The Contrapositive of Countable Choice for Inhabited Sets of Naturals

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Abstract: Within a fairly weak formal theory of numbers and number-theoretic sequences we give a direct proof of the contrapositive of countable finite choice for decidable predicates. Our proof is at the same time a proof of a stronger form of it. In that way we think that we improve a proof given by Diener and Schuster. Within the same theory we prove properties of inhabited sets of naturals satisfying the general contrapositive of countable choice. Extending our base theory with the continuity principle, we prove that each such set is finite. In that way we generalize a result of Veldman, who proved, actually within the same extension, the finiteness of these sets, supposing additionally their decidability.
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1 Introduction

In this paper, we work within Veldman's formal system of Basic Intuitionistic Mathematics (BIM), presented in [Veldman 2011a]. BIM, which is similar to the system H in [Howard and Kreisel 1966], is a minimal formal theory of numbers and number-theoretic sequences like Kleene's system M, studied in [Moschovakis and Vafeiadou 2011], and the system EL of elementary analysis [Troelstra and van Dalen 1988]. BIM can be seen as a formalization of a proper part of Bishop's (informal) constructive mathematics (BISH), [Bishop 1967] or [Bishop and Bridges 1985]. Actually, all proofs within BIM can be read as proofs within BISH.

Following [Veldman 2011a] we briefly describe BIM. The language $\mathcal{L}$ of BIM is a two-sorted language of numerical variables $l, m, n, ...$, and number-theoretic functions $\alpha, \beta, \gamma, ...$. There are two constants in $\mathcal{L}$: a numerical constant 0, and a function constant $\overline{0}$ naming the constant zero sequence. The successor function is named by S, the pairing function by J, and the projection functions by K, L, respectively. Numerical variables and constants are terms and a new term is obtained from already constructed ones and a function symbol. There is an equality symbol, $=_0$, for numerical terms and another one, $=_1$, for function terms. An equality between numerical terms or between function terms is a basic formula. Formulas of $\mathcal{L}$ are obtained by using connectives, numerical quantifiers and the function quantifiers. The logic of BIM is two-sorted intuitionistic predicate logic. The axioms of BIM are the following:

1. Extensionality: $\forall_{\alpha,\beta}(\alpha =_1 \beta \leftrightarrow \forall_n(\alpha(n) =_0 \beta(n)))$.

2. Constants:
$\forall_n(\neg(S(n) = 0))$,
$\forall_{m,n}(S(m) = S(n) \to m = n)$,
$\forall_n(\overline{0}(n) = 0)$,
$\forall_{m,n,l}(K(J(m,n)) = m \;\wedge\; L(J(m,n)) = n \;\wedge\; J(K(l), L(l)) = l)$.

3. Composition: $\forall_{\alpha,\beta}\exists_\gamma\forall_n(\gamma(n) = \alpha(\beta(n)))$.

4. Primitive Recursion:
$$\forall_{\alpha,\beta}\exists_\gamma\forall_{m,n}(\gamma(m,0) = \alpha(m) \;\wedge\; \gamma(m, S(n)) = \beta(m,n,\gamma(m,n))).$$

5. Full induction: If φ is any formula of $\mathcal{L}$, then
$$\varphi(0) \to \forall_n(\varphi(n) \to \varphi(S(n))) \to \forall_n(\varphi(n)).$$

6. Decidable Countable Choice (AC^d_{00}):
$$\forall_\alpha(\forall_m\exists_n(\alpha(m,n) = 0) \to \exists_\gamma\forall_m(\alpha(m,\gamma(m)) = 0)).$$

Note that $\alpha(m,n)$ is an abbreviation for $\alpha(J(m,n))$. Addition to BIM of constants for the primitive recursive functions and relations together with their defining equations, as it is the case in Kleene's system M, is possible, but not necessary here.

For simplicity we shall use a more relaxed "set-theoretical" writing for our formulas, susceptible to the following conventions. A *decidable subset* A of $\mathbb{N}$, $A \subseteq^d \mathbb{N}$, is a formula $\varphi(n)$ for which $\forall_n(\varphi(n) \vee \neg\varphi(n))$. We write $A(n) \leftrightarrow n \in A \leftrightarrow \varphi(n)$. Such a set can also be described by some sequence α satisfying $\forall_n(\alpha(n) = 0 \vee \alpha(n) = 1)$, and we write $n \in A \leftrightarrow \alpha(n) = 0$. A *subset* A of $\mathbb{N}$, $A \subseteq \mathbb{N}$, is a formula $\varphi(n)$ for which we cannot, in general, decide for each n whether $\varphi(n)$ or $\neg\varphi(n)$ is the case. A subset A of $\mathbb{N}$ is called *inhabited* if $\exists_n(n \in A)$. If A is identified to some $\varphi(n)$, then
$$n = \min(A) \leftrightarrow \varphi(n) \wedge \forall_{m<n}(\neg\varphi(m)).$$
A *decidable subset* A of $\mathbb{N} \times \mathbb{N}$, $A \subseteq^d \mathbb{N} \times \mathbb{N}$, is a formula $\varphi(n,m)$, which abbreviates $\varphi(J(m,n))$, such that $\forall_{n,m}(\varphi(n,m) \vee \neg\varphi(n,m))$. A *subset* A of $\mathbb{N} \times \mathbb{N}$, $A \subseteq \mathbb{N} \times \mathbb{N}$, is a formula $\varphi(n,m)$ for which the previous disjunction does not generally hold. Sets $m = \{0, 1, ..., m-1\}$ and functions $f : m \to K$, where $K \subseteq \mathbb{N}$, are defined as appropriate subsets of $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, respectively. If $f : m \to K$, the notions of f being 1-1 or onto K are defined as usual. A subset K of $\mathbb{N}$ is called *finite*, $K \subseteq^{\mathrm{fin}} \mathbb{N}$, if there is a natural number m and an 1-1

