A constructive function-theoretic approach to topological compactness

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Abstract

We introduce 2-compactness, a constructive function-theoretic alternative to topological compactness, based on the notions of Bishop space and Bishop morphism, which are constructive function-theoretic alternatives to topological space and continuous function, respectively. We show that the notion of Bishop morphism is reduced to uniform continuity in important cases, overcoming one of the obstacles in developing constructive general topology posed by Bishop. We prove that 2-compactness generalizes metric compactness, namely that the uniformly continuous real-valued functions on a compact metric space form a 2-compact Bishop topology. Among other properties of 2-compact Bishop spaces, the countable Tychonoff compactness theorem is proved for them. We work within BISH^*, Bishop's informal system of constructive mathematics BISH equipped with inductive definitions with rules of countably many premises, a system strongly connected to Martin-Löf's Type Theory.

Categories and Subject Descriptors F.4.1 [*Mathematical Logic*]

Keywords constructive topology, compactness, Bishop spaces

1. Introduction

1.1 The problem of constructivizing general topology according to Bishop

In [5], p.28, Bishop described the problem of constructivizing general topology as follows.

> The constructivization of general topology is impeded by two obstacles. First, the classical notion of a topological space is not constructively viable. Second, even for metric spaces the classical notion of a continuous function is not constructively viable; the reason is that there is no constructive proof that a (pointwise) continuous function from a compact metric space to $\mathbb{R}$ is uniformly continuous. Since uniform continuity for functions on a compact space is the useful concept, pointwise continuity (no longer useful for proving uniform continuity) is left with no useful function to perform. Since uniform continuity cannot be formulated in the context of a general topological space, the latter concept also is left with no useful function to perform.

Later research in constructive algebraic topology in the context of formal topology, has shown that we need to have a more general constructive theory of topological spaces. In [27], p.237, Palmgren notes the following.

> To be able to make certain quotient and glueing constructions it is necessary to have a constructive theory of more general topological spaces than metric spaces.

If there is though a notion of space which does not copy or follow the pattern of the notion of topological space, and if the corresponding notion of morphism between two such spaces, despite the fact that uniform continuity is not part of its definition, is reduced to uniform continuity in important cases, then one could hope to overcome the two obstacles mentioned by Bishop in the constructivization of topology.

It is remarkable that although Bishop never elaborated a constructive approach to general topology, he introduced in [2] two constructive alternatives to the notion of topological space, the notion of *neighborhood space*, mainly studied by Ishihara in [18]- [20], and the notion of *function space*, or, as we call it, of *Bishop space*, studied by Bridges in [12], by Ishihara in [20], and the author in [28]-[30]. The first is a set-theoretic alternative, while the second is a function-theoretic one, and for that suits better to constructive study. Although Bishop explicitly suggested in [2], p.70, to put the emphasis on Bishop spaces instead of on neighborhood spaces, he did not develop their theory.

Here we hope to show that the notion of Bishop space is not only constructively viable, but also an appropriate notion of space for the development of 2-compactness, a computational reconstruction of topological compactness. The Bishop morphisms, defined in the next section, are the arrows in the category of Bishop spaces $\mathbf{Bis}$, they correspond to the continuous functions between topological spaces, and play a crucial role in the definition of 2-compactness.

The theory of Bishop spaces (TBS) is an approach to constructive point-function topology. Points are accepted from the beginning, hence it is not a point-free approach to topology. Most of its notions are function-theoretic, as set-theoretic notions are avoided, or play a secondary role to its development, or, when used, they can be viewed as appropriate predicates. It is constructive, as we work within Bishop's informal system of constructive mathematics BISH^*, where

$$\mathrm{BISH}^* := \mathrm{BISH} + \text{inductive definitions with rules of countably many premises},$$

and BISH is described in [6] and [7]. The system BISH^* is the working system of [2], since the notion of Borel set, on which the measure theory found in [2] is based on, is inductively defined with

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the use of rules of countably many premises. The system BISH is the working system of [6], since the measure theory found there is based on the earlier theory of profiles given by Bishop and Cheng in [4] which avoids the use of inductive definitions.

As possible formalizations of BISH^* one can consider Myhill's formal system CST^*, an extension of his system CST that formalizes BISH and accommodates inductive definitions (see [24]), or CZF + REA + DC, where Aczel's regular extension axiom REA accommodates inductive definitions in CZF (see [21]). We consider though more interesting a direct formalization of TBS into some form of Martin-Löf's Type Theory. The use of intuitionistic logic, the inductive and function-theoretic character of TBS facilitate such a formalization, which would serve the extraction of the computational content of the formalized version of TBS, as this is understood in theoretical computer science. Here though we describe the computational meaning of TBS within BISH^*.

1.2 The problem of compactness constructively

Within classical mathematics (CLASS) the compactness of a topological space X amounts to the Heine-Borel property i.e., the existence of a finite subcover for every open cover of X. Within BISH compactness is a thorny issue, since there are metric spaces which are classically compact but that cannot be shown within BISH, as they are not compact in an extension of BISH. For example, Kleene's proof of the existence of a primitive recursive infinite binary tree without an infinite path implies the failure of König's lemma and the Heine-Borel property for the classically compact Cantor space $2^{\mathbb{N}}$ within recursive mathematics (RUSS) (see [1], p.68).

Sequential compactness is constructively also not very useful, since classically sequential compact sets, like 2, are not constructively sequential compact. For example, a *fleeing property* on $\mathbb{N}$ is a formula ϕ such that $\forall_n(\phi(n) \vee \neg\phi(n))$ and for which we cannot prove neither $\exists_n(\phi(n))$ nor $\forall_n(\neg\phi(n))$. Note that we can always provide such properties using some unsolved so far mathematical problem regarding the decimal expansion of π. If

$$\alpha(n) = \begin{cases} 1 & \text{, if } \exists_{m \leq n}(\phi(m)) \\ 0 & \text{, otherwise} \end{cases}$$

then α has no convergent subsequence, since that would imply that $\exists_n(\phi(n)) \vee \forall_n(\neg\phi(n))$. Despite that, a constructive "at most" notion of sequential compactness and a constructive "almost" notion of sequential compactness are considered in [8] and [9], respectively.

For compactness in metric spaces Bishop used Brouwer's notion of a complete and totally bounded metric space, a notion classically equivalent to the Heine-Borel property for metric spaces. It is in this sense that the Cantor space $2^{\mathbb{N}}$ endowed with the (constructively well-defined) standard metric

$$\rho(\alpha, \beta) := \inf\{2^{-n} \mid \overline{\alpha}(n) = \overline{\beta}(n)\},$$

for every $\alpha, \beta \in 2^{\mathbb{N}}$, where $\overline{\alpha}(n)$ denotes the n-th initial segment of α, is a compact metric space in BISH. Bishop didn't address the question of constructive compactness in a more general setting as he saw no reason to go beyond metric spaces.

According to Diener in [15], pp.15-6, an ideal constructive notion of general compactness should exhibit the following properties:

(i) it would be defined for, and in the language of, topological spaces,
(ii) it would be classically equivalent to the Heine-Borel property,
(iii) within BISH a complete and totally bounded metric/uniform space would satisfy this notion, and
(iv) we would be able to prove deep and meaningful theorems assuming that the underlying space satisfies this notion.

In [15] Diener defined the notion of neat compactness taken with that of neat completeness as a candidate for such notion in a pre-apartness space. His proof in [15], pp.29-31, of the existence in RUSS of a uniform structure on $2^{\mathbb{N}}$ that induces the usual topology but it is not totally bounded, shows that there is no such ideal notion of compactness.

Within TBS we need to define a constructive notion of compactness such that:

(i*) it would be defined for, and in the language of, Bishop spaces. The function-theoretic character of TBS forces us to find a function-theoretic characterization of compactness.
(ii*) Since our objects are Bishop spaces, it is not possible to have an equivalence to the Heine-Borel property, a property designed for topological spaces, but if a Bishop space is compact within $\mathbf{Bis}$, this should reflect a kind of compactness as this is understood for topological spaces.
(iii*) within BISH^* a complete and totally bounded metric space would satisfy this notion, in the only way that makes sense within TBS: a metric space X endowed with the Bishop topology of uniformly continuous functions satisfies this notion.
(iv*) we would be able to prove deep and meaningful theorems assuming that the underlying Bishop space satisfies this notion.

Regarding (i*) the question of a function-theoretic characterization of topological compactness is not recent. Mrówka's classical result in [23], according to which a topological space X is compact if and only if for every topological space Y the projection $\pi_Y : X \times Y \to Y$ "parallel" to the compact factor X is a closed map, is almost such a characterization, since the concept of a closed map is set-theoretic. It was this characterization which inspired a categorical treatment of compactness (see e.g., [13], p.410). In the (non-constructive) work of Escardó [17] and (the constructive) work of Taylor [32] compactness of a topological space X is characterized by the continuity (understood in the standard set-theoretic way) of an appropriate functional of type $S^X \to S$, where S is the Sierpinski space. For the, as expected non function-theoretic, treatment of compactness in the theory of apartness spaces and in formal topology see e.g., [11] and [31], respectively.

Here we introduce the notion of a 2-compact Bishop space as a constructive function-theoretic alternative to the classical notion of a compact topological space and we prove the following results:

(a) Theorem 2.13, a consequence of Bishop's version of the Tietze theorem for metric spaces, according to which the reciprocal of an element of a Bishop topology which is bounded away from zero is also an element of this topology.

(b) Theorem 3.8, according to which a uniformly continuous function $e : 2^{\mathbb{N}} \to X$, where X is a compact metric space, is a morphism between the corresponding Bishop spaces.

