Families of Sets in Bishop Set Theory

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Abstract

We develop the theory of set-indexed families of sets within the informal Bishop Set Theory (BST), a reconstruction of Bishop's theory of sets,. The latter is the informal theory of sets and functions underlying Bishop-style constructive mathematics (BISH) and it is developed in Chapter 3 of Bishop's seminal book *Foundations of Constructive Analysis* [9] and in Chapter 3 of *Constructive Analysis* [19] that Bishop co-authored with Bridges.

In the Introduction we briefly present the relation of Bishop's set theory to the set-theoretic and type-theoretic foundations of mathematics, and we describe the features of BST that "complete" Bishop's theory of sets. These are the explicit use of the class "universe of sets", a clear distinction between sets and classes, the explicit use of dependent operations, and the concrete formulation of various notions of families of sets.

In Chapter 2 we present the fundamentals of Bishop's theory of sets, extended with the features which form BST. The universe $\mathbb{V}_0$ of sets is implicit in Bishop's work, while the notion of a dependent operation over a non-dependent assignment routine from a set to $\mathbb{V}_0$ is explicitly mentioned, although in a rough way. These concepts are necessary to a concrete definition of a set-indexed family of sets, the main object of our study, which is only mentioned by Bishop.

In Chapter 3 we develop the basic theory of set-indexed families of sets and of family-maps between them. We study the exterior union of a family of sets Λ, or the $\sum$-set of Λ, and the set of dependent functions over Λ, or the $\prod$-set of Λ. We prove the distributivity of $\prod$ over $\sum$ for families of sets indexed by a product of sets, which is the translation of the type-theoretic axiom of choice into BST. Sets of sets are special set-indexed families of sets that allow "lifting" of functions on the index-set to functions on them. The direct families of sets and the set-relevant families of sets are introduced. The index-set of the former is a directed set, while the transport maps of the latter are more than one and appropriately indexed. With the use of the introduced universe $\mathbb{V}_0^{\mathrm{im}}$ of sets and impredicative sets we study families of families of sets, the next rung of the ladder of set-like objects in $\mathbb{V}_0^{\mathrm{im}}$.

In Chapter 4 we develop the basic theory of set-indexed families of subsets and of the corresponding family-maps between them. In contrast to set-indexed families of sets, the properties of which are determined "externally" through their transport maps, the properties of a set-indexed family $\Lambda(X)$ of subsets of a given set X are determined "internally" through the embeddings of the subsets of $\Lambda(X)$ to X. The interior union of $\Lambda(X)$ is the internal analogue to the $\sum$-set of a set-indexed family of sets Λ, and the intersection of $\Lambda(X)$ is the internal analogue to the $\prod$-set of Λ. Families of sets over products, sets of subsets, and direct families of subsets are the internal analogue to the corresponding notions for families of sets. Set-indexed families of partial functions and set-indexed families of complemented subsets, together with their corresponding family-maps, are studied.

In Chapter 5 a form of proof-relevance is added to BISH through BST, which is both separate from its standard mathematical part, and also expressible in it. The distinctive feature of intensional Martin-Löf Type Theory (MLTT) is its proof-relevance, the fact that proof-objects are considered as "first-class citizens". The various kinds of moduli, like the moduli of uniform continuity, of convergence etc., which witness that a function is uniformly continuous, a sequence converges etc., form a trace of proof-relevance in BISH. We make the algorithmic content of several constructive proofs explicit by defining a BHK-interpretation of certain formulas of BISH within BST. We define the notion of a set with a proof-relevant equality and the notion of a Martin-Löf set, which translates the first level of the identity type of intensional MLTT. As a result, notions and facts from homotopy type theory are translated in BISH.

In Chapter 6 we connect various notions and results from the theory of families of sets and subsets to the theory of Bishop spaces, a function-theoretic approach to constructive topology. Associating in an appropriate way to each set $\lambda_0(i)$ of an I-family of sets Λ a Bishop topology F_i, a spectrum $S(\Lambda)$ of Bishop spaces is generated. The $\sum$-set and the $\prod$-set of a spectrum $S(\Lambda)$ are equipped with canonical Bishop topologies. A direct spectrum of Bishop spaces is a family of Bishop spaces associated to a direct family of sets. The direct and inverse limits of direct spectra of Bishop spaces are studied. Direct spectra of Bishop subspaces are also examined. Many Bishop topologies used in this chapter are defined inductively within the extension BISH^* of BISH with inductive definitions with rules of countably many premises.

In Chapter 7 we study the Borel and Baire sets within Bishop spaces as a constructive counterpart to the study of Borel and Baire algebras within topological spaces. As we use the inductively defined least Bishop topology, and as the Borel and Baire sets over a family of F-complemented subsets are defined inductively, we work again within BISH^*. In contrast to the classical theory, we show that the Borel and the Baire sets of a Bishop space coincide. Finally, our reformulation within BST of the Bishop-Cheng definition of a measure space and of an integration space, based on the notions of families of complemented subsets and of families of partial functions, facilitates a predicative reconstruction of the originally impredicative Bishop-Cheng measure theory.

Papers related to this Thesis. Section 3.6 is included, in a slightly different form, in [95], most of the material of Chapter 6 is found in [96], sections 7.1 and 7.2 are included in [92], most of Chapter 5 is found in [101], and sections 7.3−7.5 are included in [102].

Chapter 1

Introduction

Bishop's theory of sets is Bishop's account of the informal theory of sets and functions that underlies Bishop-style constructive mathematics BISH. We briefly present the relation of this theory to the set-theoretic and type-theoretic foundations of mathematics. Bishop Set Theory (BST) is our "completion" of Bishop's theory of sets with a universe of sets, with a clear distinction between sets and classes, with an explicit use of dependent operations, and with a concrete formulation of various notions of families of sets. We explain how the theory of families of sets within BST that is elaborated in this Thesis is used, in order to reveal proof-relevance in BISH, to develop the theory of spectra of Bishop spaces, and to reformulate predicatively the fundamental notions of the impredicative Bishop-Cheng measure theory.

1.1 Bishop's theory of sets

The theory of sets underlying Bishop-style constructive mathematics (BISH) was only sketched in Chapter 3 of Bishop's seminal book [9]. Since Bishop's central aim in [9] was to show that a large part of advanced mathematics can be done within a constructive and computational framework that does not contradict the classical practice, the inclusion of a detailed account of the set-theoretic foundations of BISH could possibly be against the effective delivery of his message.

The Bishop-Cheng measure theory, developed in [18], was very different from the measure theory of [9], and the inclusion of an enriched version of the former into [19], the book on constructive analysis that Bishop co-authored with Bridges later, affected the corresponding Chapter 3 in two main respects. First, the inductively defined notion of the set of Borel sets generated by a given family of complemented subsets of a set X, with respect to a set of real-valued functions on X, was excluded, as unnecessary, and, second, the operations on the complemented subsets of a set X were defined differently, and in accordance to the needs of the new measure theory.

Yet, in both books many issues were left untouched, a fact that often was a source of confusion. In many occasions, especially in the measure theory of [18] and [19], the powerset was treated as a set, while in the measure theory of [9], Bishop generally avoided the powerset by using appropriate families of subsets instead. In later works of Bridges and Richman, like [20] and [76], the powerset was clearly used as a set, in contrast though, to the predicative spirit of [9].

The concept of a family of sets indexed by a (discrete) set, was asked to be defined in [9]

(Exercise 2, p. 72), and a definition, attributed to Richman, was given in [19] (Exercise 2, p. 78). An elaborate study though, of this concept within BISH is missing, despite its central character in the measure theory of [9], its extensive use in the theory of Bishop spaces [88] and in abstract constructive algebra [76]. Actually, in [76] Richman introduced the more general notion of a family of objects of a category indexed by some set, but the categorical component in the resulting mixture of Bishop's set theory and category theory was not explained in constructive terms[1].

Contrary to the standard view on Bishop's relation to formalisation, Bishop was very interested in it. In [12], p. 60, he writes:

> Another important foundational problem is to find a formal system that will efficiently express existing predictive mathematics. I think we should keep the formalism as primitive as possible, starting with a minimal system and enlarging it only if the enlargement serves a genuine mathematical need. In this way the formalism and the mathematics will hopefully interact to the advantage of both.

Actually, in [12] Bishop proposed Σ, a variant of Gödel's T, as a formal system for BISH. In the last two pages of [12] he sketched very briefly how Σ can be presented as a functional programming language, like fortran and algol. In p. 72 he also added:

> It would be interesting to take Σ as the point of departure for a reasonable programming language, and to write a compiler.

Bishop's views on a full-scale program on the foundations of mathematics are realised in a more developed form in his, unfortunately, unpublished papers [10] and [11]. In the first, Bishop elaborated a version of dependent type theory with one universe, in order to formalise BISH. This was the first time that some form of type theory is used to formalise constructive mathematics.

As Martin-Löf explains in [71], p. 13, he got access to Bishop's book only shortly after his own book on constructive mathematics [71] was finished. Bishop's book [9] also motivated his version of type theory. Martin-Löf opened his first published paper on type theory ([72], p. 73) as follows.

> The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book of Bishop.

The type-theoretic interpretation of Bishop's set theory into the theory of setoids (see especially the work of Palmgren [81]-[87]) has become nowadays the standard way to understand *Bishop sets* (as far as I know, this is a term due to Palmgren). A setoid is a type A in a fixed universe $\mathcal{U}$ equipped with a term $\simeq\colon A \to A \to \mathcal{U}$ that satisfies the properties of an equivalence relation. The identity type of Martin-Löf's intensional type theory (MLTT) (see [74]), expresses, in a proof-relevant way, the existence of the least reflexive relation on a type, a fact with no counterpart in Bishop's set theory. As a consequence, the free setoid on a type is definable (see [85], p. 90), and the presentation axiom in setoids is provable (see Note 1.3.2). Moreover, in MLTT the families of types over a type I is the type $I \to \mathcal{U}$, which belongs to the successor universe $\mathcal{U}'$ of $\mathcal{U}$. In Bishop's set theory though, where only one universe of sets is implicitly used, the set-character of the totality of all families of sets indexed

[1] This was done e.g., in the the formulation of category theory in homotopy type theory (Chapter 9 in [124]).

by some set I is questionable from the predicative point of view (see our comment after the Definition 3.1.3).

The quest $\boldsymbol{Q}$ of finding a formal system suitable for Bishop's system of informal constructive mathematics BISH dominated the foundational studies of the 1970's. Myhill's system CST, introduced in [80], and later Aczel's CZF (see [1]), Friedman's system B, developed in [51], and Feferman's system of explicit mathematics T_0 (see [48] and [49]), are some of the systems related to $\boldsymbol{Q}$, but soon developed independently from it. These systems were influenced a lot from the classical Zermelo-Fraenkel set theory, and could be described as "top-down" approaches to the goal of $\boldsymbol{Q}$, as they have many "unexpected" features with respect to BISH. Using Feferman's terminology from [49], these formal systems are not completely faithful to BISH. If T is a formal theory of an informal body of mathematics M, Feferman gave in [49] the following definitions.

(i) T is *adequate* for M, if every concept, argument, and result of M is represented by a (basic or defined) concept, proof, and a theorem, respectively, of T.

(ii) T is *faithful* to M, if every basic concept of T corresponds to a basic concept of M and every axiom and rule of T corresponds to or is implicit in the assumptions and reasoning followed in M (i.e., T does not go beyond M conceptually or in principle).

In [5], p. 153, Beeson called T *suitable* to M, if T is adequate for M and faithful to M.

Beeson's systems S and S_0 in [5], and Greenleaf's system of liberal constructive set theory LCST in [55] were dedicated to $\boldsymbol{Q}$. Especially Beeson tried to find a faithful and adequate formalisation of BISH, and, by including a serious amount of proof relevance, his systems stand in between the set-theoretic, proof-irrelevant point of view and the type-theoretic, proof-relevant point of view.

All aforementioned systems though, were not really "tested" with respect to BISH. Only very small parts of BISH were actually implemented in them, and their adequacy for BISH was mainly a claim, rather than a shown fact. The implementation of Bishop's constructivism within a formal system for it was taken seriously in the type-theoretic formalisations of BISH, and especially in the work of Coquand (see e.g., [37] and [40]), Palmgren (see e.g., [62] and the collaborative work [39]), the Nuprl research group of Constable (see e.g., [36]), and of Sambin and Maietti within the Minimalist Foundation (see [111] and [70]).

1.2 Bishop Set Theory (BST) and Bishop's theory of sets

Bishop set theory (BST) is an informal, constructive theory of totalities and assignment routines that serves as a "completion" of Bishop's theory of sets. Its first aim is to fill in the "gaps", or highlight the fundamental notions that were suppressed by Bishop in his account of the set theory underlying BISH. Its second aim is to serve as an intermediate step between Bishop's theory of sets and a suitable, in Beeson's sense, formalisation of BISH. To assure faithfulness, we use concepts or principles that appear, explicitly or implicitly, in BISH. Next we describe briefly the features of BST that "complete" Bishop's theory of sets.

1. Explicit use of a universe of sets. Bishop used a universe of sets only implicitly. E.g., he "roughly" describes in [9], p. 72, a set-indexed family of sets as

> ... a rule which assigns to each t in a discrete set T a set $\lambda(t)$.

Every other rule, or assignment routine mentioned by Bishop is from one given totality, the domain of the rule, to some other totality, its codomain. The only way to make the rule of a family of sets compatible with this pattern is to employ a totality of sets. In [10] Bishop explicitly used a universe in his type theory. Here we use the totality $\mathbb{V}_0$ of sets, which is defined in an open-ended way, and it contains the primitive set $\mathbb{N}$ and all defined sets. $\mathbb{V}_0$ itself is not a set, but a class. It is a notion instrumental to the definition of dependent operations, and of a set-indexed family of sets.