function $f : m \to K$, which is onto K. We say that $\alpha \in K^{\mathbb{N}}$, where $K \subseteq \mathbb{N}$, if $\forall_n(\alpha(n) \in K)$. Sometimes we also use capital letters N, Λ to denote specific natural numbers.

Thus, we may write AC_{00}^d in the form

$$\forall_n \exists_m (A(n,m)) \to \exists_\alpha \forall_n (A(n, \alpha(n))),$$

where $A \subseteq^d \mathbb{N} \times \mathbb{N}$. Trivially, AC_{00}^d is equivalent to $\mathrm{AC}_{00}!$, which is AC_{00}, general countable choice, with (unique existence) hypothesis $\forall_n \exists!_m (A(n,m))$.

In the next sections our study includes the following formulas:

1. (CCC_K) $\forall_{\alpha \in K^{\mathbb{N}}} \exists_n (A(n, \alpha(n))) \to \exists_N \forall_{i \in K} (A(N, i))$, where $K \subseteq \mathbb{N}$ and $A \subseteq \mathbb{N} \times K$.
2. (CCC_K^d) It is like CCC_K, except that $A \subseteq^d \mathbb{N} \times K$.
3. ($\forall$PEM) $\forall_\gamma (\gamma \neq \overline{0} \vee \gamma = \overline{0})$.
4. (str-$\forall$PEM) $\forall_\gamma (\exists_n (\gamma(n) \neq 0) \vee \forall_n (\gamma(n) = 0))$.
5. (LPO) $\forall_{\gamma \in 2^{\mathbb{N}}} (\exists_n (\gamma(n) = 1) \vee \forall_n (\gamma(n) = 0))$.
6. (Σ_1^0-PEM) $\exists_n (P(n)) \vee \forall_n (\neg P(n))$, where $P \subseteq^d \mathbb{N}$.
7. (Σ_2^0-PEM) $\exists_n \forall_m (P(n,m)) \vee \forall_n (\neg \forall_m (P(n,m)))$, where $P \subseteq^d \mathbb{N} \times \mathbb{N}$.
8. (Σ_2^0-DNE) $\neg\neg \exists_n \forall_m (P(n,m)) \to \exists_n \forall_m (P(n,m))$, where $P \subseteq^d \mathbb{N} \times \mathbb{N}$.

The formula CCC_K is the contrapositive of countable choice for K, CCC_K^d is the decidable contrapositive of countable choice for K, str-$\forall$PEM is a strong version of the form $\forall$PEM of the principle of excluded middle, LPO is the limited principle of omniscience, Σ_1^0-PEM is the principle of excluded middle for Σ_1^0-formulas, Σ_2^0-PEM is the principle of excluded middle for Σ_2^0-formulas, and Σ_2^0-DNE is the double negation for Σ_2^0-formulas.

2 An informative direct proof of CCC_m^d

Adding to BIM a generalized form of continuity principle, AC_{10}, and the decidable fan theorem, FAN^d, we get a fragment of formal intuitionism INT (for their exact formulations see [Veldman 2008]). Veldman, in [Veldman 1982], showed that

$$\mathrm{BIM} + \mathrm{AC}_{01} + \mathrm{FAN}^d \vdash \mathrm{CCC}_2,$$

and similarly

$$\mathrm{BIM} + \mathrm{AC}_{01} + \mathrm{FAN}^d \vdash \mathrm{CCC}_m,$$

for each $m \geq 2$. In the same paper, p.518, Veldman claimed that his reasoning "also goes through in case we do not know that A is a decidable subset of $\mathbb{N} \times 2$. If A is, indeed, a decidable subset of $\mathbb{N} \times 2$, we may argue with less circumstance".

Indeed, [Diener and Schuster 2010] proved CCC_2^d through a very weak 'form' of the fan theorem, FAN_{Δ}^p, which they also proved to be a consequence of AC_{00}^d. What we show next is that CCC_2^d, or more generally CCC_m^d, can be proved in an even simpler and more informative way, and no connection to the fan theorem is necessary[1]. Actually, we prove a stronger form of CCC_m^d.

First we prove a proposition of independent interest, from which CCC_m^d is derived as an immediate corollary.

