Constructive uniformities of pseudometrics and Bishop topologies

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We develop the first steps of a constructive theory of uniformities given by pseudometrics and study its relation to the constructive theory of Bishop topologies. Both these concepts are constructive, function-theoretic alternatives to the notion of a topology of open sets. After motivating the constructive study of uniformities of pseudometrics we present their basic theory and we prove a Stone-Čech theorem for them. We introduce the f-uniform spaces and we prove a Tychonoff embedding theorem for them. We study the uniformity of pseudometrics generated by some Bishop topology and the pseudo-compact Bishop topology generated by some uniformity of pseudometrics. Defining the large uniformity on reals we prove a "large" version of the Tychonoff embedding theorem for f-uniform spaces and we show that the notion of morphism between uniform spaces captures Bishop continuity. We work within BISH*, Bishop's informal system of constructive mathematics BISH extended with inductive definitions with rules of countably many premisses.

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1 Uniformities of pseudometrics in constructive topology

A uniformity of pseudometrics was the first notion of uniformity, which was introduced by Weil in [47] as a natural generalization of the notion of a metric. Shortly after, Tukey's uniformity of coverings and Bourbaki's unifomities of entourages were introduced in [45] and [8], respectively. Classically, these notions of uniform space are equivalent. As it is mentioned in [22], p.43, "Weil's original approach was rather unwieldy and was soon replaced by (the) two others". Despite this prevailed view, today uniformities in the language of pseudometrics are still studied classically (see, for example, [29]). Moreover, as this is shown in the classic book [21], the notion of uniformity which suits better to the classical theory of $C(X)$ is that of Weil's. In [21], p.216, Gillman and Jerison remark the following:

> From our point of view, the most efficient approach to uniform spaces is by way of pseudometrics, as they provide us with a large supply of

> continuous functions Accordingly, we *define* a uniform structure to be a family of pseudometrics (satisfying appropriate closure conditions). This enables us to give complete proofs relatively quickly of all the facts about uniform spaces that are needed here.

Uniformities given by entourages have been studied extensively within the constructive theory of apartness spaces, developed mainly by Bridges and Vîţă in [13] (see also [14] for more recent results[1]). This notion of uniformity is a set-theoretic one, which fits to the set-theoretic character of the notion of an apartness space. As far as we know, uniformities of coverings have not been studied constructively yet.

The constructive study of uniformities given by pseudometrics has a more complex history[2]. Bishop defined a uniform space through pseudometrics[3] in [4], pp.110-1, and this definition was repeated in [7], pp.124-5. Although some fundamental properties of uniform spaces were given in [4] in the form of exercises, Bishop expressed a negative view towards the development of a constructive theory of uniformities given by pseudometrics. In [4], pp.349-51, Bishop writes the following comment.

> A uniform space at first sight appears to be a natural and fruitful concept of a topological space. In fact, this is not the case. For instance, just to construct a compact uniform space X, such that the assumption that X is metrizable leads to a contradiction, seems to be a hard problem. ... Of course, important constructively defined uniform spaces that are not necessarily metrizable exist: every locally convex space has a natural uniform structure. At first glance, the concept of a locally convex space would appear to be important for constructive mathematics, since examples exist in profusion. However, in most cases of interest it seems to be unnecessary to make use of any deep facts from the general theory of convex spaces.

To Bishop's latter argument Bridges and Vîţă respond in [12], p.127, saying that

> ... the development of constructive analysis (in particular, aspects of the

[1] Richman has also studied constructively such uniformities in his unpublished work [41].

[2] We confine our account to the study of uniformities of pseudometrics within Bishop-style constructive mathematics. For the study of uniformities of pseudometrics in formal topology we refer to [17], [19], and [24].

[3] Bishop's definition is more general than the one we use here (see Definition 2.4), although the property (D_2) added here and also found in the classical literature, see for example [21], p.217, is incorporated in Bishop's definition of a morphism between uniform spaces, and it corresponds to the closure of a Bishop topology of functions under uniform limits (see clause (BS_4) of Definition 4.2).

> theory of operators) in recent years has greatly benefited from such a general theory[4] . . . .

To justify his former argument Bishop explains in [4], p.350, why the most obvious expected example of a non-metrizable compact uniform space i.e., the product uniform space $X = [0, 1]^S$, where S is an uncountable set, cannot be shown to be compact uniform space, since it cannot be shown to be totally bounded uniform space i.e., totally bounded with respect to all additions of the pseudometrics in the uniform structure of X. In our view though, this problem does not necessarily imply that the concept of a uniformity of pseudometrics is unnatural or unfruitful. Rather it forces one to find a notion of compact uniformity of pseudometrics that does not copy the definition of a compact metric and at the same time is reduced to it when the uniform space is a metric one. Such an enterprise with respect to compactness has been shown fruitful in formal topology (see [30]), and in the theory of Bishop spaces (see [36]). As we show in [33], [38], and [39], the constructive theory of metric spaces has also benefited from the general theory of Bishop spaces. For example, the fact that a non-zero bounded multiplicative linear functional on $C(K)$, where K is a compact metric space, is determined by some point of K (Proposition 8.25, in [7], p.382) is proved in [7] within the theory of normed spaces, while in [39] is a corollary within the theory of Bishop spaces. Note also that in Bishop's attempt to reconstruct some portion of general topology constructively, found in his unpublished manuscript [6], uniform spaces of pseudometrics play an important role, as *ecclesiastical spaces*, the main objects under study, are such uniform spaces equipped with a *heirarchy*, an appropriate collection of subsets of the main set.

In [13], p.178, it is commented that classical results such as, for example, that a uniformity of entourages is induced by a family of pseudometrics, or that a uniformity with a countable base of entourages is induced by a single pseudometric (see [9], Chapter IX for a classical proof of these facts), are not expected to hold constructively. This cannot be seen though as an argument against the constructive study of uniformities of pseudometrics, since this is very often the case with constructive studies of concepts which have already been treated classically. In [3], p.975, it is noted that the hypothesis that the discrete uniformity $L(X) = \{U \subseteq X \times X \mid \Delta \subseteq U\}$, where $\Delta(X) = \{(x, x) \in X \times X \mid x \in X\}$, is induced by a set of pseudometrics D on X i.e., for every $U \in L$ there exist $d_1, \ldots d_n \in D$ and $\epsilon > 0$ such that

$$\{(x, y) \in X \times X \mid \forall_{1 \leq j \leq n}(d_j(x, y) \leq \epsilon)\} \subseteq U,$$

[4] They mean a general theory of locally convex spaces, presented in section 5.4 of [12]. See also the Thesis [44] of Spitters for contributions in this theory.

implies the weak limited principle of omniscience in the form $\forall_{a,b\in\mathbb{R}}(a = b \vee \neg(a = b))$. Again, this fact cannot be considered as an argument against the development of the constructive theory of uniformities of pseudometrics, since the aim of such a theory is not to capture all classical results governing the relation between uniformities of entourages and uniformities of pseudometrics, a relation which is based on the fact that classically set-theoretic and function-theoretic objects are treated similarly. In constructive mathematics, though, function-theoretic objects behave better than set-theoretic ones.

What we want to emphasize here is that as the constructive study of uniformities of entourages fits to the constructive study of apartness spaces, the constructive study of uniformities of pseudometrics fits to the constructive study of Bishop spaces. As we try to show in the rest of this paper, uniformities of pseudometrics and Bishop topologies share the following characteristics.

(1) Both notions are function-theoretic.

(2) Their definitions have similar structure and induce similar function-theoretic notions of morphisms.

(3) They posses an intrinsic inductive character, which is represented in the concepts of the least uniformity generated by a given set of pseudometrics and of the least Bishop topology generated by a given set of real-valued functions.

(4) Their theories can be developed in parallel and within the same system BISH^*, Bishop's informal system of constructive mathematics BISH (see [4], [7], [2], [10], and [12]) extended with inductive definitions with rules of countably many premisses.

In [28] Myhill proposed the formal theory CST of sets and functions to codify BISH. He also took Bishop's inductive definitions in [4] (of Borel set and of function space, here called Bishop space) at face value and showed that the existence and disjunction properties of CST persist in the extended with inductive definitions system CST^*, which can be considered as a formalization of BISH^*. As another formalization of BISH^* one can consider the system CZF + REA + DC, where Aczel's regular extension axiom REA accommodates inductive definitions in CZF (see [26]) and DC denotes the axiom of dependent choice (see [10], p.12). Here we describe the computational meaning of the theory of uniformities of pseudometrics (and of Bishop topologies) within the informal system BISH^*.

2 Basic notions and facts

We present some first definitions and results necessary to the rest of the paper.

Definition 2.1 A *setoid* is a pair $(X, =_X)$, where X is a set and $=_X$ is an equivalence relation on X. It is called *inhabited*, if there is $x_0 \in X$. If $(X, =_X), (Y, =_Y)$ are setoids[5], a *function* f from X to Y is an operation such that $x =_X y \to f(x) =_Y f(y)$, for every $x, y \in X$ (see [7], p.15). We denote by $\mathbb{F}(X, Y)$ the set of functions from X to Y, which is equipped with the pointwise equality, by $\mathbb{F}(X)$ the set of all functions from X to $\mathbb{R}$, where $\mathbb{R}$ is equipped with the standard equality (see [7], p.18), and by $\mathbb{F}^*(X)$ the set of bounded elements of $\mathbb{F}(X)$. If $f, g \in \mathbb{F}(X)$, we define $f \leq g := \forall_{x \in X}(f(x) \leq g(x))$. If $a \in \mathbb{R}$, we denote by $\overline{a}_X$ the constant function on X with value a, and their set by $\mathrm{Const}(X)$.

Within the theory of uniform spaces of pseudometrics the main objects of study are the pseudometrics on X, while within the theory of Bishop spaces the main objects of study are the functions of type $X \to \mathbb{R}$. For the rest of this paper X, Y denote inhabited setoids.

Definition 2.2 A *pseudometric* on X is a mapping $d : X \times X \to [0, +\infty)$ such that $x = y \to d(x, y) = 0, d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$, for every $x, y, z \in X$. We denote by $\mathbb{D}(X)$ the set of all pseudometrics on X. If d is a pseudometric on X, the pair (X, d) is called a *pseudometric space*. A pseudometric d on X is called *bounded*, if there exists some $M > 0$ such that $d \leq \overline{M}_{X \times X}$. We denote by $\mathbb{D}^*(X)$ the set of bounded pseudometrics on X. If f is a function of type $X \to \mathbb{R}$, the pseudometric d_f *induced* by f is defined by

$$d_f(x, y) := d_{\mathbb{R}}(f(x), f(y)) = |f(x) - f(y)|,$$

for every $x, y \in X$. The constant function $\overline{0}_{X \times X}$ on $X \times X$ is also a pseudometric, which we call the *zero pseudometric* on X. A pseudometric d on X is called *non-zero*, if there exist $x_0, y_0 \in X$ such that $d(x_0, y_0) > 0$. If $d \in \mathbb{D}(X)$ and $x_0 \in X$, the *pseudodistance at* x_0 with respect to d is the mapping $d_{x_0} : X \to [0, \infty)$, defined by $x \mapsto d(x, x_0)$, for every $x \in X$. A pseudometric d is called a *metric*, if $d(x, y) = 0 \to x = y$, for every $x, y \in X$, and then the structure (X, d) is called a *metric space*.

One could write the first definitional clause of a pseudometric as $\forall_{x \in X}(d(x, x) = 0)$, avoiding in this way to mention some equality on X. Since this is required though in

[5]Usually we use for simplicity a single equality symbol for two setoids avoiding subscripts.

the definition of a metric and of a separating set of pseudometrics (see Definition 3.1), we include the setoid structure of X in Definition 2.2. If d_1, d_2 are two pseudometrics on X, it is immediate to see that $d_1 + d_2$ and $d_1 \vee d_2$ are pseudometrics on X, where $(d_1 \vee d_2)(x, y) = d_1(x, y) \vee d_2(x, y)$, for every $x, y \in X$, and $a \vee b = \max\{a, b\}$, for every $a, b \in \mathbb{R}$. Addition and multiplication of real-valued functions are defined pointwisely.

Definition 2.3 If $d, e \in \mathbb{D}(X)$, $\Delta \subseteq \mathbb{D}(X)$, and $\delta, \epsilon > 0$, we define:

$$U(d, \delta, e, \epsilon) := \forall_{x,y \in X}\big(d(x, y) \leq \delta \to e(x, y) \leq \epsilon\big).$$

$$U(\Delta, e) := \forall_{\epsilon > 0}\exists_{\delta > 0}\exists_{d \in \Delta}(U(d, \delta, e, \epsilon)),$$

$$\overline{\Delta} := \{e \in \mathbb{D}(X) \mid U(\Delta, e)\}.$$

We call $\overline{\Delta}$ the *pseudometric closure* of Δ, while if $\overline{\Delta} = \Delta$, we say that Δ is *pseudometrically closed*.

If (X, ρ) is a metric space, $f : X \to \mathbb{R}$ is uniformly continuous with modulus of continuity ω_f i.e.,

$$\forall_{x,y \in X}\big(\rho(x, y) \leq \omega_f(\epsilon) \to |f(x) - f(y)| \leq \epsilon\big),$$

and g is just a function of type $X \to \mathbb{R}$, then the condition $U(\{d_f\}, d_g)$ implies the uniform continuity of g; let $\epsilon > 0$ and $\delta > 0$ be such that $U(d_f, \delta, d_g, \epsilon)$. If $x, y \in X$, then

$$\rho(x, y) \leq \omega_f(\delta) \to |f(x) - f(y)| \leq \delta \to |g(x) - g(y)| \leq \epsilon$$

i.e., $\omega_g(\epsilon) = \omega_f(\delta)$.

Definition 2.4 A subset D of $\mathbb{D}(X)$ is a *uniformity* D on X, if

(D_0) $\overline{0}_{X \times X} \in D$.

(D_1) $d_1, d_2 \in D \to d_1 \vee d_2 \in D$.

(D_2) $e \in \mathbb{D}(X) \to U(D, e) \to e \in D$.

A *uniform space* is a pair $\mathcal{D} = (X, D)$, where D is a uniformity on X. A uniformity D on X, or a uniform space $\mathcal{D}$, are called *bounded*, if $D \subseteq \mathbb{D}^*(X)$.

Clearly, $\{\overline{0}_{X \times X}\}$ and $\mathbb{D}(X)$ are uniformities on X that we call the *trivial* and the *discrete* uniformity on X, respectively. If D is a uniformity on X, then $\{\overline{0}\} \subseteq D \subseteq \mathbb{D}(X)$. It is immediate to see that if D_1, D_2 are uniformities on X, then $D_1 \cap D_2$ is a uniformity on X. The next proposition expresses the independence of (D_1) and (D_2).

Proposition 2.5 *(i) There exists a $\vee$-closed, not pseudometrically closed $\Delta \subseteq \mathbb{D}(X)$.*
(ii) There exist X and a pseudometrically closed $\Delta \subseteq \mathbb{D}(X)$ that is not $\vee$-closed.

Proof (i) Let $\Delta = \{d\}$, where d is a non-zero pseudometric on X. Since $d \vee d = d$, Δ is $\vee$-closed, and since $U(d, \frac{\epsilon}{2}, d + d, \epsilon)$, for every $\epsilon > 0$, we get $d + d \in \overline{\Delta} \setminus \Delta$.
(ii) Let $X = \{x_1, x_2, x_3\}$ be a set with three elements. Let $f, g : X \to \mathbb{R}$ such that $f(x_1) = f(x_2)$, $|g(x_1) - g(x_2)| = \epsilon_{12} > 0$, and $g(x_1) = g(x_3)$, $|f(x_1) - f(x_3)| = \epsilon_{13} > 0$. By definition

$$d_f \vee d_g \in \overline{\{d_f, d_g\}} \leftrightarrow \forall_{\epsilon > 0} \exists_{\delta > 0} (U(d_f, \delta, d_f \vee d_g, \epsilon) \vee U(d_g, \delta, d_f \vee d_g, \epsilon)).$$

We suppose that $d_f \vee d_g \in \overline{\{d_f, d_g\}}$ and we apply the above condition on some $\epsilon > 0$ such that $\epsilon < \epsilon_{12} \wedge \epsilon_{13}$. If $U(d_f, \delta, d_f \vee d_g, \epsilon)$ is the case, then for x_1, x_2 we have that $|f(x_1) - f(x_2)| = 0 \leq \delta$ and $|f(x_1) - f(x_2)| \vee |g(x_1) - g(x_2)| = \epsilon_{12} \leq \epsilon$, which is a contradiction. If $U(d_g, \delta, d_f \vee d_g, \epsilon)$ is the case, then for x_1, x_3 we have that $|g(x_1) - g(x_3)| = 0 \leq \delta$ and $|f(x_1) - f(x_3)| \vee |g(x_1) - g(x_3)| = \epsilon_{13} \leq \epsilon$, which is a contradiction. Hence $d_f \vee d_g \notin \overline{\{d_f, d_g\}}$. □

Definition 2.6 If $d \in \mathbb{D}(X)$ and $a > 0$ the *truncation* of d by a is the mapping $d \wedge \overline{a}_{X \times X}$, where $(d \wedge \overline{a}_{X \times X})(x, y) = d(x, y) \wedge \overline{a}_{X \times X}(x, y) = d(x, y) \wedge a$, for every $x, y \in X$, and $a \wedge b := \min\{a, b\}$, for every $a, b \in \mathbb{R}$.