2. Clear distinction between sets and classes. A class is a totality defined through a membership condition in which a quantification over $\mathbb{V}_0$ occurs. The powerset $\mathcal{P}(X)$ of a set X, the totality $\mathcal{P}^{\mathbb{I}\mathbb{I}}(X)$ of complemented subsets of a set X, and the totality $\mathfrak{F}(X, Y)$ of partial functions from a set X to a set Y are characteristic examples of classes. A class is never used here as the domain of an assignment routine, only as a codomain of an assignment routine.

3. Explicit use of dependent operations. The standard view, even among practicioners of Bishop-style constructive mathematicians, is that dependency is not necessary to BISH. Dependent functions though, do appear explicitly in Bishop's definition of the intersection $\bigcap_{t\in T} \lambda(t)$ of a family λ of subsets of some set X indexed by an inhabited set T (see [9], p. 65, and [19], p. 70). We show that the elaboration of dependency within BISH is only fruitful to it. Dependent functions are not only necessary to the definition of products of families of sets indexed by an arbitrary set, but as we show throughout this Thesis in many areas of constructive mathematics. Some form of dependency is also formulated in Bishop's type theory [10]. The somewhat "silent" role of dependency within Bishop's set theory is replaced by a central role within BST.

4. Elaboration of the theory of families of sets. With the use of the universe $\mathbb{V}_0$, of the notion of a non-dependent assignment routine λ_0 from an index-set I to $\mathbb{V}_0$, and of a certain dependent operation λ_1, we define explicitly in Definition 3.1.1 the notion of a family of sets indexed by I. Although an I-family of sets is a certain function-like object, it can be understood also as an object of a one level higher than that of a set. The corresponding notion of a "function" from an I-family Λ to an I-family M is that of a family-map. Operations between sets generate operations between families of sets and their family-maps. If the index-set I is a directed set, the corresponding notion of a family of sets over it is that of a direct family of sets. The constructions for families of sets can be generalised appropriately for families of families of sets (see Section 3.10). Families of subsets of a given set X over an index-set I are special I-families that deserve an independent treatment. Families of equivalence classes, families of partial functions, families of complemented subsets and direct families of subsets are some of the variations of set-indexed families of subsets that are studied here and have many applications in constructive mathematics.

Here we apply the general theory of families of sets, in order:

I. To reveal proof-relevance in BISH. Classical mathematics is proof-irrelevant, as it is indifferent to objects that "witness" a relation or a more complex formula. On the other extreme, Martin-Löf type theory is proof-relevant, as every element of a type A is a proof of the "proposition" A. Bishop's presentation of BISH was on purpose closer to the proof-irrelevance of classical mathematics, although a form of proof-relevance was evident in the use of several notions of moduli (of convergence, of uniform continuity, of uniform differentiability etc.). Focusing on membership and equality conditions for sets given by appropriate existential formulas we define certain families of proof-sets that provide a BHK-interpretation within BST of formulas that correspond to the standard atomic formulas of a first order theory. With the machinery of the general theory of families of sets this BHK-interpretation within BST is extended to complex formulas. Consequently, we can associate to many formulas ϕ of BISH a set $\mathtt{Prf}(\phi)$ of "proofs" or witnesses of ϕ. Abstracting from several examples of totalities in BISH we define the notion of a set with a proof-relevant equality, and of a Martin-Löf set, a special case of the former, the equality of which corresponds to the identity type of a type in intensional MLTT. Through the concepts and results of BST notions and facts of MLTT and its extensions (either with the axiom of function extensionality, or with Vooevodsky's axiom of univalence) can be translated into BISH. While Bishop's theory of sets is standardly understood through its translation to MLTT (see e.g., [39]), the development of BST offers a (partial) translation in the converse direction.

II. To develop the theory of spectra of Bishop spaces. A Bishop space is a constructive, function-theoretic alternative to the notion of a topological space. A Bishop topology F on a set X is a subset of the real-valued function $\mathbb{F}(X)$ on X that includes the constant functions and it is closed under addition, composition with Bishop continuous functions $\mathrm{Bic}(\mathbb{R})$ from $\mathbb{R}$ to $\mathbb{R}$, and uniform limits. Hence, in contrast to topological spaces, continuity of real-valued functions is a primitive notion and a concept of open set comes a posteriori. A Bishop topology on a set can be seen as an abstract and constructive approach to the ring of continuous functions $C(X)$ of a topological space X. Associating appropriately a Bishop topology to the set $\lambda_0(i)$ of a family of sets over a set I, for every $i \in I$, the notion of a spectrum of Bishop spaces is defined. If I is a directed set, we get a direct spectrum. The theory of direct spectra of Bishop spaces and their limits is developed in Chapter 6, in analogy to the classical theory of spectra of topological spaces and their limits. The constructive theory of spectra of other structures, like groups, or rings, or modules, can be developed along the same lines.

III. To reformulate predicatively the basics of Bishop-Cheng measure theory. The standard approach to measure theory (see e.g., [123], [57]) is to take measure as a primitive notion, and to define integration with respect to a given measure. An important alternative, and, as argued by Segal in [118] and [119], a more natural approach to measure theory, is to take the integral on a certain set of functions as a primitive notion, extend its definition to an appropriate, larger set of functions, and then define measure at a later stage. This is the idea of the Daniell integral, defined by Daniell in [43], which was taken further by Weil, Kolmogoroff, and Carathéodory (see [127], [67], and [29], respectively).

In the general framework of constructive-computable mathematics, there are many approaches to measure and probability theory. There is an extended literature in intuitionistic measure theory (see e.g., [59]), in measure theory within the computability framework of Type-2 Theory of Effectivity (see e.g., [46]), in Russian constructivism (especially in the work of Šanin [112] and Demuth [21]), in type theory, where the main interest lies in the creation of probabilistic programming (see e.g., [8]), and recently also in homotopy type theory (see [47]), where homotopy type theory (see [124]) is applied to probabilistic programming.

Within BISH, measure and probability theory have taken two main directions. The first direction, developed by Bishop and Cheng in [18] and by Chan in [30]−[34], is based on the notion of integration space, a constructive version of the Daniell integral, as a starting point of constructive measure theory. Following the aforementioned spirit of classical algebraic integration theory, Bishop and Cheng defined first the notion of an integrable function through the notion of an integration space, and afterwords the measure of an integrable set. In their definition of integration space though, Bishop and Cheng used the impredicative concept $\mathfrak{F}(X)$ of all partial functions from a set X to $\mathbb{R}$. Such a notion makes the extraction of the computational content of CMT and the implementation of CMT in some programming language impossible. The second direction to constructive measure theory, developed by Coquand, Palmgren and Spitters in [38], [121] and [41], is based on the recognition of the above problem of the Bishop-Cheng theory and of the advantages of working within the abstract, algebraic, and point-free framework of Boolean rings or of vector lattices. In analogy to Segal's notion of a probability algebra, the starting notion is a boolean ring equipped with an inequality and a measure function, which is called a measure ring, on which integrable and measurable functions can be defined. One can show that the integrable sets of Bishop-Cheng form a measure ring. In general, the second direction to constructive measure theory is considered technically and conceptually simpler.

In Chapter 7 we reconstruct the Bishop-Cheng notion of measure space within BST, where a set of measurable sets is not an appropriate set of complemented subsets, as it is usually understood, but an appropriate set-indexed family of complemented subsets. This fact is acknowledged by Bishop in [12], but it is completely suppressed later by him and his collaborators (Cheng and Chan). A similar indexing appears in a predicative formulation of the Bishop-Cheng notion of an integration space.

The notions of a set-indexed family of sets and of a set-indexed family of subsets of a given set are shown here to be important tools in the precise formulation of abstract notions in constructive mathematics. Avoiding them, makes the reading of constructive mathematics easier and very close to the reading of classical mathematics. Using them, makes the writing of constructive mathematics more precise, and seriously enriches its content.

As the fundamental notion of a family of sets can be described both in categorical and type-theoretic terms, many notions and constructions from category theory and dependent type theory are represented in BST. While category theory and standard set-theory, or dependent type theory and standard set-theory do not match perfectly, large parts of category theory and dependent type theory are reflected naturally in Bishop Set Theory (see also section 8.1).

1.3 Notes

Note 1.3.1. Regarding the exact time that Bishop's unpublished papers [10] and [11] were written, it was difficult to find an answer. Bishop's scheme of presenting a formal system for BISH and of elaborating its implementation in some functional programming language is found both in [12] and in Bishop's unpublished papers. The first is Bishop's contribution to the proceedings of the Buffalo meeting in 1968 that were published in [66]. As Per Martin-Löf informed me, Bishop was not present at the meeting. The presentation of the formal system Σ and its presentation as a programming language in [12] is very sketchy. Instead, the presentation of the type theory for BISH in [10], and its presentation as a programming language in [11] is an elaborated enterprise. I have heard a story of an unsuccessful effort of Bishop to publish [10], due to some parallels between [10] and de Bruijn's work. According to that story, Bishop was unwilling to pursue the publication of his type-theoretic formalism after that rejection. In any event, Bishop's unpublished papers must have been written between 1967 and 1970. Maybe, the period between 1968 and 1969 is a better estimation. In October 1970 Bishop and Cheng sent to the editors of the Memoirs of the American Mathematical

Society their short monograph [18], a work that deviates a lot from the predicative character of [9]. In my view, the papers [10] and [11] do not fit to Bishop's period after 1970.

Note 1.3.2 (The presentation axiom for setoids)**.** If $A : \mathcal{U}$, then, by Martin-Löf's J-rule, $=_A$ is the least reflexive relation on A, and $\varepsilon A := (A, =_A)$ is the *free setoid* on A. According to the universal property of a free setoid, for every setoid $\mathcal{B} := (B, \sim_B)$ and every function $f : A \to B$, there is a *setoid-map* $\varepsilon f\colon A \to \mathcal{B}$ such that the following left diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \\ A \arrow[ur, dashed, "\varepsilon f"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, two heads, "f"] & B. \\ P \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

To show this, let $(\varepsilon f)(a) := f(a)$, and since $=_B$ is the least reflexive relation on B, we get $a =_A a' \Rightarrow (\varepsilon f)(a) =_B (\varepsilon f)(a')$, hence $f(a) \sim_B f(a')$. A setoid $\mathcal{A}$ is a *choice setoid*, if every $f : X \twoheadrightarrow A$, has a right inverse i.e., there $g\colon A \to X$ such that $f \circ g = \mathrm{id}_A$. With the use of the type-theoretic axiom of choice (see [124], section 1.6) one can show that the free setoid $(A, =_A)$ is a choice setoid. Using the identity map, every setoid $\mathcal{A}$ is the quotient of the free setoid on A, hence every setoid is the quotient of a choice setoid. If $\mathcal{C}$ is a category, an object P of $\mathcal{C}$ is called *projective*, if for every objects A, B of $\mathcal{C}$ and every arrow $f : A \twoheadrightarrow B$ and $g\colon P \to B$, there is $h\colon P \to A$ such that the above right diagram commutes. A category $\mathcal{C}$ satisfies the presentation axiom, if for every object C in $\mathcal{C}$ there is $f : P \twoheadrightarrow C$, where P is projective. For the relation between the presentation axiom and various choice principles see [103]. It is immediate to show that a projective setoid is a choice setoid. For the converse, and following [39], p. 74, let $(P, \sim_P)$ be a choice setoid. To show that it is a projective, we need to define a setoid-map h, given setoid maps f and g as above. Let

$$Q := \sum_{(a,p):A \times P} f(a) =_B g(p),$$

and let the projections $p_1 : Q \to A$,, where $p_1(a, p, e) := a$, and $p_2\colon Q \to P$, where $p_2(a, p, e) := p$. By the definition of Q we get $f \circ p_1 = g \circ p_2$. Since $p_2\colon Q \twoheadrightarrow P$ and P is a choice set, there is $k\colon P \to Q$ such that $p_2 \circ k = \mathrm{id}_P$. If $h := p_1 \circ k$, then

$$\begin{tikzcd} & & A \arrow[r, two heads, "f"] & B \\ P \arrow[urr, bend left, "h"] \arrow[r, "k"'] & Q \arrow[ur, "p_1"] \arrow[r, two heads, "p_2"'] & P \arrow[ur, "g"'] & \end{tikzcd}$$

$f \circ (p_1 \circ k) = (f \circ p_1) \circ k = (g \circ p_2) \circ k = g \circ (p_2 \circ k) = g \circ \mathrm{id}_P = g$. Consequently, every setoid is the surjective image of a choice setoid, hence of a projective setoid.

Note 1.3.3. A very first and short presentation of BST is found in [95], where there we write CSFT instead of BST. In [95] we also expressed dependency through the universe of functions $\mathbb{V}_1$ i.e., the totality of triplets (A, B, f), where A, B are sets and f is a function from A to B. Since dependent operations are explicitly used by Bishop e.g., in the definition of the intersection $\bigcap_{t \in T} \lambda(t)$ of a T-family of subsets $(\lambda(t))_{t \in T}$ of a set X, while $\mathbb{V}_1$ is neither explicitly, nor implicitly, mentioned, we use here the former concept.

Note 1.3.4. As it is noted by Palmgren in [82], p. 35, in ZF, and also in its constructive version CZF, a family of sets is represented by the fibers of a function $\lambda \colon B \to I$, where the fibers $\lambda_i := \{b \in B \mid \lambda(b) = i\}$ of λ, for every $i \in I$, represent the sets of the family. Hence the notion of a family of sets is reduced to that of a set. As this reduction rests on the replacement scheme, such a reduction is not possible neither in MLTT nor in BST.