Proposition 1. *If $A \subseteq^d \mathbb{N} \times \mathbb{N}$, then*

$$\mathrm{BIM} \vdash \forall_{m \geq 1} \exists_{e_m^A : \mathbb{N} \to m} \forall_{n \in \mathbb{N}} (A(n, e_m^A(n)) \to A(n, 0) \wedge \ldots \wedge A(n, m-1)).$$

Proof. For $m = 1$ and $A \subseteq^d \mathbb{N} \times \mathbb{N}$ the proposition holds trivially. We just define $e_1^A(n) = 0$ for each n. Suppose now that we have defined for each m and A some sequence e_m^A satisfying for each n the implication

$$A(n, e_m^A(n)) \to A(n, 0) \wedge \ldots \wedge A(n, m-1).$$

Through e_m^A we define e_{m+1}^A by

$$e_{m+1}^A(n) = \begin{cases} e_m^A(n) & \text{, if } A(n, m) \\ m & \text{, if } \neg A(n, m). \end{cases}$$

We show that for each n

$$A(n, e_{m+1}^A(n)) \to A(n, 0) \wedge A(n, 1) \wedge \ldots \wedge A(n, m-1) \wedge A(n, m).$$

If $A(n, e_{m+1}^A(n))$, then by decidability of A we distinguish between two possible cases.
If $A(n, m)$, then $e_{m+1}^A(n) = e_m^A(n)$, and by hypothesis $A(n, e_{m+1}^A(n))$ we get $A(n, e_m^A(n))$. Hence, by the induction, $A(n, 0) \wedge A(n, 1) \wedge \ldots \wedge A(n, m-1)$. Thus we have shown that $A(n, m) \to A(n, 0) \wedge A(n, 1) \wedge \ldots \wedge A(n, m-1)$.
If $\neg A(n, m)$, then $e_{m+1}^A(n) = m$, and hypothesis $A(n, e_{m+1}^A(n))$ becomes $A(n, m)$, which contradicts our supposition $\neg A(n, m)$.
Therefore $A(n, m)$ holds, so, by modus ponens, we obtain also $A(n, 0) \wedge A(n, 1) \wedge \ldots \wedge A(n, m-1)$.

[1] Just before submitting this paper Wim Veldman sent to me his pre-print [Veldman 2011b], in which he proves CCC_2^d in a straightforward way too. Although the nuclear idea of these independently given proofs is the same, his is elaborated differently.

By the definition of the previous proof $e_1^A(n) = \overline{0}$ for each A, while e_2^A and e_3^A, for example, have the form

$$e_2^A(n) = \begin{cases} 0 & \text{if } A(n,1) \\ 1 & \text{if } \neg A(n,1) \end{cases}$$

and

$$e_3^A(n) = \begin{cases} 0 & \text{if } A(n,1) \wedge A(n,2) \\ 1 & \text{if } \neg A(n,1) \wedge A(n,2) \\ 2 & \text{if } \neg A(n,2), \end{cases}$$

respectively. If we define

$$N(A,m) = \{N \in \mathbb{N} : A(N,0) \wedge A(N,1) \wedge ... \wedge A(N,m-1)\},$$

then the above proof shows that

$$\{n \in \mathbb{N} : A(n, e_m^A(n))\} \subseteq N(A,m).$$

But $N \in N(A,m)$ implies that $A(N, e_m^A(N))$, since $e_m^A(N) \in \{0,1,...m-1\}$. Therefore we get

$$N(A,m) = \{n \in \mathbb{N} : A(n, e_m^A(n))\}.$$

Proposition 2. *If $A, B \subseteq^d \mathbb{N} \times \mathbb{N}$, then*

$$\mathrm{BIM} \vdash \forall_{m \geq 1}(A \restriction_{\mathbb{N} \times m} = B \restriction_{\mathbb{N} \times m} \rightarrow e_m^A = e_m^B).$$

Proof. For $m = 1$ it holds trivially, since $e_m^A = e_m^B = \overline{0}$. Consider that the proposition is true for some $m > 1$ and let $A, B \subseteq^d \mathbb{N} \times \mathbb{N}$, such that $A \restriction_{\mathbb{N} \times (m+1)} = B \restriction_{\mathbb{N} \times (m+1)}$. Since $A(n,m) \leftrightarrow B(n,m)$, then $e_{m+1}^A(n) = e_{m+1}^B(n)$, for each n.

Thus, if A is a decidable predicate on $\mathbb{N} \times m$, for some $m \geq 1$, we define

$$e_m^A = e_m^{A^*},$$

where A^* is any fixed decidable extension of A on $\mathbb{N} \times \mathbb{N}$. By Proposition 2, e_m^A is independent of the choice of A^* extending A, and the implication

$$A^*(n, e_m^{A^*}(n)) \rightarrow A^*(n,0) \wedge A^*(n,1) \wedge ... \wedge A^*(n,m-1)$$

becomes

$$A(n, e_m^A(n)) \rightarrow A(n,0) \wedge A(n,1) \wedge ... \wedge A(n,m-1).$$

Proposition 3. $\mathrm{BIM} \vdash \forall_{m \geq 1}(\mathrm{CCC}_m^d)$.

Proof. Consider any $m \geq 1$ and some decidable predicate A on $\mathbb{N} \times m$ for which the hypothesis $\forall_{\alpha \in m^{\mathbb{N}}} \exists_{n \in \mathbb{N}}(A(n, \alpha(n)))$ of CCC_m^d holds. Applying it to e_m^A, there exists some n such that $A(n, e_m^A(n))$. By Proposition 1 we get $A(n,0) \wedge A(n,1) \wedge ... \wedge A(n,m-1)$.