Proposition 2.7 *If $\mathcal{D} = (X, D)$ is a uniform space and $e \in \mathbb{D}(X)$, the following hold.*
(D_3) $a > 0 \to d \in D \to \overline{a}_{X \times X} d \in D$.
(D_4) $e \leq d \to d \in D \to e \in D$.
(D_5) *D is inhabited.*
(D_6) $d_1, d_2 \in D \to d_1 + d_2 \in D$.
(D_7) $a > 0 \to d \in D \to d \wedge \overline{a}_{X \times X} \in D$,
(D_8) *If $d \in D$ and $x_0 \in X$, the pseudometric $d_{d_{x_0}}$ on X induced by d_{x_0} is in D.*

Proof (D_3) If $\epsilon > 0$, $x, y \in X$, and $d(x, y) \leq \frac{\epsilon}{a}$, then $(\overline{a}_{X \times X} d)(x, y) \leq \epsilon$ i.e., $\overline{a}_{X \times X} d \in \overline{\{d\}} \subseteq \overline{D} = D$.
(D_4) If $\epsilon > 0$ and $x, y \in X$, then if $d(x, y) \leq \epsilon$, then $e(x, y) \leq \epsilon$ i.e., $e \in \overline{\{d\}} \subseteq \overline{D} = D$.
(D_5) Immediately by (D_0).
(D_6) Since $d_1, d_2 \leq d_1 \vee d_2$, $d_1 + d_2 \leq \overline{2}_{X \times X}(d_1 \vee d_2)$, and we use (D_3) and (D_4).
(D_7) The triangle inequality, the only less trivial condition in showing $d \wedge \overline{a}_{X \times X} \in \mathbb{D}(X)$, follows from the property $(b + c) \wedge a = (b \wedge a) + (c \wedge a)$ of reals. If $x, y \in X$ and $\epsilon > 0$, then $d(x, y) \leq \frac{a}{2} \wedge \epsilon \to (d \wedge \overline{a}_{X \times X})(x, y) = d(x, y) \leq \epsilon$, and we use (D_2).
(D_8) If $x_1, x_2 \in X$, we have that

$$d_{d_{x_0}}(x_1, x_2) = |d_{x_0}(x_1) - d_{x_0}(x_2)| = |d(x_1, x_0) - d(x_2, x_0)| \leq d(x_1, x_2),$$

therefore $d_{d_{x_0}} \leq d$. By (D_4) we get $d_{d_{x_0}} \in D$. □

Since $d_1 \vee d_2 \leq d_1 + d_2$, one could replace (D_1) with (D_6). Moreover, (D_0) is equivalent to (D_5). One can turn the definitional clauses (D_0), (D_1) and (D_2) of a uniformity into inductive rules and define the least uniformity generated by some given set of pseudometrics D_0. This notion is central to the development of the constructive study of uniformities of pseudometrics[6].

Definition 2.8 If $D_0 \subseteq \mathbb{D}(X)$, the *least uniformity* $\coprod D_0$ generated by D_0 is defined by the following inductive rules:

$$\frac{d_0 \in D_0}{d_0 \in \coprod D_0}, \quad \frac{}{\overline{0}_{X\times X} \in \coprod D_0},$$

$$\frac{d_1, d_2 \in \coprod D_0}{d_1 \vee d_2 \in \coprod D_0},$$

$$\frac{(d \in \coprod D_0 \wedge \delta > 0 \wedge U(d, \delta, e, \epsilon))_{\epsilon>0}}{e \in \coprod D_0}.$$

If D is a uniformity on X, $D_0 \subseteq \mathbb{D}(X)$, and $D = \coprod D_0$, we call D_0 a *subbase* for D.

The most complex inductive rule in Definition 2.8 can be replaced by the following rule

$$\frac{d_1 \in \coprod D_0 \wedge \delta_1 > 0 \wedge U(d_1, \delta_1, e, 1), \ \ d_2 \in \coprod D_0 \wedge \delta_2 > 0 \wedge U(d_2, \delta_2, e, \frac{1}{2}), \dots}{e \in \coprod D_0},$$

which has countably many premisses. Definition 2.8 induces the following induction principle $\mathrm{Ind}_{\coprod D_0}$ on $\coprod D_0$: if P is any property on $\mathbb{D}(X)$, then

$$\begin{aligned}
&\forall_{d_0 \in D_0}(P(d_0)) \to \\
&P(\overline{0}_{X\times X}) \to \\
&\forall_{d_1, d_2 \in \coprod D_0}\big(P(d_1) \to P(d_2) \to P(d_1 \vee d_2)\big) \to \\
&\forall_{e \in \coprod D_0}\big(\forall_{\epsilon>0}\exists_{\delta>0}\exists_{d \in \coprod D_0}(P(d) \ \wedge \ U(d, \delta, e, \epsilon)) \to P(e)\big) \to \\
&\forall_{d \in \coprod D_0}(P(d)).
\end{aligned}$$

Definition 2.9 A property P on $\mathbb{D}(X)$ is $\coprod$-*lifted*, if

$$\forall_{d_0 \in D_0}\big(P(d_0) \to \forall_{d \in \coprod D_0}(P(d)\big),$$

while it is *lifted to the closure*, if for every $D_0 \subseteq \mathbb{D}(X)$

$$\forall_{d_0 \in D_0}\big(P(d_0) \to \forall_{d \in \overline{D_0}}(P(d)\big).$$

[6] In the constructive theory of uniformities of entourages (see [13]) such a notion cannot be defined.

Definition 2.10 If D is a uniformity on X, a $\Delta \subseteq D$ is called a *base* for D, if $D \subseteq \overline{\Delta}$.

Since $\Delta \subseteq D \to \overline{\Delta} \subseteq \overline{D} = D$, we get that Δ is a base for D if and only if $D = \overline{\Delta}$. Note that since D is inhabited, a base Δ for D is also inhabited, while a subbase need not be inhabited e.g., $\coprod \emptyset = \{\overline{0}_{X \times X}\}$. The following two propositions are easy to show.

Proposition 2.11 *If $\Delta \subseteq \mathbb{D}(X)$ is $\vee$-closed, then $\overline{\Delta}$ is $\vee$-closed.*

Proposition 2.12 *Let (X, D) be a uniform space and D_0 an inhabited subbase for D.*
(i) If D_0 is $\vee$-closed, then D_0 is a base for $\coprod D_0$.
(ii) The set $\Delta(D_0) = \{\bigvee_{i=1}^{n} d_{0i} \mid d_{0i} \in D_0, 1 \leq i \leq n, n \in \mathbb{N}\}$ is a base[7] *for D.*
(iii) The set of bounded pseudometrics $D^* = D \cap \mathbb{D}^*(X)$ of D is a base for D.
(iv) If Δ is a base for D and $a > 0$, $\Delta \wedge \overline{a}_{X \times X} = \{d \wedge \overline{a}_{X \times X} \mid d \in \Delta\}$ is a base for D.

By Proposition 2.12(iii), although D^* contains the zero pseudometric on X and it is $\vee$-closed, it is not in general pseudometrically closed, since if it was, every uniformity on X would be bounded, which, of course, is not the case. The fact that D^* is not a uniformity reveals a difference between the notion of a Bishop topology, where the bounded elements F^* of a Bishop topology F form a Bishop topology (see Definition 4.2), and the notion of uniformity of pseudometrics.

Definition 2.13 If $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$ are uniform spaces, a function $h : X \to Y$ is a *morphism* from $\mathcal{D}$ to $\mathcal{E}$ if and only if $\forall_{e \in E}(e \odot h \in D)$, where the *pseudo-composition* operation $\odot$ of the pseudometric e and the function h is defined[8] by

$$e \odot h := e \circ h^{[2]},$$

$$h^{[2]}(x_1, x_2) := (h(x_1), h(x_2)),$$

for every $x_1, x_2 \in X$ i.e., the following diagram commutes

$$\begin{tikzcd} X \times X \arrow[r, "h^{[2]}"] \arrow[dr, "D \ni e \odot h"'] & Y \times Y \arrow[d, "e \in E"] \\ & {[0, \infty)}. \end{tikzcd}$$

[7] The generation of a base out of a subbase for Bishop spaces is more complex (see [33]).
[8] It is immediate to see that $e \odot h$ is a pseudometric on X.

We denote by $\mathrm{Mor}(\mathcal{D}, \mathcal{E})$ the morphisms between $\mathcal{D}$ and $\mathcal{E}$. If $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, it is called *open*, if $\forall_{d \in D} \exists_{e \in E}(d = e \odot h)$, an *isomorphism* between $\mathcal{D}$ and $\mathcal{E}$, if it is a bijection and $h^{-1} \in \mathrm{Mor}(\mathcal{E}, \mathcal{D})$, and a *set-epimorphism*, if h is a surjection, while the *induced mapping* H of h is the function $H : E \to D$, defined by $H(e) := e \odot h$, for every $e \in E$. The morphisms between uniform spaces are the arrows in the category of uniform spaces **Unif**, where the identity arrow for $\mathcal{D}$ is the identity function id_X of X.

The proof of the next proposition is straightforward.

Proposition 2.14 *Suppose that $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$, $\mathcal{B} = (Z, B)$ are uniform spaces, $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, $e_1, e_2 \in E$, $\epsilon, \delta > 0$, $Z \subseteq E$, and $Z \odot h := \{\zeta \odot h \mid \zeta \in Z\}$.*
(i) $\overline{0}_{Y \times Y} \odot h = \overline{0}_{X \times X}$.
(ii) $(e_1 \vee e_2) \odot h = (e_1 \odot h) \vee (e_2 \odot h)$.
(iii) $(e_1 + e_2) \odot h = (e_1 \odot h) + (e_2 \odot h)$.
(iv) $U(e_1, \delta, e_2, \epsilon) \to U(e_1 \odot h, \delta, e_2 \odot h, \epsilon)$.
(v) If $e \in \overline{Z}$, then $e \odot h \in \overline{Z \odot h}$.
(vi) If h is a set-epimorphism, then h is an isomorphism if and only if h is open.
(vii) If $g \in \mathrm{Mor}(\mathcal{E}, \mathcal{B})$, then $(b \odot g) \odot h = b \odot (g \circ h)$, for every $b \in B$.
(viii) If h is a set-epimorphism and $e_1 \odot h = e_2 \odot h$, then $e_1 = e_2$.

If (X, T) is a topological space, the set $C(X)$ of real-valued continuous functions on X is a ring and a lattice. To this structure of $C(X)$ corresponds the notion of a ring and lattice homomorphism. The algebraic and lattice structure of a uniformity D on some X can be described by the signature

$$\big(\overline{0}_{X \times X}, \vee, +, (U_{\epsilon,\delta})_{\epsilon,\delta > 0}\big),$$

where $(D, \overline{0}_{X \times X}, \vee)$ is a semi-lattice with bottom, $(D, +, \overline{0}_{X \times X})$ is an abelian monoid with unit, and for every $\epsilon, \delta > 0$ the relation $U_{\epsilon,\delta} \subseteq D \times D$ is defined by $U_{\epsilon,\delta}(d, e) = U(d, \delta, e, \epsilon)$. To this structure of D corresponds a natural notion of homomorphism. By Proposition 2.14 the induced mapping H of some $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ is such a homomorphism.

Definition 2.15 If $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$ are uniform spaces, a function $\Phi : D \to E$ is called a *uniformity homomorphism*, if it preserves $\overline{0}_{X \times X}$, $\vee$, $+$, and

$$U_{\epsilon,\delta}(d_1, d_2) \to U_{\epsilon,\delta}(\Phi(d_1), \Phi(d_2)),$$

for every $d_1, d_2 \in D$ and $\delta, \epsilon > 0$. If Φ is a bijection and Φ^{-1} is a uniformity homomorphism, then Φ is called a *uniformity isomorphism*.

Proposition 2.16 ($\coprod$-lifting of morphisms) *If $\mathcal{D} = (X, D)$ and $\mathcal{E} = (Y, \coprod E_0)$ are uniform spaces, then $h : X \to Y \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ if and only if $\forall_{e_0 \in E_0}(e_0 \odot h \in D)$,*

$$\begin{tikzcd} X \times X \arrow[r, "h^{[2]}"] \arrow[dr, "D \ni e_0 \odot h"'] & Y \times Y \arrow[d, "e_0 \in E_0"] \\ & {[0, \infty)}. \end{tikzcd}$$

Proof We show inductively that $\forall_{e \in \coprod E_0}(e \odot h \in D)$. This is immediate, if $e \in E_0$, or $e = \overline{0}_{Y \times Y}$. For the case $e_1 \vee e_2$ we use Proposition 2.14(ii) and the inductive hypotheses on e_1 and e_2. If $\epsilon, \delta > 0$, the property $U(e', \delta, e, \epsilon) \to U(e' \odot h, \delta, e \odot h, \epsilon)$, where $e' \in \coprod E_0$ such that $e' \odot h \in D$, is shown by Proposition 2.14(iv). $\square$

Lemma 2.17 (Well-definability lemma) *Let $h : X \to Y$ be a surjection, $Z \subseteq \mathbb{D}(Y)$, and $d \in \mathbb{D}(X)$. If $d \in \overline{Z \odot h}$, the function $d^{\#} : Y \times Y \to \mathbb{R}$, defined by*

$$d^{\#}(y_1, y_2) = d^{\#}(h(x_1), h(x_2)) := d(x_1, x_2),$$

$$\begin{tikzcd} X \times X \arrow[r, "h^{[2]}"] \arrow[dr, "d"'] & Y \times Y \arrow[d, dashed, "d^{\#}"] \\ & {[0, \infty)} \end{tikzcd}$$

for every $y_1, y_2 \in Y$, is a well-defined pseudometric on Y i.e.,

$$\forall_{x_1, x_2, x_3, x_4 \in X}\big(h(x_1) = h(x_3) \to h(x_2) = h(x_4) \to d(x_1, x_2) = d(x_3, x_4)\big).$$

Proof Let $x_1, x_2, x_3, x_4 \in X$ with $h(x_1) = h(x_3) = y_1$ and $h(x_2) = h(x_4) = y_2$. Since $d \in \overline{Z \odot h}$, for every $\epsilon > 0$ there are $\delta > 0$ and $\zeta \in Z$ such that $U(\zeta \odot h, \delta, d, \epsilon)$ i.e.,

$$\forall_{x, x' \in X}\big(\zeta(h(x), h(x')) \leq \delta \to d(x, x') \leq \epsilon\big).$$

Let $\epsilon > 0$. There exist $\delta > 0$ and $\zeta \in Z$ such that $\forall_{x, x' \in X}\big(\zeta(h(x), h(x')) \leq \delta \to d(x, x') \leq \frac{\epsilon}{2}\big)$. Since by hypothesis $\zeta(h(x_1), h(x_3)) = \zeta(h(x_2), h(x_4)) = 0$, we get $d(x_1, x_3) \leq \frac{\epsilon}{2}$ and $d(x_2, x_4) \leq \frac{\epsilon}{2}$. Hence

$$|d(x_1, x_2) - d(x_3, x_4)| \leq d(x_1, x_3) + d(x_2, x_4) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $|d(x_1, x_2) - d(x_3, x_4)| = 0$. That $d^{\#}$ is a pseudometric on Y follows by the fact that d is a pseudometric on X and the surjectivity of h. $\square$

Proposition 2.18 *(i) If $h : X \to Y$ is a surjection, $\Delta \subseteq \mathbb{D}(X)$, and $Z \subseteq \mathbb{D}(Y)$, then*

$$\forall_{d \in \Delta} \exists_{\zeta \in Z}(d = \zeta \odot h) \to \forall_{e \in \overline{\Delta}} \exists_{\zeta^* \in \overline{Z}}(e = \zeta^* \odot h).$$

(ii) Let $\mathcal{D} = (X, D)$, $\mathcal{E} = (Y, E)$ be uniform spaces, $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ a surjection, and let Δ be a base for D.

(a) (Lifting of openness to the closure of a base) If $\forall_{d \in \Delta} \exists_{e \in E}(d = e \odot h)$, then h is open and $H^{-1}(\Delta)$ is a base for E.

(b) ($\coprod$-lifting of openness) If D_0 is a subbase for D such that $\forall_{d_0 \in D_0} \exists_{e \in E}(d_0 = e \odot h)$, then $\forall_{d \in D} \exists_{e \in E}(d = e \odot h)$.