Chapter 2

Fundamentals of Bishop Set Theory

We present the basic elements of BST, a reconstruction of Bishop's informal theory of sets, as this is developed in chapters 3 of [9] and [19]. The main new features of BST, with respect to Bishop's account, are the explicit use of the universe $\mathbb{V}_0$ of sets and the elaboration of the study of dependent operations over a non-dependent assignment routines from a set to $\mathbb{V}_0$. The first notion is implicit in Bishop's work, while the second is explicitly mentioned, although in a rough way. These concepts are necessary to the concrete definition of a set-indexed family of sets, the main object of our study, which is only roughly mentioned by Bishop. The various notions of families of sets introduced later, depend on the various notions of sets, subsets and assignment routines developed in this chapter.

2.1 Primitives

The logical framework of BST is first-order intuitionistic logic with equality (see [116], chapter 1). This primitive equality between terms is denoted by $s := t$, and it is understood as a *definitional*, or *logical*, equality. I.e., we read the equality $s := t$ as "the term s is by definition equal to the term t". If ϕ is an appropriate formula, for the standard axiom for equality $[a := b \ \& \ \phi(a)] \Rightarrow \phi(b)$ we use the notation $[a := b \ \& \ \phi(a)] :\Rightarrow \phi(b)$. The equivalence notation $:\Leftrightarrow$ is understood in the same way. The set $(\mathbb{N} =_{\mathbb{N}}, \neq_{\mathbb{N}})$ of natural numbers, where its canonical equality is given by $m =_{\mathbb{N}} n :\Leftrightarrow m := n$, and its canonical inequality by $m \neq_{\mathbb{N}} n :\Leftrightarrow \neg(m =_{\mathbb{N}} n)$, is primitive. The standard Peano-axioms are associated to $\mathbb{N}$.

A global operation $(\cdot, \cdot)$ of pairing is also considered primitive. I.e., if s, t are terms, their pair (s, t) is a new term. The corresponding equality axiom is $(s, t) := (s', t') :\Leftrightarrow s := s' \ \& \ t := t'$. The n-tuples of given terms, for every n larger than 2, are definable. The global projection routines $\mathbf{pr}_1(s, t) := s$ and $\mathbf{pr}_2(s, t) := t$ are also considered primitive. The corresponding global projection routines for any n-tuples are definable.

An undefined notion of mathematical construction, or algorithm, or of finite routine is considered as primitive. The main primitive objects of BST are totalities and assignment routines. Sets are special totalities and functions are special assignment routines, where an assignment routine is a a special finite routine. All other equalities in BST are equalities on totalities defined though an equality condition. A predicate on a set X is a bounded formula $P(x)$ with x a free variable ranging over X, where a formula is bounded, if every quantifier occurring in it is over a given set.

2.2 Totalities

Definition 2.2.1. (i) *A primitive set* $\mathbb{A}$ *is a totality with a given membership* $x \in \mathbb{A}$*, and a given equality* $x =_{\mathbb{A}} y$*, that satisfies axiomatically the properties of an equivalence relation. The set* $\mathbb{N}$ *of natural numbers is the only primitive set considered here.*

(ii) *A (non-inductive)defined totality* X *is defined by a membership condition* $x \in X :\Leftrightarrow \mathcal{M}_X(x)$*, where* $\mathcal{M}_X$ *is a formula with* x *as a free variable. If* X, Y *are defined totalities with membership conditions* $\mathcal{M}_X$ *and* $\mathcal{M}_Y$*, respectively, we define* $X := Y :\Leftrightarrow \big[\mathcal{M}_X(x) :\Leftrightarrow \mathcal{M}_Y(x)\big]$*, and in this case we say that* X *and* Y *are definitionally equal.*

(iii) *There is a special "open-ended" defined totality* $\mathbb{V}_0$*, which is called the universe of sets.* $\mathbb{V}_0$ *is not defined through a membership-condition, but in an open-ended way. When we say that a defined totality* X *is considered to be a set we "introduce"* X *as an element of* $\mathbb{V}_0$*. We do not add the corresponding induction, or elimination principle, as we want to leave open the possibility of adding new sets in* $\mathbb{V}_0$*.*

(iv) *A defined preset* X*, or simply, a preset, is a defined totality* X *the membership condition* $\mathcal{M}_X$ *of which expresses a construction that can, in principle, be carried out in a finite time. Formally this is expressed by the requirement that no quantification over* $\mathbb{V}_0$ *occurs in* $\mathcal{M}_X$*.*

(v) *A defined totality* X *with equality, or simply, a totality* X *with equality is a defined totality* X *equipped with an equality condition* $x =_X y :\Leftrightarrow \mathcal{E}_X(x, y)$*, where* $\mathcal{E}_X(x, y)$ *is a formula with free variables* x *and* y *that satisfies the conditions of an equivalence relation i.e.,* $\mathcal{E}_X(x, x)$ *and* $\mathcal{E}_X(x, y) \Rightarrow \mathcal{E}_X(y, x)$*, and* $[\mathcal{E}_X(x, y)\ \&\ \mathcal{E}_X(y, z)] \Rightarrow \mathcal{E}_X(x, y)$*. Two defined totalities with equality* $(X, =_X)$ *and* $(Y, =_Y)$ *are definitionally equal, if* $\mathcal{M}_X(x) :\Leftrightarrow \mathcal{M}_Y(x)$ *and* $\mathcal{E}_X(x, y) :\Leftrightarrow \mathcal{E}_Y(x, y)$*.*

(vi) *A defined set is a preset with a given equality.*

(vii) *A set is either a primitive set, or a defined set.*

(viii) *A totality is a class, if it is the universe* $\mathbb{V}_0$*, or if quantification over* $\mathbb{V}_0$ *occurs in its membership condition.*

Definition 2.2.2. *If* X, Y *are sets, their product* $X \times Y$ *is the defined totality with equality*

$$(x, y) \in X \times Y :\Leftrightarrow x \in A\ \&\ y \in B,$$

$$z \in X \times Y :\Leftrightarrow \exists_{x \in A} \exists_{y \in B} \big(z := (x, y)\big).$$

$X \times Y$ is considered to be a set, and its membership condition is written simpler as follows:

$$(x, y) =_{X \times Y} (x', y') :\Leftrightarrow x =_X x'\ \&\ y =_Y y'.$$

Definition 2.2.3. *A bounded formula on a set* X *is called an extensional property on* X*, if*

$$\forall_{x, y \in X} \big([x =_X y\ \&\ P(x)] \Rightarrow P(y)\big).$$

The totality X_P *generated by* $P(x)$ *is defined by* $x \in X_P :\Leftrightarrow x \in X\ \&\ P(x)$*,*

$$x \in X_P :\Leftrightarrow x \in X\ \&\ P(x),$$

and the equality of X_P *is inherited from the equality of* X*. We also write* $X_P := \{x \in X \mid P(x)\}$*. The totality* X_P *is considered to be a set, and it is called the extensional subset of* X *generated by* P*.*

Using the properties of an equivalence relation, it is immediate to show that an equality condition $\mathcal{E}_X(x,y)$ on a totality X is an extensional property on the product $X \times X$ i.e., $[(x,y) =_{X\times Y} (x',y') \ \& \ x =_X y] \Rightarrow x' =_X y'$. Let the following extensional subsets of $\mathbb{N}$:

$$\mathbb{1} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0\} := \{0\},$$

$$\mathbb{2} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0 \ \vee x =_{\mathbb{N}} 1\} := \{0,1\}.$$

Since $n =_{\mathbb{N}} m :\Leftrightarrow n := m$, the property $P(x) :\Leftrightarrow x =_{\mathbb{N}} 0 \ \vee x =_{\mathbb{N}} 1$ is extensional.

Definition 2.2.4. *If $(X, =_X)$ is a set, its diagonal is the extensional subset of $X \times X$*

$$D(X, =_X) := \{(x,y) \in X \times X \mid x =_X y\}.$$

If $=_X$ is clear from the context, we just write $D(X)$.

Definition 2.2.5. *Let X be a set. An inequality on X, or an apartness relation on X, is a relation $x \neq_X y$ such that the following conditions are satisfied:*
(Ap_1) $\forall_{x,y\in X}\big(x =_X y \ \& \ x \neq_X y \Rightarrow \bot\big)$.
(Ap_2) $\forall_{x,y\in X}\big(x \neq_X y \Rightarrow y \neq_X x\big)$.
(Ap_3) $\forall_{x,y\in X}\big(x \neq_X y \Rightarrow \forall_{z\in X}(z \neq_X x \ \vee \ z \neq_X y)\big)$.
We write $(X, =_X, \neq_X)$ to denote the equality-inequality structure of a set X, and for simplicity we refer the set $(X, =_X, \neq_X)$. The set $(X, =_X, \neq_X)$ is called discrete, if

$$\forall_{x,y\in X}\big(x =_X y \ \vee \ x \neq_X y\big).$$

An inequality $\neq_X$ on X is called tight, if $\neg(x \neq_X y) \Rightarrow x =_X y$, for every $x, y \in X$.

Remark 2.2.6. *An inequality relation $x \neq_X y$ is extensional on $X \times X$.*

Proof. We show that if $x, y \in X$ such that $x \neq y$, and if $x', y' \in X$ such that $x' =_X x$ and $y' =_X y$, then $x' \neq y'$. By Ap_3 we have that $x' \neq x$, which is excluded from Ap_1, or $x' \neq y$, which has to be the case. Hence, $y' \neq x'$, or $y' \neq y$. Since the last option is excluded similarly, we conclude that $y' \neq x'$, hence $x' \neq y'$. ☐

If $\neq_X$ is an inequality on X, and $P(x)$ is an extensional property on X, then X_P inherits the inequality from X. Since $n \neq_{\mathbb{N}} m :\Leftrightarrow \neg(n =_{\mathbb{N}} m)$, the sets $\mathbb{N}$, $\mathbb{1}$, and $\mathbb{2}$ are discrete. Clearly, if $(X, =_X, \neq_X)$ is discrete, then $\neq_X$ is tight.

Remark 2.2.7. *Let the sets $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$.*
(i) *The canonical inequality on $X \times Y$ induced by $\neq_X$ and $\neq_Y$, which is defined by*

$$(x,y) \neq_{X\times Y} (x',y') :\Leftrightarrow x \neq_X x' \ \vee \ y \neq_Y y',$$

for every (x,y) and $(x',y') \in X \times Y$, is an inequality on $X \times Y$.
(ii) *If $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$ are discrete, then $(X \times Y, =_{X\times Y}, \neq_{X\times Y})$ is discrete.*

Proof. The proof of (i) is immediate. To show (ii), let $(x,y), (x',y') \in X \times Y$. By our hypothesis $x =_X x' \ \vee \ x \neq_X x'$ and $y =_Y y' \ \vee \ y \neq_Y y'$. If $x =_X x'$ and $y =_Y y'$, then $(x,y) =_{X\times Y} (x',y')$. In any other case we get $(x,y) \neq_{X\times Y} (x',y')$. ☐

Uniqueness of an element of a set X with respect to some property $P(x)$ on X means that all elements of X having this property are $=_X$-equal. We use the following abbreviation:

$$\exists_{!x\in X}P(x) :\Leftrightarrow \exists_{x\in X}\big(P(x) \ \& \ \forall_{z\in X}\big(P(z) \Rightarrow z =_X x\big)\big).$$

Definition 2.2.8. *Let $(X, =_X)$ be a set.*
(i) *X is inhabited, if $\exists_{x\in X}\big(x =_X x\big)$.*
(ii) *X is a singleton, or contractible, or a (-2)-set, if $\exists_{x_0\in X}\forall_{x\in X}\big(x_0 =_X x\big)$. In this case, x_0 is called a centre of contraction for X.*
(iii) *X is a subsingleton, or a mere proposition, or a (-1)-set, if $\forall_{x,y\in X}\big(x =_X y\big)$.*
(iv) *The truncation of $(X, =_X)$ is the set $(X, {}_{||}{=_X}_{||})$, where*

$$x \ {}_{||}{=_X}_{||} \ y :\Leftrightarrow x =_X x \ \& \ y =_X y.$$

We use the symbol $||X||$ to denote that the set X is equipped with the truncated equality ${}_{||}{=_X}_{||}$.

Clearly, $x \ {}_{||}{=_X}_{||} \ y$, for every $x, y \in X$, and $(X, {}_{||}{=_X}_{||})$ is a subsingleton.

2.3 Non-dependent assignment routines

Definition 2.3.1. *Let X, Y be totalities. A non-dependent assignment routine f from X to Y, in symbols $f : X \rightsquigarrow Y$, is a finite routine that assigns an element y of Y to each given element x of X. In this case we write $f(x) := y$. If $g : X \rightsquigarrow Y$, let*

$$f := g :\Leftrightarrow \forall_{x\in X}\big(f(x) := g(x)\big).$$

If $f := g$, we say that f and g are definitionally equal. If $(X, =_X)$ and $(Y, =_Y)$ are sets, an operation from X to Y is a non-dependent assignment routine from X to Y, while a function from X to Y, in symbols $f : X \to Y$, is an operation from X to Y that respects equality i.e.,

$$\forall_{x,x'\in X}\big(x =_X x' \Rightarrow f(x) =_Y f(x')\big).$$

If $f : X \rightsquigarrow Y$ is a function from X to Y, we say that f is a function, without mentioning the expression "from X to Y". A function $f : X \to Y$ is an embedding, in symbols $f : X \hookrightarrow Y$, if

$$\forall_{x,x'\in X}\big(f(x) =_Y f(x') \Rightarrow x =_X x'\big).$$

Let the sets $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$. A function $f : X \to Y$ is strongly extensional, if

$$\forall_{x,x'\in X}\big(f(x) \neq_Y f(x') \Rightarrow x \neq_X x'\big).$$

If $\simeq_X$ is another equality on X, we use a new symbol e.g., X^*, for the same totality X. When we write $f : X^* \to Y$, then f is a function from X, equipped with the equality $\simeq_X$, to Y.