Hence, the hypothesis of CCC^d_m guarantees that $N(A, m)$ is inhabited. If str-CCC^d_m is the following *strong* form of CCC^d_m

$$\exists_n(A(n, e^A_m(n))) \to \exists_N \forall_{i\in m}(A(N, i)),$$

where $A \subseteq^d \mathbb{N} \times m$, then the previous proof is actually a proof of the following proposition.

Proposition 4. $\mathrm{BIM} \vdash \forall_{m\geq 1}(\text{str-}\mathrm{CCC}^d_m)$.

Obviously, in order to prove the conclusion of CCC^d_m, it suffices to assume that $\exists_n(A(n, e^A_m(n)))$, for given $A \subseteq^d \mathbb{N} \times m$. Hence, the hypothesis of CCC^d_m restricted only to one sequence for each A is enough to obtain its conclusion. Moreover, this restricted hypothesis alone proves the full hypothesis of CCC^d_m. If α is any sequence in K and $A \subseteq^d \mathbb{N} \times m$, then necessarily $A(N, \alpha(N))$, where N is the natural number determined in the conclusion of CCC^d_m. Whatever the value of $\alpha(N)$ is, we have $A(N, \alpha(N))$, since $\alpha(N) \in m$.

Proposition 5. *If* $A \subseteq^d \mathbb{N} \times \mathbb{N}$*, then*

$$\mathrm{BIM} \vdash \forall_{m\geq 1}(A(n, e^A_m(n)) \to e^A_m(n) = 0).$$

Proof. For $m = 1$ the implication follows trivially, since $e^A_1 = \overline{0}$. Suppose that this holds for the $m > 1$ case and let $A(n, e^A_{m+1}(n))$. By Proposition 1 we have $A(n, 0) \wedge A(n, 1) \wedge ... \wedge A(n, m)$. Thus, by the definition of e^A_{m+1}, $e^A_{m+1}(n) = e^A_m(n) = 0$.

Hence, we conclude that

$$N(A, m) = \{n \in \mathbb{N} : A(n, e^A_m(n))\} \subseteq \{n \in \mathbb{N} : e^A_m(n) = 0\}.$$

The last inclusion is not, in general, an equality. For example, if we consider $A \subseteq^d \mathbb{N} \times 1$, where $A(n, 0) \leftrightarrow n = 3$, then $N(A, 1) = \{3\}$, while $e^A_1 = \overline{0}$.

With respect to the proof of CCC^d_2 given in [Diener and Schuster 2010] we may conclude the following:

1. The fact that Veldman, in [Veldman 1982], used the undecidable version of the fan theorem to prove CCC_m does not entail that the fan theorem, or something similar to it, is necessary for a proof of CCC^d_m.
2. The introduction of the sequences e^A_m not only provides the inhabitedness of $N(A, m)$, given the hypothesis of CCC^d_m, but also a characterization of all its elements. Moreover, by restricting the hypothesis of CCC^d_m to them, we prove a stronger form of CCC^d_m.

3. An implementation of our proof of CCC_m^d in the interactive proof system MINLOG has been given by Helmut Schwichtenberg (see [Min]). Although TCF, the formal theory on which MINLOG rests, is based on minimal logic, everything that we prove here can be proved within an appropriate fragment of TCF.

3 An inhabited set K satisfying CCC_K is finite

To prove (our main) Proposition 17, we start from two results in [Veldman 1982].

Proposition 6. *(Veldman 1982)* $\mathrm{BIM} \vdash \mathrm{CCC}_{\mathbb{N}}^d \to \text{str-}\forall\mathrm{PEM}$.

Trivially, str-$\forall$PEM $\to$ LPO. Since LPO is a taboo for all varieties of constructivism, the previous result shows that $\mathrm{CCC}_{\mathbb{N}}^d$ is unacceptable within BIM. We need to add an axiom to BIM, in order to achieve the negation of $\mathrm{CCC}_{\mathbb{N}}^d$.

Let CP denote the continuity principle, according to which, if $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a function on the intuitionistic Baire space $\mathbb{N}^{\mathbb{N}}$, then

$$\forall_\alpha \exists_n \forall_\beta (\overline{\beta}(n) = \overline{\alpha}(n) \to \varphi(\beta) = \varphi(\alpha)),$$

where $\overline{\alpha}(n)$ is the initial segment of α of length n. The principle CP is not accepted within BISH, where a rule-based notion of sequence is used. Of course, CP is not valid in classical mathematics, while it is justified in INT through the on-going or incomplete character of the elements of the intuitionistic Baire space. Using CP, one gets directly the negation of $\forall$PEM i.e.

$$\mathrm{BIM} + \mathrm{CP} \vdash \neg\forall\mathrm{PEM}.$$

Proposition 6 and the fact that str-$\forall$PEM $\to$ $\forall$PEM suffice for the proof of the following proposition.

