\overset{\mathrm{def}}{=} \mathrm{Appr}_n(\mathrm{Appr}_m(f)(\mathrm{Appr}_n(x))) \\
&\overset{\mathrm{def}}{=} \mathrm{Appr}_n(\mathrm{Appr}_m(f(\mathrm{Appr}_m(\mathrm{Appr}_n(x))))) \\
&\overset{(*)}{=} \mathrm{Appr}_n(\mathrm{Appr}_m(f(\mathrm{Appr}_{\min(n,m)}(x)))) \\
&\overset{(**)}{=} \mathrm{Appr}_{\min(n,m)}(f(\mathrm{Appr}_{\min(n,m)}(x))) \\
&\overset{\mathrm{def}}{=} \mathrm{Appr}_{\min(n,m)}(f)(x),
\end{aligned}$$

where equality $\overset{(*)}{=}$ is justified by the condition (ii) of a limit space with approximations on X, and equality $\overset{(**)}{=}$ is justified by the condition (ii) on Y[11].

[11] If we had used conditions (iia) and (iib) instead of (ii), then (iia) in $X \to Y$ requires (iia) in X and (iib) in Y, while (iib) in $X \to Y$ requires (iib) in X and (iia) in Y.

Condition (iic) alone is satisfied in the same way i.e.,

$$\begin{aligned}\mathrm{Appr}_n(\mathrm{Appr}_n(f))(x) &= \mathrm{Appr}_n(\mathrm{Appr}_n(f)(\mathrm{Appr}_n(x)))\\ &= \mathrm{Appr}_n(\mathrm{Appr}_n(f(\mathrm{Appr}_n(\mathrm{Appr}_n(x)))))\\ &= \mathrm{Appr}_n(\mathrm{Appr}_n(f(\mathrm{Appr}_n(x))))\\ &= \mathrm{Appr}_n(f(\mathrm{Appr}_n(x)))\\ &= \mathrm{Appr}_n(f)(x)\end{aligned}$$

(iii) First we show that if $\mathrm{Appr}_n(Y)$ in inhabited, then $\mathrm{Appr}_n(X \to Y)$ is also inhabited. If $y \in Y$, then for the constant function $\hat{y}$ we have that

$$\begin{aligned}\mathrm{Appr}_n(\hat{y})(x) &= \mathrm{Appr}_n(\hat{y}(\mathrm{Appr}_n(x)))\\ &= \mathrm{Appr}_n(y).\end{aligned}$$

Hence,

$$\mathrm{Appr}_n(\hat{y}) = \widehat{\mathrm{Appr}_n(y)}$$

i.e., the nth approximation of a constant function $\hat{y}$ is the constant function $\widehat{\mathrm{Appr}_n(y)}$. If y inhabits $\mathrm{Appr}_n(Y)$, then the constant function $\mathrm{Appr}_n(\hat{y}) = \widehat{\mathrm{Appr}_n(y)} = \hat{y}$ inhabits $\mathrm{Appr}_n(X \to Y)$. I.e., in this case the nth approximation of $\hat{y}$ is identical to it.
To prove the finiteness of $\mathrm{Appr}_n(X \to Y)$ we show that the nth-approximation of a function in the function limit space acts equally on its input and on the nth-approximation of it i.e.,

$$\text{(v)}\quad \mathrm{Appr}_n(f)(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(f)(x),$$

since

$$\begin{aligned}\mathrm{Appr}_n(f)(\mathrm{Appr}_n(x)) &= \mathrm{Appr}_n(f(\mathrm{Appr}_n(\mathrm{Appr}_n(x))))\\ &\overset{\text{iic}}{=} \mathrm{Appr}_n(f(\mathrm{Appr}_n(x)))\\ &= \mathrm{Appr}_n(f)(x).\end{aligned}$$

Now it is obvious why $\mathrm{Appr}_n(X \to Y) = \{\mathrm{Appr}_n(f) \mid f \in X \to Y\}$ is a finite set: because of (v) the function $\mathrm{Appr}_n(f) : X \to Y$ is determined by its restriction

$$\mathrm{Appr}_n(f)_{|\mathrm{Appr}_n(X)} : \mathrm{Appr}_n(X) \to \mathrm{Appr}_n(Y)$$

i.e.,

$$\mathrm{Appr}_n(f)_{|\mathrm{Appr}_n(X)} = \mathrm{Appr}_n(g)_{|\mathrm{Appr}_n(X)} \to \mathrm{Appr}_n(f) = \mathrm{Appr}_n(g).$$