If X is a set, the *identity map* id_X on X is the operation $\mathrm{id}_X : X \rightsquigarrow X$, defined by $\mathrm{id}_X(x) := x$, for every $x \in X$. Clearly, id_X is an embedding, which is strongly extensional, if $\neq_X$ is a given inequality on X. If Y is also a set, the projection maps $\mathtt{pr}_X$ and $\mathtt{pr}_Y$ on X and Y, respectively, are the operations $\mathtt{pr}_X : X \times Y \rightsquigarrow X$ and $\mathtt{pr}_Y : X \times Y \rightsquigarrow Y$, where

$$\mathtt{pr}_X(x, y) := \mathbf{pr}_1(x, y) := x \ \ \& \ \ \mathtt{pr}_Y(x, y) := \mathbf{pr}_2(x, y) := y; \qquad (x, y) \in X \times Y.$$

Clearly, the operations $\mathtt{pr}_X$ and $\mathtt{pr}_Y$ are functions, which are strongly extensional, if $\neq_X, \neq_Y$ are inequalities on X, Y, and $\neq_{X \times Y}$ is the canonical inequality on $X \times Y$ induced from them. After introducing the universe $\mathbb{V}_0$ of sets in section 2.4, we shall define non-dependent assignment routines from a set to a totality, like $\mathbb{V}_0$, which is not considered to be a set. In most of the cases the non-dependent assignment routines defined here have a set as a domain.There are cases though, see e.g., Definitions 2.6.5 2.6.6, 4.2.1, and 4.3.1, where a non-dependent assignment routine is defined on a totality, before showing that this totality is a set. *We never define a non-dependent assignment routine from a class to a totality.*

Let the operation $m^* \colon \mathbb{R} \rightsquigarrow \mathbb{Q}$, defined by $m^*(a) := q_m$, where a real number a is a regular sequence of rational numbers $(q_n)_n$ (see [19], p. 18), and q_m is the m-term of this sequence. for some fixed m. The operation m^* is an example of an operation, which is not a function, since unequal real numbers, with respect to the definition of $=_{\mathbb{R}}$ in [19], p. 18, may have equal m-terms in $\mathbb{Q}$. To define a function $f \colon X \to Y$, *first* we define the operation $f \colon X \rightsquigarrow Y$, and *afterwords* we prove that f is a function (from X to Y).

The *composition* $g \circ f$ of the operations $f \colon X \rightsquigarrow Y$ and $g \colon Y \rightsquigarrow Z$ is the operation $g \circ f \colon X \rightsquigarrow Z$, defined by $(g \circ f)(x) := g(f(x))$, for every $x \in X$. Clearly, $g \circ f$ is a function, if f and g are functions. If $h \colon Z \rightsquigarrow W$, notice the following definitional equalities

$$f \circ \mathrm{id}_X := f, \qquad \mathrm{id}_Y \circ f := f, \qquad h \circ (g \circ f) := (h \circ g) \circ f.$$

A diagram commutes always with respect to the equalities of the related sets. E.g., the commutativity of the following diagram is the equality $e(f(x)) =_W g(h(x))$, for every $x \in X$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "e"] \\ Z \arrow[r, "g"'] & W. \end{tikzcd}$$

Definition 2.3.2. *Let X, Y be sets, and $\neq_Y$ an inequality on Y. The totality $\mathbb{O}(X, Y)$ of operations from X to Y is equipped with the following canonical equality and inequality:*

$$f =_{\mathbb{O}(X,Y)} g :\Leftrightarrow \forall_{x \in X}\big(f(x) =_Y f(x)\big),$$

$$f \neq_{\mathbb{O}(X,Y)} g :\Leftrightarrow \exists_{x \in X}\big(f(x) \neq_Y f(x)\big).$$

The totality $\mathbb{O}(X, Y)$ is considered to be a set. The set $\mathbb{F}(X, Y)$ of functions from X to Y is defined by separation on $\mathbb{O}(X, Y)$ through the extensional property $P(f) :\Leftrightarrow \forall_{x,x' \in X}\big(x =_X x' \Rightarrow f(x) =_Y f(x')\big)$. The equality $=_{\mathbb{F}(X,Y)}$ and the inequality $\neq_{\mathbb{F}(X,Y)}$ are inherited from $=_{\mathbb{O}(X,Y)}$ and $\neq_{\mathbb{O}(X,Y)}$, respectively.

Remark 2.3.3. *Let the sets $(X =_X)$ and $(Y, =_Y, \neq_Y)$. If $f \colon X \to Y$, let $x_1 \neq_X^f x_2 :\Leftrightarrow f(x_1) \neq_Y f(x_2)$, for every $x_1, x_2 \in X$.*
(i) *$x_1 \neq_X^f x_2$ is an inequality on X.*
(ii) *If $(Y, =_Y, \neq_Y)$ is discrete, then $(X =_X, \neq_X^f)$ is discrete if and only if f is an embedding.*
(ii) *If $\neq_Y$ is tight, then $\neq_X^f$ is tight if and only if f is an embedding.*

Proof. (i) Conditions (Ap_1)-(Ap_3) for $\neq^f_X$ are reduced to conditions (Ap_1)-(Ap_3) for $\neq_Y$.
(ii) If $(X =_X, \neq^f_X)$ is discrete, let $f(x_1) =_X f(x_2)$, for some $x_1, x_2 \in X$. Since the possibility $x_1 \neq^f_X x_2 :\Leftrightarrow f(x_1) \neq_Y f(x_2)$ is impossible, we conclude that $x_1 =_X x_2$. If f is an embedding, and since $f(x_1) =_X f(x_2)$ or $f(x_1) \neq_Y f(x_2)$, either $x_1 =_X x_2$, or $x_1 \neq^f_X x_2$.
(iii) If $\neq^f_X$ is tight, and $f(x_1) =_X f(x_2)$, then $\neg(x_1 \neq^f_X x_2)$, hence $x_1 =_X x_2$. If f is an embedding and $\neg(x_1 \neq^f_X x_2) \Leftrightarrow \neg\big(f(x_1) \neq_Y f(x_2)\big)$, then $f(x_1) =_X f(x_2)$, and $x_1 =_X x_2$. $\square$

Definition 2.3.4. *A function $f\colon X \to Y$ is called surjective, if $\forall_{y\in Y}\exists_{x\in X}\big(f(x) =_Y y\big)$. A function $g\colon Y \to X$ is called a modulus of surjectivity for f, if the following diagram commutes*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

If g is a modulus of surjectivity for f, we also say that f is a retraction and Y is a retract of X. If $y \in Y$, the fiber $\mathtt{fib}^f(y)$ of f at y is the following extensional subset of X

$$\mathtt{fib}^f(y) := \{x \in X \mid f(x) =_Y y\}.$$

A function $f\colon X \to Y$ is contractible, if $\mathtt{fib}^f(y)$ is contractible, for every $y \in Y$. If $\neq_Y$ is an inequality on Y, the cofiber $\mathtt{cofib}^f(y)$ of f at y is the following extensional subset of X

$$\mathtt{cofib}^f(y) := \{x \in X \mid f(x) \neq_Y y\}.$$

2.4 The universe of sets

The totality of all sets is the universe $\mathbb{V}_0$ of sets, equipped with the canonical equality

$$X =_{\mathbb{V}_0} Y :\Leftrightarrow \exists_{f\in\mathbb{F}(X,Y)}\exists_{g\in\mathbb{F}(Y,X)}\big(g \circ f = \mathrm{id}_X \ \& \ f \circ g = \mathrm{id}_Y\big)$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r, "g"] \arrow[rr, bend left, "\mathrm{id}_Y"] & X \arrow[r, "f"'] & Y. \end{tikzcd}$$

In this case we write $(f, g) : X =_{\mathbb{V}_0} Y$. If $X, Y \in \mathbb{V}_0$ such that $X =_{\mathbb{V}_0} Y$, we define the set

$$\mathtt{PrfEql}_0(X, Y) := \big\{(f, g) \in \mathbb{F}(X, Y) \times \mathbb{F}(Y, X) \mid (f, g) : X =_{\mathbb{V}_0} Y\big\}$$

of all objects that "witness", or "realise", or prove the equality $X =_{\mathbb{V}_0} Y$. The equality of $\mathtt{PrfEql}_0(X, Y)$ is the canonical one i.e., $(f, g) =_{\mathtt{PrfEql}_0(X,Y)} (f', g') :\Leftrightarrow f =_{\mathbb{F}(X,Y)} f' \ \& \ g =_{\mathbb{F}(Y,X)} g'$. Notice that, in general, not all elements of $\mathtt{PrfEql}_0(X, Y)$ are equal. As in [124], Example 3.1.9, if $X := Y := \mathbb{2} := \{0, 1\}$, then $(\mathrm{id}_{\mathbb{2}}, \mathrm{id}_{\mathbb{2}}) \in \mathtt{PrfEql}_0(\mathbb{2}, \mathbb{2})$, and if $\mathtt{sw}_{\mathbb{2}} : \mathbb{2} \to \mathbb{2}$ maps 0 to 1 and 1 to 0, then $(\mathtt{sw}_{\mathbb{2}}, \mathtt{sw}_{\mathbb{2}}) \in \mathtt{PrfEql}_0(\mathbb{2}, \mathbb{2})$, while $\mathtt{sw}_{\mathbb{2}} \neq \mathrm{id}_{\mathbb{2}}$.

It is expected that the proof-terms in $\mathtt{PrfEql}_0(X, Y)$ are compatible with the properties of the equivalence relation $X =_{\mathbb{V}_0} Y$. This means that we can define a distinguished proof-term $\mathtt{refl}(X) \in \mathtt{PrfEql}_0(X, X)$ that proves the reflexivity of $X =_{\mathbb{V}_0} Y$, an operation $^{-1}$, such that if $(f,g) : X =_{\mathbb{V}_0} Y$, then $(f,g)^{-1} : Y =_{\mathbb{V}_0} X$, and an operation of "composition" $*$ of proof-terms, such that if $(f,g) : X =_{\mathbb{V}_0} Y$ and $(h,k) : Y =_{\mathbb{V}_0} Z$, then $(f,g) * (h,k) : X =_{\mathbb{V}_0} Z$. If $h \in \mathbb{F}(Y,W)$ and $k \in \mathbb{F}(W,Y)$, let

$$\texttt{refl}(X) := \big(\mathrm{id}_X, \mathrm{id}_X\big) \ \ \& \ \ (f,g)^{-1} := (g,f) \ \ \& \ \ (f,g) * (h,k) := (h \circ f, g \circ k).$$

It is immediate to see that these operations satisfy the *groupoid laws*:

(i) $\texttt{refl}(X) * (f,g) =_{\texttt{PrfEql}_0(X,Y)} (f,g)$ and $(f,g) * \texttt{refl}(Y) =_{\texttt{PrfEql}_0(X,Y)} (f,g)$.

(ii) $(f,g) * (f,g)^{-1} =_{\texttt{PrfEql}_0(X,X)} \texttt{refl}(X)$ and $(f,g)^{-1} * (f,g) =_{\texttt{PrfEql}_0(Y,Y)} \texttt{refl}(Y)$.

(iii) $\big((f,g) * (h,k)\big) * (s,t) =_{\texttt{PrfEql}_0(X,W)} (f,g) * \big((h,k) * (s,t)\big)$.

Moreover, the following *compatibility condition* is satisfied:

(iv) If $(f,g), (f',g') \in \texttt{PrfEql}_0(X,Y)$ and $(h,k), (h',k') \in \texttt{PrfEql}_0(Y,Z)$, then if $(f,g) =_{\texttt{PrfEql}_0(X,Y)} (f',g')$ and $(h,k) =_{\texttt{PrfEql}_0(Y,Z)} (h',k')$, then $(f,g) * (h,k) =_{\texttt{PrfEql}_0(X,Z)} (f',g') * (h',k')$.

Proposition 2.4.1. *Let X, Y be sets, $f \in \mathbb{F}(X,Y)$ and $g \in \mathbb{F}(Y,X)$. If $(f,g) \colon X =_{\mathbb{V}_0} Y$, then the set $\texttt{fib}^f(y)$ is contractible, for every $y \in Y$.*

Proof. If $y \in Y$, then $g(y) \in \texttt{fib}^f(y)$, as $f(g(y)) =_Y \mathrm{id}_Y(y) := y$. If $x \in X$, $x \in \texttt{fib}^f(y) :\Leftrightarrow f(x) =_Y y$, and $x =_X g(f(x)) =_X g(y)$ i.e., $g(y)$ is a centre of contraction for $\texttt{fib}^f(y)$. $\square$

Definition 2.4.2. *Let X, Y be sets. The evaluation map $\mathrm{ev}_{X,Y} : \mathbb{F}(X,Y) \times X \rightsquigarrow Y$ is defined by $\mathrm{ev}_{X,Y}(f,x) := f(x)$, for every $f \in \mathbb{F}(X,Y)$ and $x \in X$.*

Proposition 2.4.3. *Let X, Y, Z be sets.*

(i) *The evaluation map $\mathrm{ev}_{X,Y}$ is a function from $\mathbb{F}(X,Y) \times X$ to Y.*

(ii) *For every function $h : Z \times X \to Y$, there is a unique function $\hat{h} : Z \to \mathbb{F}(X,Y)$ such that for every $z \in Z$ and $x \in X$ $\mathrm{ev}_{X,Y}\big(\hat{h}(z), x\big) =_Y h(z,x)$.*

Proof. (i) By definition $(f,x) =_{\mathbb{F}(X,Y)\times X} (f',x')$ if and only if $f =_{\mathbb{F}(X,Y)} f'$ and $x =_X x'$. Hence $\mathrm{ev}_{X,Y}(f,x) := f(x) =_Y f'(x) =_Y f'(x') := \mathrm{ev}_{X,Y}(f',x')$.