Proposition 7. *(Veldman 1982)* $\mathrm{BIM} + \mathrm{CP} \nvdash \mathrm{CCC}_{\mathbb{N}}^d$. *Therefore* $\mathrm{BIM} + \mathrm{CP} \nvdash \mathrm{CCC}_{\mathbb{N}}$*, and* $\mathrm{BIM} \nvdash \mathrm{CCC}_{\mathbb{N}}$.

We cannot expect, though, that $\mathrm{BIM} \vdash \neg\mathrm{CCC}_{\mathbb{N}}$. For, $\mathrm{CCC}_{\mathbb{N}}$ is classically equivalent to the contrapositive of AC_{00}, so $\mathrm{CLASS} \vdash \mathrm{CCC}_{\mathbb{N}}$ (that is classical mathematics); and BIM–just like BISH [Bridges and Richman 1987] p.2–is consistent with CLASS.

The equivalence in the next proposition shows that Proposition 6 can be written as

$$\mathrm{BIM} \vdash \mathrm{CCC}_{\mathbb{N}}^d \to \Sigma_1^0\text{-PEM}.$$

Although this is proved in [Berardi 2006], it is just a reformulation of Proposition 6.

Proposition 8. BIM $\vdash$ str-$\forall$PEM $\leftrightarrow$ Σ^0_1-PEM.

Proof. ($\rightarrow$) Suppose $P(n)$ is decidable. Then it is standard to define the following sequence $\gamma : \mathbb{N} \rightarrow 2$ by

$$\gamma(n) = \begin{cases} 0 & \text{if } \neg P(n) \\ 1 & \text{if } P(n). \end{cases}$$

Obviously str-$\forall$PEM for γ implies Σ^0_1-PEM for P.

($\leftarrow$) If $\gamma \in \mathbb{N}^{\mathbb{N}}$, then we define the decidable predicate P, by $P(n) \leftrightarrow \gamma(n) \neq 0$. Since $\neg(\gamma(n) \neq 0) \rightarrow \gamma(n) = 0$, Σ^0_1-PEM for P implies str-$\forall$PEM for γ.

Note that if $K \subseteq^d \mathbb{N}$ and P is decidable on K, then we can prove likewise that, within BIM, CCC_K implies the corresponding Σ^0_1-PEM formula to K. For non-decidable, inhabited sets K satisfying CCC_K this is proved inside the proof of Proposition 16.

Proposition 9. BIM $\vdash \mathrm{CCC}^d_{\mathbb{N}} \rightarrow \Sigma^0_2$-PEM.

Proof. By Propositions 6 and 8, it suffices to show that the predicate

$$Q(n) \leftrightarrow \forall_m(P(n, m))$$

is decidable. By the decidable predicate $P(n, m)$ we define again the sequence

$$\alpha^n(m) = \begin{cases} 1 & \text{if } \neg P(n, m) \\ 0 & \text{if } P(n, m). \end{cases}$$

By str-$\forall$PEM, either there is some m such that $\alpha^n(m) = 1$–i.e. $\neg P(n, m)$ and hence $\neg(\forall_m(P(n, m)))$–or else $\forall_m(\alpha^n(m) = 0)$–i.e. $\forall_m(P(n, m))$.

We can give another proof of a result of [Ishihara and Schuster 2011], using Proposition 9. The left implication of the next equivalence is proved in [Berardi 2006], while the right one is proved in [Ishihara and Schuster 2011] differently, within the formal system EL, and for quantifier-free predicates P.

Proposition 10. BIM $\vdash \mathrm{CCC}^d_{\mathbb{N}} \leftrightarrow \Sigma^0_2$-DNE.

Proof. ($\leftarrow$) Let A be a decidable subset of $\mathbb{N}^2$ such that $\forall_\alpha \exists_n(A(n, \alpha(n)))$. It suffices to show that $\neg\neg(\exists_N \forall_i(A(N, i)))$. If we suppose that $\neg(\exists_N \forall_i(A(N, i)))$, then $\forall_N(\neg(\forall_i(A(N, i)))$. This implies that $\forall_N(\neg\neg(\exists_i(\neg A(N, i))))$ and by the implication Σ^0_2-DNE $\rightarrow$ Σ^0_1-DNE we conclude that $\forall_N \exists_i(\neg A(N, i))$. By AC^d_{00} we get $\exists_\alpha \forall_n(\neg A(n, \alpha(n)))$, a fact which contradicts the hypothesis of $\mathrm{CCC}_{\mathbb{N}}(A)$ for the sequence α.

($\rightarrow$) Suppose that P is a decidable predicate such that $\neg\neg(\exists_n \forall_m(P(n, m)))$. In view of Proposition 9, we rule out the possibility that $\forall_n(\neg(\forall_m(P(n, m))))$. But the latter implies that $\neg(\exists_n \forall_m(P(n, m)))$, which contradicts our initial supposition.

The next three propositions show that CCC_K partially replaces decidability for an inhabited set K.