Proof (i) If $e \in \overline{\Delta}$ and $\epsilon >$, there exist $d \in \Delta$ and $\delta > 0$ such that $U(d, \delta, e, \epsilon)$. Since $\Delta \subseteq Z \odot h$, we have that $e \in \overline{\Delta} \subseteq \overline{Z \odot h}$, by the well-definability lemma we have that $e^\# \in \mathbb{D}(Y)$ and $e = e^\# \odot h$. If $\zeta \in Z$ such that $d = \zeta \odot h$, then

$$U(d, \delta, e, \epsilon) \leftrightarrow U(\zeta \odot h, \delta, e^\# \odot h, \epsilon).$$

Since h is onto Y, we get $U(\zeta, \delta, e^\#, \epsilon)$. Since $\epsilon > 0$ is arbitrary, $e^\#$ is in $\overline{Z}$ i.e., the required element ζ^* of $\overline{Z}$ is $e^\#$.
(ii) (a) This lifting follows from (i) for $Z = E$. If $e \in E$, then $e \odot h \in D$, and since Δ is a base for D, if $\epsilon > 0$, there are $\delta > 0$, $d \in \Delta$ such that $U(d, \delta, e \odot h, \epsilon)$. By hypothesis there is $e' \in E$ with $d = e' \odot h$, hence $U(e' \odot h, \delta, e \odot h, \epsilon)$, and consequently $U(e', \delta, e, \epsilon)$. Since $e' \in H^{-1}(\Delta)$ and $\epsilon > 0$ is arbitrary, $H^{-1}(\Delta)$ is a base for E.
(b) This lifting follows from (a) and the fact that for every element $\bigvee_{i=1}^{n} d_{0i}$ of the base $\Delta(D_0)$ for D (Proposition 2.12)(ii) there exist $e_1, \ldots, e_n \in E$ such that

$$\bigvee_{i=1}^{n} d_{0i} = \bigvee_{i=1}^{n} (e_i \odot h) = \left(\bigvee_{i=1}^{n} e_i \right) \odot h.$$

□

The $\coprod$-lifting of openness is used in the proof of Theorems 3.15 and 6.6.

Definition 2.19 If $\mathcal{D} = (X, D)$ and $\mathcal{E} = (Y, E)$ are uniform spaces, the *product* uniform space is the pair $\mathcal{D} \times \mathcal{E} = (X \times Y, D \times E)$, where

$$D \times E := \coprod \big[\{d \odot \pi_1 \mid d \in D\} \cup \{e \odot \pi_2 \mid e \in E\}\big] =: \coprod_{d \in D}^{e \in E} d \odot \pi_1, e \odot \pi_2,$$

π_1 is the projection map of $X \times Y$ on X and π_2 is the projection map of $X \times Y$ on Y. If $A \subseteq X$ is inhabited, the *relative* uniform space on A is the pair $\mathcal{D}_{|A} = (A, D_{|A})$, where

$$D_{|A} := \coprod \{d_{|A \times A} \mid d \in D\} =: \coprod_{d \in D} d_{|A \times A}.$$

An isomorphism h between $\mathcal{D}$ and $\mathcal{E}_{|h(X)}$ is called a *uniform embedding* of $\mathcal{D}$ into $\mathcal{E}$.

According to Beeson [2], p.44, if A is a rule which associates to every element i of a set I a set A_i, the *infinite product* $\prod_{i \in I} A_i$ is defined by

$$\prod_{i \in I} A_i := \Big\{ f \in \mathbb{F}(I, \bigcup_{i \in I} A(i)) \mid \forall_{i \in I}(f(i) \in A(i)) \Big\},$$

where the exterior union $\bigcup_{i \in I} A_i$ is defined by Richman (see Ex. 2 in [7], p.78). If A associates to every element of I the set X, we denote the product $\prod_{i \in I} X$ by X^I. Since $X^I = \mathbb{F}(I, X)$, the exterior union is avoided in this case.

Definition 2.20 If $\mathbb{X}$ is a rule which associates to every element i of a set I a setoid $(X_i, =_i)$, and $\mathbb{D}$ is a rule which associates to every element i of I a set $D_i \subseteq \mathbb{D}(X_i)$, such that $\mathcal{D}_i = (X_i, D_i)$ is a uniform space, the *I-product of the uniform spaces $\mathcal{D}_i$* is the pair

$$\prod_{i \in I} \mathcal{D}_i := \bigg(\prod_{i \in I} X_i, \coprod_{d \in D_i}^{i \in I} d \odot \varpi_i \bigg),$$

where ϖ_i is the i-th projection function from $\prod_{i \in I} X_i$ to X_i i.e., $\varpi_i(f) = f(i)$, for every $f \in \prod_{i \in I} X_i$ and every $i \in I$. If $\mathcal{D} = (X, D)$ is a uniform space and to each element of I the sets X and D are associated, we denote the *I-product of $\mathcal{D}$* by

$$\mathcal{D}^I := \bigg(X^I, \coprod_{d \in D}^{i \in I} d \odot \varpi_i \bigg).$$

It is easy to see that $D \times E$ is the least uniformity on $X \times Y$ such that π_1, π_2 are in $\mathrm{Mor}(\mathcal{D} \times \mathcal{E}, \mathcal{D})$ and in $\mathrm{Mor}(\mathcal{D} \times \mathcal{E}, \mathcal{E})$, respectively, and that $\mathcal{D} \times \mathcal{E}$ satisfies the universal property of the product. The following two propositions are easy to show.

Proposition 2.21 *If $D_0 \subseteq \mathbb{D}(X)$ and $E_0 \subseteq \mathbb{D}(Y)$, and $A \subseteq X$ inhabited, then*

$$\begin{aligned} \coprod D_0 \times \coprod E_0 &= \coprod \big[\{ d_0 \odot \pi_1 \mid d_0 \in D_0 \} \cup \{ e_0 \odot \pi_2 \mid e_0 \in E_0 \} \big] \\ &=: \coprod_{d_0 \in D_0}^{e_0 \in E_0} d_0 \odot \pi_1, e_0 \odot \pi_2, \end{aligned}$$

$$\Big(\coprod D_0 \Big)_{|A} = \coprod \{ d_{0|A \times A} \mid d_0 \in D_0 \} =: \coprod_{d_0 \in D_0} d_{0|A \times A}.$$

Proposition 2.22 *Let $\mathcal{D} = (X, D)$ be a uniform space, $x_0 \in X$, and $d \in \mathbb{D}(X)$.*
(i) The mappings ${}_{x_0}i : X \to X \times X$ and $i_{x_0} : X \to X \times X$, defined by $x \mapsto (x, x_0)$, and $x \mapsto (x_0, x)$ for every $x \in X$, respectively, are uniform embeddings of $\mathcal{D}$ into $\mathcal{D} \times \mathcal{D}$.
(ii) If $d \odot \pi_1 \in D \times D$, or if $d \odot \pi_2 \in D \times D$, then $d \in D$.

It is easy to see that the previous equalities hold for the I-product of uniform spaces too. The next fact is also immediate to show.

Proposition 2.23 *Let $\mathcal{D} = (X, D)$ and $\mathcal{E} = (Y, E)$ be uniform spaces and $h : X \to Y$.*
(i) $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ if and only if $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E}_{h(X)})$.
(ii) If h is an open morphism from $\mathcal{D}$ to $\mathcal{E}$, h is open as a morphism from $\mathcal{D}$ to $\mathcal{E}_{|h(X)}$.

If $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E}_{|h(X)})$ is open, it is not necessarily open as an element of $\mathrm{Mor}(\mathcal{D}, \mathcal{E})$; if $d \in D$ and $e' \in E_{|h(X)}$ with $d = e' \odot h$, it is not necessary that $e' = e_{|h(X) \times h(X)}$, for some $e \in E$.

Definition 2.24 If (X, ρ) is a metric space, we call the uniformity

$$D(\rho) = \coprod \{\rho\} := \coprod \rho$$

on X the *metric* uniformity on X generated by ρ, and $\mathcal{D}(\rho) = (X, D(\rho))$ the *metric* uniform space generated by ρ. The uniform space

$$\mathcal{R} = \big(\mathbb{R}, D(d_{\mathbb{R}})\big)$$

is the *uniform space of reals*. An I-product $\mathcal{R}^I$ of $\mathcal{R}$ is called a *Euclidean* uniform space.

If $I = n := \{1, \ldots, n\}$, then by Proposition 2.21 we have that

$$D(d_{\mathbb{R}})^n = \Big(\coprod d_{\mathbb{R}}\Big)^n = \coprod d_{\mathbb{R}} \odot \pi_1, \ldots, d_{\mathbb{R}} \odot \pi_n = \coprod d_{\pi_1}, \ldots, d_{\pi_n},$$

since $d_{\mathbb{R}} \odot \pi_i = d_{\pi_i}$, for every $i \in n$. In the classical literature, see e.g., [21], p.224, and in the constructive one, see [7], p.124, an element of $\mathrm{Mor}(\mathcal{D}, \mathcal{E})$ is called a uniformly continuous function. Because of Proposition 2.25(ii) the notion of a morphism between uniform spaces is a generalization of a uniformly continuous function between metric spaces. As we show though in Theorem 6.9, the notion of morphism between uniform spaces can also be reduced to other notions of continuity, like Bishop continuity. Next proposition has an immediate proof.

Proposition 2.25 *Let (X, ρ), (Y, σ) be metric spaces and $h : X \to Y$.*
(i) $\coprod \rho = \overline{\{\rho\}}$.
(ii) $h \in \mathrm{Mor}(\mathcal{D}(\rho), \mathcal{D}(\sigma))$ if and only if h is uniformly continuous.

Definition 2.26 If $\mathcal{D} = (X, D)$ is a uniform space, we denote by $\mathcal{M}(\mathcal{D})$ the set $\mathrm{Mor}(\mathcal{D}, \mathcal{R})$ and by $\mathcal{M}^*(\mathcal{D})$ the bounded elements of $\mathrm{Mor}(\mathcal{D}, \mathcal{R})$.

The next fact follows easily.

Proposition 2.27 *Let $\mathcal{D} = (X, D)$ be a uniform space and $h : X \to \mathbb{R}$.*
(i) $h \in \mathcal{M}(\mathcal{D}) \leftrightarrow d_h \in D$.
(ii) $\mathcal{D}$ is bounded if and only if $\mathcal{M}(\mathcal{D}) = \mathcal{M}^*(\mathcal{D})$.

By Proposition 2.27(i), the fact that $d_{\overline{a}_X} = \overline{0}_{X \times X}$, for (3), and (D_8), for (4), we get

$$(1) \qquad \mathcal{M}(\mathcal{D}) = \{h \in \mathbb{F}(X) \mid d_h \in D\},$$

$$(2) \quad \mathcal{M}^*(\mathcal{D}) = \{h \in \mathbb{F}^*(X) \mid d_h \in D\},$$

$$(3) \qquad \mathrm{Const}(X \times X) \subseteq \mathcal{M}^*(\mathcal{D}),$$

$$(4) \qquad \{d_x \mid d \in D, x \in X\} \subseteq \mathcal{M}(\mathcal{D}).$$

The next result, which is found as an exercise in [21], p.237, and is included here for the sake of completeness, has its analogue in the theory of Bishop spaces, namely that a Bishop topology is the set of Bishop morphisms from the Bishop space to the Bishop space of reals (see footnote in Definition 4.2). Its proof is straightforward.

Proposition 2.28 *Let $\mathcal{D} = (X, D)$ be a uniform space and $e \in \mathbb{D}(X)$.*
(i) $e \in D$ if and only if $e \in \mathcal{M}(\mathcal{D} \times \mathcal{D})$.
(ii) $e \in D^*$ if and only if $e \in \mathcal{M}^*(\mathcal{D} \times \mathcal{D})$.

The hypothesis $e \in \mathbb{D}(X)$ in the formulation of Proposition 2.28 is used in the proof of both implications of case (i), and it is also necessary, since the constant maps are in $\mathcal{M}^*(\mathcal{D} \times \mathcal{D})$, but, except from $\overline{0}_{X \times X}$, they don't satisfy the properties of a pseudometric.

Definition 2.29 If $h : X \times X \to \mathbb{R}$ and $x_0 \in X$, we define the functions $h^{\Delta}, h_{x_0}, {}_{x_0}h : X \to \mathbb{R}$, by $h^{\Delta}(x) := h(x, x), h_{x_0}(x) := h(x_0, x), {}_{x_0}h(x) := h(x, x_0)$, for every $x \in X$, respectively. If $g : X \to Y$, we define $g^{[2]} : X \times X \to Y \times Y$ by $g^{[2]}(x_1, x_2) = (g(x_1), g(x_2))$, for every $x_1, x_2 \in X$.

The next proposition follows easily.

Proposition 2.30 *Let $\mathcal{D} = (X, D)$, $\mathcal{E} = (Y, E)$ be uniform spaces and $x_0 \in X$.*
(i) If $h : X \times X \to \mathbb{R}$ such that $h \in \mathcal{M}(\mathcal{D} \times \mathcal{D})$, the maps h^{Δ}, h_{x_0} and ${}_{x_0}h$ are in $\mathcal{M}(\mathcal{D})$.
(ii) If $g : X \to Y$, then $g \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ if and only if $g^{[2]} \in \mathrm{Mor}(\mathcal{D}^2, \mathcal{E}^2)$.

3 Separating uniformities

Definition 3.1 If D is a uniformity on X, its *canonical* point-point apartness relation $\bowtie_D$ on X is defined, for every $x, y \in X$ by

$$x \bowtie_D y := \exists_{d \in D}(d(x, y) > 0).$$

If $\bowtie_D$ is tight[9], then D is called a *tight* uniformity. A subset Δ of $\mathbb{D}(X)$ is a *separating* set of pseudometrics, if

$$\forall_{x,y \in X}\big(\forall_{d \in \Delta}(d(x, y) = 0) \to x = y\big).$$

If D is separating, we call $\mathcal{D}$ *separated by* D, or simply *separated*.

Next characterization of tightness follows immediately, while the easy to show Proposition 3.3 implies that a metric uniformity $D(\rho)$ is separating.

Proposition 3.2 *If D is a uniformity on X, then D is tight if and only D is separating.*

Proposition 3.3 *If D_0 is a subbase of a uniformity D on X, then D is separating if and only if $\forall_{x,y \in X}(\forall_{d_0 \in D_0}(d_0(x, y) = 0) \to x = y)$.*

Corollary 3.4 *If $(\mathcal{D} = (X, D)$ is a uniform space and Δ is a base for D, then D is separating if and only if Δ is separating.*

Next proposition is also easy to show.

Proposition 3.5 *Let $\mathcal{D} = (X, D)$ and $\mathcal{E} = (Y, E)$ be uniform spaces.*
(i) If h is an isomorphism between $\mathcal{D}$, $\mathcal{E}$ and D is separating, then E is separating.
(ii) If $A \subseteq X$ is inhabited and D is separating, then $D_{|A}$ is separating.
(iii) $D \times E$ is separating if and only if D, E are separating.

Definition 3.6 If $\mathcal{D} = (X, D)$ is a uniform space and $\phi : X \to Y$ is a surjection, the *quotient uniformity* D_ϕ on Y with respect to ϕ is defined by

$$D_\phi := \{e \in \mathbb{D}(Y) \mid e \odot \phi \in D\},$$

and the *quotient uniform space* with respect to ϕ is the pair $\mathcal{D}_\phi = (Y, D_\phi)$.

[9] A point-point apartness relation $\bowtie$ on X is called *tight*, if $\forall_{x,y \in X}(\neg(x \bowtie y) \to x = y)$. The equivalent formulation of the tightness of $\bowtie_D$, given in Proposition 3.2, is part of Bishop's definition of an equalizing family of pseudometrics found in [6]. In the classical literature, see e.g., [21], the term Hausdorff uniformity is used instead. Here we use similar terms for the corresponding notions within the theory of Bishop spaces (see Definition 4.2).

That D_ϕ is a uniformity on Y is shown through Proposition 2.14(i), (ii) and (iv), since for these equalities h need not be a morphism, just a function from X to Y.

Proposition 3.7 *Suppose that $\mathcal{D} = (X, D)$, $\mathcal{B} = (Z, B)$ are uniform spaces, E is a uniformity on Y, and $\phi : X \to Y$ is a surjection.*
(i) D_ϕ is the largest uniformity on Y with respect to which ϕ is a morphism.
(ii) A function $h : Y \to Z$ is in $\mathrm{Mor}(\mathcal{D}_\phi, \mathcal{B})$ if and only if $h \circ \phi \in \mathrm{Mor}(\mathcal{D}, \mathcal{B})$.
(iii) If ϕ is an open morphism with respect to D and E, then $E = D_\phi$.