But then

$$|\mathrm{Appr}_n(X \to Y)| \leq |\mathrm{Appr}_n(Y)^{\mathrm{Appr}_n(X)}|.$$

(iv) If we fix $f \in X \to Y$ and $f_n \in (X \to Y)^{\mathbb{N}}$ we show that $\lim_n(f, f_n) \to \lim_n(f, \mathrm{Appr}_n(f_n))$, where the hypothesis, by definition, amounts to

$$\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(f(x), f_n(x_n)))$$

and the conclusion to

$$\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(f(x), \mathrm{Appr}_n(f_n)(x_n))).$$

We fix $x \in X$ and $x_n \in X^{\mathbb{N}}$ such that $\lim(x, x_n)$. By condition (iv) on X we get that $\lim(x, x_n) \to \lim(x, \mathrm{Appr}_n(x_n))$, while by the definition of $\lim(f, f_n)$ on x and the sequence $\mathrm{Appr}_n(x_n)$ we have that $\lim(f(x), f_n(\mathrm{Appr}_n(x_n)))$. By condition (iv) on Y we get that

$$\lim(f(x), \mathrm{Appr}_n(f_n(\mathrm{Appr}_n(x_n)))) \leftrightarrow \lim(f(x), \mathrm{Appr}_n(f_n)(x_n)).$$

□

Note that the category **Gappr** is cartesian closed, since in the proof of condition (iii) we used only condition (iic) on X. Next result is proved similarly, and explains why the functions Appr_n in the arrow case are defined as above.

Proposition 11 (BISH)**.** *If $(X, \lim, (\mathrm{Appr}_n)_{n \in \mathbb{N}})$ is a limit space with approximations and $(Y, \lim)$ is a limit space, and if for each n we define on $X \to Y$*

$$f \mapsto \mathrm{Appr}_n{}'(f) \quad \textit{and} \quad \mathrm{Appr}_n{}'(f)(x) := f(\mathrm{Appr}_n(x)),$$

then $(X \to Y, \lim, (\mathrm{Appr}_n{}')_{n \in \mathbb{N}})$ is a limit space with approximation functions satisfying conditions (i), (ii) and (iv), where lim *is the already defined* lim*-relation on $X \to Y$.*

Although $(X \to Y, \lim, (\mathrm{Appr}_n{}')_{n \in \mathbb{N}})$ satisfies condition (v), what we conclude is that $|\mathrm{Appr}_n(X \to Y)| \leq |Y^{\mathrm{Appr}_n(X)}|$, therefore, we cannot form a cartesian closed category of limit spaces using the approximation functions $\mathrm{Appr}_n{}'$.

4 Examples

The prime example of a hierarchy of limit spaces with approximations, on which the definition of **Appr** was actually based, is the hierarchy $\mathrm{Ct}_{\mathbb{N}}(\rho)$ of countable functionals over $\mathbb{N}$. If our type system is $\iota = \mathbb{N} \mid \rho \to \sigma$, then we define the following hierarchy of limit spaces:

$$\mathrm{Ct}(\iota) := (\mathbb{N}, \lim_{\mathcal{T}_{\mathrm{di}}}),$$

$$\mathrm{Ct}(\rho \to \sigma) := (\mathrm{Ct}(\rho) \to \mathrm{Ct}(\sigma), \lim_{\rho \to \sigma}),$$

where $\mathcal{T}_{\mathrm{di}}$ is the discrete topology on $\mathbb{N}$, and $\lim_{\rho \to \sigma}$ is defined as in the case of a function limit space. To each limit space $(\mathrm{Ct}(\rho), \lim_\rho)$ the following approximation functions are added:

$$\mathrm{Appr}_{n,\iota}(m) = \min(n, m),$$

while if $F \in \mathrm{Ct}(\rho \to \sigma)$ and $f \in \mathrm{Ct}(\rho)$ we define

$$F \mapsto \mathrm{Appr}_{n,\rho \to \sigma}(F),$$

$$\mathrm{Appr}_{n,\rho \to \sigma}(F)(f) = \mathrm{Appr}_{n,\sigma}(F(\mathrm{Appr}_{n,\rho}(f))).$$

Corollary 12 (BISH)**.** *The structure* $\mathbb{A}_\rho = (\mathrm{Ct}(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n\in\mathbb{N}})$ *is a limit space with approximations, for each* ρ. *Moreover, there exists an enumerable dense subset* D_ρ *in* $(\mathrm{Ct}(\rho), \mathcal{T}_{\lim_\rho})$*, for each* ρ.