Proposition 11. *Suppose that K is an inhabited subset of $\mathbb{N}$ satisfying CCC_K. Then*

$$\mathrm{BIM} \vdash \exists_n (n = \min(K)).$$

Proof. Suppose that $k_0 \in K$. We define $A \subseteq \mathbb{N} \times K$ by

$$A(n, k) \leftrightarrow n \in K \wedge n \leq k.$$

Obviously A is decidable only if K is decidable. We show that A satisfies the premiss of CCC_K. If $\alpha \in K^{\mathbb{N}}$, we define the sequence $\alpha^* = (\alpha^n(k_0))_{n=1}^{\infty}$, where $\alpha^0(k_0) = k_0$, and $\alpha^{n+1}(k_0) = \alpha(\alpha^n(k_0))$. Thus,[2] there exists an index $i \leq \alpha^*(0) = k_0$ such that $\alpha^i(k_0) \leq \alpha^{i+1}(k_0) = \alpha(\alpha^i(k_0))$. Obviously, $A(\alpha^i(k_0), \alpha(\alpha^i(k_0)))$. Hence, A satisfies the premiss of CCC_K. The conclusion of CCC_K, $\exists_N \forall_{i \in K}(A(N, i))$, expresses exactly that N is the minimum of K.

Proposition 12. *If $K \subseteq \mathbb{N}$, $k_0, k_1 \in K$ and $k_0 \neq k_1$, then*

$$\mathrm{BIM} \vdash \mathrm{CCC}_K \to \mathrm{CCC}_{K \setminus \{k_0\}}.$$

Proof. Let $B \subseteq \mathbb{N} \times (K \setminus \{k_0\})$ satisfy

$$\forall_{\beta \in (K \setminus \{k_0\})^{\mathbb{N}}} \exists_n (B(n, \beta(n))),$$

the hypothesis of $\mathrm{CCC}_{K \setminus \{k_0\}}$. Define a predicate $A \subseteq \mathbb{N} \times K$ by

$$A = B \cup \{(n, k_0) : n \in \mathbb{N}\}.$$

We show that A satisfies the hypothesis of CCC_K. If $\alpha \in K^{\mathbb{N}}$, then we define through α a sequence $\beta \in (K \setminus \{k_0\})^{\mathbb{N}}$ by

$$\beta(n) = \begin{cases} \alpha(n) & \text{if } \alpha(n) \neq k_0 \\ k_1 & \text{if } \alpha(n) = k_0. \end{cases}$$

By the hypothesis of $\mathrm{CCC}_{K \setminus \{k_0\}}$ we know that $\exists_n (B(n, \beta(n)))$. For this specific n we split cases. If $\alpha(n) \neq k_0$, then $\beta(n) = \alpha(n)$, and by the definition of A, $B(n, \beta(n))$ is written as $A(n, \alpha(n))$. If $\alpha(n) = k_0$, then we get automatically $A(n, \alpha(n))$. The conclusion of CCC_K provides N such that $A(N, i)$ for each $i \in K$. Therefore $B(N, j)$ holds for each $j \in K \setminus \{k_0\}$.

[2] Here we use the simplest case of Dickson's lemma, that is, for any $\gamma : \mathbb{N} \to \mathbb{N}$ there exists $i \leq \gamma(0)$ such that $\gamma(i) \leq \gamma(i + 1)$. Within BIM this has a simple inductive proof on the value of the term $\gamma(0)$.

We define $K \subseteq \mathbb{N}$ to be *weakly decidable*, $\mathrm{wd}(K)$, if K has a minimum k_0 and

$$\forall_{k \in \mathbb{N} \setminus \{k_0\}}(k \in K \to \forall_{l \in \mathbb{N}}(k_0 < l < k \to l \in K \vee l \notin K)).$$

Proposition 13. *Suppose that K is an inhabited subset of $\mathbb{N}$ satisfying* CCC_K. *Then*

$$\mathrm{BIM} \vdash \mathrm{wd}(K).$$

Proof. Since by Proposition 11 we know that K has a minimum k_0, we prove $\mathrm{wd}(K)$ by finding all the elements of K between k_0 and k, the natural number which inhabits K.
If $k_0 = k$, the conclusion is derived in a trivial way. If $k_0 \neq k$, as in the hypothesis of $\mathrm{wd}(K)$, then by Proposition 12 we get that the inhabited set $K \setminus \{k_0\}$ satisfies $\mathrm{CCC}_{K \setminus \{k_0\}}$. Therefore there is a minimum k_1 of $K \setminus \{k_0\}$. If $k_1 = k$, then k_0, k are all the elements of K between k_0 and k, while, if $k_1 \neq k$, we go on as previously. Obviously this procedure, if repeated at most k-number of times, provides all the required elements of K.

Veldman, in [Veldman 1982], called a subset K of $\mathbb{N}$ *transparent*, $\mathrm{tr}(K)$, if

$$\forall_{\Lambda \in \mathbb{N}}(\forall_{k \in K}(k \leq \Lambda) \ \vee \ \exists_{k \in K}(k > \Lambda)).$$

Veldman proved that, within BIM, a decidable set K satisfying CCC_K is transparent. Here, following his argument, we replace decidability of K with inhabitedness of K.