(c) Theorem 3.10, according to which the Bishop topology of the uniformly continuous real-valued functions on a compact metric space is 2-compact i.e., 2-compactness generalizes metric compactness.

(d) Proposition 3.11, according to which the Bishop topology on the Cantor space is equal to the uniformly continuous real-valued functions on it.

(e) Theorem 3.12, according to which the set of Bishop morphisms between the Cantor space, seen as a Bishop space, and the Bishop space of the natural numbers endowed with the discrete topology is equal to the uniformly continuous functions between the corresponding metric spaces.

(f) Corollary 3.15, according to which the set of Bishop morphisms between the Cantor space, seen as a Bishop space, and itself, or a compact metric space endowed with the Bishop topology of the uniformly continuous real-valued functions on it, is equal to the uniformly continuous functions between the corresponding metric spaces.

(g) Proposition 4.4, the countable Tychonoff compactness theorem for 2-compact Bishop spaces.

Results (d)-(f) show that an appropriate set of Bishop morphisms between some fundamental Bishop spaces is equal to the set of the uniformly continuous functions between the corresponding metric spaces. These results are instances of the fact that the notion of Bishop morphism is reduced to uniform continuity and indicate that working with Bishop spaces and Bishop morphisms one can overcome the aforementioned by Bishop second difficulty in the constructivization of general topology. One can provide more such reducibility results, like Bridges's backward uniform continuity theorem (Theorem 3.13), or Bridges's forward uniform continuity theorem (see [12] for its proof in BISH with the antithesis of Specker's theorem, and [28] for its interpretation as a reducibility result).

Apart from dealing within TBS with Bishop's obstacles in the constructivization of general topology, and in connection to Palmgren's ascertainment on the need of certain quotient constructions, there is a very simple and natural notion of quotient Bishop topology, which behaves exactly like the classical notion (see [28]). This notion though does not occur here.

2. Basic definitions and facts

In this section we give all basic definitions and results necessary to the rest of the paper. For all proofs not included here we refer to [28]. The next definition is a slight variation of the definition given by Bishop and Bridges in [6], p.85.

Definition 2.1. *A bounded subset B of an inhabited metric space X is a triplet (B, x_0, M), where $x_0 \in X$, $B \subseteq X$, and $M > 0$ is a bound for $B \cup \{x_0\}$. We write $\mathbb{B}(X)(B)$ to indicate that (B, x_0, M) is bounded.*

If (B, x_0, M) is a bounded subset of X then $B \subseteq \mathcal{B}(x_0, M)$, where $\mathcal{B}(x_0, M)$ is the open sphere of radius M about x_0, and $(\mathcal{B}(x_0, M), x_0, 2M)$ is also a bounded subset of X. In other words, a bounded subset of X is included in an inhabited bounded subset of X which is also metric-open i.e., it includes an open ball of every element of it, a fact used in the proof of Lemma 2.16.

Definition 2.2. *We denote the set of all functions of type $X \to \mathbb{R}$ by $\mathbb{F}(X)$, the constant function on X with value $a \in \mathbb{R}$ by $\overline{a}$ and their set by $\mathrm{Const}(X)$. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called Bishop continuous, or simply continuous, if it is uniformly continuous on every bounded subset B of $\mathbb{R}$ i.e., for every bounded subset B of $\mathbb{R}$ and for every $\epsilon > 0$ there exists $\omega_{\phi,B}(\epsilon) > 0$ such that*

$$\forall_{x,y \in B}(|x - y| \leq \omega_{\phi,B}(\epsilon) \to |\phi(x) - \phi(y)| \leq \epsilon),$$

where the function $\omega_{\phi,B} : \mathbb{R}^+ \to \mathbb{R}^+$, $\epsilon \mapsto \omega_{\phi,B}(\epsilon)$, is called a modulus of continuity for ϕ on B. Two continuous functions $(\phi_1, (\omega_{\phi_1,B})_B)$, $(\phi_2, (\omega_{\phi_2,B})_B)$ are equal, if $\phi_1(x) = \phi_2(x)$, for every $x \in \mathbb{R}$, and we denote their set by $\mathrm{Bic}(\mathbb{R})$. The set of real-valued continuous functions defined on some $Y \subseteq \mathbb{R}$ which are uniformly continuous on every bounded subset of Y is denoted by $\mathrm{Bic}(Y)$.

At first sight it seems that the definition of Bishop continuity rests on quantification over the power set $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ i.e.,

$$\mathrm{Bic}(\mathbb{R})(\phi) \leftrightarrow \forall_{B \in \mathcal{P}(\mathbb{R})}(\mathbb{B}(\mathbb{R})(B) \to \phi_{|B} \text{ uniformly continuous}).$$

It suffices though to quantify over $\mathbb{N}$ i.e.,

$$\mathrm{Bic}(\mathbb{R})(\phi) \leftrightarrow \forall_{n \in \mathbb{N}}(\phi_{|[-n,n]} \text{ uniformly continuous}),$$

since a bounded subset of $\mathbb{R}$ is by definition a triplet (B, x_0, M) and since $B \subseteq (x_0 - M, x_0 + M)$, for some $M > 0$, we get that $(x_0 - M, x_0 + M) \subseteq [-n, n]$, where $n = \max\{N_1, N_2\}$ and $N_1, N_2 \in \mathbb{N}$ such that $N_1 > x_0 + M$ and $-N_2 < x_0 - M$ by the Archimedean property of reals. Hence, the uniform continuity of ϕ on $[-n, n]$ implies its uniform continuity on B.

Definition 2.3. *A locally compact metric space is an inhabited metric space (X, d) each bounded subset of which is included in a compact subset of X. A function $f : X \to \mathbb{R}$ is called Bishop continuous, or simply continuous, if f is uniformly continuous on every bounded subset of X i.e., there is a map $\omega_{f,B} : \mathbb{R}^+ \to \mathbb{R}^+$, $\epsilon \mapsto (\omega_{f,B})(\epsilon)$, for every bounded subset B of X, the modulus of continuity of f on B. We denote by $\mathrm{Bic}(X)$ the set of all Bishop continuous functions from X to $\mathbb{R}$. Equality on $\mathrm{Bic}(X)$ is defined as in the definition of $\mathrm{Bic}(\mathbb{R})$.*

As in the case of $\mathrm{Bic}(\mathbb{R})$ at first it seems that the above definition requires quantification over the power set $\mathcal{P}(X)$ of X i.e.,

$$\mathrm{Bic}(X)(f) \leftrightarrow \forall_{B \in \mathcal{P}(X)}(\mathbb{B}(X)(B) \to f_{|B} \text{ uniformly continuous}).$$

One easily avoids such a quantification since, if x_0 inhabits X, then for every bounded subset $(B, {x_0}', M)$ of X we have that there is some $n \in \mathbb{N}$ such that $n > 0$ and

$$B \subseteq [d_{x_0} \leq \overline{n}] = \{x \in X \mid d(x_0, x) \leq n\};$$

if $x \in B$, then $d(x, x_0) \leq d(x, {x_0}') + d({x_0}', x_0) \leq M + d({x_0}', x_0)$, therefore $x \in [d_{x_0} \leq \overline{n}]$, for some $n > M + d({x_0}', x_0)$. Hence, we can write

$$\mathrm{Bic}(X)(f) \leftrightarrow \forall_{n \in \mathbb{N}}(f_{|[d_{x_0} \leq \overline{n}]} \text{ uniformly continuous}),$$

and $[d_{x_0} \leq \overline{n}]$ is trivially a bounded subset of X.

Definition 2.4. *If $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, then*

$$U(g, f, \epsilon) :\leftrightarrow \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) :\leftrightarrow \forall_{\epsilon > 0}\exists_{g \in \Phi}(U(g, f, \epsilon)).$$

Definition 2.5. *A Bishop space is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$, a Bishop topology, or simply a topology, satisfies the following conditions:*

(BS_1) $a \in \mathbb{R} \to \overline{a} \in F$.
(BS_2) $f \in F \to g \in F \to f + g \in F$.
(BS_3) $f \in F \to \phi \in \mathrm{Bic}(\mathbb{R}) \to \phi \circ f \in F$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F \ni \phi \circ f"'] & \mathbb{R} \arrow[d, "\phi \in \mathrm{Bic}(\mathbb{R})"] \\ & \mathbb{R}. \end{tikzcd}$$

(BS_4) $f \in \mathbb{F}(X) \to U(F, f) \to f \in F$.

Bishop used the term *function space* for $\mathcal{F}$ and topology for F. Since the former is used in many different contexts, we prefer the term Bishop space for $\mathcal{F}$, while we use the latter, as the *topology of functions* F on X corresponds nicely to the standard *topology of opens* $\mathcal{T}$ on X. Actually, a Bishop topology induces a canonical topology of opens with $U(f) := \{x \in X \mid f(x) > 0\}$, where $f \in F$, as basic open sets. One can show classically that the canonical topology of F is completely regular[1].

We will show in this paper that the concept of Bishop space is constructively viable, since using it, we can overcome Bishop's obstacles in the constructivization of general topology that was mentioned in the Introduction.

A Bishop topology F is a ring and a lattice[2]; by BS_2 and BS_3 if $f, g \in F$, then $f \cdot g = \frac{(f+g)^2 - f^2 - g^2}{2}$, $f \vee g = \max\{f, g\} =$

[1] It is an open question, if a completely regular topology of opens is the canonical topology of some Bishop topology of functions.

[2] It is even an f-algebra, a fact that connects TBS to the the theory of Gelfand duality for real C*-algebras and for Riesz spaces (see [14]).