Proof (i) This is immediate from the definition of a morphism between uniform spaces.
(ii) By Proposition 2.14(vii) we have that

$$\begin{aligned} h \in \mathrm{Mor}(\mathcal{D}_\phi, \mathcal{B}) &\leftrightarrow \forall_{b \in B}(b \odot h \in D_\phi) \\ &\leftrightarrow \forall_{b \in B}((b \odot h) \odot \phi \in D) \\ &\leftrightarrow \forall_{b \in B}(b \odot (h \circ \phi) \in D) \\ &\leftrightarrow h \circ \phi \in \mathrm{Mor}(\mathcal{D}, \mathcal{B}). \end{aligned}$$

(iii) Since $\phi \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, by (i) we get $E \subseteq D_\phi$. If $d \in D_\phi$ i.e., $d \odot \phi \in D$, then by the supposed openness of ϕ there is some $e \in E$ such that $d \odot \phi = e \odot \phi$. By Proposition 2.14(viii) we get $d = e$, and hence $D_\phi \subseteq E$. □

Proposition 3.8 *If $\mathcal{D} = (X, D)$ is a uniform space, we define*

$$x_1 \sim x_2 :\Leftrightarrow \forall_{d \in D}(d(x_1, x_2) = 0),$$

for every $x_1, x_2 \in X$. Let $X/\!\sim$ be the set of all equivalence classes of the equivalence relation $\sim$, let $\pi : X \to X/\!\sim$ be the map defined by $x \mapsto [x]_\sim$, for every $x \in X$, and $\mathbb{D}_\sim = (X/\!\sim, D_\pi)$ the quotient uniform space with respect to π.
(i) For every $d \in D$, the mapping $\tilde{d} : X/\!\sim \times X/\!\sim \to \mathbb{R}$, defined by $\tilde{d}([x_1]_\sim, [x_2]_\sim) = d(x_1, x_2)$, for every $x_1, x_2 \in X$, is a well-defined pseudometric on $X/\!\sim$ that is in D_π.
(ii) π is an open morphism from $\mathcal{D}$ to $\mathbb{D}_\sim$.
(iii) The map $\sim : D \to D_\pi$, defined by $d \mapsto \tilde{d}$, for every $d \in D$, is a uniformity epimorphism.

Proof (i) and (ii) If $d \in D$, $\tilde{d}$ is well-defined; if $x_1, x_2, x_3, x_4 \in X$ with $x_1 \sim x_3$ and $x_2 \sim x_4$, then $|d(x_1, x_2) - d(x_3, x_4)| \leq d(x_1, x_3) + d(x_2, x_4) = 0$, hence $\tilde{d}([x_1]_\sim, [x_2]_\sim) = d(x_1, x_2) = d(x_3, x_4 = \tilde{d}([x_3]_\sim, [x_4]_\sim)$. The fact that $\tilde{d}$ is a pseudometric on $X/\!\sim$ is trivial. Since $(\tilde{d} \odot \pi)(x_1, x_2) = \tilde{d}(\pi(x_1), \pi(x_2)) = d(x_1, x_2)$, for every $x_1, x_2 \in X$, we get $\tilde{d} \odot \pi = d$, therefore $\tilde{d} \in D_\pi$. The last equality shows that π is an open morphism.
(iii) First we show that it is a surjection; if $e \in D_\pi$ i.e., $e \odot \pi \in D$, then $\widetilde{e \odot \pi} = e$,

since $\widetilde{(e \odot \pi)}([x_1]_\sim, [x_2]_\sim) = (e \odot \pi)(x_1, x_2) = e([x_1]_\sim, [x_2]_\sim)$, for every $x_1, x_2 \in X$. The properties of a uniformity homomorphism follow immediately for $\tilde{}$. $\square$

The next result shows that it suffices to work with uniform spaces with separating uniformities. Its proof is a translation of the classical Stone-Čech theorem for topological spaces, which expresses a similar sufficiency of the completely regular topological spaces (see [46], p.6). Note that in [21], p.219, a different result motivated the sufficiency of uniform spaces with a separating uniformity.

Theorem 3.9 (Stone-Čech theorem for uniform spaces) *If $\mathcal{D} = (X, D)$ is a uniform space, there exists a uniform space $\rho\mathcal{D} = (\rho X, \rho D)$ and a mapping $\tau_X : X \to \rho X \in \mathrm{Mor}(\mathcal{D}, \rho\mathcal{D})$ such that the following hold.*
(i) The uniformity ρD is separating.
(ii) The induced mapping $T_X : \rho D \to D$ of τ_X is a uniformity isomorphism.
(iii) If $d \in D$, there is a unique $\rho d \in \rho D$ such that the following diagram commutes

$$\begin{tikzcd} X \times X \arrow[r, "{\tau_X}^{[2]}"] \arrow[dr, "d"'] & \rho X \times \rho X \arrow[d, "\rho d"] \\ & \mathbb{R}. \end{tikzcd}$$

Proof Let $\rho X := X/\sim$, $\rho D := D_\pi$ and $\tau_X = \pi$, where $\sim$ is defined in Proposition 3.8.
(i) By Proposition 3.8, if $x_1, x_2 \in D$, we have that

$$\forall_{\tilde{d} \in D_\pi}(\tilde{d}([x_1]_\sim, [x_2]_\sim) = 0) \leftrightarrow \forall_{d \in D}(d(x_1, x_2) = 0) \leftrightarrow x_1 \sim x_2 \leftrightarrow [x_1]_\sim = [x_2]_\sim.$$

(ii) By Proposition 3.8(iii) every element of D_π is of the form $\tilde{d}$, for some $d \in D$, hence the induced mapping Π of π is defined by $\Pi(\tilde{d}) = \tilde{d} \odot \pi = d$. The fact that Π is a uniformity homomorphism follows immediately. Its inverse is the uniformity homomorphism $\tilde{}$, defined in Proposition 3.8(iii), since $d \overset{\sim}{\mapsto} \tilde{d} \overset{\Pi}{\mapsto} d$ and $\tilde{d} \overset{\Pi}{\mapsto} d \overset{\sim}{\mapsto} \tilde{d}$, for every $d \in D$. We define $T_X = \Pi$.
(iii) It follows immediately, if we define $\rho d = \tilde{d}$. $\square$

Proposition 3.10 *Let $\mathcal{D} = (X, D)$ be a uniform space.*
(i) If D_0 is a subbase for D, then $\rho D_0 = \{\rho d_0 \mid d_0 \in D_0\}$ is a subbase for ρD.
(ii) If Δ is a base D, then $\rho\Delta = \{\rho d \mid d \in \Delta\}$ is a base for ρD.

Proof (i) We show that $\rho \coprod D_0 = \coprod \rho D_0$. Since $\rho d_0 \in \rho D$, for every $d_0 \in D_0$, $\coprod \rho D_0 \subseteq \rho D$. By a simple induction on $\coprod D_0$ we get $\{\rho d \mid d \in \coprod D_0\} \subseteq \coprod \rho D_0$.

(ii) If $d \in D$ and $\epsilon > 0$, there are $\delta > 0$, $d' \in \Delta$ with $U(d', \delta, d, \epsilon)$. Since $\rho d'([x_1]_\sim, [x_2]_\sim) = d'(x_1, x_2)$ and $\rho d([x_1]_\sim, [x_2]_\sim) = d(x_1, x_2)$, for every $x_1, x_2 \in X$, we get $U(\rho d', \delta, \rho d, \epsilon)$. $\square$

Definition 3.11 We call $\Phi \subseteq \mathbb{F}(X)$ *separating*, if

$$\forall_{x,y \in X}(\forall_{f \in \Phi}(f(x) = f(y)) \to x = y).$$

The set Φ induces the equivalence relation $\approx$ on X defined by $x_1 \approx x_2 := \forall_{f \in \Phi}(f(x) = f(y))$, for every $x_1, x_2 \in X$. If $f \in \Phi$, the map $\rho f : X/\approx \to \mathbb{R}$, defined by $\rho f([x]_\approx) = f(x)$, is, by the definition of $\approx$, well-defined. Moreover, we define $\rho\Phi := \{\rho f \mid f \in \Phi\}$.

Definition 3.12 A uniform space $\mathbb{D} = (X, D)$ is called *functionally determined*, or an $\mathtt{f}$-*uniform space*, and D is called an $\mathtt{f}$-*uniformity*, if there exists $\Phi \subseteq \mathbb{F}(X)$ such that

$$D = \coprod_{f \in \Phi} d_f.$$

In this case we say that Φ *determines* D, or that Φ is a *determining family* for D. We denote by $\mathtt{f}$-**Unif** the full subcategory[10] of $\mathtt{f}$-uniform spaces of **Unif**.

If $\Phi \subseteq \mathbb{F}(X)$ and $\sim$ is the equivalence relation on X generated by the family of pseudometrics $\{d_f \mid f \in \Phi\}$, then for every $x_1, x_2 \in X$ we have that

$$\begin{aligned} x_1 \sim x_2 &:= \forall_{f \in \Phi}(d_f(x, y) = 0) \\ &\leftrightarrow \forall_{f \in \Phi}(|f(x) - f(y)| = 0) \\ &\leftrightarrow \forall_{f \in \Phi}(f(x) = f(y)) \\ &:= x_1 \approx x_2 \end{aligned}$$

Proposition 3.13 *Let $\mathcal{D} = (X, \coprod_{f \in \Phi} d_f)$, $\mathcal{E} = (Y, \coprod_{g \in \Theta} d_g)$ be $\mathtt{f}$-uniform spaces.*
(i) The product $\mathcal{D} \times \mathcal{E}$ is an $\mathtt{f}$-uniform space.
(ii) If $A \subseteq X$ is inhabited, the relative space $\mathcal{D}_{|A}$ is an $\mathtt{f}$-uniform space.
(iii) $\coprod_{f \in \Phi} d_f$ is separating if and only if Φ is separating.

Proof (i) and (ii) Since $\Phi \subseteq \mathbb{F}(X)$ and $\Theta \subseteq \mathbb{F}(Y)$, if we define the the sets

$$\Phi \circ \pi_1 := \{f \circ \pi_1 \mid f \in \Phi\} \subseteq \mathbb{F}(X \times Y),$$

$$\Theta \circ \pi_2 := \{g \circ \pi_2 \mid g \in \Theta\} \subseteq \mathbb{F}(X \times Y),$$

$$\Phi_{|A} := \{f_{|A} \mid f \in \Phi\} \subseteq \mathbb{F}(A),$$

[10] For all categorical notions mentioned here we refer to [1], or [27].

$$\left(\coprod_{f\in\Phi} d_f\right) \times \left(\coprod_{g\in\Theta} d_g\right) = \coprod_{f\in\Phi}^{g\in\Theta} d_f \odot \pi_1, d_g \odot \pi_2$$
$$= \coprod_{f\in\Phi}^{g\in\Theta} d_{f\circ\pi_1}, d_{g\circ\pi_2}$$
$$= \coprod_{f'\in\Phi\circ\pi_1}^{g'\in\Theta\circ\pi_2} d_{f'}, d_{g'},$$

$$\left(\coprod_{f\in\Phi} d_f\right)_{|A} = \coprod_{f\in\Phi} (d_f)_{|A\times A} = \coprod_{f\in\Phi} d_{(f_{|A})} = \coprod_{f''\in\Phi_{|A}} d_{f''}.$$

(iii) By Proposition 3.3 and the equivalences $\forall_{f\in\Phi}(d_f(x,y)=0) \leftrightarrow \forall_{f\in\Phi}(|f(x)-f(y)| = 0) \leftrightarrow \forall_{f\in\Phi}(f(x)=f(y))$. □

Proposition 3.14 *Let $\mathcal{D}=(X,D), \mathcal{E}=(Y,E)$ be* `f`*-uniform spaces.*
(i) $\rho\mathcal{D}$ is an `f`*-uniform space.*
(ii) $\rho(\mathcal{D}\times\mathcal{E}) = \rho\mathcal{D}\times\rho\mathcal{E}$.
(iii) If $A\subseteq X$ is inhabited, then $\rho(\mathcal{D}_{|A}) = (\rho\mathcal{D})_{|A}$.

Proof Let $D=\coprod_{f\in\Phi} d_f$ and $E=\coprod_{g\in\Theta} d_g$, $x_1,x_2,x_3,x_4\in X$, and $f\in\Phi$.
(i) Working as in the proof of Proposition 3.10(i) we get $\rho\coprod_{f\in\Phi} d_f = \coprod_{f\in\Phi}\rho d_f$. Since

$$\begin{aligned}(\rho d_f)([x_1]_\sim,[x_2]_\sim) &= d_f(x_1,x_2)\\ &= |f(x_1)-f(x_2)|\\ &= |(\rho f)([x_1]_\sim)\\ &= (\rho f)([x_2]_\sim)|\\ &= d_{\rho f}([x_1]_\sim,[x_2]_\sim),\end{aligned}$$

we get $\rho d_f = d_{\rho f}$, for every $f\in\Phi$, hence $\rho\coprod_{f\in\Phi} d_f = \coprod_{f\in\Phi}\rho d_f = \coprod_{\rho f\in\rho\Phi} d_{\rho f}$.
(ii) We have that

$$\begin{aligned}\rho(d_f\odot\pi_1)(([x_1]_\sim,[x_2]_\sim),([x_3]_\sim,[x_4]_\sim)) &= \rho d_{f\circ\pi_1}(([x_1]_\sim,[x_2]_\sim),([x_3]_\sim,[x_4]_\sim))\\ &= d_{f\circ\pi_1}((x_1,x_2),(x_3,x_4))\\ &= |f(x_1)-f(x_3)|\\ &= d_f(x_1,x_3)\\ &= \rho d_f([x_1]_\sim,[x_3]_\sim)\\ &= (\rho d_f\odot\pi_1)(([x_1]_\sim,[x_2]_\sim),([x_3]_\sim,[x_4]_\sim))\end{aligned}$$

i.e., $\rho(d_f \odot \pi_1) = \rho d_f \odot \pi_1$. Similarly we get $\rho(d_g \odot \pi_2) = \rho d_g \odot \pi_2$. Using (i) we get

$$\begin{aligned}
\rho(D \times E) &= \rho\Bigg(\coprod_{f\in\Phi}^{g\in\Theta} d_f \odot \pi_1, d_g \odot \pi_2\Bigg) \\
&= \coprod_{f\in\Phi}^{g\in\Theta} \rho(d_f \odot \pi_1), \rho(d_g \odot \pi_2) \\
&= \coprod_{f\in\Phi}^{g\in\Theta} \rho d_f \odot \pi_1, \rho d_g \odot \pi_2 \\
&= \Bigg(\coprod_{f\in\Phi} \rho d_f\Bigg) \times \Bigg(\coprod_{g\in\Theta} \rho d_g\Bigg) \\
&= \rho D \times \rho E.
\end{aligned}$$

(iii) If $a_1, a_2 \in A$, then

$$\begin{aligned}
\rho d_{(f_{|A})}([a_1]_\sim, [a_2]_\sim) &= d_{(f_{|A})}(a_1, a_2) \\
&= |f(a_1) - f(a_2)| \\
&= d_f(a_1, a_2) \\
&= \rho d_f([a_1]_\sim, [a_2]_\sim) \\
&= (\rho d_f)_{|A\times A}([a_1]_\sim, [a_2]_\sim)
\end{aligned}$$

i.e., $\rho d_{(f_{|A})} = (\rho d_f)_{|A\times A}$. Hence by Proposition 3.13(ii) we have that

$$\rho(D_{|A}) = \rho\Bigg(\coprod_{f\in\Phi} d_{(f_{|A})}\Bigg) = \coprod_{f\in\Phi} \rho d_{(f_{|A})} = \coprod_{f\in\Phi} (\rho d_f)_{|A\times A} = \Bigg(\coprod_{f\in\Phi} \rho d_f\Bigg)_{|A} = (\rho D)_{|A}.$$

□

Theorem 3.15 (Tychonoff embedding theorem for $\mathtt{f}$-uniform spaces) *If $\mathcal{D} = (X, \coprod_{f\in\Phi} d_f)$ is an $\mathtt{f}$-uniform space, then $\mathcal{D}$ is separated if and only if $\mathcal{D}$ is uniformly embedded into the Euclidean uniform space $\mathcal{R}^\Phi$.*

Proof By Proposition 3.13(iii) $\mathcal{D}$ is separated if and only if Φ is separating. If $\mathcal{D}$ is separated, we define the mapping $\varepsilon_X : X \to \mathbb{R}^\Phi$

$$x \mapsto \hat{x},$$

$$\hat{x}(f) := f(x),$$

for every $x \in X$ and $f \in \Phi$. By Proposition 2.21 for a Φ-product we get

$$D(d_{\mathbb{R}})^{\Phi} = \Big(\coprod d_{\mathbb{R}}\Big)^{\Phi} = \coprod_{f\in\Phi} d_{\mathbb{R}} \odot \varpi_f = \coprod_{f\in\Phi} d_{\varpi_f},$$

$$\big(D(d_{\mathbb{R}})^{\Phi}\big)_{|\varepsilon_X(X)} = \coprod_{f\in\Phi} (d_{\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)}.$$

If $x, y \in X$ such that $\varepsilon_X(x) = \varepsilon_X(y) \leftrightarrow \forall_{f\in\Phi}(f(x) = f(y))$, then $x = y$, since Φ is separating. By the $\coprod$-lifting of morphisms we have that

$$\varepsilon_X \in \mathrm{Mor}(\mathcal{D}, \mathcal{R}^{\Phi}_{|\varepsilon_X(X)}) \leftrightarrow \forall_{f\in\Phi}\Bigg((d_{\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X \in \coprod_{f\in\Phi} d_f\Bigg),$$

which holds, since

$$(d_{\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X = d_{\varpi_f} \odot \varepsilon_X = d_f,$$

for every $f \in \Phi$; if $x, y \in X$,

$$(d_{\varpi_f} \odot \varepsilon_X)(x, y) = d_{\varpi_f}(\hat{x}, \hat{y}) = |\varpi_f(\hat{x}) - \varpi_f(\hat{y})| = |\hat{x}(f) - \hat{y}(f)| = |f(x) - f(y)| = d_f(x, y).$$

Since $d_f = (d_{\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X$, for every $f \in \Phi$, by the $\coprod$-lifting of openness ε_X is an open morphism from $\mathcal{D}$ to $(\mathcal{R}^{\Phi})_{|\varepsilon_X(X)}$ i.e., a uniform embedding of $\mathcal{D}$ into $\mathcal{R}^{\Phi}$. □

In the previous theorem we avoided the exterior union of sets. If $X = \mathbb{R}^n$, then $D(d_{\mathbb{R}})^n = \coprod d_{\pi_1}, \ldots, d_{\pi_n}$ i.e., $\Phi = \{\pi_1, \ldots, \pi_n\}$ determines $D(d_{\mathbb{R}})^n$. If $\vec{x} \in \mathbb{R}^n$, for the embedding $\varepsilon_{\mathbb{R}^n}$ we have that $\hat{\vec{x}}(\pi_i) = \pi_i(\vec{x}) = x_i$ i.e., if we identify Φ with n, then $\varepsilon_{\mathbb{R}^n}$ is identified with the identity function on $\mathbb{R}^n$. Next corollaries are translations of the corresponding results for topological spaces (see [46], pp.6-7).