Proof. In case $\rho = \iota$ it is direct that $(\mathrm{Ct}(\iota), \lim_\iota, (\mathrm{Appr}_{n,\iota})_{n\in\mathbb{N}})$ is a limit space with approximations: Appr_n is $\lim_{\mathcal{T}_{\mathrm{di}}}$-continuous i.e., $\lim_{\mathcal{T}_{\mathrm{di}}}(m, m_l)$ implies that $\lim_{\mathcal{T}_{\mathrm{di}}}(\mathrm{Appr}_n(m), \mathrm{Appr}_n(m_l))$, since the hypothesis amounts to the sequence m_l being eventually the constant sequence m, therefore the sequence $\mathrm{Appr}_n(m_l)$ is eventually the constant sequence $\mathrm{Appr}_n(m)$. Condition (ii) holds automatically, while for condition (iii) we see that $\mathrm{Appr}_n(\mathbb{N}) = \{0, 1, \dots, n\}$. Condition (iv) is written as $\lim_{\mathcal{T}_{\mathrm{di}}}(m, m_l) \to \lim_{\mathcal{T}_{\mathrm{di}}}(m, \mathrm{Appr}_l(m_l))$. Since the premiss says that the sequence m_l is after some index l_0 constantly m, then for $l \geq \max(l_0, m)$ we get that the sequence $\mathrm{Appr}_l(m_l)$ is constantly m. The fact that $(\mathrm{Ct}(\rho \to \sigma), \lim_{\rho\to\sigma}, (\mathrm{Appr}_{n,\rho\to\sigma})_{n\in\mathbb{N}})$ is a limit space with approximations is a direct consequence of Theorem 10. Moreover, as a consequence of Proposition 6 we get that the set $D_\rho = \bigcup_{n\in\mathbb{N}} \mathrm{Appr}_n(\mathrm{Ct}(\rho))$ is an enumerable dense subset of $(\mathrm{Ct}(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ. $\square$

Although for our next example we could work with any compact subset of $\mathcal{N}$, which is the body of a fan on $\mathbb{N}$, we work with the simplest such subset, the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$. As it is well-known, the family

$$B(\overline{\alpha}(k)) = \{\beta \in \mathcal{C} \mid \overline{\beta}(k) = \overline{\alpha}(k)\} = \{\beta \in \mathcal{C} \mid \overline{\alpha}(k) \prec \beta\},$$

where $u \prec \alpha$ means that the sequence α extends $u \in 2^{<\mathbb{N}}$ and $\overline{\alpha}(k)$ is the k-initial segment of α, is a countable base of a topology $\mathcal{T}$ on $\mathcal{C}$. The space $(\mathcal{C}, \mathcal{T})$ is a T_1, compact space with a countable base of clopen sets, and without isolated points[12]. Consequently, we have that

$$\begin{aligned}\lim_{\mathcal{T}}(\alpha, \alpha_n) &\leftrightarrow \forall_k \exists_{n_0} \forall_{n\geq n_0}(\alpha_n(k) = \alpha(k)) \\ &\leftrightarrow \forall_k \exists_{n_0} \forall_{n\geq n_0}(\overline{\alpha_n}(k) = \overline{\alpha}(k)),\end{aligned}$$

for each $\alpha \in \mathcal{C}$ and $\alpha_n \in \mathcal{C}^{\mathbb{N}}$. We define the approximation functions $\mathrm{Appr}_n : \mathcal{C} \to \mathcal{C}$ by

$$\alpha \mapsto \mathrm{Appr}_n(\alpha),$$

$$\mathrm{Appr}_n(\alpha)(i) = \begin{cases} \alpha(i) & \text{, if } i \leq n \\ 0 & \text{, if } i > n \end{cases}$$

i.e.,

$$\mathrm{Appr}_n(\alpha) = \overline{\alpha}(n+1) * \overline{0},$$

where $\overline{0}$ denotes the constant sequence 0, and $u * \alpha$ denotes the concatenation of the finite sequence u and the infinite sequence α.

Proposition 13 (BISH)**.** *The structure* $\mathbb{A} = (\mathcal{C}, \lim_{\mathcal{T}}, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *is a limit space with approximations.*

[12] As shown by Brouwer, these properties characterize the topological space $(\mathcal{C}, \mathcal{T})$.