$\frac{f+g+|f-g|}{2}$, $f \wedge g = \min\{f, g\} = \frac{f+g-|f-g|}{2}$ and $|f| \in F$, since $|\mathrm{id}_{\mathbb{R}}| \in \mathrm{Bic}(\mathbb{R})$, where $\mathrm{id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$. The sets $\mathrm{Const}(X)$ and $\mathbb{F}(X)$ are topologies on X, called the *trivial* and the *discrete* topology, respectively. If F is a topology on X, then $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$. It is straightforward to see that $\mathbb{F}_b(X) := \{f \in \mathbb{F}(X) \mid f \text{ is bounded}\}$ is a topology on X, and if $\mathcal{F} = (X, F)$ is a Bishop space, then $\mathcal{F}_b = (X, F_b)$ is a Bishop space, where $F_b = \mathbb{F}_b(X) \cap F$. If X is a locally compact metric space, it is easy to see that $\mathrm{Bic}(X)$ is a topology on X. Since $\mathbb{R}$ with its standard metric is locally compact, the structure $\mathcal{R} = (\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$ is the *Bishop space of reals*. A topology F on X induces the *canonical apartness relation* on X defined, for every $x, y \in X$, by

$$x \bowtie_F y :\leftrightarrow \exists_{f \in F}(f(x) \bowtie_{\mathbb{R}} f(y)),$$

where $a \bowtie_{\mathbb{R}} b \leftrightarrow a > b \vee a < b \leftrightarrow |a-b| > 0$, for every $a, b \in \mathbb{R}$. It is easy to see that $a \bowtie_{\mathbb{R}} b \leftrightarrow a \bowtie_{\mathrm{Bic}(\mathbb{R})} b$.

One of the most important features of the notion of a Bishop space is its implicit inductive character, as conditions BS_1-BS_4 can be seen as inductive rules.

Definition 2.6. *The least topology $\mathcal{F}(F_0)$ generated by a set $F_0 \subseteq \mathbb{F}(X)$, called a subbase of $\mathcal{F}(F_0)$, is defined by the following inductive rules:*

$$\frac{f_0 \in F_0}{f_0 \in \mathcal{F}(F_0)}\ , \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathcal{F}(F_0)}\ , \quad \frac{f, g \in \mathcal{F}(F_0)}{f + g \in \mathcal{F}(F_0)}\ ,$$

$$\frac{f \in \mathcal{F}(F_0),\ \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \mathcal{F}(F_0)}\ , \quad \frac{(g \in \mathcal{F}(F_0),\ U(g, f, \epsilon))_{\epsilon > 0}}{f \in \mathcal{F}(F_0)}\ .$$

The most complex inductive rule above can be replaced by the rule

$$\frac{g_1 \in \mathcal{F}(F_0) \ \wedge\ U(g_1, f, \frac{1}{2}),\ g_2 \in \mathcal{F}(F_0) \ \wedge\ U(g_2, f, \frac{1}{2^2}),\ \ldots}{f \in \mathcal{F}(F_0)}\ ,$$

which has the "structure" of Brouwer's F-inference with countably many conditions in its premiss (see e.g., [22]). The above rules induce the following induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$:

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \rightarrow \\
&\forall_{a \in \mathbb{R}}(P(\overline{a})) \rightarrow \\
&\forall_{f, g \in \mathcal{F}(F_0)}(P(f) \rightarrow P(g) \rightarrow P(f+g)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)} \forall_{\phi \in \mathrm{Bic}(\mathbb{R})}(P(f) \rightarrow P(\phi \circ f)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)}(\forall_{\epsilon > 0} \exists_{g \in \mathcal{F}(F_0)}(P(g) \ \wedge\ U(g, f, \epsilon)) \rightarrow P(f)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)}(P(f)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. Hence, starting with a constructively acceptable subbase F_0 the generated least topology $\mathcal{F}(F_0)$ is a constructively graspable set of functions exactly because of the corresponding principle $\mathrm{Ind}_{\mathcal{F}}$. Despite the seemingly set-theoretic character of the notion of a Bishop space the core of TBS is the study of the inductively generated Bishop spaces. The identity function $\mathrm{id}_{\mathbb{R}}$ on $\mathbb{R}$ belongs to $\mathrm{Bic}(\mathbb{R})$ and it is easy to see that $\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}})$.

If P is a property on $\mathbb{F}(X)$, we say that P is *lifted* from a subbase F_0 to the generated Bishop space $\mathcal{F}(F_0)$, if

$$\forall_{f_0 \in F_0}(P(f_0)) \rightarrow \forall_{f \in \mathcal{F}(F_0)}(P(f)).$$

It is easy to see inductively that boundedness is a lifted property, a fact used e.g., in the proof of Proposition 4.8. The next lifting has a straightforward inductive proof, and it is used in the proof of Proposition 3.11.

Proposition 2.7 (lifting of uniform continuity)**.** *Suppose that (X, d) is a metric space and $F_0 \subseteq \mathbb{F}(X)$, such that every $f_0 \in F_0$ is bounded and uniformly continuous on X. Then every $f \in \mathcal{F}(F_0)$ is uniformly continuous on X.*

Definition 2.8. *Let $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $A \subseteq X$ is inhabited. The relative Bishop space of $\mathcal{F}$ on A is the structure $\mathcal{F}_{|A} = (A, F_{|A})$, where $F_{|A} := \mathcal{F}(\{f_{|A} \mid f \in F\})$. The product of $\mathcal{F}$ and $\mathcal{G}$ is the structure $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where $F \times G := \mathcal{F}(\{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\})$.*

It is easy to see that the product of Bishop spaces satisfies the universal property of products. If F_0 is a subbase of F and G_0 is a subbase of G, it is easy to see inductively that $F_{|A} = \mathcal{F}(\{f_{0|A} \mid f_0 \in F_0\})$ and $\mathcal{F}(F_0) \times \mathcal{F}(G_0) = \mathcal{F}(\{f_0 \circ \pi_1 \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\})$. Consequently, $\mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\{\mathrm{id}_{\mathbb{R}} \circ \pi_1\} \cup \{\mathrm{id}_{\mathbb{R}} \circ \pi_2\}) = \mathcal{F}(\{\pi_1, \pi_2\})$. The arbitrary product of Bishop spaces is defined similarly.

Within TBS "continuity" is represented in a simple and purely function-theoretic way.

Definition 2.9. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a Bishop morphism, or simply a morphism, from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \rightarrow Y$ such that $\forall_{g \in G}(g \circ h \in F)$*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g \circ h"'] & Y \arrow[d, "g \in G"] \\ & \mathbb{R}. \end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of morphisms from $\mathcal{F}$ to $\mathcal{G}$ and by $\mathbf{Bis}$ *the category of Bishop spaces with arrows the Bishop morphisms. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is onto Y, then h is called a set-epimorphism, and we denote their set by $\mathrm{setEpi}(\mathcal{F}, \mathcal{G})$. We call h open, if $\forall_{f \in F} \exists_{g \in G}(f = g \circ h)$, and an isomorphism between $\mathcal{F}$ and $\mathcal{G}$, if it is 1-1 and onto Y such that $h^{-1} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$.*

Clearly, $\mathrm{Const}(X, Y) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, if F is a topology on X, then $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$, and a bijection $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is an isomorphism if and only if h is open. If G_0 is a subbase of G, it is easy to see inductively that

$$h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \leftrightarrow \forall_{g_0 \in G_0}(g_0 \circ h \in F),$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g_0 \circ h"'] & Y \arrow[d, "g_0 \in G_0"] \\ & \mathbb{R}. \end{tikzcd}$$

a fundamental property that we call the *lifting of morphisms*. In [29] we showed the following lifting, that is used here in the proof of Corollary 4.6.

Proposition 2.10 (Lifting of openness)**.** *If $\mathcal{F} = (X, \mathcal{F}(F_0))$, $\mathcal{G} = (Y, G)$ and $h \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$, then $\forall_{f_0 \in F_0} \exists_{g \in G}(f_0 = g \circ h) \rightarrow \forall_{f \in \mathcal{F}(F_0)} \exists_{g \in G}(f = g \circ h)$.*

Using the definition of a continuous function on a locally compact metric space, given in [6] p.110, Bishop's formulation of the Tietze theorem for metric spaces becomes as follows.