Corollary 3.16 *If $\mathcal{D} = (X, D)$ is a uniform space, $\mathcal{E} = (Y, \coprod_{g\in\Theta} d_g)$ is a separated* `f`*-uniform space and $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, there exists a mapping $\rho h : \rho X \to Y \in \mathrm{Mor}(\rho\mathcal{D}, \mathcal{E})$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[dr, "h"'] & \rho X \arrow[d, "\rho h"] \\ & Y. \end{tikzcd}$$

Proof If e_Y is the Tychonoff embedding of $\mathcal{E}$ into $\mathcal{R}^{\Theta}$, we define $\mu : \rho X \to \mathbb{R}^{\Theta}$ by

$$[x]_{\sim} \mapsto \varepsilon_Y(h(x)) = \widehat{h(x)},$$

$$\widehat{h(x)}(g) = g(h(x)),$$

Since $D(d_{\mathbb{R}})^{\Theta} = \bigvee_{g\in\Theta} d_{\varpi_g}$, by the $\bigvee$-lifting of morphisms $\mu \in \mathrm{Mor}(\rho\mathcal{D}, \mathcal{R}^{\Theta}) \leftrightarrow \forall_{g\in\Theta}(d_{\varpi_g} \odot \mu \in \rho D)$. By Theorem 3.9 we have that

$$\rho(d_g \odot h) \circ \tau_X^{[2]} = d_g \circ h^{[2]},$$

and if $x_1, x_2 \in X$ and $g \in \Theta$, we have that

$$\begin{aligned}
(d_{\varpi_g} \odot \mu)([x_1]_{\sim}, [x_2]_{\sim}) &= d_{\varpi_g}(\widehat{h(x_1)}, \widehat{h(x_2)}) \\
&= |\varpi_g(\widehat{h(x_1)}) - \varpi_g(\widehat{h(x_2)})| \\
&= |\widehat{h(x_1)}(g) - \widehat{h(x_2)}(g)| \\
&= |g(h(x_1)) - g(h(x_2))| \\
&= (d_g \odot h)(x_1, x_2) \\
&= \big(\rho(d_g \odot h)\big)([x_1]_{\sim}, [x_2]_{\sim})
\end{aligned}$$

i.e., $d_{\varpi_g} \odot \mu = \rho(d_g \odot h) \in \rho D$. We define $\rho h := \varepsilon_Y^{-1} \circ \mu$, and if $x \in X$, then $(\varepsilon_Y^{-1} \circ \mu)([x]_{\sim}) = \varepsilon_Y^{-1}(\varepsilon_Y(h(x)) = h(x)$. □

Corollary 3.17 *If $\mathcal{D} = (X, D)$ is a uniform space, $\mathcal{E} = (Y, \bigvee_{g\in\Theta} d_g)$ is an* $\mathtt{f}$*-uniform space, $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, then there exists a mapping $\rho h : \rho X \to \rho Y \in \mathrm{Mor}(\rho\mathcal{D}, \rho\mathcal{E})$ such that the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\tau_X"'] & Y \arrow[d, "\tau_Y"] \\
\rho X \arrow[r, "\rho h"'] & \rho Y.
\end{tikzcd}$$

Proof By Proposition 3.14(i) $\rho\mathcal{E}$ is an $\mathtt{f}$-uniform space. Since $\tau_Y \circ h \in \mathrm{Mor}(\mathcal{D}, \rho\mathcal{E})$, as a composition of morphisms, by Corollary 3.16 we have that

$$\rho(\tau_Y \circ h) \circ \tau_X = \tau_Y \circ h,$$

therefore $\rho h := \rho(\tau_Y \circ h)$ is the required mapping. □

Proposition 3.18 *The full subcategory* $\mathtt{f}$**-Unif** *of* **Unif** *is reflective in* **Unif**.

Proof If $\mathcal{D} = (X, D)$ is in **Unif**, we take the $\mathtt{f}$-uniform space $\mathtt{f}\text{-}\mathcal{D} = (X, \bigvee_{h\in\mathcal{M}(\mathcal{D})} d_h)$. By Proposition 2.27(i) $\bigvee_{h\in\mathcal{M}(\mathcal{D})} d_h \subseteq D$. The identity function $\mathrm{id}_X : X \to X$ is in $\mathrm{Mor}(\mathcal{D}, \mathtt{f}\text{-}\mathcal{D})$, since if $h \in \mathcal{M}(\mathcal{D})$, $d_h \odot \mathrm{id}_X = d_h \in D$. If $\mathcal{E} = (Y, \bigvee_{f\in\Phi} d_f)$ is an $\mathtt{f}$-uniform space and $\theta \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, then $u = \theta$ is the unique mapping which makes the following diagram to commute

$$
\begin{tikzcd}
(X, D) \arrow[r, "\mathrm{id}_X"] \arrow[dr, "\theta"'] & (X, \coprod_{h \in \mathcal{M}(\mathcal{D})} d_h) \arrow[d, dashed, "u"] \\
& (Y, \coprod_{f \in \Phi} d_f).
\end{tikzcd}
$$

It remains to show (see [27], p.91) that $u \in \mathrm{Mor}(\mathtt{f}\text{-}\mathcal{D}, \mathcal{E})$ i.e., $\forall_{f \in \Phi}(d_f \odot u \in \coprod_{h \in \mathcal{M}(\mathcal{D})} d_h)$. If $f \in \Phi$, then $f \circ u = f \circ \theta$, and since $\theta \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, $d_f \odot \theta = d_{f \circ \theta} = d_{f \circ u} = d_f \odot u \in D$, therefore $f \circ u \in \mathcal{M}(\mathcal{D})$. Consequently, $d_f \odot u \in \coprod_{h \in \mathcal{M}(\mathcal{D})} d_h$. $\square$

One can show similarly that the correspondence $\lambda(X, D) = (X, \coprod_{h \in \mathcal{M}(\mathcal{D})} d_h)$ and $\lambda(\theta) = \theta$, for every $\theta \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, is a covariant functor from **Unif** to `f`-**Unif**.

Theorem 3.19 *If $\mathcal{D} = (X, D)$ is a separated uniform space, $\mathcal{E} = (Y, E)$ is a uniform space, and $\tau \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$, the following are equivalent:*
(i) τ is open.
(ii) The induced mapping $T : E \to D$ of τ is onto D.
(iii) τ is a uniform embedding of $\mathcal{D}$ into $\mathcal{E}$ such that

$$\mathcal{E}_{\tau(X)} = \{e_{|\tau(X) \times \tau(X)} \mid e \in E\}.$$

Proof The equivalence between (i) and (ii) is immediate. We suppose that τ is open, and we show first that τ is an injection. If $x_1, x_2 \in X$ such that $\tau(x_1) = \tau(x_2)$, we show that $\forall_{d \in D}(d((x_1, x_2) = 0)$, so that, since D is separating, $x_1 = x_2$. If $d \in D$, by hypothesis there exists $e \in E$ such that $d = e \odot \tau$, hence $d(x_1, x_2) = (e \odot \tau)(x_1, x_2) = e(\tau(x_1), \tau(x_2)) = 0$. Since τ is open as a morphism from $\mathcal{D}$ to $\mathcal{E}$, by Proposition 2.23(ii) it is open as a morphism from $\mathcal{D}$ onto $\mathcal{E}_{|\tau(X)}$ i.e., it is a uniform embedding from $\mathcal{D}$ into $\mathcal{E}$. Clearly, $\{e_{|\tau(X) \times \tau(X)} \mid e \in E\} \subseteq \mathcal{E}_{\tau(X)}$. The inclusion

$$\coprod_{e \in E} e_{|\tau(X) \times \tau(X)} \subseteq \{e_{|\tau(X) \times \tau(X)} \mid e \in E\}$$

follows immediately by showing that $\{e_{|\tau(X) \times \tau(X)} \mid e \in E\}$ is a uniformity. Clearly, $\overline{0} = \overline{0}_{|\tau(X) \times \tau(X)}$. If $e_1, e_2 \in E$, then

$$e_{1|\tau(X) \times \tau(X)} \vee e_{2|\tau(X) \times \tau(X)} = (e_1 \vee e_2)_{|\tau(X) \times \tau(X)}.$$

If $e \in E$ and $e' \in \mathbb{D}(\tau(X))$ such that $U(e_{|\tau(X) \times \tau(X)}, \delta, e', \epsilon)$, then $U(e \odot \tau, \delta, e' \odot \tau, \epsilon)$. Since this is the case for every $\epsilon > 0$, we get $e' \odot \tau \in D$. By hypothesis there exists $e'' \in E$ such that $e' \odot \tau = e'' \odot \tau = e''_{|\tau(X) \times \tau(X)} \odot \tau$, hence by Proposition 2.14(viii)

$e' = e''_{|\tau(X)\times\tau(X)}$ i.e., $e' \in \{e_{|\tau(X)\times\tau(X)} \mid e \in E\}$. For the converse implication, since τ is an isomorphism between $\mathcal{D}$ and $\mathcal{E}_{\tau(X)}$, if $d \in D$, there exists $e' \in \{e_{|\tau(X)\times\tau(X)} \mid e \in E\}$ such that $d = e' \odot \tau$ i.e., there exists $e \in E$ such that $d = e_{|\tau(X)\times\tau(X)} \odot \tau = e \odot \tau$ i.e., τ is open as a morphism from $\mathcal{D}$ to $\mathcal{E}$. □

4 From Bishop spaces to uniform spaces

In this section we study the relationship between a Bishop space and its generated uniform space. First we give a definition that corresponds to Definition 2.3 using the letter U for both relations $U(d, \delta, e, \epsilon)$ and $U(g,f, \epsilon)$ to stress the similarity in the development of the theories of uniformities of pseudometrics and of Bishop topologies.

Definition 4.1 If $f, g \in \mathbb{F}(X)$, $\Phi \subseteq \mathbb{F}(X)$, and $\epsilon > 0$, the *uniform closure* $\overline{\Phi}$ of Φ is

$$\overline{\Phi} := \{f \in \mathbb{F}(X) \mid U(\Phi,f)\},$$

$$U(\Phi,f) := \forall_{\epsilon>0}\exists_{g\in\Phi}(U(g,f,\epsilon)),$$

$$U(g,f,\epsilon) := \forall_{x\in X}(|f(x) - g(x)| \leq \epsilon).$$

We denote by $\mathrm{B}(\mathbb{R})$ the set of all Bishop continuous functions of type $\mathbb{R} \to \mathbb{R}$ i.e., those which are uniformly continuous one every bounded subset B of $\mathbb{R}$ with a modulus of continuity $\omega_{\phi,B}(\epsilon)$, for every $\epsilon > 0$ i.e.,

$$\forall_{x,y\in B}\big(|x - y| \leq \omega_{\phi,B}(\epsilon) \to |\phi(x) - \phi(y)| \leq \epsilon\big).$$

Definition 4.2 A *Bishop space* is a pair (X, F), where $F \subseteq \mathbb{F}(X)$ is a *Bishop topology* of functions on X satisfying the following conditions:

(BS_1) $a \in \mathbb{R} \to \overline{a}_X \in F$.
(BS_2) $f, g \in F \to f + g \in F$.
(BS_3) $f \in F \to \phi \in \mathrm{B}(\mathbb{R}) \to \phi \circ f \in F$.
(BS_4) $\overline{F} = F$.

If $\mathcal{G} = (Y, G)$ is a Bishop space, a function $h : X \to Y$ is a *Bishop morphism*, if

$$\forall_{g\in G}(g \circ h \in F),$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g \circ h"'] & Y \arrow[d, "g \in G"] \\ & \mathbb{R}. \end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ the set of all Bishop morphisms from $\mathcal{F}$ to $\mathcal{G}$. The Bishop morphisms are the arrows in the category of Bishop spaces **Bis**. The Bishop space

$$\mathcal{R} := (\mathbb{R}, \mathrm{B}(\mathbb{R}))$$

is called the *Bishop space of reals*[11], and $\mathrm{B}^*(\mathbb{R})$ denotes the set of bounded elements of $\mathrm{B}(\mathbb{R})$. We use the notations $\mathcal{M}(\mathcal{F})$ for $\mathrm{Mor}(\mathcal{F},\mathcal{R})$ and $\mathcal{M}^*(\mathcal{F})$ for the bounded elements of $\mathcal{M}(\mathcal{F})$. A topology F is called *pseudo-compact*, if every element of F is a bounded function. We denote by F^* the topology of bounded elements of F.

It is immediate to see that $\mathrm{Const}(X)$ and $\mathbb{F}(X)$ are topologies on X, which we call the *trivial* and the *discrete* topology on X, respectively, and that if F is a topology on X, then $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$. Moreover, if F is a topology on X, $F^* = F \cap \mathcal{F}^*(X)$ is a topology on X. A Bishop topology F is a ring and a lattice; since $|\mathrm{id}_{\mathbb{R}}| \in \mathrm{B}(\mathbb{R})$, where $\mathrm{id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$, by BS_3 we get that if $f \in F$ then $|f| \in F$. By BS_2 and BS_3 we also get that if $f, g \in F$, then $f \cdot g, f \vee g, f \wedge g \in F$ (see [7], p.77).

Definition 4.3 The *Bishop closure* of F_0, or the *least topology* $\bigvee F_0$ generated by some $F_0 \subseteq \mathbb{F}(X)$, is defined by the following inductive rules:

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}\,, \quad \frac{a \in \mathbb{R}}{\overline{a}_X \in \bigvee F_0}\,, \quad \frac{f, g \in \bigvee F_0}{f + g \in \bigvee F_0}\,,$$

$$\frac{f \in \bigvee F_0,\ \phi \in \mathrm{B}(\mathbb{R})}{\phi \circ f \in \bigvee F_0}\,, \quad \frac{(g \in \bigvee F_0,\ U(g,f,\epsilon))_{\epsilon > 0}}{f \in \bigvee F_0}\,.$$

We call F_0 a *subbase* of $\bigvee F_0$, and we also call $\bigvee F_0$ the *Bishop closure* of F_0.

Note that if F_0 is inhabited, then the rule of the inclusion of the constant functions is redundant to the rule of closure under composition with $\mathrm{B}(\mathbb{R})$. The most complex inductive rule above can be replaced by the rule

$$\frac{g_1 \in \bigvee F_0\ \wedge\ U(g_1, f, \frac{1}{2}),\ \ g_2 \in \bigvee F_0\ \wedge\ U(g_2, f, \frac{1}{2^2}),\ \ldots}{f \in \bigvee F_0}\,.$$

[11] We use for simplicity the same symbol $\mathcal{R}$ for the uniform space of reals and for the Bishop space for reals, as the meaning of the symbol is going to be clear in every context. What corresponds to Proposition 2.28(i) is that if F is a topology on X, then $F = \mathrm{Mor}(\mathcal{F},\mathcal{R})$.

The above rules induce the following induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\bigvee F_0$:

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \to \\
&\forall_{a \in \mathbb{R}}(P(\overline{a}_X)) \to \\
&\forall_{f,g \in \bigvee F_0}(P(f) \to P(g) \to P(f+g)) \to \\
&\forall_{f \in \bigvee F_0}\forall_{\phi \in \mathrm{B}(\mathbb{R})}(P(f) \to P(\phi \circ f)) \to \\
&\forall_{f \in \bigvee F_0}(\forall_{\epsilon > 0}\exists_{g \in \bigvee F_0}(P(g) \ \wedge \ U(g,f,\epsilon)) \to P(f)) \to \\
&\forall_{f \in \bigvee F_0}(P(f)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. Through $\mathrm{Ind}_{\mathcal{F}}$ one shows the $\bigvee$-lifting of Bishop morphisms: a function $h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_0)$ if and only if

$$\forall_{g_0 \in G_0}(g_0 \circ h \in F),$$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "F \ni g_0 \circ h"'] & Y \arrow[d, "g_0 \in G_0"] \\
& \mathbb{R}.
\end{tikzcd}$$

Definition 4.4 If F is a topology on X an $F_0 \subseteq F$ such that $F = \bigvee F_0$ is called a *subbase* for F. A $\Phi \subseteq F$ such that $\overline{\Phi} = F$, is called Φ a *base* for F. A topology F is called *tight*, if the canonical point-point apartness relation induced by F defined by $x \bowtie_F y :\leftrightarrow \forall_{f \in F}(f(x) = f(y))$, for every $x, y \in X$, is tight. A Bishop space with a tight topology is called a *separated* Bishop space.