(ii) For every $z \in Z$, we define the assignment routine $\hat{h}$ from Z to $\mathbb{F}(X,Y)$ by $z \mapsto \hat{h}(z)$, where $\hat{h}(z)$ is the assignment routine from X to Y, defined by $\hat{h}(z)(x) := h(z,x)$, for every $x \in X$. First we show that $\hat{h}(z)$ is a function from X to Y; if $x =_X x'$, then $(z,x) =_{Z\times X} (z,x')$, hence $\hat{h}(z)(x) := h(z,x) =_Y h(z,x') := \hat{h}(z)(x')$. Next we show that the assignment routine $\hat{h}$ is a function from Z to $\mathbb{F}(X,Y)$; if $z =_Z z'$, then, if $x \in X$, and since then $(z,x) =_{Z\times X} (z',x)$, we have that $\hat{h}(z)(x) := h(z,x) =_Y h(z',x) := \hat{h}(z')(x)$. Since $x \in X$ is arbitrary, we conclude that $\hat{h}(z) =_{\mathbb{F}(X,Y)} \hat{h}(z')$. Since $\mathrm{ev}_{X,Y}\big(\hat{h}(z), x\big) := \hat{h}(z)(x) := h(z,x)$, we get the strong from of the required equality $\mathrm{ev}_{X,Y} \circ (\hat{h} \times 1_X) := h$. If $g : Z \to \mathbb{F}(X,Y)$ satisfying the required equality, and if $z \in Z$, then, for every $x \in X$ we have that $g(z)(x) := \mathrm{ev}_{X,Y}\big(g(z), x\big) =_Y h(z,x) =_Y \mathrm{ev}_{X,Y}\big(\hat{h}(z), x\big) := \hat{h}(z)(x)$, hence $g(z) =_{\mathbb{F}(X,Y)} \hat{h}(z)$. $\square$

2.5 Dependent operations

Definition 2.5.1. *Let I be a set and $\lambda_0 \colon I \rightsquigarrow \mathbb{V}_0$ a non-dependent assignment routine from I to $\mathbb{V}_0$. A dependent operation Φ over λ_0, in symbols*

$$\Phi \colon \bigcurlywedge_{i \in I} \lambda_0(i),$$

is an assignment routine that assigns to each element i in I an element $\Phi(i)$ in the set $\lambda_0(i)$. If $i \in I$, we call $\Phi(i)$ the i-component of Φ, and we also use the notation $\Phi_i := \Phi(i)$. An assignment routine is either a non-dependent assignment routine, or a dependent operation over some non-dependent assignment routine from a set to the universe. If $\Psi\colon \bigcurlywedge_{i\in I}\lambda_0(i)$, let $\Phi := \Psi :\Leftrightarrow \forall_{i\in I}\big(\Phi_i := \Psi_i\big)$. If $\Phi := \Psi$, we say that Φ and Ψ are definitionally equal.

Let the non-dependent assignment routines $\lambda_0\colon I \rightsquigarrow \mathbb{V}_0, \mu_0\colon I \rightsquigarrow \mathbb{V}_0, \nu_0\colon I \rightsquigarrow \mathbb{V}_0$ and $\kappa_0\colon I \rightsquigarrow \mathbb{V}_0$. Let $\mathbb{F}(\lambda_0,\mu_0)\colon I \rightsquigarrow \mathbb{V}_0$ be defined by $\mathbb{F}(\lambda_0,\mu_0)(i) := \mathbb{F}(\lambda_0(i),\mu_0(i)$, for every $i \in I$. The identity operation Id_{λ_0} over λ_0 is the dependent operation

$$\mathrm{Id}_{\lambda_0}\colon \bigcurlywedge_{i\in I}\mathbb{F}(\lambda_0(i),\mu_0(i)) \quad \mathrm{Id}_{\lambda_0}(i) := \mathrm{id}_{\lambda_0(i)}; \quad i \in I.$$

Let $\Psi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\mu_0(i),\nu_0(i))$ and $\Phi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\lambda_0(i),\mu_0(i))$. Their *composition* $\Psi\circ\Phi$ is defined by

$$\Psi\circ\Phi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\lambda_0(i),\nu_0(i)) \quad (\Psi\circ\Phi)_i := \Psi_i\circ\Phi_i; \quad i \in I.$$

If $\Xi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\nu_0(i),\kappa_0(i))$, notice the following definitional equalities

$$\Phi\circ\mathrm{Id}_{\lambda_0} := \Phi, \quad \mathrm{Id}_{\mu_0}\circ\Phi := \Phi, \quad \Xi\circ(\Psi\circ\Phi) := (\Xi\circ\Psi)\circ\Phi.$$

Definition 2.5.2. *If I is a set and $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, let $\mathbb{A}(I,\lambda_0)$ be the totality of dependent operations over λ_0, equipped with the canonical equality:*

$$\Phi =_{\mathbb{A}(I,\lambda_0)} \Psi :\Leftrightarrow \forall_{i\in I}\big(\Phi_i =_{\lambda_0(i)} \Psi_i\big).$$

The totality $\mathbb{A}(I,\lambda_0)$ is considered to be a set. If $\neq_{\lambda_0(i)}$ is an inequality on $\lambda_0(i)$, for every $i \in I$, the canonical inequality $\neq_{\mathbb{A}(I,\lambda_0)}$ on $\mathbb{A}(I,\lambda_0)$ is defined by $\Phi \neq_{\mathbb{A}(I,\lambda_0)} \Psi :\Leftrightarrow \exists_{i\in I}\big(\Phi_i \neq_{\lambda_0(i)} \Psi_i\big)$.

Clearly, $\Phi =_{\mathbb{A}(I,\lambda_0)} \Psi$ is an equivalence relation, and $\Phi \neq_{\mathbb{A}(I,\lambda_0)} \Psi$ is an inequality relation. If $i \in I$, the *i-projection map* on $\mathbb{A}(I,\lambda_0)$ is the operation $\mathtt{pr}_i^{\lambda_0}\colon \mathbb{A}(I,\lambda_0) \rightsquigarrow \lambda_0(i)$, defined by $\mathtt{pr}_i^{\lambda_0}(\Phi) := \Phi_i$, for every $i \in I$. The operation $\mathtt{pr}_i^{\lambda_0}$ is a function. If $\Phi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\lambda_0(i),\mu_0(i))$, a *modulus of surjectivity* for Φ is a dependent operation $\Psi\colon \bigcurlywedge_{i\in I}\mathbb{F}(\mu_0(i),\lambda_0(i)$ such that $\Phi\circ\Psi =_{\mathbb{A}(I,\mathbb{F}(\lambda_0,\lambda_0)} \mathrm{Id}_{\lambda_0}$. In this case, Ψ_i is a modulus of surjectivity for Φ, for every $i \in I$. If $f\colon X \to Y$, let $\mathtt{fib}^f\colon Y \rightsquigarrow \mathbb{V}_0$ be defined by $y \mapsto \mathtt{fib}^f(y)$, for every $y \in Y$. If f is contractible, then by Definition 2.3.4 every fiber $\mathtt{fib}^f(y)$ of f is contractible. A *modulus of centres of contraction* for a contractible function f is a dependent operation $\mathtt{centre}^f\colon \bigcurlywedge_{y\in Y}\mathtt{fib}^f(y)$, such that $\mathtt{centre}^f_y := \mathtt{centre}^f(y)$ is a centre of contraction for f.

2.6 Subsets

Definition 2.6.1. *Let X be a set. A subset of X is a pair (A, i_A^X), where A is a set and $\imath_A^X\colon A \hookrightarrow X$ is an embedding of A into X. If (A, i_A^X) and (B, i_B^X) are subsets of X, then A is a subset of B, in symbols $(A, i_A^X) \subseteq (B, i_B^X)$, or simpler $A \subseteq B$, if there is $f\colon A \to B$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i_A^X"'] & & B \arrow[dl, hook', "i_B^X"] \\ & X. & \end{tikzcd}$$

In this case we use the notation $f\colon A \subseteq B$. Usually we write A instead of (A, i^X_A). The totality of the subsets of X is the powerset $\mathcal{P}(X)$ of X, and it is equipped with the equality

$$(A, i^X_A) =_{\mathcal{P}(X)} (B, i^X_B) :\Leftrightarrow A \subseteq B \;\&\; B \subseteq A.$$

If $f\colon A \subseteq B$ and $g\colon B \subseteq A$, we write $(f, g)\colon A =_{\mathcal{P}(X)} B$.

Since the membership condition for $\mathcal{P}(X)$ requires quantification over $\mathbb{V}_0$, the totality $\mathcal{P}(X)$ is a class. Clearly, $(X, \mathrm{id}_X) \subseteq X$. If X_P is an extensional subset of X (see Definition 2.2.3), then $(X_P, i^X_P) \subseteq X$, where $i^X_P\colon X_P \rightsquigarrow X$ is defined by $i^X_P(x) := x$, for every $x \in X_P$.

Proposition 2.6.2. *If $A, B \subseteq X$, and $f, g : A \subseteq B$, then f is an embedding, and $f =_{\mathbb{F}(A,B)} h$*

$$\begin{tikzcd}
A \arrow[rr, hook, bend left, "f"] \arrow[rr, hook, bend right, "h"'] \arrow[dr, hook, "i^X_A"'] & & B \arrow[dl, hook', "i^X_B"] \\
& X. &
\end{tikzcd}$$

Proof. If $a, a' \in A$ such that $f(a) =_B f(a')$, then $i^X_B(f(a)) =_X i^X_B(f(a')) \Leftrightarrow i^X_A(a) =_X i^X_A(a')$, which implies $a =_A a'$. Moreover, if $i^X_B(f(a)) =_X i^X_A(a) =_X i^X_B(h(a))$, then $f(a) = h(a)$. $\square$

The "internal" equality of subsets implies their "external" equality as sets i.e., $(f, g) : A =_{\mathcal{P}(X)} B \Rightarrow (f, g) : A =_{\mathbb{V}_0} B$. If $a \in A$, then $i^X_A(g(f(a))) =_X i^X_B(f(a)) = i^X_A(a)$, hence $g(f(a)) =_A a$, and then $g \circ f =_{\mathbb{F}(A,A)} \mathrm{id}_A$. Similarly we get $f \circ g =_{\mathbb{F}(B,B)} \mathrm{id}_B$. Let the set

$$\mathtt{PrfEql}_0(A, B) := \big\{(f, g) \in \mathbb{F}(A, B) \times \mathbb{F}(B, A) \mid f\colon A \subseteq B \;\&\; g\colon B \subseteq A\big\},$$

equipped with the canonical equality of pairs as in the case of $\mathtt{PrfEql}_0(X, Y)$. Because of the Proposition 2.6.2, the set $\mathtt{PrfEql}_0(A, B)$ is a subsingleton i.e.,

$$(f, g)\colon A =_{\mathcal{P}(X)} B \;\&\; (f', g')\colon A =_{\mathcal{P}(X)} B \Rightarrow (f, g) = (f', g').$$

If $f \in \mathbb{F}(A, B), g \in \mathbb{F}(B, A), h \in \mathbb{F}(B, C)$, and $k \in \mathbb{F}(C, B)$, let $\mathtt{refl}(A) := \big(\mathrm{id}_A, \mathrm{id}_A\big)$ and $(f, g)^{-1} := (g, f)$, and $(f, g) * (h, k) := (h \circ f, g \circ k)$, and the properties (i)-(iv) for $\mathtt{PrfEql}_0(A, B)$ hold by the equality of all their elements.

Corollary 2.6.3. *Let the set $(X, =_X, \neq_X)$ and $\big(A, =_A, i^X_A, \neq^{i^X_A}_A\big) \subseteq X$, where the canonical inequality $\neq^{i^X_A}_A$ on A is given by $a \neq^{i^X_A}_A a' :\Leftrightarrow i^X_A(a) \neq_X i^X_A(a')$, for every $a, a' \in A$. If $(X, =_X, \neq_X)$ is discrete, then $\big(A, =_A, i^X_A, \neq^{i^X_A}_A\big)$ is discrete, and if $\neq_X$ is tight, $\neq^{i^X_A}_A$ is tight.*

Proof. Since i^X_A is an embedding, it follows immediately from Remark 2.3.3. $\square$

Remark 2.6.4. *If P, Q are extensional properties on the set X, then*

$$X_P =_{\mathcal{P}(X)} X_Q \Leftrightarrow \forall_{x \in X}\big(P(x) \Leftrightarrow Q(x)\big).$$

Proof. The implication ($\Leftarrow$) is immediate to show, since the corresponding identity maps witness the equality $X_P =_{\mathcal{P}(X)} X_Q$. For the converse implication, let $(f, g) : X_P =_{\mathcal{P}(X)} X_Q$. Let $x \in X$ such that $P(x)$. By the commutativity of the following outer diagram

$$\begin{tikzcd} X_P \arrow[rr, hook, bend left, "f"] \arrow[dr, hook, "i_P^X"'] & & X_Q \arrow[ll, hook, bend left, "g"] \arrow[dl, hook, "i_Q^X"] \\ & X & \end{tikzcd}$$

we get $f(x) := i_Q^X(f(x)) =_X i_P^X(x) := x$, and by the extensionality of Q and the fact that $Q(f(x))$ holds we get $Q(x)$. By the commutativity of the above inner diagram and the extensionality of P we get similarly the inverse implication. $\square$

Definition 2.6.5. *If $(A, i_A^X), (B, i_B^X) \subseteq X$, their union $A \cup B$ is the totality defined by*

$$z \in A \cup B :\Leftrightarrow z \in A \ \vee \ z \in B,$$

equipped with the non-dependent assignment routine[1] $i_{A\cup B}^X : A \cup B \rightsquigarrow X$, *defined by*

$$i_{A\cup B}^X(z) := \begin{cases} i_A^X(z) & , z \in A \\ \\ i_B^X(z) & , z \in B. \end{cases}$$

If $z, w \in A \cup B$, we define $z =_{A\cup B} w :\Leftrightarrow i_{A\cup B}^X(z) =_X i_{A\cup B}^X(w)$.