Theorem 2.11. *Let Y be a locally compact subset of a metric space X and $I \subset \mathbb{R}$ an inhabited compact interval. Let $f : Y \rightarrow I$ be uniformly continuous on the bounded subsets of Y. Then there exists a function $g : X \rightarrow I$ which is uniformly continuous on the bounded subsets of X, and which satisfies $g(y) = f(y)$, for every $y \in Y$.*

Corollary 2.12. *If Y is a locally compact subset of $\mathbb{R}$ and $g : Y \rightarrow I \in \mathrm{Bic}(Y)$, where $I \subset \mathbb{R}$ is an inhabited compact interval, then there exists a function $\phi : \mathbb{R} \rightarrow I \in \mathrm{Bic}(\mathbb{R})$ which satisfies $\phi(y) = g(y)$, for every $y \in Y$.*

Theorem 2.13. *Suppose that (X, F) is a Bishop space and $f \in F$ such that $f(x) \geq c > 0$, for every $x \in X$. Then, $\frac{1}{f} \in F$.*

Proof. If $c > 0$, the interval $[c, +\infty)$ is a locally compact subset of $(\mathbb{R}, d_{\mathbb{R}})$, where $d_{\mathbb{R}}(x, y) := |x - y|$, since a bounded subset of $[c, +\infty)$ is bounded above by some $M > c$; if $B \subseteq [c, +\infty)$ is a bounded subset, then there is some $x_0 \in [c, +\infty]$ such that $\exists_{M'>0}\forall_{x,y\in B\cup\{x_0\}}(|x - y| \leq M')$. Hence, for every $x \in B$ we have that $c \leq x = |x| \leq |x - x_0| + |x_0| \leq M' + |x_0| = M' + x_0 = M$. Since $M' > 0$ and $x_0 \geq c$, we get that $M > c$, therefore B is included in the compact subset $[c, M]$ of $[c, +\infty)$. Next we consider the inverse function $^{-1} : [c, +\infty) \to [0, \frac{1}{c}]$, $x \mapsto \frac{1}{x}$, which is uniformly continuous on the bounded subsets of $[c, +\infty)$; the identity function x is bounded away from 0 on every compact subinterval of $[c, +\infty)$, since there is actually a common c for which $|x| = x \geq c$, for every x in the compact subinterval, hence (by Proposition (4.7) in [6] p.39) x^{-1} is uniformly continuous on the compact subsets of $[c, +\infty)$, therefore it is uniformly continuous on the bounded subsets of $[c, +\infty)$. Since the range of this inverse function is included in the inhabited compact interval $[0, \frac{1}{c}]$, by the Corollary 2.12 there exists a function $\phi : \mathbb{R} \to [0, \frac{1}{c}]$ such that $\phi(x) = \frac{1}{x}$, for every $x \in [c, +\infty)$, and $\phi \in \mathrm{Bic}(\mathbb{R})$. If $f \in F$ such that $f \geq \overline{c}$, then by BS_3 the function $\phi \circ f \in F$, and since $\forall_{x\in X}(\phi(f(x)) = \frac{1}{f(x)})$, we conclude that $\phi \circ f = \frac{1}{f} \in F$. $\square$

Corollary 2.14. *Suppose that (X, F) is a Bishop space and $f \in F$ such that $|f(x)| \geq c > 0$, for every $x \in X$. Then, $\frac{1}{f} \in F$.*

Proof. If $x \in X$ and $|f(x)| \geq c$, then $|f(x)|^2 = |f(x)||f(x)| = |f(x)^2| = f(x)^2 \geq c^2 > 0$. Since F is closed under multiplication and $f \in F$, we have that $f^2 \in F$ and by Theorem 2.13 we get that $\frac{1}{f^2} \in F$, therefore $f \cdot \frac{1}{f^2} = \frac{1}{f} \in F$. $\square$

Next follows a basic lemma of constructive analysis (which without the uniqueness property is shown in [6], pp.91-2, while the uniqueness property is included in [27], p.238) and a useful generalization of it.

Lemma 2.15. *If $D \subseteq X$ is a dense subset of the metric space X, Y is a complete metric space, and $f : D \to Y$ is uniformly continuous with modulus of continuity ω, there exists a unique uniform continuous extension $g : X \to Y$ of f with modulus of continuity $\frac{1}{2}\omega$.*

Lemma 2.16. *Suppose that X is an inhabited metric space, $D \subseteq X$ is dense in X and Y is a complete metric space. If $f : D \to Y$ is uniformly continuous on every bounded subset of D, then there exists a unique extension $g : X \to Y$ of f which is uniformly continuous on every bounded subset of X with modulus of continuity*

$$\omega_{g,B}(\epsilon) = \frac{1}{2}\omega_{f,B\cap D}(\epsilon),$$

for every inhabited, bounded and metric-open subset B of X. If f is bounded by some $M > 0$, then g is also bounded by M.

Proposition 2.17. *For the discrete topologies on $\mathbb{N}$ and $2 = \{0, 1\}$ we have that $\mathbb{F}(\mathbb{N}) = \mathrm{Bic}(\mathbb{N}) = \mathcal{F}(\mathrm{id}_{\mathbb{N}})$ and $\mathbb{F}(2) = \mathrm{Bic}(2) = \mathcal{F}(\mathrm{id}_2)$, where $\mathrm{id}_{\mathbb{N}}, \mathrm{id}_2$ are the identity inclusions of $\mathbb{N}$ and 2 into $\mathbb{R}$, respectively.*

Proof. If $g : \mathbb{N} \to \mathbb{R}$, then working as in the proof of Proposition 3(iv) in [30], we construct a function ϕ^* which is uniformly continuous on every bounded subset B of $\mathbb{Q}_+$ and extends g ($\phi^*(q) = \gamma_n(q), q \in [n, n+1)$, and $\gamma_n(\mathbb{Q}\cap(n, n+1))$ is the set of rational values in the linear segment between $g(n)$ and $g(n+1)$). By Lemma 2.16 ϕ^* is extended to some $\phi \in \mathrm{Bic}(\mathbb{R})$ which also extends g. If $f : 2 \to \mathbb{R}$, then f is extended to a function $\hat{f} : \mathbb{Q} \to \mathbb{Q}$ such that $\hat{f} \in \mathrm{Bic}(\mathbb{Q})$ by linearly connecting $f(0)$ and $f(1)$, while $\hat{f}$ is constant $f(0)$ on every $q \leq 0$, and constant $f(1)$ on every $q \geq 1$. By Lemma 2.15 there is an extension of $\hat{f}$ which is in $\mathrm{Bic}(\mathbb{R})$. $\square$

Definition 2.18. *The Cantor space $\mathcal{C}$ and the Baire space $\mathfrak{N}$ are the following Bishop spaces*

$$\mathcal{C} = (2^{\mathbb{N}}, \bigvee_{n\in\mathbb{N}} \pi_n), \quad \mathcal{N} = (\mathbb{N}^{\mathbb{N}}, \bigvee_{n\in\mathbb{N}} \varpi_n),$$

$$\bigvee_{n\in\mathbb{N}} \pi_n := \prod_{n\in\mathbb{N}} \mathrm{Bic}(2) = \mathcal{F}(\{\mathrm{id}_2 \circ \pi_n = \pi_n \mid n \in \mathbb{N}\}),$$

$$\bigvee_{n\in\mathbb{N}} \varpi_n := \prod_{n\in\mathbb{N}} \mathrm{Bic}(\mathbb{N}) = \mathcal{F}(\{\mathrm{id}_{\mathbb{N}} \circ \varpi_n = \varpi_n \mid n \in \mathbb{N}\}),$$

where $\bigvee_{n\in\mathbb{N}} \pi_n$ is called the Cantor (Bishop) topology on $\mathcal{C}$, $\bigvee_{n\in\mathbb{N}} \varpi_n$ the Baire (Bishop) topology on $\mathcal{N}$, and

$$\pi_n(\alpha) = \alpha(n),$$

$$\varpi_n(\beta) = \beta(n),$$

for every $n \in \mathbb{N}$, $\alpha \in 2^{\mathbb{N}}$ and $\beta \in \mathbb{N}^{\mathbb{N}}$, respectively. If I is an arbitrary set, the I-Boolean Bishop space is the space

$$(2^I, \bigvee_{i\in I} \varpi_i),$$

where

$$\bigvee_{i\in I} \varpi_i := \prod_{i\in I} \mathrm{Bic}(2),$$

and ϖ_i denotes the i-th projection function on 2^I, for every $i \in I$.

Clearly, if $\alpha, \beta \in 2^{\mathbb{N}}$, then $\alpha \bowtie_\rho \beta \leftrightarrow \alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta$, where ρ is the metric on $2^{\mathbb{N}}$ mentioned in the subsection 1.2 and $\alpha \bowtie_\rho \beta$ is the apartness relation induced by ρ.

3. 2-compactness generalizes metric compactness

Definition 3.1. *If X is a metric space and $A \subseteq X$, A is called a compact image, if there exists some compact metric space K and a uniformly continuous function $h : K \to X$ such that $h(K) = A$.*

Since within BISH every compact metric space is the compact image of the Cantor space $2^{\mathbb{N}}$ (see [7], p.106) and uniform continuity is preserved under composition of functions, we get that A is a compact image if and only if there is some uniformly continuous function $h : 2^{\mathbb{N}} \to A$ which is onto A. Classically, a compact image is also compact, something which is not the case within BISH. For example, $(0, 1]$ is a compact image in RUSS, since there is a uniformly continuous function defined on $[0, 1]$ which is onto $(0, 1]$ (see [7], p.129). The same example shows that it is not the case within BISH that a compact image is always a closed subset. Since though total boundedness is preserved by uniformly continuous functions (see [6], p.94), a compact image is totally bounded. The notion of compact image is central to Bishop's study of continuity of functions between abstract metric spaces (see [3] and [12]), and it is generalized within TBS as follows.

Definition 3.2. *A Bishop space $\mathcal{F} = (X, F)$ is called 2-compact, if there is some set I and a set-epimorphism $e : 2^I \to X$ from the I-Boolean space to $\mathcal{F}$.*

Note that classically a metric space X is totally bounded if and only if there exists some uniformly continuous bijection $e : B \to X$, where B is a subset of the Cantor set[3] (see [25], p.153). Since id_{2^I}

[3] In [28] we show that the Cantor set i.e., the reals in $[0, 1]$ that do not require the use of the digit 1 in their triadic expansion, endowed with the relative Bishop topology of $\mathrm{Bic}([0, 1]) = C_u([0, 1])$, is isomorphic, as a Bishop space, to the Cantor space.

is a morphism from the I-Boolean space onto itself, the I-Boolean space is 2-compact.

If X is a compact metric space, it is easy to see that the set $C_u(X)$ of all uniformly continuous real-valued functions on X is a Bishop topology on X, which is called the *uniform* topology on X. In this section we show that every compact metric space endowed with its uniform topology is 2-compact (Theorem 3.10). It is in this sense that 2-compactness generalizes metric compactness.