As expected, a topology F is tight if and only if F it is separating (Definition 3.11).

Definition 4.5 Let $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ be Bishop spaces, $A \subseteq X$ is inhabited, and $\phi : X \to Y$ is onto Y. The *product* Bishop space $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$ of $\mathcal{F}$ and $\mathcal{G}$, *relative* Bishop space $\mathcal{F}_{|A} = (A, F_{|A})$ on A, and the *quotient topology* G_ϕ on Y are defined, respectively, by

$$F \times G := \bigvee \big[\{f \circ \pi_1, \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}\big] =: \bigvee_{f \in F}^{g \in G} f \circ \pi_1, g \circ \pi_2$$

$$F_{|A} = \bigvee \{f_{|A} \mid f \in F\} =: \bigvee_{f \in F} f_{|A}.$$

$$F_\phi := \{g \in \mathbb{F}(Y) \mid g \circ \phi \in F\}.$$

As in the case of uniform spaces one shows inductively that

$$\bigvee F_0 \times \bigvee G_0 := \bigvee \big[\{f_0 \circ \pi_1, \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\}\big]$$
$$=: \bigvee_{\substack{g_0 \in G_0 \\ f_0 \in F_0}} f_0 \circ \pi_1, g_0 \circ \pi_2,$$
$$\big(\bigvee F_0\big)_{|A} = \bigvee\{f_{0|A} \mid f_0 \in F_0\} =: \bigvee_{f_0 \in F_0} f_{0|A}.$$

Next proposition, the proof of which is omitted as straightforward, shows the relation between the elements of $\mathbb{F}(X)$ and their induced pseudometrics.

Proposition 4.6 *Suppose that $f, g \in \mathbb{F}(X)$, $a, c > 0$, $b \in \mathbb{R}$, and $\phi \in \mathrm{B}(\mathbb{R})$.*
(i) $d_{\overline{a}_X} = \overline{0}_{X\times X}$.
(ii) $d_{f+g} \leq d_f + d_g$.
(iii) $d_{f+\overline{b}_X} = d_f$.
(iv) $d_{\overline{a}_X h} = \overline{a}_X d_h$.
(v) $U(g, f, \frac{\epsilon}{3}) \rightarrow U(d_g, \frac{\epsilon}{3}, d_f, \epsilon)$.
(vi) If f is bounded, then $U(d_f, \omega_{\phi, f(X)}(\epsilon), d_{\phi\circ f}, \epsilon)$.
(vii) If $|f| \geq \overline{c}_X$, then $d_{\frac{1}{f}} \leq \frac{1}{c^2} d_f$.

Next proposition describes the "canonical" uniform space of pseudometrics generated by some Bishop space and it has a categorical formulation.

Proposition 4.7 *Let $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ be Bishop spaces. The uniform space generated by $\mathcal{F}$ is the pair $\mathcal{D}(\mathcal{F}) = (X, D(F))$, where*

$$D(F) := \coprod\{d_f \mid f \in F\} =: \coprod_{f\in F} d_f.$$

(i) The mapping τ which sends $\mathcal{F}$ to $\tau(\mathcal{F}) = \mathcal{D}(\mathcal{F})$ and a function $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ to $\tau(h) = h \in \mathrm{Mor}(\tau(\mathcal{F}), \tau(\mathcal{G}))$ is a covariant functor from **Bis** *to* `f`**-Unif**.
(ii) $D(\mathrm{Const}(X)) = \{\overline{0}_{X\times X}\}$, and $D(\mathbb{F}(X)) = \coprod_{f\in\mathbb{F}(X)} d_f$.
(iii) F is separating if and only if $D(F)$ is separating.

Proof (i) If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $h \in \mathrm{Mor}(\tau(\mathcal{F}), \tau(\mathcal{G})) \leftrightarrow \forall_{g\in G}(d_g \odot h \in D(F))$. If $g \in G$, and since $d_g = d_{\mathbb{R}} \odot g$, by Proposition 2.14(vii) we have that $d_g \odot h = (d_{\mathbb{R}} \odot g) \odot g = d_{\mathbb{R}} \odot (g \circ h) = d_{g\circ h}$. Since $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, $g \circ h \in F$, hence $d_{g\circ h} \in D(F)$. It is immediate that $\tau(\mathrm{id}_X) = \mathrm{id}_{\tau(X)}$ and if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}), h' \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$, where $\mathcal{H} = (Z, H)$ is a Bishop space, then $\tau(h' \circ h) = \tau(h') \circ \tau(h)$.

(ii) This is immediate by Proposition 4.6(i) and by the definition of $D(F)$.
(iii) By Proposition 3.3 $D(F)$ is separating if and only if $\forall_{x,y\in X}(\forall_{f\in F}(d_f(x,y)=0) \to x=y) \leftrightarrow \forall_{x,y\in X}(\forall_{f\in F}(f(x)=f(y)) \to x=y)$, the separating property of F. $\square$

Proposition 4.8 *If F_0 is a subbase of a pseudo-compact topology F on X, then*

$$D(\bigvee F_0) = \coprod_{f_0\in F_0} d_{f_0}.$$

Proof It suffices to show that $\coprod_{f\in F} d_f \subseteq \coprod_{f_0\in F_0} d_{f_0}$. This we show using induction on $\bigvee F_0$. For the cases $f=\overline{a}_X$, where $a\in\mathbb{R}$, $f=f_1+f_2$, and $U(g,f,\frac{\epsilon}{3})$ we use the inductive hypotheses, the basic properties of a uniformity and Proposition 4.6(i), (ii), and (iii), respectively. If $f=\phi\circ g$, where $\phi\in \mathrm{B}(\mathbb{R})$ and $g\in\bigvee F_0$ such that $d_g\in\coprod_{f_0\in F_0} d_{f_0}$, then by Proposition 4.6(iv) we get $U(d_g,\omega_{\phi,g(X)}(\epsilon),d_{\phi\circ g},\epsilon)$, and since $\epsilon>0$ is arbitrary we conclude that $d_{\phi\circ g}\in\coprod_{f_0\in F_0} d_{f_0}$. $\square$

Corollary 4.9 *(i) If τ is restricted to the full subcategory of pseudo-compact Bishop spaces* $\mathbf{Bis}^*$ *of* $\mathbf{Bis}$*, then τ preserves products and subspaces.*
(ii) If $\mathcal{F}=(X,F)$ is a Bishop space and $\mathcal{F}_\phi=(Y,F_\phi)$ is the quotient Bishop space with respect to the surjection $\phi: X\to Y$, then $D(F_\phi)\subseteq D(F)_\phi$.

Proof (i) Let $\mathcal{F}=(X,F),\mathcal{G}=(Y,G)$ be in $\mathbf{Bis}^*$ and $A\subseteq X$ inhabited. Since boundedness of functions is a lifted property from a subbase for a Bishop topology to the topology itself, $\mathcal{F}\times\mathcal{G}$ and $\mathcal{F}_{|A}$ are in $\mathbf{Bis}^*$. Since $d_f\odot\pi_1=d_{f\circ\pi_1}$ and $d_g\odot\pi_2=d_{g\circ\pi_2}$,

$$\begin{aligned}
D(F\times G) &= \coprod_{f\in F}^{g\in G} d_{f\circ\pi_1}, d_{g\circ\pi_2}\\
&= \coprod_{f\in F}^{g\in G} d_f\odot\pi_1, d_g\odot\pi_2\\
&= \left(\coprod_{f\in F} d_f\odot\pi_1\right)\times\left(\coprod_{g\in G} d_g\odot\pi_2\right)\\
&= D(F)\times D(G),
\end{aligned}$$

Since $d_{(f_{|A})}=(d_f)_{|(A\times A)}$, we have that

$$D(F_{|A}) = \coprod_{f\in F} d_{(f_{|A})} = \coprod_{f\in F} (d_f)_{|(A\times A)} = D(F)_{|A}.$$

(ii) By definition

$$D(F_\phi) = \coprod_{g \in F_\phi} d_g = \coprod_{g \in \mathbb{F}(Y), g\circ\phi \in F} d_g,$$

$$D(F)_\phi = \{e \in \mathbb{D}(Y) \mid e \odot \phi \in D(F)\}.$$

If $g \in \mathbb{F}(Y)$ with $g \circ \phi \in F$, then $d_g \odot \phi = d_{g\circ\phi} \in \coprod_{f \in F} d_f = D(F)$. □

Corollary 4.10 *Let $\mathcal{F} = (X, F)$ be a separated Bishop space.*
(i) The uniform space $\tau(\mathcal{F})$ is embedded into the Euclidean uniform space $\mathcal{R}^F$.
(ii) If F_0 is a subbase for F, $\tau(\mathcal{F})$ is embedded into the Euclidean uniform space $\mathcal{R}^{F_0}$.

Proof (i) By Proposition 4.7(iii) $\tau(\mathcal{F})$ is separated, and we use Theorem 3.15.
(ii) By Proposition 4.8 $D(F) = \coprod_{f_0 \in F_0} d_{f_0}$. It is easy to show that F is separating if and only if F_0 is separating, and then we use Theorem 3.15. □

Proposition 4.11 *Let (X, F) be a Bishop space, $f, f' \in F$, and $h \in \mathbb{F}(X)$ a positively non-constant function i.e., $h(x_0) \bowtie_{\mathbb{R}} h(y_0) \leftrightarrow |h(x_0) - h(y_0)| > 0$, for some $x_0, y_0 \in X$.*
(i) If $d_h = d_f$, then $h \in F$.
(ii) If $d_h = d_f \vee d_{f'}$, then $h \in F$.

Proof Let $g := h - \overline{h(x_0)}_X$.
(i) Since $|g| = \left|h - \overline{h(x_0)}_X\right| = \left|f - \overline{f(x_0)}_X\right| \in F$, $|g|^2 = g^2 \in F$. Moreover,

$$g - \overline{g(y_0)}_X = (h - \overline{h(x_0)}_X) - \overline{h(y_0) - h(x_0)}_X = h - \overline{h(x_0)}_X - \overline{h(y_0)}_X + \overline{h(x_0)}_X = h - \overline{h(y_0)}_X.$$

Hence

$$\left|g - \overline{g(y_0)}_X\right| = \left|h - \overline{h(y_0)}_X\right| = \left|f - \overline{f(y_0)}_X\right| \in F,$$

$$|g - \overline{g(y_0)}_X|^2 = \left(g - \overline{g(y_0)}_X\right)^2 = g^2 - 2g\overline{g(y_0)}_X + \overline{g(y_0)}_X^2 \in F.$$

Since $g^2, \overline{g(y_0)}_X^2 \in F$, we get $-2g\overline{g(y_0)}_X \in F$. Since $g(y_0) = (h(y_0) - h(x_0)) \bowtie_{\mathbb{R}} 0$, we get $g \in F$, hence $h \in F$.
(ii) Since $|g| = \left|h - \overline{h(x_0)}_X\right| = \left|f - \overline{f(x_0)}_X\right| \vee \left|g - \overline{g(x_0)}_X\right| \in F$, we work as in (i). □

As in the case of uniform spaces one can show that any Bishop topology F on some X is isomorphic as an algebra and a lattice to a separating topology ρF on ρX. If we define the equivalence relation $x_1 \approx x_2 \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2))$, for every $x_1, x_2 \in X$ (Definition 3.11), and if $\tau = \pi : X \to X/\approx$, where $x \mapsto [x]_\approx$, then if $\rho X = X/\approx$ is endowed with the quotient Bishop topology $\rho F = \{\rho f \mid f \in F\} = G_\pi$, where $(\rho f)([x]_\approx) = f(x)$, for every $[x]_\approx \in \rho X$, the following theorem is proved (see also [32]).

Theorem 4.12 (Stone-Čech theorem for Bishop spaces) *If $\mathcal{F} = (X, F)$ is a Bishop space, there is a Bishop space $\rho\mathcal{F} = (\rho X, \rho F)$ and a mapping $\tau_X : X \to \rho X \in \mathrm{Mor}(\mathcal{F}, \rho\mathcal{F})$ such that the following hold.*
(i) The topology ρF is separating.
(ii) The induced mapping $T_X : \rho F \to F$ of τ_X is an algebra and lattice isomorphism.
(iii) If $\in F$, there is a unique $\rho f \in \rho F$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[dr, "f"'] & \rho X \arrow[d, "\rho f"] \\ & \mathbb{R}. \end{tikzcd}$$

Proposition 4.13 *If $\mathcal{F} = (X, F)$ is a Bishop space, then $\rho\mathcal{D}(\mathcal{F}) = \mathcal{D}(\rho\mathcal{F})$.*

Proof By definition $\rho\mathcal{D}(\mathcal{F}) = (\rho X, \rho D(F))$ and $\mathcal{D}(\rho\mathcal{F}) = (\rho X, D(\rho F))$, where since $\forall_{f\in F}(f(x_1) = f(x_2)) \leftrightarrow \forall_{f\in F}(d_f(x_1, x_2) = 0)$, for every $x_1, x_2 \in X$, the carrier sets ρX in both structures are the same and therefore the same notation is justified. Moreover, $(\rho d_f)([x_1]_\approx, [x_2]_\approx) = d_f(x_1, x_2) = |f(x_1) - f(x_2)| = |(\rho f)([x_1]_\approx) = (\rho f)([x_2]_\approx)| = d_{\rho f}([x_1]_\approx, [x_2]_\approx)$ i.e., $\rho d_f = d_{\rho f}$, for every $f \in F$. By Proposition 3.14(i) we get

$$\rho D(F) = \rho\left(\coprod_{f\in F} d_f\right) = \coprod_{f\in F} \rho d_f = \coprod_{f\in F} d_{\rho f} = D(\{\rho f \mid f \in F\}) = D(\rho F).$$

$\square$

Defining the notion of topological embedding of a Bishop spaces into another, and the notion of a Euclidean Bishop space $\mathcal{R}^I$ in the obvious way, the same embedding $e_X : X \to \mathbb{R}^F$ together with the corresponding $\bigvee$-lifting of openness for Bishop morphisms show the Tychonoff embedding theorem for Bishop spaces (see [32]).

Theorem 4.14 (Tychonoff embedding theorem for Bishop spaces) *If $\mathcal{F} = (X, F)$ is a Bishop space, F is separating if and only if $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^F$.*

Proposition 4.15 *If $\mathcal{F} = (X, F)$ is a Bishop space, the following diagram commutes*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varepsilon_X"] \arrow[d, "\tau"'] & (\mathcal{R}^F)_{\epsilon_X(X)} \arrow[d, "\tau"] \\ \mathcal{D}(\mathcal{F}) \arrow[r, "\varepsilon_X"'] & \mathcal{D}((\mathcal{R}^F)_{\epsilon_X(X)}). \end{tikzcd}$$

Proof By definition and by our remark after Definition 4.5 we have that

$$(\mathrm{B}(\mathbb{R})^F)_{\varepsilon_X(X)} = \bigvee_{f\in F} (\varpi_f)_{|\varepsilon_X(X)},$$

while by Proposition 4.8 we have that

$$D((\mathrm{B}(\mathbb{R})^F)_{\varepsilon_X(X)}) = \coprod_{f\in F} d_{[(\varpi_f)_{|\varepsilon_X(X)}]}.$$

Since $d_{[(\varpi_f)_{|\varepsilon_X(X)}]} = (d_{\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)}$, we get that $D((\mathrm{B}(\mathbb{R})^F)_{\varepsilon_X(X)}) = (\mathcal{R}^F)_{|\varepsilon_X(X)}$. $\square$

5 From uniform spaces to Bishop spaces

In this section we study a pseudo-compact Bishop topology generated by a uniformity.

Proposition 5.1 *Let $\mathcal{D} = (X, D)$ be a uniform space.*
(i) $a \in \mathbb{R} \to \overline{a}_X \in \mathcal{M}(\mathcal{D})$.
(ii) $h_1, h_2 \in \mathcal{M}(\mathcal{D}) \to h_1 + h_2 \in \mathcal{M}(\mathcal{D})$.
(iii) $\overline{\mathcal{M}(\mathcal{D})} = \mathcal{M}(\mathcal{D})$, where $\overline{\mathcal{M}(\mathcal{D})}$ is the uniform closure of $\mathcal{M}(\mathcal{D})$.
(iv) If $h \in \mathcal{M}(\mathcal{D})$ and $a > 0$, then $\overline{a}_X h \in \mathcal{M}(\mathcal{D})$.
(v) If $h \in \mathcal{M}(\mathcal{D})$ and $c > 0$ such that $|h| \geq \overline{c}_X$, then $\frac{1}{h} \in \mathcal{M}(\mathcal{D})$.
(vi) If $h \in \mathcal{M}^*(\mathcal{D}) \to \phi \in \mathrm{B}(\mathbb{R}) \to \phi \circ h \in \mathcal{M}^*(\mathcal{D})$.
(vii) $\mathcal{M}^*(\mathcal{D})$ is a pseudo-compact Bishop topology on X.