Clearly, $=_{A\cup B}$ is an equality on $A\cup B$, which is considered to be a set, $i_{A\cup B}^X$ is an embedding of $A\cup B$ into X, and the pair $\big(A\cup B, i_{A\cup B}^X\big)$ is a subset of X. Note that if P, Q are extensional properties on X, then $X_P \cup X_Q := X_{P\vee Q}$, since $z \in X_{P\vee Q} :\Leftrightarrow (P \vee Q)(z) :\Leftrightarrow P(z)$ or $Q(z) \Leftrightarrow$: $z \in X_P \cup X_Q$, and the inclusion map $i : X_P \cup X_Q \hookrightarrow X$ is the identity, as it is for $X_{P\vee Q}$ (see Definition 2.2.1). If $\neq_X$ is a given inequality on X, the canonical inequality on $A \cup B$ is determined in Corollary 2.6.3.

Definition 2.6.6. *If $(A, i_A^X), (B, i_B^X) \subseteq X$, their intersection $A \cap B$ is the totality defined by separation on $A \times B$ as follows:*

$$A \cap B := \{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\}.$$

Let the non-dependent assignment routine $i_{A\cap B}^X : A \cap B \rightsquigarrow X$, defined by $i_{A\cap B}^X(a, b) := i_A^X(a)$, for every $(a, b) \in A \cap B$. If (a, b) and (a', b') are in $A \cap B$, let

$$(a, b) =_{A\cap B} (a', b') :\Leftrightarrow i_{A\cap B}^X(a, b) =_X i_{A\cap B}^X(a', b') :\Leftrightarrow i_A^X(a) =_X i_A^X(a').$$

We write $A \between B$ to denote that the intersection $A \cap B$ is inhabited.

[1]Here we define a non-dependent assignment routine on the totality $A \cup B$, without knowing beforehand that $A \cup B$ is a set. It turns out that $A \cup B$ is set, but for that we need to define $i_{A\cup B}^X$ first.

Clearly, $=_{A\cap B}$ is an equality on $A\cap B$, which is considered to be a set, $i^X_{A\cap B}$ is an embedding of $A\cap B$ into X, and $\big(A\cap B, i^X_{A\cap B}\big)$ is a subset of X. If $\neq_X$ is a given inequality on X, the canonical inequality on $A\cap B$ is determined in Corollary 2.6.3. If P, Q are extensional properties on X, then $X_P \cap X_Q$ has elements in $X\times X$, while $X_{P\wedge Q}$ has elements in X, hence the two subsets are not definitionally equal. Next we show that they are "externally" equal i.e., equal in $\mathbb{V}_0$.

Remark 2.6.7. *If P, Q are extensional properties on the set X, then $X_{P\wedge Q} =_{\mathbb{V}_0} X_P \cap X_Q$.*

Proof. Since the inclusion maps corresponding to X_P and X_Q are the identities, let $f : X_{P\wedge Q} \to X_P \cap X_Q$ with $f(z) := (z, z)$, for every $z \in X_{P\wedge Q}$, and let $g : X_P \cap X_Q \to X_{P\wedge Q}$ with $g(a, b) := a$, for every $(a, b) \in X_P \cap X_Q$. Hence, $f(g(a, b)) := f(a) := (a, a)$, and since $(a, b) \in X_P \cap X_Q$, we have by definition that $P(a), Q(b)$ and $a =_X b$, hence $(a, a) =_{X\times X} (a, b)$. If $z \in X_{P\wedge Q}$, then $g(f(z)) := g(z, z) := z$. $\square$

Clearly, $X \cap X =_{\mathcal{P}(X)} X$, while $\mathtt{pr}_A \colon (A\cap B, i^X_{A\cap B}) \subseteq (A, i_A)$ and the identity map $e_A \colon A \to A\cup B$ witnesses the inequality $(A, i^X_A) \subseteq (A\cup B, i^X_{A\cup B})$

$$\begin{tikzcd}
A\cap B \arrow[r, hook, bend left, "\mathtt{pr}_A"] \arrow[dr, hook, "i^X_{A\cap B}"'] & A \arrow[r, hook, bend left, "e_A"] \arrow[d, hook, "i^X_A"'] & A\cup B \arrow[dl, hook', "i^X_{A\cup B}"] \\
& X &
\end{tikzcd}$$

The following properties of the union and intersection of subsets are easy to show.

Proposition 2.6.8. *Let A, B and C be subsets of the set X.*
(i) *$A\cup B =_{\mathcal{P}(X)} B\cup A$ and $A\cap B =_{\mathcal{P}(X)} B\cap A$.*
(ii) *$A\cup (B\cup C) =_{\mathcal{P}(X)} (A\cup B)\cup C$ and $A\cap (B\cap C) =_{\mathcal{P}(X)} (A\cap B)\cap C$.*
(iii) *$A\cap (B\cup C) =_{\mathcal{P}(X)} (A\cap B)\cup (A\cap C)$ and $A\cup (B\cap C) =_{\mathcal{P}(X)} (A\cup B)\cap (A\cup C)$.*

Definition 2.6.9. *Let X, Y be sets, $(A, i^X_A)(C, i^X_C) \subseteq X$, $e \colon (A, i^X_A) \subseteq (C, i^X_C)$, $f \colon C \to Y$, and $(B, i^Y_B) \subseteq Y$. The restriction $f_{|A}$ of f to A is the function $f_{|A} := f \circ e$*

$$\begin{tikzcd}
A \arrow[r, hook, "e"] \arrow[rr, bend right, "f_{|A}"'] & C \arrow[r, "f"] & Y.
\end{tikzcd}$$

The image $f(A)$ of A under f is the pair $f(A) := (A, f_{|A})$, where A is equipped with the equality $a =_{f(A)} a' :\Leftrightarrow f_{|A}(a) =_Y f_{|A}(a')$, for every $a, a' \in A$. We denote $\{f(a) \mid a \in A\} := f(A)$. The pre-image $f^{-1}(B)$ of B under f is the set

$$f^{-1}(B) := \{(c, b) \in C\times B \mid f(c) =_Y i^Y_B(b)\}.$$

Let $i^C_{f^{-1}(B)} \colon f^{-1}(B) \hookrightarrow C$, defined by $i^C_{f^{-1}(B)}(c, b) := c$, for every $(c, b) \in f^{-1}(B)$. The equality of the extensional subset $f^{-1}(B)$ of $C\times B$ is inherited from the equality of $C\times B$.

Clearly, the restriction $f_{|A}$ of $f\colon X \to Y$ to $(A, \imath_A) \subseteq X$ is the function $f_A := f \circ i^X_A$. It is immediate to show that $f(A) \subseteq Y$ and $f^{-1}(B) \subseteq C$. Notice that

$$(A, i^X_A) =_{\mathcal{P}(X)} (B, i^X_B) \Rightarrow i^X_A(A) =_{\mathcal{P}(X)} i^X_B(B),$$

since, if $(f, g) : (A, i^X_A) =_{\mathcal{P}(X)} (B, i^X_B)$, then $i^X_A(a) =_X i^X_B(f(a))$ and $i^X_B(b) =_X i^X_A(g(b))$, for every $a \in A$ and $b \in B$, respectively. If $\neq_Y$ is a given inequality on Y, the canonical inequality on $f(A)$ is determined in Corollary 2.6.3. Similarly, if $\neq_X$ is an inequality on X, $f\colon X \to Y$, and $(B, i^Y_B) \subseteq Y$, the canonical inequality on $f^{-1}(B)$ is given by $(x, b) \neq_{f^{-1}(B)} (x', b') :\Leftrightarrow x \neq_X x'$, and not by the canonical inequality on $X \times B$.

Proposition 2.6.10. *Let X, Y be sets, A, B subsets of X, C, D subsets of Y, and $f : X \to Y$.*
(i) *$f^{-1}(C \cup D) =_{\mathcal{P}(X)} f^{-1}(C) \cup f^{-1}(D)$.*
(ii) *$f^{-1}(C \cap D) =_{\mathcal{P}(X)} f^{-1}(C) \cap f^{-1}(D)$.*
(iii) *$f(A \cup B) =_{\mathcal{P}(Y)} f(A) \cup f(B)$.*
(iv) *$f(A \cap B) =_{\mathcal{P}(Y)} f(A) \cap f(B)$.*
(v) *$A \subseteq f^{-1}(f(A))$.*
(vi) *$f(f^{-1}(C) \cap A) =_{\mathcal{P}(Y)} C \cap f(A)$, and $f(f^{-1}(C)) =_{\mathcal{P}(Y)} C \cap f(X)$.*

Proposition 2.6.11. *Let $(A, i^X_A), (B, i^X_B), (A', i^X_{A'}), (B, i^X_{B'}) \subseteq X$, such that $A =_{\mathcal{P}(X)} A'$ and $B =_{\mathcal{P}(X)} B'$. Let also $(C, i^Y_C), (C', i^Y_{C'}), (D, i^Y_D) \subseteq Y$, such that $C =_{\mathcal{P}(Y)} C'$, and let $f\colon X \to Y$.*
(i) *$A \cap B =_{\mathcal{P}(X)} A' \cap B'$, and $A \cup B =_{\mathcal{P}(X)} A' \cup B'$.*
(ii) *$f(A) =_{\mathcal{P}(Y)} f(A')$, and $f^{-1}(C) =_{\mathcal{P}(X)} f^{-1}(C')$.*
(iii) *$(A \times C, i^X_A \times i^Y_C) \subseteq X \times Y$, where the map $i^X_A \times i^Y_C\colon A \times C \hookrightarrow X \times Y$ is defined by*

$$(i^X_A \times i^Y_C)(a, c) := \big(i^X_A(a), i^Y_C(c)\big); \qquad (a, c) \in A \times C.$$

(iv) *$A \times C =_{\mathcal{P}(X \times Y)} A' \times C'$.*
(v) *$A \times (C \cup D) =_{\mathcal{P}(X \times Y)} (A \times C) \cup (A \times D)$.*
(vi) *$A \times (C \cap D) =_{\mathcal{P}(X \times Y)} (A \times C) \cap (A \cap D)$.*

Proof. All cases are straightforward to show. $\square$

2.7 Partial functions

Definition 2.7.1. *Let X, Y be sets. A partial function from X to Y is a triplet (A, i^X_A, f^Y_A), where $(A, i^X_A) \subseteq X$, and $f^Y_A \in \mathbb{F}(A, Y)$. Often, we use only the symbol f^Y_A instead of the triplet (A, i^X_A, f^Y_A), and we also write $f^Y_A\colon X \rightharpoonup Y$. If (A, i^X_A, f^Y_A) and (B, i^X_B, f^Y_B) are partial functions from X to Y, we call f^Y_A a subfunction of f^Y_B, in symbols $(A, i^X_A, f^Y_A) \leq (B, i^X_B, f^Y_B)$, or simpler $f^Y_A \leq f^Y_B$, if there is $e_{AB}\colon A \to B$ such that the following inner diagrams commute*

$$\begin{tikzcd}
A \arrow[rr, "e_{AB}"] \arrow[dr, hook, "i^X_A"'] \arrow[ddr, bend right, "f^Y_A"'] & & B \arrow[dl, hook', "i^X_B"] \arrow[ddl, bend left, "f^Y_B"] \\
& X & \\
& Y. &
\end{tikzcd}$$

In this case we use the notation $e_{AB}\colon f_A^Y \leq f_B^Y$. *The totality of partial functions from* X *to* Y *is the partial function space* $\mathfrak{F}(X,Y)$, *and it is equipped with the equality*

$$(A, i_A^X, f_A^Y) =_{\mathfrak{F}(X,Y)} (B, i_B^X, f_B^Y) :\Leftrightarrow f_A^Y \leq f_B^Y \ \& \ f_B^Y \leq f_A^Y.$$

If $e_{AB}\colon f_A^Y \leq f_B^Y$ *and* $e_{BA}\colon f_B^Y \leq f_A^Y$, *we write* $(e_{AB}, e_{BA}) : f_A^Y =_{\mathfrak{F}(X,Y)} f_B^Y$.