Proposition 14. *Suppose that K is an inhabited subset of $\mathbb{N}$ satisfying* CCC_K. *Then*

$$\mathrm{BIM} \vdash \mathrm{tr}(K).$$

Proof. For each $\Lambda \in \mathbb{N}$ we define the predicate $A^\Lambda \subseteq K^2$ by

$$A^\Lambda(k, m) \leftrightarrow k > \Lambda \ \vee \ m \leq \Lambda.$$

If k_0 inhabits K and $\alpha \in K^{\mathbb{N}}$, we prove the hypothesis of CCC_K. If $\alpha(k_0) \leq \Lambda$, then $A(k_0, \alpha(k_0))$. If $\alpha(k_0) > \Lambda$, then $A(\alpha(k_0), \alpha(\alpha(k_0)))$. Therefore there is some $N \in K$ such that $A(N, k)$ for each $k \in K$. If $N > \Lambda$, then Λ is less than an element of K, while if $N \leq \Lambda$, then Λ is an upper bound for K.

Proposition 15. *Suppose that K is an inhabited subset of $\mathbb{N}$ satisfying* CCC_K. *Then*

$$\mathrm{BIM} \vdash \neg(K \subseteq^{\mathrm{fin}} \mathbb{N}) \to \mathrm{CCC}_{\mathbb{N}}.$$

Proof. It suffices to show that K is an infinite countable set, i.e. a set for which there is some $e : \mathbb{N} \to K$, which is 1-1 and onto K. Then, it is trivial to see that the hypothesis CCC_K is automatically translatable to $\mathrm{CCC}_{\mathbb{N}}$. We prove countability of K as follows:

By Proposition 11, we can find $k_0 = \min K$. By Proposition 13, we can find all elements of K between k_0 and k. Let

$$[k_0, k] = \{\min(K) = k_0, k_1, \ldots, k_{n-1} = k\}$$

be this set. The next step is to show that hypothesis of CCC_K, taken with the assumption that K is infinite, entail the (strong) existence of an element N of K not belonging to $[k_0, k]$. If that is proved, then for each n we repeat the previous procedure so many times until we find a unique element of K corresponding to n. Of course, every element of K is eventually found. Then, by $\mathrm{AC}_{00}!$ we obtain the existence of e.

In order to find such an N we define $A \subseteq \mathbb{N} \times K$ by

$$A = \{(k_i, k_j) : i, j \in \{0, 1, ..., n-1\}\} \cup \{(n, k) : n \in K \setminus [k_0, k]\}.$$

To show that A satisfies the premiss of CCC_K let $\alpha \in K^{\mathbb{N}}$. We consider the values $\alpha(k_0), ..., \alpha(k_{n-1})$. Either $\alpha(k_i) \in [k_0, k]$ and therefore $A(k_i, \alpha(k_i))$ holds, or else $\alpha(k_i) \notin [k_0, k]$. In that case we get $A(\alpha(k_i), \alpha(\alpha(k_i)))$, since $\alpha(k_i) \in K \setminus [k_0, k]$. By the conclusion of CCC_K there is some N such that $A(N, k)$ for each $k \in K$. Then $N \notin [k_0, k]$, since each element k_i of $[k_0, k]$ satisfies $A(k_i, k)$ only for the finitely many elements of $[k_0, k]$, and $N \in [k_0, k]$ would mean that $K = [k_0, k]$, i.e. K is finite, which contradicts the hypothesis that K is infinite. Therefore N is a new element of K.

Hence, the hypothesis that the inhabited set K satisfying CCC_K is infinite turns out to be also a taboo within BIM, therefore unacceptable. As in the case of $\mathrm{CCC}_{\mathbb{N}}$, it will be CP again which will provide its negation. Firstly, we prove another proposition within BIM.

According to [Escardó 2011], a subset K of $\mathbb{N}$ is called an *omniscience set*, $\mathrm{omn}(K)$, if it satisfies the omniscience principle (OP), according to which

$$\forall_{\gamma \in 2^K} (\exists_n (\gamma(n) = 1) \ \vee \ \forall_n (\gamma(n) = 0)).$$

If $K = \mathbb{N}$, then OP becomes LPO.

Proposition 16. *Suppose that K is an inhabited subset of $\mathbb{N}$ satisfying CCC_K. Then*

1. $\mathrm{BIM} \vdash \exists_{k \in K}(P(k)) \ \vee \ \forall_{k \in K}(\neg P(k))$*, where $P(k)$ is a decidable predicate on K. As a consequence we get* $\mathrm{omn}(K)$*.*

2. BIM $\vdash \exists_k \forall_m (P(k,m)) \ \vee \ \forall_k(\neg\forall_m(P(k,m)))$, *where* $P(k,m)$ *is a decidable predicate on* K^2.

3. BIM $\vdash \neg\neg\exists_k\forall_m(P(k,m)) \to \exists_k\forall_m(P(k,m))$, *where* $P(k,m)$ *is a decidable predicate on* K^2.

Proof. We adjust previous results on $\mathbb{N}$ to K.