Lemma 3.3. *The function $\phi : 2^{\mathbb{N}} \to \mathbb{R}$, defined, for every $\alpha \in 2^{\mathbb{N}}$, by*

$$\phi(\alpha) = \begin{cases} 1 & \text{, if } \alpha(0) \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$

belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$. The similarly defined function on $\mathbb{N}^{\mathbb{N}}$ belongs to the Baire topology $\bigvee_{n\in\mathbb{N}} \varpi_n$.

Proof. We present the proof for the case of the Cantor topology, but we write it so that it includes the case of the Baire topology too. First we note that ϕ is well-defined, since $\alpha(0) \in 2$. By BS_4 it suffices to show that $U(\bigvee_{n\in\mathbb{N}} \pi_n, \phi)$. For that we show that there is some $g \in \bigvee_{n\in\mathbb{N}} \pi_n$ such that $U(g, \phi, \epsilon)$, for every $0 < \epsilon < 1$; if $\epsilon' > 0$, there exists some $n \in \mathbb{N}$ such that $n > 0$ and $\epsilon' > \frac{1}{n}$ (see [10], p.27). Since $\frac{1}{n} < 1$, if we have that $U(g, \phi, \frac{1}{n})$, we get trivially that $U(g, \phi, \epsilon')$. If we fix some $\epsilon \in (0, 1)$, we consider any real σ such that $0 < \sigma \leq \frac{\epsilon}{1-\epsilon}$. In this case we get that

$$|\frac{1}{1+\sigma} - 1| = 1 - \frac{1}{1+\sigma} = \frac{\sigma}{1+\sigma} \leq \epsilon.$$

We also have that

$$\forall_{n\geq 1}(\frac{1}{1+\sigma} \leq \frac{n}{n+\sigma} < 1).$$

We define the function

$$g := \frac{\pi_0}{\pi_0 + \overline{\sigma}} \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since $\pi_0 + \overline{\sigma} \geq \overline{\sigma} \in \bigvee_{n\in\mathbb{N}} \pi_n$, therefore by Theorem 2.13 its inverse $\frac{1}{\pi_0+\overline{\sigma}}$ is in $\bigvee_{n\in\mathbb{N}} \pi_n$ too. Next we show that

$$U(g, \phi, \epsilon) := \forall_{\alpha\in 2^{\mathbb{N}}}(|g(\alpha) - \phi(\alpha)| = |\frac{\alpha(0)}{\alpha(0)+\sigma} - \phi(\alpha)| \leq \epsilon).$$

If $\alpha(0) = 0$, $\phi(\alpha) = g(\alpha) = 0$, and we are done. If $\alpha(0) \neq 0 \leftrightarrow \alpha(0) = n \geq 1$, then[4]

$$\begin{aligned} |\frac{\alpha(0)}{\alpha(0)+\sigma} - \phi(\alpha)| &= |\frac{n}{n+\sigma} - 1| \\ &= 1 - \frac{n}{n+\sigma} \\ &\leq 1 - \frac{1}{1+\sigma} \\ &= \frac{\sigma}{1+\sigma} \\ &\leq \epsilon. \end{aligned}$$

□

The following lemmas are formulated here only for the Cantor space, although their proofs are automatically applicable to the case of the Baire space too.

Lemma 3.4. *If $\alpha \in 2^{\mathbb{N}}$ and $i \geq 1$, the function $\theta_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, defined by*

$$\theta_{\alpha,i}(\beta) = \begin{cases} 1 & \text{, if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 0 & \text{, otherwise} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$, belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$.

Proof. We show by induction on i that $\forall_{i\geq 1}(\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n))$. First we show that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,1} \in \bigvee_{n\in\mathbb{N}} \pi_n)$. On the set 2 we define the operation $\dot{-}$ by the rules $0 \dot{-} 1 = 1 \dot{-} 0 = 1$ and $1 \dot{-} 1 = 0 \dot{-} 0 = 0$ i.e., $j \dot{-} k = |j - k|$, for every $j, k \in 2$. We fix some $\alpha \in 2^{\mathbb{N}}$ and we show that if ϕ is the element of $\bigvee_{n\in\mathbb{N}} \pi_n$ from Lemma 3.3, and if $S_\alpha : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $S_\alpha(\beta) = \alpha \dot{-} \beta$, where $(\alpha \dot{-} \beta)(n) = \alpha(n) \dot{-} \beta(n)$, for every $n \in \mathbb{N}$, we have that

$$\theta_{\alpha,1} = (\overline{1} - \phi) \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since $\overline{1} - \phi \in \bigvee_{n\in\mathbb{N}} \pi_n$ and $S_\alpha \in \mathrm{Mor}(\mathcal{C}, \mathcal{C})$, therefore by the definition of the Bishop morphism we get that $(\overline{1} - \phi) \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$. To show that $S_\alpha \in \mathrm{Mor}(\mathcal{C}, \mathcal{C})$ it suffices by the lifting of morphisms to show that $\pi_n \circ S_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$, which is true, since $\pi_n \circ S_\alpha = |\overline{\pi_n(\alpha)} - \pi_n| \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$. It is straightforward to check that $\theta_{\alpha,1}(\beta) = ((\overline{1} - \phi) \circ S_\alpha)(\beta)$; recall only that $\overline{\alpha}(1) = \overline{\beta}(1) \leftrightarrow \alpha(0) = \beta(0)$. Next we suppose that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n)$ and we show that $\forall_{\alpha\in 2^{\mathbb{N}}}(\theta_{\alpha,i+1} \in \bigvee_{n\in\mathbb{N}} \pi_n)$. For that we consider the shifting function $s_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, defined by $s_i(\beta)(n) = \beta(n + i)$, for every $n \in \mathbb{N}$. Again we have that $s_i \in \mathrm{Mor}(\mathcal{C}, \mathcal{C})$, since $\pi_n \circ s_i = \pi_{n+i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, for every $n \in \mathbb{N}$. Moreover,

$$\theta_{\alpha,i+1} = \theta_{\alpha,i} \cdot (\theta_{s_i(\alpha),1} \circ s_i) \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since $\theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$ by the inductive hypothesis on α, while $\theta_{s_i(\alpha),1} \in \bigvee_{n\in\mathbb{N}} \pi_n$, by the case $i = 1$ on the sequence $s_i(\alpha)$, and $\theta_{s_i(\alpha),1} \circ s_i \in \bigvee_{n\in\mathbb{N}} \pi_n$ by the definition of a Bishop morphism. Since $\theta_{s_i(\alpha),1}(s_i(\beta)) = 1$, if $\alpha(i) = \beta(i)$ and $\theta_{s_i(\alpha),1}(s_i(\beta)) = 0$, otherwise, it is immediate to see that $\theta_{\alpha,i}(\beta)\theta_{s_i(\alpha),1}(s_i(\beta)) = \theta_{\alpha,i+1}(\beta)$. □

Lemma 3.5. *If $\alpha \in 2^{\mathbb{N}}$ and $i \geq 1$, the function $\eta_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, defined by*

$$\eta_{\alpha,i}(\beta) = \begin{cases} 2^{-i} & \text{, if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 3 & \text{, otherwise} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$, belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$.

Proof. First we define the functions $\theta^*_{\alpha,i}, \theta^{**}_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$, where

$$\theta^*_{\alpha,i}(\beta) = \begin{cases} 1 & \text{, if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 3 & \text{, otherwise} \end{cases}$$

$$\theta^{**}_{\alpha,i}(\beta) = \begin{cases} 1 & \text{, if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 2 & \text{, otherwise} \end{cases}$$

for every $\beta \in 2^{\mathbb{N}}$. If we show that $\theta^{**}_{\alpha,i}, \theta^*_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, then we have that

$$\eta_{\alpha,i} = (2^{-i}\theta^*_{\alpha,i})(\theta^{**}_{\alpha,i})^i \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since a topology is closed under products and powers. For the above equality we have that $(2^{-i}\theta^*_{\alpha,i})(\beta)(\theta^{**}_{\alpha,i})^i(\beta) = 2^{-i}1^i = 2^{-i}$, if $\overline{\alpha}(i) = \overline{\beta}(i)$, and $(2^{-i}\theta^*_{\alpha,i})(\beta)(\theta^{**}_{\alpha,i})^i(\beta) = (2^{-i}3)2^i = 3$, otherwise. If $\theta_{\alpha,i}$ is the function defined in Lemma 3.4, we get that

$$\theta^{**}_{\alpha,i} = \overline{2} - \theta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

$$\theta^*_{\alpha,i} = \theta^{**}_{\alpha,i} + (\overline{1} - \theta_{\alpha,i}) \in \bigvee_{n\in\mathbb{N}} \pi_n.$$

□

[4] Note that the last equivalence works simultaneously for both the Cantor and the Baire space.