Proof. (i) We fix some $k \in \mathbb{N}$ and we show that $\lim_n(\alpha, \alpha_n)$ implies that $\lim_n(\mathrm{Appr}_k(\alpha), \mathrm{Appr}_k(\alpha_n))$. If we take n_0 the index for which all sequences α_n, for $n \geq n_0$, have the same $(k+1)$-initial segment with α, then for all such n, by the definition of Appr_k, we get that $\mathrm{Appr}_k(\alpha) = \mathrm{Appr}_k(\alpha_n)$, and then the conclusion $\lim_n(\mathrm{Appr}_k(\alpha), \mathrm{Appr}_k(\alpha_n))$ follows automatically.
(ii) We consider first $n \leq m$. We have that

$$\begin{aligned}
\mathrm{Appr}_n(\mathrm{Appr}_m(\alpha)) &= \overline{\mathrm{Appr}_m(\alpha)}(n+1) * \overline{0} \\
&= \overline{\overline{\alpha}(m+1) * \overline{0}}(n+1) * \overline{0} \\
&= \overline{\alpha}(n+1) * \overline{0} \\
&= \mathrm{Appr}_n(\alpha).
\end{aligned}$$

If $n > m$ we have that

$$\begin{aligned}
\mathrm{Appr}_n(\mathrm{Appr}_m(\alpha)) &= \overline{\mathrm{Appr}_m(\alpha)}(n+1) * \overline{0} \\
&= \overline{\overline{\alpha}(m+1) * \overline{0}}(n+1) * \overline{0} \\
&= \overline{\alpha}(m+1) * \overline{0} \\
&= \mathrm{Appr}_m(\alpha).
\end{aligned}$$

(iii) For each n the set $\mathrm{Appr}_n(\mathcal{C})$ is finite, since it is equipollent to the finite set of the nodes of the Cantor tree $2^{<\mathbb{N}}$ of length $n+1$.
(iv) We fix $\alpha \in \mathcal{C}$, $\alpha_n \in \mathcal{C}^{\mathbb{N}}$ and we show that $\lim(\alpha, \alpha_n) \to \lim(\alpha, \mathrm{Appr}_n(\alpha_n))$, where the conclusion amounts to $\forall_k \exists_{n_0} \forall_{n \geq n_0} (\mathrm{Appr}_n(\alpha_n)(k) = \alpha(k))$. We fix some $k \in \mathbb{N}$. For that k we know by the definition of the premiss that

$$\exists_{n_0(k)} \forall_{n \geq n_0(k)} (\alpha_n(k) = \alpha(k)).$$

We define $n_0 = \max(n_0(k), k)$ and we consider some $n \geq n_0$. Then

$$\mathrm{Appr}_n(\alpha_n) = \overline{(\alpha_n)}(n+1) * \overline{0} = (\alpha_n(0), \ldots, \alpha_n(k), \ldots, \alpha_n(n)) * \overline{0},$$

hence $\mathrm{Appr}_n(\alpha_n)(k) = \alpha(k)$. $\square$

Proposition 6 guarantees that the set $D = \bigcup_{n \in \mathbb{N}} \mathrm{Appr}_n(\mathcal{C})$ is an enumerable dense subset of $\mathcal{C}$, a fact already known, of course, by the study of the initial topology $\mathcal{T}$ on $\mathcal{C}$. What our general study of **Appr** adds is that, if our type system is $\iota = \mathcal{C} \mid \rho \to \sigma$, and if we define the following hierarchy of limit spaces

$$\mathcal{C}(\iota) := (\mathcal{C}, \lim_{\mathcal{T}}),$$

$$\mathcal{C}(\rho \to \sigma) := (\mathcal{C}(\rho) \to \mathcal{C}(\sigma), \lim_{\rho \to \sigma}),$$

and supply these spaces with the approximation functions $\mathrm{Appr}_{n,\iota}$ as defined above, and the arrow functions $\mathrm{Appr}_{n,\rho\to\sigma}$, we get directly the following corollary:

Corollary 14 (BISH)**.** *The structure $\mathbb{A}_\rho = (\mathcal{C}(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n\in\mathbb{N}})$ is a limit space with approximations, for each ρ. Moreover, there exists an enumerable dense subset D_ρ in $(\mathcal{C}(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ.*

Note that we can prove similarly to Proposition 13 that the Baire space $\mathcal{N}$ can be seen as a limit space with approximations, except that $\mathrm{Appr}_n(\mathcal{N})$ is countable, and hence the density theorem cannot be applied directly in the hierarchy over $\mathcal{N}$ (enumerability of the dense set is lost, but not its existence, since density relies only on conditions (iii) and (iv)). It is direct to see classically that the topological space $(\mathcal{C}, \mathcal{T})$ is sequential, although by the metrizability of $\mathcal{C}$ this is a special case of Remark 18.