Since the membership condition for $\mathfrak{F}(X,Y)$ requires quantification over $\mathbb{V}_0$, the totality $\mathfrak{F}(X,Y)$ is a class. Clearly, if $f\colon X \to Y$, then $(X, \mathrm{id}_X, f) \in \mathfrak{F}(X,Y)$. If $(e_{AB}, e_{BA}) : f_A^Y =_{\mathfrak{F}(X,Y)} f_B^Y$, then $(e_{AB}, e_{BA}) : A =_{\mathcal{P}(X)} B$, and $(e_{AB}, e_{BA}) : A =_{\mathbb{V}_0} B$. Let the set

$$\mathtt{PrfEql}_0(f_A^Y, f_B^Y) := \big\{(f,g) \in \mathbb{F}(A,B) \times \mathbb{F}(B,A) \mid f\colon f_A^Y \leq f_B^Y \ \& \ g\colon f_B^Y \leq f_A^Y\big\},$$

equipped with the canonical equality of the product. All the elements of $\mathtt{PrfEql}_0(f_A^Y, f_B^Y)$ are equal to each other. If $f \in \mathbb{F}(A,B), g \in \mathbb{F}(B,A), h \in \mathbb{F}(B,C)$, and $k \in \mathbb{F}(C,B)$, let

$$\mathtt{refl}(f_A^Y) := \big(\mathrm{id}_A, \mathrm{id}_A\big) \ \ \& \ \ (f,g)^{-1} := (g,f) \ \ \& \ \ (f,g) * (h,k) := (h \circ f, g \circ k),$$

and the groupoid-properties for $\mathtt{PrfEql}_0(f_A^Y, f_B^Y)$ hold by the equality of its elements.

Proposition 2.7.2. *Let* $(A, i_A^X, f_A^Y) \in \mathfrak{F}(X,Y)$ *and* $(B, i_B^Y, g_B^Z) \in \mathfrak{F}(Y,Z)$. *Their composition*

$$g_B^Z \odot f_A^Y := \Big(\big(f_A^Y\big)^{-1}(B),\ i_A^X \circ e^A_{(f_A^Y)^{-1}(B)},\ \big(g_B^Z \circ f_A^Y\big)^Z_{(f_A^Y)^{-1}(B)}\Big), \quad \textit{where}$$

$$\big(f_A^Y\big)^{-1}(B) := \big\{(a,b) \in A \times B \mid f_A^Y(a) =_Y i_B^Y(b)\big\},$$

$$e^A_{(f_A^Y)^{-1}(B)}\colon \big(f_A^Y\big)^{-1}(B) \hookrightarrow A, \quad (a,b) \mapsto a; \quad (a,b) \in \big(f_A^Y\big)^{-1}(B),$$

$$\big(g_B^Z \circ f_A^Y\big)^Z_{(f_A^Y)^{-1}(B)}(a,b) := g_B^Z(b); \quad (a,b) \in \big(f_A^Y\big)^{-1}(B),$$

is a partial function that belongs to $\mathfrak{F}(X,Z)$. *If* $(A, i_A^X, i_A^X) \in \mathfrak{F}(X,X), (B, i_B^Y, i_B^Y) \in \mathfrak{F}(Y,Y)$, *and* $(C, i_C^Z, h_C^W) \in \mathfrak{F}(Z,W)$, *the following properties hold:*

(i) $f_A^Y \odot i_A^X =_{\mathfrak{F}(X,Y)} f_A^Y$ *and* $i_B^Y \odot f_A^Y =_{\mathfrak{F}(X,Y)} f_A^Y$.

(ii) $\big(h_C^W \odot g_B^Z\big) \odot f_A^X =_{\mathfrak{F}(X,Z)} h_C^W \odot \big(g_B^Z \odot f_A^X\big)$.

Proof. (i) We show only the first equality and for the second we work similarly. By definition

$$f_A^Y \odot i_A^X := \Big(\big(i_A^X\big)^{-1}(A),\ i_A^X \circ e^A_{(i_A^X)^{-1}(A)},\ \big(f_A^Y \circ i_A^X\big)^Y_{(i_A^X)^{-1}(A)}\Big), \quad \text{where}$$

$$\big(i_A^X\big)^{-1}(A) := \big\{(a,a') \in A \times A \mid i_A^X(a) =_X i_A^X(a')\big\},$$

$$e^A_{(i_A^X)^{-1}(A)}\colon \big(i_A^X\big)^{-1}(A) \hookrightarrow A, \quad (a,a') \mapsto a; \quad (a,a') \in \big(i_A^X\big)^{-1}(A),$$

$$\big(f_A^Y \circ i_A^X\big)(a,a') := f_A^Y(a'); \quad (a,a') \in \big(i_A^X\big)^{-1}(A).$$

Let the operations $\phi\colon A \rightsquigarrow \big(i_A^X\big)^{-1}(A)$, defined by $\phi(a) := (a,a)$, for every $a \in A$, and $\theta\colon \big(i_A^X\big)^{-1}(A) \rightsquigarrow A$, defined by $\theta(a,a') := a$, for every $(a,a') \in \big(i_A^X\big)^{-1}(A)$. It is immediate to show that ϕ and θ are well-defined functions. It is straightforward to show the commutativity of the following inner diagrams

$$\begin{tikzcd}
A \arrow[rr, bend left, "\phi"] \arrow[dr, bend right, "i_A^X"'] \arrow[ddr, bend right=60, "f_A^Y"'] & & \big(i_A^X\big)^{-1}(A) \arrow[ll, bend left, "\theta"] \arrow[dl, bend left, "i_A^X \circ e^A_{(i_A^X)^{-1}(A)}"] \arrow[ddl, bend left=60, "f_A^Y \circ i_A^X"] \\
& X & \\
& Y. &
\end{tikzcd}$$

(ii) We have that $h_C^W \odot g_B^Z := \left(\big(g_B^Z\big)^{-1}(C),\ i_B^Y \circ e^B_{(g_B^Z)^{-1}(C)},\ \big(h_C^W \odot g_B^Z\big)^W_{(g_B^Z)^{-1}(C)}\right)$, where

$$\big(g_B^Z\big)^{-1}(C) := \big\{(b,c) \in B \times C \mid g_B^Z(b) =_Z i_C^Z(c)\big\},$$

$$e^B_{(g_B^Z)^{-1}(C)} \colon \big(g_B^Z\big)^{-1}(C) \hookrightarrow B, \quad (b,c) \mapsto b; \quad (b,c) \in \big(g_B^Z\big)^{-1}(C),$$

$$\big(h_C^W \circ g_B^Z\big)(b,c) := h_C^W(c); \quad (b,c) \in \big(g_B^Z\big)^{-1}(C).$$

Hence, $\big(h_C^W \odot g_B^Z\big) \odot f_A^X := \left(D, i_A^X \circ e_D^A, \big[\big(h_C^W \circ g_B^Z\big) \circ f_Z^X\big]^W_D\right)$, where

$$D := \big(f_A^Y\big)^{-1}\big[\big(g_B^Z\big)^{-1}(C)\big] := \left\{(a,d) \in A \times \big[\big(g_B^Z\big)^{-1}(C)\big] \mid f_A^Y(a) =_Y \big(i_B^Y \circ e^B_{(g_B^Z)^{-1}(C)}\big)(d)\right\},$$

with $d := (b,c) \in B \times C$ such that $g_B^Z(b) =_Z i_C^Z(c)$. The map $e_D^A \colon D \hookrightarrow A$ is defined by the rule $(a,d) \mapsto a$, for every $(a,d) \in D$, and

$$\big[\big(h_C^W \circ g_B^Z\big) \circ f_A^Y\big](a,d) := \big(h_C^W \circ g_B^Z\big)(d) := h_C^W(c); \quad (a,d) := (a,(b,c)) \in D.$$

Moreover, $h_C^W \odot \big(g_B^Z \odot f_A^Y\big) := \left(E,\ i_A^X \circ e^A_{(f_A^Y)^{-1}(B)} \circ e_E^{(f_A^Y)^{-1}(B)},\ \big[h_C^W \circ \big(g_B^Z \circ f_Z^X\big)\big]^W_E\right)$, where

$$E := \left[\big(g_B^Z \circ f_A^Y\big)^Z_{(f_A^Y)^{-1}(B)}\right]^{-1}(C) := \left\{(u,c) \in \big[\big(f_A^Y\big)^{-1}(B)\big] \times C \mid (g_B^Z \circ f_A^Y)(u) =_Z i_C^Z(c)\right\},$$

$e_E^{(f_A^Y)^{-1}(B)} \colon E \hookrightarrow \big((f_A^Y)\big)^{-1}(B)$ is defined by the rule $(u,c) \mapsto u$, for every $(u,c) \in E$, and

$$\big[h_C^W \circ \big(g_B^Z \circ f_A^Y\big)\big](u,c) := h_C^W(c); \quad (u,c) \in E.$$

Let the operations $\phi \colon D \rightsquigarrow E$, defined by $\phi(a,(b,c)) := ((a,b),c)$, for every $(a,(b,c)) \in D$, and $\theta \colon E \rightsquigarrow D$, defined by $\theta((a,b),c) := (a,(b,c))$, for every $((a,b),c) \in E$. It is straightforward to show that ϕ and θ are well-defined functions, and that the following inner diagrams commute

$$\begin{tikzcd}
D \arrow[rr, bend left, "\phi"] \arrow[dr, bend right, "i_A^X \circ e_D^A"'] \arrow[ddr, bend right=60, "\big(h_C^W \circ g_B^Z\big) \circ f_A^Y"'] & & E \arrow[ll, bend left, "\theta"] \arrow[dl, bend left, "i_A^X \circ e^A_{(f_A^Y)^{-1}(B)} \circ e_E^{(f_A^Y)^{-1}(B)}"] \arrow[ddl, bend left=60, "h_C^W \circ \big(g_B^Z \circ f_A^Y\big)"] \\
& X & \\
& W. &
\end{tikzcd}$$

□

The next proposition is straightforward to show.

Proposition 2.7.3. *Let* $(A, i_A^X, f_A^Y), (B, i_B^X, f_B^X) \in \mathfrak{F}(X, Y)$

$$\begin{tikzcd} A \arrow[r, hook, "i_A^X"] \arrow[dr, "f_A^Y"'] & X & B \arrow[l, hook', "i_B^X"'] \arrow[dl, "f_B^Y"] \\ & Y. & \end{tikzcd}$$

Their left $f_A^Y \cap_l f_B^Y$ *and right intersection* $f_A^Y \cap_r f_B^Y$ *are the partial functions*

$$f_A^Y \cap_l f_B^Y := \Big(A \cap B,\ i_{A\cap B}^X,\ \big(f_A^Y \cap_l f_B^Y\big)_{A\cap B}^Y\Big), \quad \textit{where}$$

$$\big(f_A^Y \cap_l f_B^Y\big)_{A\cap B}^Y(a,b) := f_A^Y(a); \qquad (a,b) \in A \cap B, \qquad \textit{and}$$

$$f_A^Y \cap_r f_B^Y := \Big(A \cap B,\ i_{A\cap B}^X,\ \big(f_A^Y \cap_r f_B^Y\big)_{A\cap B}^Y\Big), \quad \textit{where}$$

$$\big(f_A^Y \cap_r f_B^Y\big)_{A\cap B}^Y(a,b) := f_B^Y(b); \qquad (a,b) \in A \cap B.$$

Their union $f_A^Y \cup f_B^Y$ *is the partial function*

$$f_A^Y \cup f_B^Y := \Big(A \cup B,\ i_{A\cup B}^X,\ \big(f_A^Y \cup f_B^Y\big)_{A\cup B}^Y\Big), \quad \textit{where}$$

$$\big(f_A^Y \cup f_B^Y\big)_{A\cup B}^Y(z) := \begin{cases} f_A^Y(z) & , z \in A \\ f_B^Y(z) & , z \in B. \end{cases}$$

(i) $f_A^Y \cap_l f_B^Y \leq f_A^Y$ *and* $f_A^Y \cap_r f_B^Y \leq f_B^Y$.
(ii) *If* $f_A^Y(a) =_Y f_B^Y(b)$, *for every* $(a,b) \in A \cap B$, *then* $f_A^Y \cap_l f_B^Y =_{\mathfrak{F}(X,Y)} f_A^Y \cap_r f_B^Y$.
(iii) $f_A^Y \leq f_A^Y \cup f_B^Y$ *and* $f_B^Y \leq f_A^Y \cup f_B^Y$.
(iv) $f_A^Y \cup f_B^Y =_{\mathfrak{F}(X,Y)} f_B^Y \cup f_A^Y$.

Definition 2.7.4. *Let the operation of multiplication on* $\mathbb{2}$, *defined by* $0 \cdot 1 := 1 \cdot 0 := 0 \cdot 0 := 0$ *and* $1 \cdot 1 := 1$. *If* $(A, i_A^X, f_A^{\mathbb{2}}), (B, i_B^X, g_B^{\mathbb{2}}) \in \mathfrak{F}(X, \mathbb{2})$, *let*

$$f_A \cdot g_B := \big(A \cap B, i_{A\cap B}^X, (f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}\big),$$

where $(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}} \colon A \cap B \to \mathbb{2}$ *is defined, for every* $(a,b) \in A \cap B$, *by*

$$(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}(a,b) := f_A^{\mathbb{2}}(a) \cdot g_B^{\mathbb{2}}(b).$$

By the equality of the product on $A \cap B$, it is immediate to show that the operation $(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}$ is a function. More generally, operations on Y induce operations on $\mathfrak{F}(X, Y)$. The above example with $Y := \mathbb{2}$ is useful to the next section.

2.8 Complemented subsets

An inequality on a set X induces a positively defined notion of disjointness of subsets of X.

Definition 2.8.1. *Let $(X, =_X, \neq_X)$ be a set, and $(A, i_A^X), (B, i_B^X) \subseteq X$. We say that A and B are disjoint with respect to $\neq_X$, in symbols $A]\![_{\neq_X} B$, if*

$$A \underset{\neq_X}{]\![} B :\Leftrightarrow \forall_{a \in A} \forall_{b \in B}\big(i_A^X(a) \neq_X i_B^X(b)\big).$$

If $\neq_X$ is clear from the context, we only write $A]\![B$.