The previous lemmas prepare the proof of the next proposition which is necessary to our proof of Theorem 3.8. If $\alpha \in 2^{\mathbb{N}}$, we denote by ρ_α the uniformly continuous function $\rho_\alpha : 2^{\mathbb{N}} \to \mathbb{R}$, where $\beta \mapsto \rho(\alpha, \beta)$, for every $\beta \in 2^{\mathbb{N}}$. We use the same notation ρ_α, if $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Proposition 3.6. *The Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ includes the set $\{\rho_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$.*

Proof. If $\alpha \in 2^{\mathbb{N}}$ is fixed and $i \geq 1$, we show first that the function $\sigma_{\alpha,i} : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\sigma_{\alpha,i}(\beta) = \begin{cases} 2^{-i} & \text{, if } \overline{\alpha}(i) = \overline{\beta}(i) \\ 2^{-m} & \text{, if } \overline{\alpha}(m) = \overline{\beta}(m) \text{ and } \alpha(m) \neq \beta(m), \end{cases}$$

belongs to the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$. Clearly, $\sigma_{\alpha,i}(\beta)$ is well-defined, since if $\overline{\alpha}(i) \neq \overline{\beta}(i)$, there is a unique $m < i$ such that $\overline{\alpha}(m) = \overline{\beta}(m)$ and $\alpha(m) \neq \beta(m)$. If $\eta_{\alpha,i}$ is the function defined in Lemma 3.5, then we have that

$$\sigma_{\alpha,i} = \bigwedge_{j=1}^{i} \eta_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n,$$

since, if $\overline{\alpha}(i) = \overline{\beta}(i)$, then $\overline{\alpha}(j) = \overline{\beta}(j)$, for every $j \leq i$, while if $\overline{\alpha}(i) \neq \overline{\beta}(i)$, then

$$\eta_{\alpha,i}(\beta) = \ldots = \eta_{\alpha,m+1}(\beta) = 3,$$

$$\eta_{\alpha,m}(\beta) = 2^{-m}, \ldots, \eta_{\alpha,1}(\beta) = 1,$$

therefore $\bigwedge_{j=1}^{i} \eta_{\alpha,i}(\beta) = 2^{-m}$. Clearly, $\sigma_{\alpha,i} \in \bigvee_{n\in\mathbb{N}} \pi_n$, since a Bishop topology is a $(\wedge, \vee)$-lattice. Next we fix some $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \epsilon$. We show that

$$U(\sigma_{\alpha,n_0}, \rho_\alpha, \epsilon) := \forall_{\beta\in 2^{\mathbb{N}}}(|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| \leq \epsilon).$$

If $\overline{\alpha}(n_0) = \overline{\beta}(n_0)$, then $\rho_\alpha(\beta) \leq 2^{-n_0}$, and $|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| = |2^{-n_0} - \rho_\alpha(\beta)| = 2^{-n_0} - \rho_\alpha(\beta) \leq 2^{-n_0} < \epsilon$. If $\overline{\alpha}(n_0) \neq \overline{\beta}(n_0)$, then $\rho_\alpha(\beta) = \sigma_{\alpha,n_0}(\beta)$ and we get that $|\sigma_{\alpha,n_0}(\beta) - \rho_\alpha(\beta)| = 0 \leq \epsilon$. Since by Lemma 3.5 we have that $\sigma_{\alpha,n_0} \in \bigvee_{n\in\mathbb{N}} \pi_n$, and $\epsilon > 0$ is arbitrarily chosen, we get that $U(\bigvee_{n\in\mathbb{N}} \pi_n, \rho_\alpha)$, therefore by the condition BS_4 we conclude that $\rho_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$. $\square$

If $\alpha \bowtie_\rho \beta$, then $\rho(\alpha, \beta) = \rho_\alpha(\beta) > 0$. Since $\rho_\alpha(\alpha) = 0$ and by Proposition 3.6 $\rho_\alpha \in \bigvee_{n\in\mathbb{N}} \pi_n$, we get immediately that $\alpha \bowtie_{\bigvee_{n\in\mathbb{N}} \pi_n} \beta$.

Definition 3.7. *If X is a metric space, the set $U_0(X)$ of distances at a point of X is defined by*

$$U_0(X) := \{d_{x_0} \mid x_0 \in X\},$$

$$d_{x_0}(x) := d(x_0, x),$$

for every $x \in X$.

Theorem 3.8. *If X is a compact metric space, $\mathcal{U}(X) = (X, C_u(X))$ is the corresponding uniform Bishop space, and $e : 2^{\mathbb{N}} \to X$ is uniformly continuous, then $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X))$.*

Proof. By Corollary 5.16 of Bishop's Stone-Weierstrass theorem for compact metric spaces (see [6], p.108), if X is a compact metric space with positive diameter, then $\mathcal{F}(U_0(X)) = C_u(X)$. Since $(2^{\mathbb{N}}, \rho)$ is compact with diameter $1 > 0$, we get that $\mathcal{F}(U_0(2^{\mathbb{N}})) = C_u(2^{\mathbb{N}})$. Because of Proposition 3.6, and since $\mathcal{F}(F_0)$ is the least topology including F_0, the Cantor topology includes the uniform topology i.e.,

$$C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n.$$

By the lifting of morphisms we have that $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X)) \leftrightarrow \forall_{x_0\in X}(d_{x_0} \circ e \in \bigvee_{n\in\mathbb{N}} \pi_n)$. Since the composition of uniformly continuous functions is a uniformly continuous function, we get that $d_{x_0} \circ e \in C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n$, for every $x_0 \in X$. $\square$

Most of the important results on compact metric spaces hold for totally bounded metric spaces too i.e., their proof does not require the property of completeness. This is the case for the Stone-Weierstrass theorem, and the fact that $C_u(X)$ is a Bishop topology on X when X is totally bounded. For the latter it suffices to explain why $C_u(X)$ satisfies condition BS_3. If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in C_u(X)$, then $\phi \circ f = \phi_{|f(X)} \circ f \in C_u(X)$, since $f(X)$ is a bounded subset of $\mathbb{R}$ and $\phi_{|f(X)}$ is uniformly continuous on $f(X)$ by the definition of $\mathrm{Bic}(\mathbb{R})$.

Corollary 3.9. *If X is a metric space and $A \subseteq X$ is a compact image, then $\mathcal{U}(A) = (A, C_u(A))$ is a 2-compact Bishop space.*

Proof. If $h : 2^{\mathbb{N}} \to A$ is a uniformly continuous function onto A, then A is totally bounded and $C_u(A)$ is a Bishop topology on A. Moreover, $h \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(A)) \leftrightarrow \forall_{g\in C_u(A)}(g \circ h \in \bigvee_{n\in\mathbb{N}} \pi_n)$, which is the case, since $g \circ h \in C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n$, for every $g \in C_u(A)$. $\square$

Theorem 3.10. *If X is a compact metric space, then the corresponding uniform Bishop space $\mathcal{U}(X) = (X, C_u(X))$ is 2-compact.*

Proof. Since a compact metric space is an inhabited space[5], there exists a uniformly continuous function e from $2^{\mathbb{N}}$ onto X (see [7], p.106 for a proof of this fact[6] in BISH). By Theorem 3.8 we get that $e \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X))$, hence $e \in \mathrm{setEpi}(\mathcal{C}, \mathcal{U}(X))$ i.e., $\mathcal{U}(X)$ is 2-compact. $\square$

Proposition 3.11. *All the elements of the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ are uniformly continuous functions, and $\bigvee_{n\in\mathbb{N}} \pi_n = C_u(2^{\mathbb{N}})$.*

Proof. First we show that $\{\pi_n \mid n \in \mathbb{N}\} \subseteq C_u(2^{\mathbb{N}})$. If we fix some $n \in \mathbb{N}$ and some $0 < \epsilon < 1$, we define $\omega_{\pi_n}(\epsilon) = 2^{-n}$. If $\alpha, \beta \in 2^{\mathbb{N}}$ such that $\rho(\alpha, \beta) < 2^{-n} \leftrightarrow \rho(\alpha, \beta) \leq 2^{-(n+1)}$, then $\overline{\alpha}(n+1) = \overline{\beta}(n+1)$, hence $\alpha(n) = \beta(n)$ and $|\pi_n(\alpha) - \pi_n(\beta)| = |\alpha(n) - \beta(n)| = 0 < \epsilon$. Since every function π_n is bounded, by Proposition 2.7 we get that $\bigvee_{n\in\mathbb{N}} \pi_n \subseteq C_u(2^{\mathbb{N}})$. Since in the proof of Theorem 3.8 we showed that $C_u(2^{\mathbb{N}}) \subseteq \bigvee_{n\in\mathbb{N}} \pi_n$, we get the required equality. $\square$

Theorem 3.12 (Fan theorem for the Cantor topology)**.** *If $2^{\mathbb{N}}$ is equipped with the Cantor metric ρ and $\mathbb{N}$ with the discrete metric $d_{\mathbb{N}}$, then*

$$\mathrm{Mor}(\mathcal{C}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))) = C_u(2^{\mathbb{N}}, \mathbb{N}),$$

where $C_u(2^{\mathbb{N}}, \mathbb{N})$ denotes the uniformly continuous functions between the corresponding metric spaces.

[5] A totally bounded space has a finite ϵ-approximation, for every $\epsilon > 0$, and by our Definition 4.2 of a finite set a totally bounded space is inhabited.

[6] Its proof requires the completeness property and cannot go through in the case of a totally bounded metric space.