Next we use the notion of a limit space with general approximations to describe a compact metric space X. This is a natural step from $\mathcal{C}$, since every such space X is the quotient of $\mathcal{C}$. Actually we are going to use a lemma proved within BISH in [3], p.105, in order to define the approximation functions on the limit space generated by the metric structure of X. This lemma is necessary to show constructively the existence of a uniform quotient map from $\mathcal{C}$ onto X.

Following [3], p.28, if (X, d) is a metric space, a set $Y \subseteq X$ is called an ϵ-approximation to X, if $\forall_{x\in X}\exists_{y\in Y}(d(x, y) < \epsilon)$. A metric space (X, d) is *totally bounded*, if for each $\epsilon > 0$ there exists some $Y \subseteq X$ such that Y is a finite ϵ-approximation to X, while it is *compact*, if it is complete and totally bounded.

Lemma 15 (BISH)**.** *If (X, d) is an inhabited compact metric space and $r \in (0, \frac{1}{2}]$, there exist sequences $(x_u)_{u\in 2^{<\mathbb{N}}}$ and $\gamma \in \mathcal{S}$ such that, for each $n \geq 1$, we have that*

(i) $\{x_u \mid |u| = \gamma(n)\}$ *is an r^n-approximation to X.*

(ii) $|u| = \gamma(n) \to \forall_{w\in 2^{<\mathbb{N}}}(d(x_u, x_{u*w}) < \frac{r^{n-2}}{1-r})$.

(iii) $|u| = \gamma(n) \to d(x, x_u) < r^{n-1} - r^{n+1} \to$

$$\to \exists_{w\in 2^{<\mathbb{N}}}(|u * w| = \gamma(n + 1) \ \wedge \ d(x, x_{u*w}) < r^{n+1}).$$

(iv) $|u| = \gamma(n) \to |u * w| < \gamma(n + 1) \to x_{u*w} = x_u$.

Note that $u * w$ denotes the concatenation of the finite sequences u, w, and that the proof of the above lemma uses for the definition of γ the principle of dependent choices on $\mathbb{N}$. The next proposition and the subsequent corollary are in BISH, if there is a way to decide whether $x \in \mathrm{Appr}_n(X)$ or not.

Proposition 16. *If (X, d) is an inhabited compact metric space, $\lim_d$ is the limit relation induced by its metric d, and $\mathrm{Appr}_n : X \to X$ is defined, for each n, by*

$$\mathrm{Appr}_n(x) = \begin{cases} x_{\min_\prec\{u\in 2^{<\mathbb{N}} \mid x_u\in \mathrm{Appr}_n(X) \ \wedge \ d(x,x_u)<r^n\}} & , \textit{if } x \notin \mathrm{Appr}_n(X) \\ x & , \textit{if } x \in \mathrm{Appr}_n(X), \end{cases}$$

where $\prec$ is any fixed total ordering on $2^{<\mathbb{N}}$, and

$$\mathrm{Appr}_n(X) = \{x_u \mid |u| = \gamma(n)\}$$

and the sequences $(x_u)_{u\in 2^{<\mathbb{N}}}$ and $\gamma \in \mathcal{S}$ are determined in Lemma 15, then the structure $\mathbb{A} = (X, \lim_d, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ is a limit space with general approximations.

Proof. The property $\mathrm{Appr}_n(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(x)$ follows automatically by the definition of $\mathrm{Appr}_n(x)$. The fact that $\mathrm{Appr}_n(X)$ is finite follows by the finiteness of the set of nodes in $2^{<\mathbb{N}}$ of fixed length $\gamma(n)$. Finally we show that $\lim_d(x, x_n) \to \lim_d(x, \mathrm{Appr}_n(x_n))$. The premiss is $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}(d(x,x_n) < \epsilon)$, while the conclusion amounts to $\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}(d(x,\mathrm{Appr}_n(x_n)) < \epsilon)$. We fix some $\epsilon > 0$, and by the unfolding of the premiss we find $n_0(\frac{\epsilon}{2})$ such that $d(x, x_n) < \frac{\epsilon}{2}$, for each $n \geq n_0(\frac{\epsilon}{2})$. Also, there is some n_1 such that $r^n < \frac{\epsilon}{2}$, for each $n \geq n_1$. For each $n \geq \max(n_0(\frac{\epsilon}{2}), n_1)$ we have that

$$\begin{aligned} d(x, \mathrm{Appr}_n(x_n)) &\leq d(x, x_n) + d(x_n, \mathrm{Appr}_n(x_n)) \\ &< \frac{\epsilon}{2} + r^n \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