Proof (i)-(vi) follow immediately by Proposition 4.6, by Proposition 2.27(i) and by Proposition 2.7. Case (vii) follows immediately from (i)-(vi). $\square$

Proposition 5.2 *Let $\mathcal{D} = (X, D)$ and $\mathcal{E} = (Y, E)$ be uniform spaces. The pseudo-compact Bishop space generated by $\mathcal{D}$ is the pair $\mathcal{F}(\mathcal{D}) = (X, F(D))$, where*

$$F(D) := \mathcal{M}^*(\mathcal{D}).$$

(i) The mapping ν which sends $\mathcal{D}$ to $\nu(\mathcal{D}) = \mathcal{F}(\mathcal{D})$ and a function $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ to $\nu(h) = h \in \mathrm{Mor}(\nu(\mathcal{D}), \nu(\mathcal{E}))$ is a covariant functor from **Unif** *to* **Bis****.*
(ii) $F(\{\overline{0}_{X\times X}\}) = \mathrm{Const}(X)$, and $F(\mathbb{D}(X)) = \mathbb{F}^*(X)$.
(iii) D is separating if and only if $F(D)$ is separating.

Proof (i) If $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E})$ i.e, if $\forall_{e\in E}(e \odot h \in D)$, then

$$\begin{aligned} h \in \mathrm{Mor}(\nu(\mathcal{D}), \nu(\mathcal{E})) &\leftrightarrow \forall_{g\in\mathcal{M}^*(\mathcal{E})}(g \circ h \in \mathcal{M}^*(\mathcal{D})) \\ &\leftrightarrow \forall_{g\in\mathbb{F}^*(Y)}(d_g \in E \to d_{g\circ h} \in D), \end{aligned}$$

which is the case, since $d_{g \circ h} = d_g \odot h$. It is immediate that $\nu(\mathrm{id}_X) = \mathrm{id}_{\nu(X)}$ and if $h \in \mathrm{Mor}(\mathcal{D}, \mathcal{E}), h' \in \mathrm{Mor}(\mathcal{E}, \mathcal{B})$, where $\mathcal{B} = (Z, B)$ is a uniform space, then $\nu(h' \circ h) = \nu(h') \circ \nu(h)$.
(ii) Since $h \in \mathcal{M}^*(X, \{\overline{0}_{X \times X}\}) \leftrightarrow d_h = \overline{0}_{X \times X}$, we show $d_h = \overline{0}_{X \times X} \leftrightarrow h \in \mathrm{Const}(X)$. The less trivial implication is $d_h = \overline{0}_{X \times X} \to h \in \mathrm{Const}(X)$. If x_0 inhabits X, and if $h(x_0) = a$, then $h = \overline{a}_X$, since if $x \in X$, then $d_h(x, x_0) = 0 \leftrightarrow h(x) = h(x_0) = a$. For the second equality we have that a bounded function $h : X \to \mathbb{R}$ is in $\mathcal{M}^*(X, \mathbb{D}(X))$ if and only if $d_h \in \mathbb{D}(X)$ if and only if $h \in \mathbb{F}^*(X)$.
(iii) Let $x, y \in X$. Suppose that D is separating and $\forall_{h \in \mathcal{M}^*(\mathcal{D})}(h(x) = h(y))$. It suffices to show $\forall_{d \in D}(d(x, y) = 0)$. If $d \in D$ and $a > 0$, then by (D_7) the truncation $d \wedge \overline{a}_{X \times X}$ of d by a is in D^*. By (D_8) we have that $d_{(d \wedge \overline{a}_{X \times X})_x} \in D$, therefore $(d \wedge \overline{a}_{X \times X})_x \in \mathcal{M}^*(\mathcal{D})$. Note that $(d \wedge \overline{a}_{X \times X})_x \in \mathbb{F}^*(X)$, since $d \wedge \overline{a}_{X \times X} \in D^*$. By our hypothesis

$$0 = (d \wedge \overline{a}_{X \times X})_x(x) = (d \wedge \overline{a}_{X \times X})_x(y) = (d \wedge \overline{a}_{X \times X})(x, y),$$

therefore $d(x, y) = 0$. If $F(D)$ is separating, by Proposition 2.27(i) we get $\forall_{d \in D}(d(x, y) = 0) \to \forall_{h \in \mathcal{M}^*(\mathcal{D})}(d_h(x, y) = 0) \leftrightarrow \forall_{h \in \mathcal{M}^*(\mathcal{D})}(h(x) = h(y)) \to x = y$. □

Proposition 5.3 *If $\mathbb{D} = (X, D)$ is a uniform space, then $\rho\mathcal{F}(\mathcal{D}) = \mathcal{F}(\rho\mathcal{D})$.*

Proof By definition we have that

$$\rho\mathcal{F}(\mathcal{D}) = \rho(X, \mathcal{M}^*(\mathcal{D})) = (\rho X, \rho\mathcal{M}^*(\mathcal{D})),$$

$$\mathcal{F}(\rho\mathcal{D}) = \mathcal{F}(\rho X, \rho D) = (\rho X, \mathcal{M}^*(\rho D)),$$

$$\rho\mathcal{M}^*(\mathcal{D}) = \{\rho h \mid h \in \mathcal{M}^*(\mathcal{D})\} = \{\rho h \mid h \in \mathbb{F}^*(X), d_h \in D\},$$

$$\mathcal{M}^*(\rho D) = \{h \in \mathbb{F}^*(\rho X) \mid d_h \in \rho D\}.$$

Note that by the proof of Proposition 5.2(iii) if $x, y \in X$, then $\forall_{h \in \mathcal{M}^*(\mathcal{D})}(h(x) = h(y)) \leftrightarrow \forall_{d \in D}(d(x, y) = 0)$, therefore the two equivalence relations $x \approx y \leftrightarrow \forall_{h \in \mathcal{M}^*(\mathcal{D})}(h(x) = h(y))$ and $x \sim y \leftrightarrow \forall_{d \in D}(d(x, y) = 0)$ are equal, and ρX is the same set, either if ρX is the carrier set of $(\rho X, \rho\mathcal{M}^*(\mathcal{D}))$, or of $(\rho X, \mathcal{M}^*(\rho D))$. First we show that $\rho\mathcal{M}^*(\mathcal{D}) \subseteq \mathcal{M}^*(\rho D)$. Let $h \in \mathbb{F}^*(X)$ such that $d_h \in D$. Since $(\rho h)([x]_\sim) = h(x)$, for every $[x]_\sim \in \rho X$, we get $\rho h \in \mathbb{F}^*(X)$. We need to show that $d_{\rho h} \in \rho D = \{\rho d \mid d \in D\}$. If $[x]_\sim, [y]_\sim \in \rho X$, we have that

$$\begin{aligned} d_{\rho h}([x]_\sim, [y]_\sim) &= |(\rho h)([x]_\sim) - (\rho h)([y]_\sim)| \\ &= |h(x) - h(y)| \\ &= d_h(x, y) \\ &= (\rho d_h)([x]_\sim, [y]_\sim) \end{aligned}$$

i.e., $d_{\rho h} = \rho d_h \in \rho D$, since by hypothesis $d_h \in D$. Next we show that $\mathcal{M}^*(\rho D) \subseteq \rho\mathcal{M}^*(\mathcal{D})$. Let $h' \in \mathbb{F}^*(\rho X)$ such that $d_{h'} \in \rho D$ i.e., $d_{h'} = \rho d$, for some $d \in D$. We define $h : X \to \mathbb{R}$ by $h(x) := h'([x]_\sim)$, for every $x \in X$. Clearly, $h \in \mathbb{F}^*(X)$, since $h' \in \mathbb{F}^*(\rho X)$. If $x, y \in X$, we have that

$$\begin{aligned} d_h(x,y) &= |h(x) - h(y)| \\ &= |h'([x]_\sim) - h'([y]_\sim)| \\ &= d_{h'}([x]_\sim, [y]_\sim) \\ &= (\rho d)([x]_\sim, [y]_\sim) \\ &= d(x,y) \end{aligned}$$

i.e., $d_h = d \in D$, hence $h \in \mathcal{M}^*(\mathcal{D})$. Moreover, if $[x]_\sim \in \rho X$, we have that $(\rho h)([x]_\sim) = h(x) = h'([x]_\sim)$ i.e., $h' = \rho h \in \rho\mathcal{M}^*(\mathcal{D})$. □

Proposition 5.4 *Let $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$ be uniform spaces.*
(i) $\mathcal{M}^*(\mathcal{D}) \times \mathcal{M}^*(\mathcal{E}) \subseteq \mathcal{M}^*(\mathcal{D} \times \mathcal{E})$.
(ii) If $A \subseteq X$ is inhabited, $\mathcal{M}^*(D)_{|A} \subseteq \mathcal{M}^*(\mathcal{D}_{|A})$.
(iii) If $D_0 \subseteq \mathbb{D}(X)$ and $M^*[D_0] := \{f_0 \in \mathbb{F}^*(X) \mid d_{f_0} \in D_0\}$, $\bigvee M^*[D_0] \subseteq F(\coprod D_0)$.

Proof (i) By definition $\mathcal{M}^*(\mathcal{D}) = \{f^* \in \mathbb{F}^*(X) \mid d_{f^*} \in D\}$, $\mathcal{M}^*(\mathcal{E}) = \{g^* \in \mathbb{F}^*(Y) \mid d_{g^*} \in E\}$, and

$$\mathcal{M}^*(\mathcal{D}) \times \mathcal{M}^*(\mathcal{E}) = \coprod_{d_{f^*} \in D}^{d_{g^*} \in E} f^* \circ \pi_1, g^* \circ \pi_2.$$

Since

$$\mathcal{M}^*(\mathcal{D} \times \mathcal{E}) = \{h \in \mathbb{F}^*(X \times Y) \mid d_h \in D \times E\},$$

$$d_{f^*} \in D \to d_{f^*} \odot \pi_1 = d_{f^* \circ \pi_1} \in D \times E, \qquad d_{g^*} \in E \to d_{g^*} \odot \pi_2 = d_{g^* \circ \pi_2} \in D \times E,$$

we get $\{f^* \circ \pi_1 \mid d_{f^*} \in D\} \cup \{g^* \circ \pi_2 \mid d_{g^*} \in E\} \subseteq \mathcal{M}^*(\mathcal{D} \times \mathcal{E})$.
(ii) By definition

$$\mathcal{M}^*(D)_{|A} = \bigvee_{d_h \in D}^{h \in \mathbb{F}^*(X)} h_{|A},$$

$$\mathcal{M}^*(\mathcal{D}_{|A}) = \left\{g \in \mathbb{F}^*(A) \mid d_g \in D_{|A} = \coprod_{d \in D} d_{|A \times A}\right\}.$$

If $h \in \mathcal{M}^*(\mathcal{D})$, then $d_h \in D$ and $d_{(h_{|A})} = (d_h)_{|A \times A}$, hence $h_{|A} \in \mathcal{M}^*(\mathcal{D}_{|A})$. Consequently, $\{h_{|A} \mid h \in \mathbb{F}^*(X), d_h \in D\} \subseteq \mathcal{M}^*(\mathcal{D}_{|A})$.
(iii) By Proposition 2.27(i) we have that

$$f \in F\big(\coprod D_0\big) = M^*\big(\coprod D_0\big) \leftrightarrow d_f \in \coprod D_0.$$

If $f_0 \in M^*[D_0]$, then $d_{f_0} \in D_0 \subseteq \coprod D_0$, hence $f_0 \in M^*\big(\coprod D_0\big)$ i.e., $M^*[D_0] \subseteq M^*\big(\coprod D_0\big)$, and $\bigvee M^*[D_0] \subseteq M^*\big(\coprod D_0\big)$. $\square$

Next proposition has an immediate proof.

Proposition 5.5 *Let $\mathcal{D} = (X, D), \mathcal{E} = (Y, E)$ be uniform spaces, and $h \in \mathbb{F}^*(X \times Y)$.*
(i) If $d \in D$ and $d_h = d \odot \pi_1$, then $h \in \mathcal{M}^*(\mathcal{D}) \times \mathcal{M}^*(\mathcal{E})$.
(ii) If $e \in E$ and $d_h = e \odot \pi_2$, then $h \in \mathcal{M}^*(\mathcal{D}) \times \mathcal{M}^*(\mathcal{E})$.

Next we relate a Bishop space $\mathcal{F}$ to $\nu(\tau(\mathcal{F}))$ and a uniform space $\mathcal{D}$ to $\tau(\nu(\mathcal{D}))$.

Proposition 5.6 *Let $\mathcal{F} = (X, F)$ be a Bishop space and $\mathcal{D} = (X, D)$ a uniform space.*
(i) $F^* \subseteq \mathcal{M}^*(\coprod_{f \in F} d_f)$.
(ii) $\coprod_{h \in \mathcal{M}^*(\mathcal{D})} d_h \subseteq D$.
(iii) If $\Phi \subseteq \mathbb{F}^*(X)$, then

$$\coprod_{h \in \mathcal{M}^*\left(\coprod_{f \in \Phi} d_f\right)} d_h = \coprod_{f \in \Phi} d_f.$$

Proof (i) If $g \in F^*$, then $d_g \in \coprod_{f \in F} d_f$, and $g \in \mathcal{M}^*(\coprod_{f \in F} d_f)$.
(ii) If $h \in \mathcal{M}^*(\mathcal{D})$, then $d_h \in D$, and the inclusion follows.
(iii) If $g \in \Phi \subseteq \mathbb{F}^*(X)$, then $d_g \in \coprod_{f \in \Phi} d_f$, hence $g \in \mathcal{M}^*(\coprod_{f \in \Phi} d_f)$ and $d_g \in \coprod_{h \in \mathcal{M}^*\left(\coprod_{f \in \Phi} d_f\right)} d_h$. Consequently,

$$\coprod_{f \in \Phi} d_f \subseteq \coprod_{h \in \mathcal{M}^*(\coprod_{f \in \Phi} d_f)} d_h.$$

The converse inclusion follows from (ii). $\square$

6 The large uniform space of reals

Definition 6.1 The pair $\tau(\mathcal{R}) = (\mathbb{R}, D(\mathrm{B}(\mathbb{R})))$, where according to Proposition 4.7

$$D(\mathrm{B}(\mathbb{R})) = \coprod_{\phi \in \mathrm{B}(\mathbb{R})} d_\phi,$$

is called the *large uniform space of reals*, and $D(\mathrm{B}(\mathbb{R}))$ the *large uniformity on reals*. An I-product $\tau(\mathcal{R})^I$ of $\tau(\mathcal{R})$ is called a *large* Euclidean uniform space.

Proposition 6.2 *The large uniformity $D(\mathrm{B}(\mathbb{R}))$ on $\mathbb{R}$ is strictly larger than the metric uniformity $D(d_{\mathbb{R}})$ and it is also separating.*

Proof Since[12] $\mathrm{id}_{\mathbb{R}} \in \mathrm{B}(\mathbb{R})$ and $d_{\mathbb{R}} = d_{(\mathrm{id}_{\mathbb{R}})}$, we get $D(d_{\mathbb{R}}) \subseteq D(\mathrm{B}(\mathbb{R}))$ i.e., $D(\mathrm{B}(\mathbb{R}))$ is

[12]It is immediate to see that $\mathrm{B}(\mathbb{R}) = \bigvee\{\mathrm{id}_{\mathbb{R}}\}$.

larger uniformity on $\mathbb{R}$ than $D(d_{\mathbb{R}})$. Since a larger uniformity of a separating one is separating, we conclude that $D(\mathrm{B}(\mathbb{R}))$ is separating. To show that $D(\mathrm{B}(\mathbb{R}))$ is strictly larger we use the fact that there are elements of $\mathrm{B}(\mathbb{R})$ which are not in $C_u(\mathbb{R})$, the set of real-valued functions that are uniformly continuous on $\mathbb{R}$, like the map $\phi_0 : \mathbb{R} \to \mathbb{R}$ defined by $\phi_0(x) = \sin(x^2)$, for every $x \in \mathbb{R}$. Hence $d_{\phi_0} \in D(\mathrm{B}(\mathbb{R}))$ but not in $D(d_{\mathbb{R}})$, since if $d_{\phi_0} \in D(d_{\mathbb{R}})$, by Proposition 2.27(i) we get $\phi_0 \in \mathcal{M}(\mathcal{R}) = \mathrm{Mor}(\mathcal{D}(d_{\mathbb{R}}), \mathcal{D}(d_{\mathbb{R}}))$ and by Proposition 2.25(ii) ϕ_0 is uniformly continuous. $\square$

Definition 6.3 If $\mathcal{D} = (X, D)$ is a uniform space, we define

$$\mathcal{M}_\tau(\mathcal{D}) := \mathrm{Mor}(\mathcal{D}, \tau(\mathcal{R})) = \{h \in \mathbb{F}(X) \mid \forall_{\phi \in \mathrm{B}(\mathbb{R})}(d_\phi \odot h = d_{\phi \circ h} \in D)\},$$
$$\mathcal{M}^*_\tau(\mathcal{D}) := \mathrm{Mor}^*(\mathcal{D}, \tau(\mathcal{R})).$$

Proposition 6.4 *Let $\mathcal{D} = (X, D)$ be a uniform space.*
(i) $\mathcal{M}_\tau(\mathcal{D}) \subseteq \mathcal{M}(\mathcal{D})$ and $\mathcal{M}^*_\tau(\mathcal{D}) \subseteq \mathcal{M}^*(\mathcal{D})$.
(ii) $a \in \mathbb{R} \to \overline{a}_X \in \mathcal{M}_\tau(\mathcal{D})$.
(iii) If $h \in \mathcal{M}_\tau(\mathcal{D}) \to \phi \in \mathrm{B}(\mathbb{R}) \to \phi \circ h \in \mathcal{M}_\tau(\mathcal{D})$.