Proof. If $\phi : 2^{\mathbb{N}} \to \mathbb{N}$, then by Proposition 2.17 and the lifting of morphisms we get

$$\begin{aligned}\phi \in \mathrm{Mor}(\mathcal{C}, (\mathbb{N}, \mathbb{F}(\mathbb{N}))) &\leftrightarrow \mathrm{id}_{\mathbb{N}} \circ \phi \in \bigvee_{n\in\mathbb{N}} \pi_n \\ &\leftrightarrow \phi \in \bigvee_{n\in\mathbb{N}} \pi_n \\ &\leftrightarrow \phi \in C_u(2^{\mathbb{N}}) = C_u(2^{\mathbb{N}}, \mathbb{R}) \\ &\leftrightarrow \phi \in C_u(2^{\mathbb{N}}, \mathbb{N}).\end{aligned}$$

For the last equivalence we need to show the equivalence between the following

$$\text{(I)} \quad \rho(\alpha, \beta) \leq \omega_\phi(\epsilon) \to |\phi(\alpha) - \phi(\beta)| \leq \epsilon$$

$$\text{(II)} \quad \rho(\alpha, \beta) \leq \omega^*_\phi(\epsilon) \to d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) \leq \epsilon,$$

where

$$d_{\mathbb{N}}(n, m) = \begin{cases} 1 & \text{, if } n \neq m \\ 0 & \text{, otherwise.} \end{cases}$$

Suppose (I) and let $\epsilon > 0$. We define $\omega^*_\phi(\epsilon) = \omega_\phi(\epsilon \wedge \frac{1}{2})$. By the constructive trichotomy ($\forall_{a,b\in\mathbb{R}}(a < b \to \forall_{x\in\mathbb{R}}(a < x \vee x < b))$, see [6], p.26) we have that $\frac{1}{2} < \epsilon \vee \epsilon < 1$. In both cases we get that $\epsilon \wedge \frac{1}{2} < 1$, hence $\rho(\alpha, \beta) \leq \omega_\phi(\epsilon \wedge \frac{1}{2}) \to |\phi(\alpha) - \phi(\beta)| \leq \epsilon \wedge \frac{1}{2} < 1 \to \phi(\alpha) = \phi(\beta) \to d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) = 0 \leq \epsilon$. Next we suppose (II) and let $\epsilon > 0$. We define $\omega_\phi(\epsilon) = \omega^*_\phi(\epsilon \wedge \frac{1}{2})$. If $\rho(\alpha, \beta) \leq \omega^*_\phi(\epsilon \wedge \frac{1}{2})$, then $d_{\mathbb{N}}(\phi(\alpha), \phi(\beta)) \leq \epsilon \wedge \frac{1}{2} < 1 \to \phi(\alpha) = \phi(\beta)$, hence $|\phi(\alpha) - \phi(\beta)| \leq \epsilon$. ☐

By Proposition 3.11 the Cantor topology $\bigvee_{n\in\mathbb{N}} \pi_n$ i.e., the set $\mathrm{Mor}(\mathcal{C}, \mathcal{R})$, "captures" exactly the set of uniformly continuous functions on $2^{\mathbb{N}}$ without a compactness assumption, while by Theorem 3.12 the Bishop morphisms between $\mathcal{C}$ and $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ "capture" the uniformly continuous functions with respect to the corresponding metrics.

A proof of the subsequent backward uniform continuity theorem of Bridges can be found in [7], p.32. Our formulation of it here is within TBS and stresses the fact that it suffices to consider composition only with the elements of $U_0(Y)$ instead with all the elements of $C_u(Y)$.

Theorem 3.13 (Backward uniform continuity theorem (BUCT))**.** *Suppose that X is a metric space, Y is a compact metric space, and $h : X \to Y$. If F is a topology on X such that $F \supseteq C_u(X)$, then the following are equivalent:*

(i) $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{U}(Y))$ such that $\forall_{g\in U_0(Y)}(g \circ h \in C_u(X))$.
(ii) h is uniformly continuous.

Since the uniform topology on a compact metric space X is exactly $C_u(X)$, we get immediately the following corollary of BUCT.

Corollary 3.14. *If X and Y are compact metric spaces, then $h : X \to Y \in \mathrm{Mor}(\mathcal{U}(X), \mathcal{U}(Y))$ if and only if h is uniformly continuous.*

The next corollary of Proposition 3.11 and BUCT is another instance of the "capture" of uniform continuity by the notion of Bishop morphism. The discrete Bishop space $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ in Theorem 3.12 is replaced by the Cantor space itself, or a compact metric space endowed with the uniform topology.

Corollary 3.15. *If X is a compact metric space, then*

$$\mathrm{Mor}(\mathcal{C}, \mathcal{C}) = C_u(2^{\mathbb{N}}, 2^{\mathbb{N}}),$$

$$\mathrm{Mor}(\mathcal{C}, \mathcal{U}(X)) = C_u(2^{\mathbb{N}}, X).$$

Proof. By the equality $C_u(2^{\mathbb{N}}) = \bigvee_{n\in\mathbb{N}} \pi_n$ and Corollary 3.14 of BUCT we get that

$$\begin{aligned} e : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \in \mathrm{Mor}(\mathcal{C}, \mathcal{C}) &\leftrightarrow \forall_{\phi\in\bigvee_{n\in\mathbb{N}}\pi_n}(\phi \circ e \in \bigvee_{n\in\mathbb{N}} \pi_n) \\ &\leftrightarrow \forall_{\phi\in C_u(2^{\mathbb{N}})}(\phi \circ e \in C_u(2^{\mathbb{N}})) \\ &\leftrightarrow e \in \mathrm{Mor}(\mathcal{U}(2^{\mathbb{N}}), \mathcal{U}(2^{\mathbb{N}})) \\ &\leftrightarrow e \text{ is uniformly continuous,}\end{aligned}$$

$$\begin{aligned} h : 2^{\mathbb{N}} \to X \in \mathrm{Mor}(\mathcal{C}, \mathcal{U}(X)) &\leftrightarrow \forall_{\phi\in C_u(X)}(\phi \circ h \in \bigvee_{n\in\mathbb{N}} \pi_n) \\ &\leftrightarrow \forall_{\phi\in C_u(X)}(\phi \circ h \in C_u(2^{\mathbb{N}})) \\ &\leftrightarrow h \text{ is uniformly continuous.}\end{aligned}$$

☐

A categorical reformulation of Theorem 3.10, Proposition 3.11 and Corollary 3.14 is that the construction $X \mapsto \mathcal{U}(X)$ is a full and faithful functor from the category of compact metric spaces into 2-compact Bishop spaces which preserves the notion of Cantor space. Our result should be compared in future work with Palmgren's full and faithful embedding of the category of locally compact metric spaces into the category of locally compact formal topologies, found in [26].

4. Properties of 2-compact Bishop spaces

In this section we show some fundamental properties of 2-compact Bishop spaces.

Proposition 4.1. *If $\mathcal{F} = (X, F)$ is a 2-compact Bishop space, $\mathcal{G} = (Y, G)$ is a Bishop space and $h : X \to Y \in \mathrm{setEpi}(\mathcal{F}, \mathcal{G})$, then $\mathcal{G}$ is 2-compact.*

Proof. If $e : 2^I \to X$ is a set-epimorphism from the I-Boolean space onto $\mathcal{F}$, then $h \circ e : 2^I \to Y$ is a set-epimorphism from the I-Boolean space onto $\mathcal{G}$, as the composition of Bishop morphisms is again a Bishop morphism. ☐

Definition 4.2. *We call a set Y finite, if there is some $n > 0$ and a bijection $j : n \to Y$, where $n = \{0, 1, \ldots, n-1\}$.*

By the previous definition a finite set is inhabited.

Proposition 4.3. *If Y is a finite set, then $(Y, \mathbb{F}(Y))$ is 2-compact.*

Proof. Suppose that there is some $n > 0$ and a bijection $j : n \to Y$, hence $Y = \{j_0, \ldots, j_{n-1}\}$. It is trivial to see that j is an isomorphism between $(Y, \mathbb{F}(Y))$ and $(n, \mathbb{F}(n))$, where $\mathbb{F}(n) = \mathcal{F}(\mathrm{id}_n)$. It suffices to show then that there exists a surjection $e : 2^n \to n$ which is a morphism from the n-Bolean space onto $(n, \mathbb{F}(n))$ i.e., $\mathrm{id}_n \circ e = e \in \bigvee_{l\in n} \varpi_l$. If $i \in n$, let $\pi_i \in 2^n$ defined as $\pi_i(l) = 1$, if $l = i$, and $\pi_i(l) = 0$, otherwise. The function $e := \sum_{l=0}^{n-1} \bar{l}\varpi_l \in \bigvee_{l\in n} \varpi_l$, and is onto n, since $e(\pi_i) = i$, for every $i \in n$. ☐

Proposition 4.4 (Countable Tychonoff theorem)**.** *If for every $n \in \mathbb{N}$ the Bishop space $\mathcal{F}_n = (X_n, F_n)$ is 2-compact, then the product $\mathcal{F} = \prod_{n\in\mathbb{N}} \mathcal{F}_n$ is 2-compact.*

Proof. By the definition of 2-compactness there exist some I_n and some $e_n : 2^{I_n} \to X_n$ such that e_n is a set-epimorphism from the I_n-Boolean space to $\mathcal{F}_n$, for every $n \in \mathbb{N}$. Without loss of generality we assume that the sets $(I_n)_n$ are pairwise disjoint, since it is straightforward to see that there is an isomorphism between the Bishop spaces 2^{I_n} and $2^{I_n \times \{n\}}$. If $I = \bigcup_{n\in\mathbb{N}} I_n$ and $X = \prod_{n\in\mathbb{N}} X_n$, we define the function

$$E : 2^I \to X,$$

$$E(\alpha) := (e_n(\alpha_{|I_n}))_{n\in\mathbb{N}},$$

$$\alpha_{|I_n} : I_n \to 2,$$

$$\alpha_{|I_n}(i) = \alpha(i),$$

for every $i \in I_n$. In order to show that E is onto X we fix some $(x_n)_{n\in\mathbb{N}} \in X$, and since there exists some $\beta_n \in 2^{I_n}$ such that $e_n(\beta_n) = x_n$, for every $n \in \mathbb{N}$, we define $\alpha \in 2^I$ by $\alpha(i) = \beta_n(i)$, where n is the unique index n for which $i \in I_n$, for every $i \in I$. In other words, $\alpha_{|I_n} = \beta_n$, for every $n \in \mathbb{N}$. Hence, $E(\alpha) = (e_n(\alpha_{|I_n}))_{n\in\mathbb{N}} = (e_n(\beta_n))_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$. By the lifting of morphisms we have that E is a morphism between the I-Boolean space and $\mathcal{F}$ if and only if

$$\forall_{n\in\mathbb{N}}\forall_{f\in F_n}((f \circ \pi_n) \circ E \in \bigvee_{i\in I} \varpi_i).$$