$\square$

Thus, if our type system is $\iota = X \mid \rho \to \sigma$, and if we define the following hierarchy of limit spaces over a fixed inhabited compact metric space (X, d)

$$X(\iota) := (X, \lim_d),$$

$$X(\rho \to \sigma) := (X(\rho) \to X(\sigma), \lim_{\rho\to\sigma}),$$

and add to these spaces the approximation functions $\mathrm{Appr}_{n,\iota}$ as defined above, and the arrow functions $\mathrm{Appr}_{n,\rho\to\sigma}$, we get directly by the fact that **Gappr** is cartesian closed the following corollary.

Corollary 17. *The structure $\mathbb{A}_\rho = (X(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n\in\mathbb{N}})$ is a limit space with general approximations, for each ρ. Moreover, there exists an enumerable dense subset D_ρ in $(X(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ.*

Of course, we could use a type system where the base types are determined by more than one compact metric spaces and have a result similar to Corollary 17.

Remark 18 (CLASS)**.** *A metric space (X, d) is a sequential space.*

Proof. It suffices to prove that $\mathcal{T}_{\lim_d} \subseteq \mathcal{T}$. By definition $\mathcal{O} \in \mathcal{T}_{\lim_d}$ if and only if $\forall_{x\in X}\forall_{x_n\in X^{\mathbb{N}}}(\lim_d(x, x_n) \to \mathrm{ev}(x_n, \mathcal{O}))$. In order to show that $\mathcal{O}$ is in the metric topology we fix some $x \in \mathcal{O}$ and we prove the existence of some $n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subseteq \mathcal{O}$. If the converse is the case i.e., if $\exists_{x\in\mathcal{O}}\forall_n\exists_{y\in B(x,\frac{1}{n})}(y \notin \mathcal{O})$, then by the appropriate choice principle we get a sequence x_n such that $x_n \in B(x, \frac{1}{n})$, and $x_n \notin \mathcal{O}$, for each n. But then $\lim_d(x, x_n)$, while x_n is always outside $\mathcal{O}$. $\square$

Kisyński's theorem (proved in [8]) suffices to prove that all limit spaces in the above (or mixed) hierarchies are topological, since all of them satisfy the uniqueness property (Theorem 4(iii)).

Theorem 19 (Kisyński 1960 (CLASS))**.** *If a limit space satisfies the uniqueness property, then it is topological.*

Of course, there are topological limit spaces, like the trivial limit space $(X, X \times X^{\mathbb{N}})$, which do not have the uniqueness property. Note that Hyland in [6], p.59, proved that if a limit space induces a Hausdorff space, then it is topological; a result, because of a remark in Section 2, weaker than Kisyński's theorem and of a much simpler proof. This shows that the transition of a result on a Hausdorff space to a limit space with the uniqueness property is not in general trivial. Next simple corollary of Kisyński's theorem is proved (directly) also in [4], p.484.

Corollary 20 (CLASS)**.** *(i) If $f : (X, \mathcal{T}_{\lim_X}) \to (Y, \mathcal{T}_{\lim_Y})$ is continuous and $(Y, \lim_Y)$ has the uniqueness property, then $f : (X, \lim_X) \to (Y, \lim_Y)$ is* lim-*continuous.*
(ii) If $(X, \lim)$ is a limit space and $(Y, \lim_Y)$ has the uniqueness property, then $\mathbb{C}(X, Y) = X \to Y$, where $\mathbb{C}(X, Y)$ denotes the set of continuous functions from X to Y w.r.t. the topologies induced by the corresponding limits.

Proof. (i) By Kisyński's theorem the limit space $(Y, \lim_Y)$ is topological, therefore by Proposition 3(ii) f is also lim-continuous.
(ii) This is derived directly from (i) and Proposition 3(i). □

Of course, part (ii) of the previous corollary applies to each function space appearing in the above hierarchies of limit spaces. Next proposition reveals the similarities between the induced topology of an abstract limit space with approximations and the topology of Cantor space.