1. We define $A \subseteq K^2$ by

$$A(k,m) \leftrightarrow \exists_{l \leq k}(P(l)) \ \vee \ \forall_{l \leq m}(\neg P(l)).$$

We prove that A satisfies $\forall_{\alpha \in K^{\mathbb{N}}} \exists_n (A(n, \alpha(n)))$, the hypothesis of CCC_K. Let $k \in K$, and fix $\alpha \in K^{\mathbb{N}}$. If $P(\alpha(k))$, then $A(\alpha(k), \alpha(\alpha(k)))$. If $\neg P(\alpha(k))$, then by the weak decidability of K we can find all elements $\min(K) = k_0 < k_1 < ... < k_{n-1} = \alpha(k)$ in K. If $\neg P(k_0), ..., \neg P(k_{n-2})$, then again $A(k, \alpha(k))$. If there is an $i \in \{0, 1, ..., n-2\}$ such that $P(k_i)$, then $A(k_i, \alpha(k_i))$. Hence there is some $\Lambda \in K$ such that $A(\Lambda, k)$ for each $k \in K$. Testing P on the elements of K less or equal than Λ, either we find a witness of P, or there is no witness of P in K.

To prove $\mathrm{omn}(K)$ from 1, we consider the predicate defined, for each $\gamma \in 2^K$, by $P^{\gamma}(k) \leftrightarrow \gamma(k) = 1$.

2. Because of 1, it suffices to show again that $Q(k) \leftrightarrow \forall_m(P(k,m))$ is decidable on K. If we fix k, then $R(m) \leftrightarrow \neg P(k,m)$ is a decidable predicate on K. By 1, if $\exists_{m \in K}(R(m))$, then for that m we get $\neg P(k,m)$, therefore $\neg Q(k)$. If $\forall_{m \in K}(\neg R(m))$, then $\forall_{m \in K}(\neg\neg P(k,m))$, and by the decidability of P we get $\forall_{m \in K}(P(k,m))$, i.e. $Q(k)$.

3. We use 2 exactly as in the proof of Proposition 10.

We call a subset K of $\mathbb{N}$ *weakly finite*, $K \subseteq^{\mathrm{wfin}} \mathbb{N}$, if it satisfies the following Σ_2^0-formula

$$\exists_{\Lambda \in K} \forall_{k \in K}(k \leq \Lambda).$$

Proposition 17. *Suppose that* K *is an inhabited subset of* $\mathbb{N}$ *satisfying* CCC_K. *Then*

$$\mathrm{BIM} + \mathrm{CP} \vdash K \subseteq^{\mathrm{fin}} \mathbb{N}.$$

Proof. Since BIM $\vdash \neg[K \subseteq^{\mathrm{fin}} \mathbb{N}] \to \mathrm{CCC}_{\mathbb{N}}$, and by Proposition 7, BIM + CP $\vdash \mathrm{CCC}_{\mathbb{N}} \to \bot$, we get

$$\mathrm{BIM} + \mathrm{CP} \vdash \neg\neg(K \subseteq^{\mathrm{fin}} \mathbb{N}).$$

Trivially, $K \subseteq^{\mathrm{fin}} \mathbb{N} \to K \subseteq^{\mathrm{wfin}} \mathbb{N}$. Therefore

$$\neg\neg(K \subseteq^{\mathrm{fin}} \mathbb{N}) \to \neg\neg(K \subseteq^{\mathrm{wfin}} \mathbb{N}).$$

Thus, we get
$$\mathrm{BIM} + \mathrm{CP} \vdash \neg\neg(K \subseteq^{\mathrm{wfin}} \mathbb{N}).$$
Also, by 3 of Proposition 16 we get
$$\mathrm{BIM} \vdash \neg\neg(K \subseteq^{\mathrm{wfin}} \mathbb{N}) \to K \subseteq^{\mathrm{wfin}} \mathbb{N},$$
which can be written also as
$$\mathrm{BIM} + \mathrm{CP} \vdash \neg\neg(K \subseteq^{\mathrm{wfin}} \mathbb{N}) \to K \subseteq^{\mathrm{wfin}} \mathbb{N}.$$
But then we derive
$$\mathrm{BIM} + \mathrm{CP} \vdash K \subseteq^{\mathrm{wfin}} \mathbb{N}.$$
Since we have proved within BIM that K is weakly decidable, we can find all its elements up to its maximum Λ. Therefore K is a finite subset of $\mathbb{N}$.

This proposition is a generalization of a result of [Veldman 1982], which can be stated again as $\mathrm{BIM} + \mathrm{CP} \vdash K \subseteq^{\mathrm{fin}} \mathbb{N}$, where K is not only an inhabited subset of $\mathbb{N}$ satisfying CCC_K, but it is *also* decidable.

Because of the weak decidability of an inhabited set K satisfying CCC_K, Veldman's original proof can also be turned into a proof of Proposition 17.

Veldman's justification that a transparent set K is properly bounded by some natural Λ remains in our setting almost exactly as it is, and it is here where CP is used. While Veldman determines the bounded set K by its decidability, we can use its weak decidability and its transparency for that. The number $\Lambda - 1$ is necessarily a bound for K, while $\Lambda - 2$ is either a bound or is less than some element of K. In the last case $\Lambda - 1$ belongs to K and is necessarily the maximum of K. Then by the weak decidability of K, we can find all its elements. In the first case we repeat the same transparency argument with $\Lambda - 3$. This procedure terminates exactly because K is inhabited. Hence the maximum of K is found in any case, and consequently, by the weak decidability of K, all its elements are found too.

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