In order to show that we define the function

$$\mathrm{cut}_n : 2^I \to 2^{I_n},$$

$$\alpha \mapsto \alpha_{|I_n},$$

for every $\alpha \in 2^I$, and we show that cut_n is a morphism between the I-Boolean space and the I_n-Boolean space i.e., $\forall_{j\in I_n}(\pi_j \circ \mathrm{cut}_n \in \bigvee_{i\in I} \varpi_i)$, for every $n \in \mathbb{N}$. Since $(\pi_j \circ \mathrm{cut}_n)(\alpha) = \mathrm{cut}_n(\alpha)(j) = \alpha_{|I_n}(j) = \alpha(j) = \varpi_j(\alpha)$, for every $\alpha \in 2^I$, we get that $\pi_j \circ \mathrm{cut}_n = \varpi_j \in \bigvee_{i\in I} \varpi_i$. If we fix some $n \in \mathbb{N}$ and some $f \in F_n$, then

$$[(f \circ \pi_n) \circ E](\alpha) = (f \circ \pi_n)((e_n(\alpha_{|I_n}))_{n\in\mathbb{N}}) =$$

$$= (f \circ e_n)(\alpha_{|I_n}) = [(f \circ e_n) \circ \mathrm{cut}_n](\alpha),$$

for every $\alpha \in 2^I$. Hence, $(f \circ \pi_n) \circ E = (f \circ e_n) \circ \mathrm{cut}_n = f \circ (e_n \circ \mathrm{cut}_n) \in \bigvee_{i\in I} \varpi_i$, since $e_n \circ \mathrm{cut}_n : 2^I \to X_n$ is a morphism between the I-Boolean space and $\mathcal{F}_n$, as a composition of morphisms, and consequently $f \circ (e_n \circ \mathrm{cut}_n) \in \bigvee_{i\in I} \varpi_i$, by the definition of a morphism and the fact that $f \in F_n$. $\square$

Definition 4.5. *The Hilbert cube $\mathcal{I}^\infty$ is the Bishop space*

$$\mathcal{I}^\infty := (I^\infty, (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^\infty}),$$

$$I^\infty := \{(x_n) \in l^2(\mathbb{N}) \mid \forall_{n\in\mathbb{N}}(|x_n| \leq \frac{1}{n})\},$$

$$l^2(\mathbb{N}) := \{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Corollary 4.6. *The Hilbert cube $\mathcal{I}^\infty$ is a 2-compact Bishop space.*

Proof. Since the topology on $[-1, 1]$ is the uniform one, by Theorem 3.10 we have that $\mathcal{I}_{(-1)1} = ([-1, 1], C_u([-1, 1]))$ is 2-compact, while by Proposition 4.4 we have that $\mathcal{I}_{(-1)1}^{\mathbb{N}}$ is 2-compact. We show that $\mathcal{I}^\infty$ is isomorphic to $\mathcal{I}_{(-1)1}^{\mathbb{N}}$, therefore by Proposition 4.1 we get that $\mathcal{I}^\infty$ is 2-compact. It suffices to show that the bijection

$$e : I^\infty \to [-1, 1]^{\mathbb{N}},$$

$$(x_1, x_2, x_3, \ldots) \mapsto (x_1, 2x_2, 3x_3, \ldots)$$

is an open morphism. By the properties of the product and relative Bishop topology, and by the lifting of morphisms we have that

$$(\mathrm{Bic}[-1, 1])^{\mathbb{N}} = \mathcal{F}(\{\mathrm{id}_{|[-1,1]} \circ \pi_n \mid n \in \mathbb{N}\}) = \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\}),$$

$$(\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^\infty} = \mathcal{F}(\{\pi_{n|I^\infty} \mid n \in \mathbb{N}\}),$$

and

$$e \in \mathrm{Mor}(\mathcal{I}^\infty, \mathcal{I}_{(-1)1}^{\mathbb{N}}) \leftrightarrow \forall_{n\in\mathbb{N}}(\pi_n \circ e \in (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}{}_{|I^\infty}).$$

Since

$$\begin{aligned}(\pi_n \circ e)((x_m)_m) &= \pi_n(x_1, 2x_2, 3x_3, \ldots)\\ &= nx_n\\ &= (\overline{n} \cdot \pi_{n|I^\infty})((x_m)_m),\end{aligned}$$

for every $(x_m)_m \in I^\infty$, we get that

$$\pi_n \circ e = \overline{n} \cdot \pi_{n|I^\infty} \in (\mathrm{Bic}(\mathbb{R}))^{\mathbb{N}}_{|I^\infty},$$

$$\pi_{n|I^\infty} = \overline{\frac{1}{n}} \cdot (\pi_n \circ e) = (\overline{\frac{1}{n}} \cdot \pi_n) \circ e,$$

and since $\overline{\frac{1}{n}} \cdot \pi_n \in (\mathrm{Bic}[-1, 1])^{\mathbb{N}}$, for every $n \in \mathbb{N}$, we conclude by the lifting of openness, Proposition 2.10, that the set-epimorphism e is open, hence e is an isomorphism between $\mathcal{I}^\infty$ and $\mathcal{I}_{(-1)1}^{\mathbb{N}}$. $\square$

Although e is the bijection used in the classical proof too, see [16], p.193, here we avoided to use the metric on the product $\mathcal{I}_{(-1)1}^{\mathbb{N}}$.

Definition 4.7. *A Bishop space (X, F) is called pseudo-compact, if every element of F is a bounded function.*

Proposition 4.8. *If $\mathcal{F} = (X, F)$ is 2-compact, then $\mathcal{F}$ is pseudo-compact.*

Proof. If $e : 2^I \to X$ is a set-epimorphism from some I-Boolean space to $\mathcal{F}$, and $f \in F$, then $f \circ e \in \bigvee_{i\in I} \varpi_i$. Since ϖ_i is bounded, for every $i \in I$, by the lifting of boundedness we get that every element of $\bigvee_{i\in I} \varpi_i$ is bounded. Hence, $f(X) = (f \circ e)(2^I)$ is a bounded subset of $\mathbb{R}$. $\square$

5. Concluding comments

In this paper we introduced 2-compactness as a constructive function-theoretic alternative to topological compactness. With respect to the properties (i*)-(iv*) that a constructive notion of compactness within TBS needs to satisfy, mentioned in subsection 1.2, we can say now the following.

(i)** 2-compactness is a function-theoretic notion of compactness since the notions of Bishop space and Bishop morphism are function-theoretic. As such, it is suitable to a formalization into some appropriate version of Type Theory.

(ii)** Since Bishop morphisms play in the category of Bishop spaces the role of continuous functions in the category of topological spaces, since the Bishop topology of a Boolean Bishop space is the Bishop product of the discrete topology on 2, and since the continuous image of a compact topological space is compact, a 2-compact Bishop space reflects indeed a kind of topological compactness.

(iii)** 2-compactness generalizes metric compactness in the sense of **(iii*)** (Theorem 3.10). Moreover, in the course of proving this we showed some fundamental reducibility results which indicate that the Bishop morphisms are reduced in well-expected cases to uniformly continuous functions overcoming one of the two main obstacles posed by Bishop in the constructivization of general topology.

(iv)** Although in section 4 we proved some fundamental properties of 2-compactness which show its resemblance to topological compactness, like the countable Tychonoff theorem for 2-compact Bishop spaces, there are properties of 2-compactness which show its difference from topological compactness. For example, the image $e(2^I)$ for some set-epimorphism $e : 2^I \to X$ is not, in general, closed in the induced canonical topological structure of X, or in the metric structure of X, if X is a metric space. This is due to the aforementioned fact in BISH that the image of a compact metric space under some uniformly continuous function is not generally a closed subset. There are also facts which indicate that 2-compactness does not behave like metric compactness. Hannes Diener suggested to us an example of a 2-compact space (X, F) for which it is not possible to accept constructively that $f(X)$ has a supremum, for some $f \in F$. Similarly, there is a metric space endowed with some 2-compact Bishop topology without being constructively a compact metric space. The reason behind such phenomena is the generality of the index set I in the definition of 2-compactness. From the classical point of view these facts seem problematic, but from the constructive point of view they are expected and show the plethora of new problems and questions that the constructive approach to mathematics generates. Recall that constructive compact metric spaces behave quite differently from the classical ones too, since, for example, in BISH a closed subset of a compact metric space is not, in general, compact. A partial constructive version of the classical fact that a closed subspace of a compact metric space is compact is Bishop's important result that if $f : X \to \mathbb{R}$ is uniformly continuous and X is compact, then the set $X(f, a) := \{x \in X \mid f(x) \leq a\}$ is compact for all but countably many reals $a > \inf\{f(x) \mid x \in X\}$ (see [6], p.98).

There are many issues requiring further study regarding 2-compactness. The characterization of the 2-compact subspaces of a 2-compact space, the isomorphism of two 2-compact spaces whenever their Bishop topologies are isomorphic as rings, and the transfer of properties of a Boolean space to a 2-compact Bishop space are some of them. We hope that the future study of 2-compact Bishop spaces will reveal new important properties of Bishop spaces and Bishop morphisms, reinforcing our conviction that TBS is a fruitful approach to constructive topology.

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