Proposition 21. *If $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ is a limit space with approximations satisfying the uniqueness property, then the following hold:*

*(i) (*CLASS*) The set $Z(\mathrm{Appr}_n, e) = \{x \in X \mid \mathrm{Appr}_n(x) = e\}$ is* lim-*clopen, for each $n \in \mathbb{N}$ and $e \in \mathrm{Appr}_n(X)$.*
*(ii) (*CLASS*) If X is a compact metric space and* $\lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}}$ *are defined as above, then the functions* Appr_n *are not in general* lim-*continuous.*
*(iii) (*BISH*) The sets of the form $Z(\mathrm{Appr}_n, e)$ form a countable basis for a topology $\mathcal{T}_Z$ on X which is included in $\mathcal{T}_{\lim}$.*
*(iv) (*CLASS*) The topological space $(X, \mathcal{T}_Z)$ is T_1 and 2nd countable.*

Proof. (i) Since Appr_n is lim-continuous, and so is the function $\hat{e} : X \to X$ with the constant value e, we get by Proposition 1(iii) that $Z(\mathrm{Appr}_n, e) = Z(\mathrm{Appr}_n, \hat{e})$ is lim-closed. To prove that $Z(\mathrm{Appr}_n, e)$ is a lim-open set we fix some $x \in \mathrm{Appr}_n(X)$ and some $x_m \subseteq X$ such that $\lim(x, x_m)$, and we show that eventually $\mathrm{Appr}_n(x_m) = \mathrm{Appr}_n(x) = e$. Of course, if $\mathrm{Appr}_n(X) = \{e\}$, we get directly what we want. Suppose next that $\mathrm{Appr}_n(X) = \{e_1, \ldots, e_{i-1}, e_i = e, e_{i+1}, \ldots, e_k\}$, and also that the sequence $\mathrm{Appr}_n(x_m)$ is not eventually constant e i.e., there are infinite terms of $\mathrm{Appr}_n(x_m)$ other than e. By the infinite pigeonhole principle there exists $e_j \neq e$, and $\alpha \in \mathcal{S}$ such that $\mathrm{Appr}_n(x_{\alpha(m)}) = e_j$, for each m, which implies $\lim(e_j, \mathrm{Appr}_n(x_{\alpha(m)}))$. On the other hand, the lim-continuity of Appr_n implies $\lim(\mathrm{Appr}_n(x), \mathrm{Appr}_n(x_m)) \leftrightarrow \lim(e, \mathrm{Appr}_n(x_m))$,

while by condition (ii) of limit spaces we get $\lim(e, \mathrm{Appr}_n(x_{\alpha(m)}))$. The uniqueness property of the limit space implies that $e_j = e$, which is absurd.
(ii) It is clear that $Z(\mathrm{Appr}_n, e) \neq \emptyset$, since by its definition $e \in \mathrm{Appr}_n(X) \leftrightarrow \exists_{x\in X}(\mathrm{Appr}_n(x) = e) \rightarrow \exists_{x\in X}(x \in Z(\mathrm{Appr}_n, e))$. Also, if $\{e\} \subsetneq \mathrm{Appr}_n(X)$, then $Z(\mathrm{Appr}_n, e) \subsetneq X$. Since (i) rests on the lim-continuity of the approximation functions Appr_n, if that was the case for these functions in the case of a compact metric space, then (i) would hold. But there are compact metric spaces, like the connected unit interval $[0, 1]$, which have no non-trivial clopen sets.
(iii) It is clear that each $x \in X$ is in $Z(\mathrm{Appr}_n, \mathrm{Appr}_n(x))$, for each n. Next we show that if $x \in Z(\mathrm{Appr}_n, e_n) \cap Z(\mathrm{Appr}_m, e_m)$, then

$$Z(\mathrm{Appr}_n, e_n) \cap Z(\mathrm{Appr}_m, e_m) = Z(\mathrm{Appr}_{\max(n,m)}, e_{\max(n,m)}).$$

We suppose that $n < m$, and we show that $Z(\mathrm{Appr}_m, e_m) \subseteq Z(\mathrm{Appr}_n, e_n)$. By our hypothesis $\mathrm{Appr}_n(x) = e_n$ and $\mathrm{Appr}_m(x) = e_m$. Next we suppose that $\mathrm{Appr}_m(y) = e_m$ and we show that $\mathrm{Appr}_n(y) = e_n$:

$$\begin{aligned}
\mathrm{Appr}_n(y) &= \mathrm{Appr}_n(\mathrm{Appr}_m(y)) \\
&= \mathrm{Appr}_n(e_m) \\
&= \mathrm{Appr}_n(\mathrm{Appr}_m(x)) \\
&= \mathrm{Appr}_n(x) \\
&= e_n.
\end{aligned}$$