Proof (i) If $h \in \mathcal{M}_\tau(\mathcal{D})$, then for $\phi = \mathrm{id}_{\mathbb{R}}$ we get $d_h \in D$ and by Proposition 2.27(i) $h \in \mathcal{M}(\mathcal{D})$. The inclusion $\mathcal{M}^*_\tau(\mathcal{D}) \subseteq \mathcal{M}^*(\mathcal{D})$ follows now immediately.
(ii) It is immediate from $\phi \circ \overline{a}_X = \overline{\phi(a)}_X$ and $d_{\overline{\phi(a)}_X} = \overline{0}_{X \times X}$.
(iii) If $\theta \in \mathrm{B}(\mathbb{R})$, then $d_\theta \odot (\phi \circ h) = d_{\theta \circ (\phi \circ h)} = d_{(\theta \circ \phi) \circ h} = d_{\theta \circ \phi} \odot h \in D$, since $\theta \circ \phi \in \mathrm{B}(\mathbb{R})$ and $h \in \mathcal{M}_\tau(\mathcal{D})$. $\square$

Proposition 6.5 *$\mathcal{M}(\mathcal{R} \times \mathcal{R})$ is strictly larger than $\mathcal{M}_\tau(\mathcal{R} \times \mathcal{R})$.*

Proof By Proposition 2.28(i) $d_{\mathbb{R}} \in \mathcal{M}(\mathcal{R} \times \mathcal{R})$. We show that $d_{\mathbb{R}}$ does not belong in $\mathcal{M}_\tau(\mathcal{R} \times \mathcal{R})$. If that was the case, then

$$\forall_{\phi \in \mathrm{B}(\mathbb{R})}(d_{\phi \circ d_{\mathbb{R}}} \in D(d_{\mathbb{R}}) \times D(d_{\mathbb{R}})).$$

Let $\phi_0(x) = \sin(x^2)$, for every $x \in \mathbb{R}$, for which we know that $d_{\phi_0} \in D(\mathrm{B}(\mathbb{R})) \setminus D(d_{\mathbb{R}})$. If $d_{\phi_0 \circ d_{\mathbb{R}}} \in D(d_{\mathbb{R}}) \times D(d_{\mathbb{R}})$, then, since by Proposition 2.22(i) the mapping ${}_0i : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, defined by $a \mapsto (a, 0)$, for every $a \in \mathbb{R}$, is in $\mathrm{Mor}(\mathcal{R}, \mathcal{R} \times \mathcal{R})$, therefore $d_{\phi_0 \circ d_{\mathbb{R}}} \odot {}_0i \in D(d_{\mathbb{R}})$. If $a, b \in \mathbb{R}$ though, we have that

$$\begin{aligned}
[d_{\phi_0 \circ d_{\mathbb{R}}} \odot {}_0i](a, b) &= d_{\phi_0 \circ d_{\mathbb{R}}}((a, 0), (b, 0)) \\
&= |\phi_0(|a - 0|) - \phi_0(|b - 0|)| \\
&= |\sin(|a|^2) - \sin(|b|^2)| \\
&= |\sin(a^2) - \sin(b^2)| \\
&= d_{\phi_0}(a, b)
\end{aligned}$$

i.e., $d_{\phi_0 \circ d_{\mathbb{R}}} \odot {}_0i = d_{\phi_0} \in D(d_{\mathbb{R}})$, which is a contradiction. $\square$

Next follows the “large” version of Theorem 3.15.

Theorem 6.6 *If $\mathcal{D} = (X, \bigvee_{f\in\Phi} d_f)$ is an* $\mathtt{f}$*-uniform space such that its determining family Φ is closed under composition with* $\mathrm{B}(\mathbb{R})$*, then $\mathcal{D}$ is separated if and only if $\mathcal{D}$ is uniformly embedded into the large Euclidean uniform space $\tau(\mathcal{R})^{\Phi}$.*

Proof If $\mathcal{D}$ is separated, the mapping $\varepsilon_X : X \to \mathbb{R}^{\Phi}$, defined in the proof of Theorem 3.15, is an injection. By Proposition 2.21 we get

$$D(\mathrm{B}(\mathbb{R}))^{\Phi} = \Big(\bigvee_{\phi\in \mathrm{B}(\mathbb{R})} d_\phi\Big)^{\Phi} = \bigvee_{\phi\in \mathrm{B}(\mathbb{R})}^{f\in\Phi} d_\phi \odot \varpi_f = \bigvee_{\phi\in \mathrm{B}(\mathbb{R})}^{f\in\Phi} d_{\phi\circ\varpi_f},$$

$$\big(D(\mathrm{B}(\mathbb{R}))^{\Phi}\big)_{|\varepsilon_X(X)} = \bigvee_{\phi\in \mathrm{B}(\mathbb{R})}^{f\in\Phi} (d_{\phi\circ\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)}.$$

By the $\bigvee$-lifting of morphisms we have that

$$\varepsilon_X \in \mathrm{Mor}(\mathcal{D}, (\tau(\mathcal{R})^{\Phi})_{|\varepsilon_X(X)}) \leftrightarrow \forall_{f\in\Phi}\forall_{\phi\in \mathrm{B}(\mathbb{R})}\bigg((d_{\phi\circ\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X \in \bigvee_{f\in\Phi} d_f\bigg).$$

If $f \in \Phi$ and $\phi \in \mathrm{B}(\mathbb{R})$, then $\phi \circ f \in \Theta$ and

$$(d_{\phi\circ\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X = d_{\phi\circ\varpi_f} \odot \varepsilon_X = d_{\phi\circ f},$$

since

$$\begin{aligned}
[(d_{\phi\circ\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X](x,y) &= d_{\phi\circ\varpi_f}(\hat{x},\hat{y}) \\
&= |\phi(\varpi_f(\hat{x})) - \phi(\varpi_f(\hat{y}))| \\
&= |\phi(\hat{x}(f)) - \phi(\hat{y}(f))| \\
&= |\phi(f(x)) - \phi(f(y))| \\
&= d_{\phi\circ f}(x,y)
\end{aligned}$$

for every $x, y \in X$. By the above equality we also get

$$d_f = d_{\mathrm{id}_{\mathbb{R}}\circ f} = d_{\mathrm{id}_{\mathbb{R}}\circ\varpi_f} \odot \varepsilon_X = (d_{\mathrm{id}_{\mathbb{R}}\circ\varpi_f})_{|\varepsilon_X(X)\times\varepsilon_X(X)} \odot \varepsilon_X,$$

for every $f \in \Phi$. By the $\bigvee$-lifting of openness ε_X is an open morphism from $\mathcal{D}$ onto $(\tau(\mathcal{R})^{\Phi})_{|\varepsilon_X(X)}$ i.e., ε_X is a uniform embedding of $\mathcal{D}$ into $\tau(\mathcal{R})^{\Phi}$. The converse follows immediately from Proposition 3.5 and the fact that $D(\mathrm{B}(\mathbb{R}))$ is separating. $\square$

Since a Bishop topology is closed under composition with $\mathrm{B}(\mathbb{R})$, we get the following.

Corollary 6.7 *If $\mathcal{F} = (X, F)$ is a separated Bishop space, then $\mathcal{D}(\mathcal{F})$ is uniformly embedded into the large Euclidean uniform space $\tau(\mathcal{R})^F$.*

One can show that the remaining properties of a Bishop topology hold for $\mathcal{M}^*_\tau(\mathcal{D})$. Although $\mathcal{M}^*_\tau(\mathcal{D})$ looks smaller than $\mathcal{M}^*(\mathcal{D})$, it turns out that the two Bishop topologies are equal, therefore we lose no bounded morphisms of type $X \to \mathbb{R}$, if we replace the metric uniformity $D(d_{\mathbb{R}})$ on $\mathbb{R}$ by the strictly larger uniformity $D(\mathrm{B}(\mathbb{R}))$.

Proposition 6.8 *If $\mathcal{D} = (X, D)$ is a uniform space, then $\mathcal{M}^*_\tau(\mathcal{D}) = \mathcal{M}^*(\mathcal{D})$.*

Proof It suffices to show $\mathcal{M}^*(\mathcal{D}) \subseteq \mathcal{M}^*_\tau(\mathcal{D})$. By definition $\mathcal{M}^*(\mathcal{D}) = \{h \in \mathbb{F}^*(X) \mid d_h \in D\}$. Since $\mathrm{B}(\mathbb{R}) = \bigvee\{\mathrm{id}_{\mathbb{R}}\}$ and $d_h = d_{\mathbb{R}} \odot h = d_{\mathrm{id}_{\mathbb{R}}} \odot h$, the hypothesis $h \in \mathcal{M}^*(\mathcal{D})$ means that h satisfies the required property $d_\phi \odot h \in D$ for the subbase $\mathrm{id}_{\mathbb{R}}$ of the topology $\mathrm{B}(\mathbb{R})$. If $a \in \mathbb{R}$, then $d_{\overline{a}_X} \odot h = \overline{0}_{X\times X} \odot h = \overline{0}_{X\times X} \in D$. Suppose next that $\phi_1, \phi_2 \in \mathrm{B}(\mathbb{R})$ such that $d_{\phi_1} \odot h \in D$ and $d_{\phi_2} \odot h \in D$. By Proposition 4.6(ii), property (D_6) and the inductive hypotheses we have that

$$d_{\phi_1+\phi_2} \odot h = d_{(\phi_1+\phi_2)\circ h} = d_{(\phi_1\circ h)+(\phi_2\circ h)} \leq d_{\phi_1\circ h} + d_{\phi_2\circ h} \in D,$$

and by (D_4) we get $d_{\phi_1+\phi_2} \odot h \in D$. If $\theta, \phi \in \mathrm{B}(\mathbb{R})$ and $d_\phi \odot h \in D$, then $d_{\theta\circ\phi} \odot h = d_{(\theta\circ\phi)\circ h} = d_{\theta\circ(\phi\circ h)}$. Since h is bounded, $\phi \circ h$ is also bounded; $h(X)$ is bounded, therefore by local compactness of $(\mathbb{R}, d_{\mathbb{R}})$ there is a compact subset K of $\mathbb{R}$ such that $h(X) \subseteq K$. Since ϕ is uniformly continuous on K we have that $\phi(h(X)) \subseteq \phi(K) \subseteq [-M, M]$, for some $M > 0$. By Proposition 4.6(iv) we get

$$U\big(d_{\phi\circ h}, \omega_{\theta,(\phi\circ h)(X)}(\epsilon), d_{\theta\circ(\phi\circ h)}, \epsilon\big).$$

Since $\epsilon > 0$ is arbitrary we get $d_{\theta\circ(\phi\circ h)} = d_\theta \odot (\phi \circ h) \in D$. Finally, if $\phi, \theta \in \mathrm{B}(\mathbb{R})$ such that $U(\phi, \theta, \frac{\epsilon}{3})$ and $d_\phi \odot h \in D$, by Proposition 4.6(iii) we get $U(d_\phi, \frac{\epsilon}{3}, d_\theta, \epsilon)$, therefore $U(d_\phi \odot h, \frac{\epsilon}{3}, d_\theta \odot h, \epsilon)$. Since $\epsilon > 0$ is arbitrary, we conclude that $d_\theta \odot h \in D$. $\square$

Next result shows that the morphism between uniform spaces "captures" Bishop continuity when the large uniform space of reals replaces the uniform space of reals.

Theorem 6.9 $\mathcal{M}(\tau(\mathcal{R})) = \mathrm{B}(\mathbb{R})$ *and* $\mathcal{M}^*(\tau(\mathcal{R})) = \mathrm{B}^*(\mathbb{R})$.

Proof By definition we have that

$$\mathcal{M}(\tau(\mathcal{R})) = \{h \in \mathbb{F}(\mathbb{R}) \mid d_h \in \coprod_{\phi\in\mathrm{B}(\mathbb{R})} d_\phi\}.$$

First we show $\mathrm{B}(\mathbb{R}) \subseteq \mathcal{M}_\tau(\tau(\mathcal{R}))$; if $\phi \in \mathrm{B}(\mathbb{R})$, then trivially $d_\phi \in \coprod_{\phi \in \mathrm{B}(\mathbb{R})} d_\phi$. Next we show that $\mathcal{M}(\tau(\mathcal{R})) \subseteq \mathrm{B}(\mathbb{R})$. We fix some bounded $B \subseteq \mathbb{R}$ and some $\epsilon > 0$. Since $d_h \in \coprod_{\phi \in \mathrm{B}(\mathbb{R})} d_\phi$, there exist $\delta > 0$, $n \in \mathbb{N}$ and $\phi_1, \ldots, \phi_n \in \mathrm{B}(\mathbb{R})$ such that

$$\big(d_{\phi_1} \vee \ldots \vee d_{\phi_n}\big)(x, y) \leq \delta \to d_h(x, y) \leq \epsilon,$$

for every $x, y \in X$. If we define

$$\omega_{h,B}(\epsilon) = \omega_{\phi_1,B}(\delta) \wedge \ldots \wedge \omega_{\phi_1,B}(\delta),$$

then, if $x, y \in B$ such that $|x - y| \leq \omega_{h,B}(\epsilon)$, we get

$$|\phi_1(x) - \phi_1(y)| \leq \delta, \ldots, |\phi_n(x) - \phi_n(y)| \leq \delta,$$

therefore $\big(d_{\phi_1} \vee \ldots \vee d_{\phi_n}\big)(x, y) \leq \delta$. Hence we get $d_h(x, y) = |h(x) - h(y)| \leq \epsilon$. The equality $\mathcal{M}^*(\tau(\mathcal{R})) = \mathrm{B}^*(\mathbb{R})$ follows from the equality $\mathcal{M}(\tau(\mathcal{R})) = \mathrm{B}(\mathbb{R})$. ◻

7 Open questions and future work

In this paper we developed the first steps of a constructive theory of uniformities given by pseudometrics and studied its relation to the constructive theory of Bishop topologies. The interplay between the theory of constructive uniform spaces of pseudometrics and the theory of Bishop topologies is analogous to the interplay between the classical theory of uniform spaces of pseudometrics and the theory of $C(X)$ (see [21], Chapter 15). The following are some of the many problems and open questions that we want to address in future work.

1. There are more than one ways to associate a Bishop topology to a given uniformity of pseudometrics. If $d \in \mathbb{D}(X)$ and D is a uniformity on X, we may define the following Bishop topologies on X

$$F_0(d) := \bigvee_{x \in X} d_x, \quad F_0(D) := \bigvee_{x \in X}^{d \in D} d_x, \quad F_0^*(D) := \bigvee_{x \in X}^{d \in D^*} d_x.$$

Their study is a natural continuation of Section 5.

2. If $\mathcal{F} = (X, F)$ is a Bishop space, an element d of $\mathbb{D}(X)$ is called *F-continuous*, if $d \in \mathcal{M}(\mathcal{F} \times \mathcal{F})$. This notion corresponds to that of a continuous pseudometric on a topological space. We denote by $\mathbb{CD}(F)$ the set of F-continuous pseudometrics on X. By the $\bigvee$-lifting of Bishop morphisms we get $d \in \mathcal{M}(\mathcal{F} \times \mathcal{F}) \leftrightarrow \mathrm{id}_{\mathbb{R}} \circ d = d \in F \times F$. Moreover, if $f \in F$, then $d_f \in \mathbb{CD}(F)$; since F is an algebra and closed under $|.|$ we get

$$d_f = |(f \circ \pi_1) - (f \circ \pi_2)| \in F \times F.$$

It would be interesting to study the algebraic and analytic properties of $\mathbb{CD}(F)$.

3. To find a function-theoretic notion of complete uniform space of pseudometrics and to determine those uniform spaces which have a completion.

4. To find a function-theoretic notion of compact uniform space and to connect it to already known notions of compact Bishop spaces found in [36] and [39].

5. To study constructively extension theorems for pseudometrics, like the classical result that a bounded element of a relative uniformity is extended to a bounded pseudometric in the uniformity of the whole space.

6. To study the uniformities of seminorms and search for appropriate notions of locally convex Bishop spaces.

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