The family of all sets of the form $Z(\mathrm{Appr}_n, e)$ is countable, as a countable union of finite sets. Since each set $Z(\mathrm{Appr}_n, e)$ is lim-open, the topology $\mathcal{T}_Z$ having them as a basis is included in $\mathcal{T}_{\lim}$.
(iv) Consider $x \in X$ and some $y \in X \setminus \{x\}$. Since $x \neq y$, there exists $n \in \mathbb{N}$ such that $\mathrm{Appr}_n(x) \neq \mathrm{Appr}_n(y)$; if $\forall_n(\mathrm{Appr}_n(x) = \mathrm{Appr}_n(y))$, then $\lim(x, \mathrm{Appr}_n(x))$, $\lim(y, \mathrm{Appr}_n(y))$, and the uniqueness property of X imply $x = y$. Thus, $y \in Z(\mathrm{Appr}_n, \mathrm{Appr}_n(y)) \subseteq (X \setminus \{x\})$. The 2nd countability of $\mathcal{T}_Z$ is due to the countability of the base $(Z(\mathrm{Appr}_n, e))_{n,e}$. $\square$

One could ask if $\mathcal{T}_{\lim} \subseteq \mathcal{T}_Z$. For example, in the base case we have that $n \in \mathrm{Appr}_{n+1}(X)$ and

$$\begin{aligned}
Z(\mathrm{Appr}_{n+1}, n) &= \{x \in \mathbb{N} \mid \mathrm{Appr}_{n+1}(x) = n\} \\
&= \{x \in \mathbb{N} \mid \min(n + 1, x) = n\} \\
&= \{n\}.
\end{aligned}$$

A result of Hyland shows that this is not the case.

Proposition 22 (CLASS). *(i)* $\mathrm{Ct}(k)$ *is not 1st countable, for each* $k > 1$.
(ii) If $(X, \lim, (\mathrm{Appr}_n)_n)$ *is a limit space with approximations, then* $\mathcal{T}_Z \subsetneq \mathcal{T}_{\lim}$.
(iii) If $(X, \lim, (\mathrm{Appr}_n)_n)$ *is a limit space with approximations, then the converse to condition (iv) of the definition of a limit space with approximations i.e.,* $\lim(x, \mathrm{Appr}_n(x_n)) \rightarrow \lim(x, x_n)$*, does not hold in general.*

Proof. (i) See [6] p.64, or [11] p.54.
(ii) If the two topologies were equal, then $\mathrm{Ct}(k)$ would be 2nd countable, something which contradicts (i).
(iii) We show that the converse to condition (iv) implies that $\mathcal{T}_{\lim} \subseteq \mathcal{T}_Z$ i.e., for each lim-open set $\mathcal{O}$ and $x \in \mathcal{O}$ there is some $Z(\mathrm{Appr}_n, e)$ such that $x \in Z(\mathrm{Appr}_n, e) \subseteq \mathcal{O}$. If this is not the case, there is a sequence $y_n \subseteq X$ such that $\mathrm{Appr}_n(y_n) = \mathrm{Appr}_n(x)$ and $y_n \notin \mathcal{O}$, for each n. Since $\lim(x, \mathrm{Appr}_n(x)) \leftrightarrow \lim(x, \mathrm{Appr}_n(y_n))$, the converse implication would imply that $\lim(x, y_n)$, therefore y_n would be eventually in $\mathcal{O}$, which is absurd.

□

5 Future work

Our aim was to show that Normann's notion of the nth approximation of a countable functional is fruitful and also compatible to a general constructive point of view.

A natural question is whether limit spaces with general approximations can include separable, non-compact metric spaces, since Lemma 15 is extended within BISH to a separable metric space and the Baire space $\mathcal{N}$ instead of $\mathcal{C}$. In that way the study of the hierarchy $\mathrm{Ct}_{\mathbb{R}}(\rho)$ of functionals over $\mathbb{R}$ would be similar to that of $\mathrm{Ct}_{\mathbb{N}}(\rho)$ (the two hierarchies have a different treatment in [13]).

In a forthcoming work we show that there is a strong connection between Normann's notion of a probabilisitic projection, introduced in [14], and a limit space with general approximations. Namely, if we add the property of positivity to Normann's notion of probabilistic projection, and it is easy to see that the probabilistic projections proved by Normann to exist are actually positive, we show that a positive probabilistic selection induces a limit space with general approximations.

Although there are constructive approaches to external computability at higher types (see e.g., the theory of computable functionals TCF in [20]), we find equally interesting the development of a constructive approach to internal computability. That combines an as much as possible constructive reconstruction of the theory of limit spaces with a possible parallel interpretation of Gödel's T, typed μ-recursion and the Kleene schemes.

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