Clearly, if $A]\![B$, then $A \cap B$ is not inhabited. The positive disjointness of subsets of X induces the notion of a complemented subset of X, and the negative notion of the complement of a set is avoided. We use bold letters to denote a complemented subset of a set.

Definition 2.8.2. *A complemented subset of a set $(X, =_X, \neq_X)$ is a pair $\boldsymbol{A} := (A^1, A^0)$, where $(A^1, i_{A^1}^X)$ and $(A^0, i_{A^0}^X)$ are subsets of X such that $A^1]\![A^0$. We call A^1 the* 1*-component of $\boldsymbol{A}$ and A^0 the* 0*-component of $\boldsymbol{A}$. If $\mathtt{Dom}(\boldsymbol{A}) := A^1 \cup A^0$ is the domain of $\boldsymbol{A}$, the indicator function, or characteristic function, of $\boldsymbol{A}$ is the operation $\chi_{\boldsymbol{A}} : \mathtt{Dom}(\boldsymbol{A}) \rightsquigarrow 2$ defined by*

$$\chi_{\boldsymbol{A}}(x) := \begin{cases} 1 & , x \in A^1 \\ 0 & , x \in A^0. \end{cases}$$

Let $x \in \boldsymbol{A} :\Leftrightarrow x \in A^1$ and $x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0$. If $\boldsymbol{A}, \boldsymbol{B}$ are complemented subsets of X, let

$$\boldsymbol{A} \subseteq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \&\ B^0 \subseteq A^0.$$

Let $\mathcal{P}^{]\![}(X)$ be their totality, equipped with the equality $\boldsymbol{A} =_{\mathcal{P}^{]\![}(X)} \boldsymbol{B} :\Leftrightarrow \boldsymbol{A} \subseteq \boldsymbol{B} \ \&\ \boldsymbol{B} \subseteq \boldsymbol{A}$. Let $\mathtt{PrfEql}_0(\boldsymbol{A}, \boldsymbol{B}) := \mathtt{PrfEql}_0(A^1, B^1) \times \mathtt{PrfEql}_0(A^0, B^0)$. A map $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a pair (f^1, f^0), where $f^1 \colon A^1 \to B^1$ and $f^0 \colon A^0 \to B^0$.

Clearly, $\boldsymbol{A} =_{\mathcal{P}^{]\![}(X)} \boldsymbol{B} \Leftrightarrow A^1 =_{\mathcal{P}(X)} B^1 \ \&\ A^0 =_{\mathcal{P}(X)} B^0$, and $\mathtt{PrfEql}_0(\boldsymbol{A}, \boldsymbol{B})$ is a subsingleton, as the product of subsingletons. Since the membership condition for $\mathcal{P}^{]\![}(X)$ requires quantification over $\mathbb{V}_0$, the totality $\mathcal{P}^{]\![}(X)$ is a class. The operation $\chi_{\boldsymbol{A}}$ is a function, actually, $\chi_{\boldsymbol{A}}$ is a partial function in $\mathfrak{F}(X, 2)$. Let $z, w \in A^1 \cup A^0$ such that $z =_{A^1 \cup A^0} w$ i.e.,

$$\left.\begin{array}{ll} i_{A^1}^X(z) & , z \in A^1 \\ & \\ i_{A^0}^X(z) & , z \in A^0 \end{array}\right\} := i_{A^1 \cup A^0}^X(z) =_X i_{A^1 \cup A^0}^X(w) := \left\{\begin{array}{ll} i_{A^1}^X(w) & , w \in A^1 \\ & \\ i_{A^0}^X(w) & , w \in A^0. \end{array}\right.$$

Let $z \in A^1$. If $w \in A^0$, then $i_{A^1}^X(z) := i_{A^1 \cup A^0}^X(z) =_X i_{A^1 \cup A^0}^X(w) := i_{A^0}^X(w)$ i.e., $(z, w) \in A^1 \cap A^0$, which contradicts the hypothesis $A^1]\![A^0$. Hence $w \in A^1$, and $\chi_{\boldsymbol{A}}(z) = \chi_{\boldsymbol{A}}(w)$. If $z \in A^0$, we proceed similarly.

Definition 2.8.3. *If $(X, =_X)$ is a set, let the inequality on X defined by*

$$x \neq_X^{\mathbb{F}(X,2)} x' :\Leftrightarrow \exists_{f \in \mathbb{F}(X,2)}\big(f(x) =_2 1 \ \&\ f(x') =_2 0\big)$$

If $f \in \mathbb{F}(X, 2)$, the following extensional subsets of X

$$\delta_0^1(f) := \{x \in X \mid f(x) =_2 1\},$$

$$\delta_0^0(f) := \{x \in X \mid f(x) =_2 0\},$$

are called detachable, or free subsets of X. Let also their pair $\boldsymbol{\delta}(f) := \big(\delta_0^1(f), \delta_0^0(f)\big)$.

Clearly, $x \neq_X^{\mathbb{F}(X,2)} x' \Leftrightarrow \exists_{f \in \mathbb{F}(X,2)}\big(f(x) \neq_2 f(x')\big)$, and $\boldsymbol{\delta}(f)$ is a complemented subset of X with respect to the inequality $\neq_X^{\mathbb{F}(X,2)}$. The characteristic function $\chi_{\boldsymbol{\delta}(f)}$ of $\boldsymbol{\delta}(f)$ is definitionally equal to f (recall that $f(x) =_2 1 :\Leftrightarrow f(x) := 1$), and $\delta_0^1(f) \cup \delta_0^0(f) = X$.

Definition 2.8.4. *If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}^{\rrbracket\llbracket}(X)$ and $\boldsymbol{C} \in \mathcal{P}^{\rrbracket\llbracket}(Y)$, let*

$$\boldsymbol{A} \cup \boldsymbol{B} := (A^1 \cup B^1, A^0 \cap B^0),$$

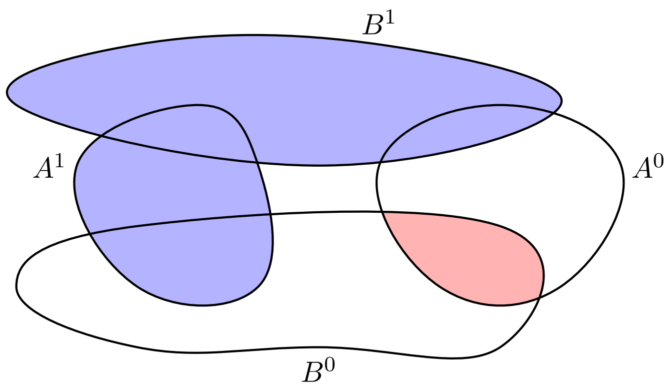
$$\boldsymbol{A} \cap \boldsymbol{B} := (A^1 \cap B^1, A^0 \cup B^0),$$

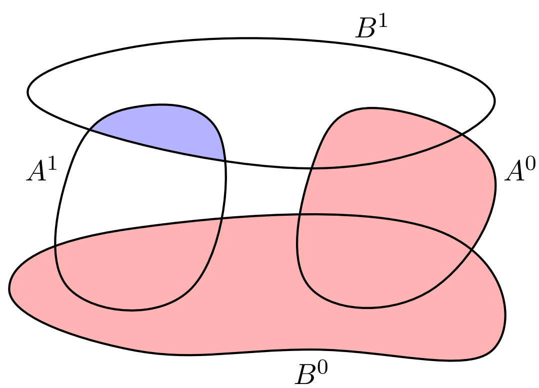
$$-\boldsymbol{A} := (A^0, A^1),$$

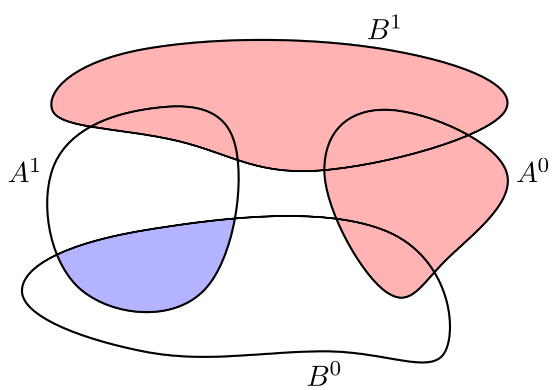
$$\boldsymbol{A} - \boldsymbol{B} := (A^1 \cap B^0, A^0 \cup B^1),$$

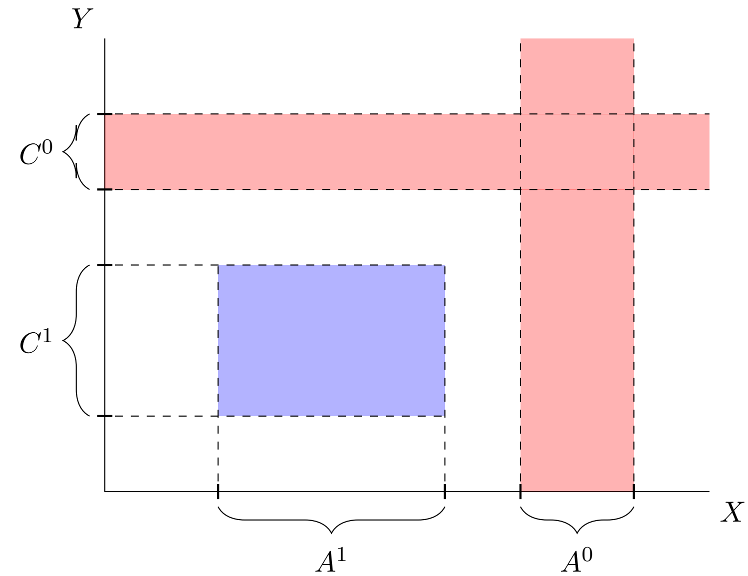
$$\boldsymbol{A} \times \boldsymbol{C} := \big(A^1 \times C^1,\ [A^0 \times Y] \cup [X \times C^0]\big).$$

The following diagrams depict $\boldsymbol{A} \cup \boldsymbol{B}$, $\boldsymbol{A} \cap \boldsymbol{B}$, $\boldsymbol{A} - \boldsymbol{B}$, and $\boldsymbol{A} \times \boldsymbol{C}$, respectively.









Remark 2.8.5. *If* $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}^{\mathbb{II}}(X)$ *and* $\boldsymbol{C} \in \mathcal{P}^{\mathbb{II}}(Y)$, *then* $\boldsymbol{A} \cup \boldsymbol{B}$, $\boldsymbol{A} \cap \boldsymbol{B}$, $-\boldsymbol{A}$, *and* $\boldsymbol{A} - \boldsymbol{B}$ *are in* $\mathcal{P}^{\mathbb{II}}(X)$ *and* $\boldsymbol{A} \times \boldsymbol{C}$ *is in* $\mathcal{P}^{\mathbb{II}}(X \times Y)$.

Proof. We show only the last membership. If $(a_1, b_1) \in A^1 \times B^1$ and $(a_0, b_0) \in A^0 \times B^0$, then $i^X_{A^1}(a_1) \neq_X i^X_{A^0}(a_0)$ and $i^Y_{B^1}(b_1) \neq_Y i^Y_{B^0}(b_0)$. By definition

$$i^{X \times Y}_{A^1 \times B^1}(a_1, b_1) := \big(i^X_{A^1}(a_1), i^Y_{B^1}(b_1)\big).$$

If $(a_0, y) \in A^0 \times Y$, then $(i^X_{A^0} \times \mathrm{id}_Y)(a_0, y) := (i^X_{A^0}(a_0), y)$, and if $(x, b_0) \in X \times B^0$, then $(\mathrm{id}_X \times i^Y_{B^0})(x, b_0) := (x, i^Y_{B^0}(b_0))$. In both cases we get the required inequality. $\square$

Remark 2.8.6. *Let* $\boldsymbol{A}, \boldsymbol{B}$ *and* $\boldsymbol{C}$ *be in* $\mathcal{P}^{\mathbb{II}}(X)$. *The following hold:*
(i) $-(-\boldsymbol{A}) := \boldsymbol{A}$.
(ii) $-(\boldsymbol{A} \cup \boldsymbol{B}) := (-\boldsymbol{A}) \cap (-\boldsymbol{B})$.
(iii) $-(\boldsymbol{A} \cap \boldsymbol{B}) := (-\boldsymbol{A}) \cup (-\boldsymbol{B})$.
(iv) $\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) =_{\mathcal{P}^{\mathbb{II}}(X)} (\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$.
(v) $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) =_{\mathcal{P}^{\mathbb{II}}(X)} (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap \boldsymbol{C})$.
(vi) $\boldsymbol{A} - \boldsymbol{B} := \boldsymbol{A} \cap (-\boldsymbol{B})$.
(vii) $\boldsymbol{A} \subseteq \boldsymbol{B} \Leftrightarrow (\boldsymbol{A} \cap \boldsymbol{B}) =_{\mathcal{P}^{\mathbb{II}}(X)} \boldsymbol{A}$.
(viii) $\boldsymbol{A} \subseteq \boldsymbol{B} \Leftrightarrow -\boldsymbol{B} \subseteq -\boldsymbol{A}$.
(ix) *If* $\boldsymbol{A} \subseteq \boldsymbol{B}$ *and* $\boldsymbol{B} \subseteq \boldsymbol{C}$, *then* $\boldsymbol{A} \subseteq \boldsymbol{C}$.

Proposition 2.8.7. *Let* $\boldsymbol{A} \in \mathcal{P}^{\mathbb{II}}(X)$ *and* $B, C \in \mathcal{P}^{\mathbb{II}}(Y)$.
(i) $\boldsymbol{A} \times (\boldsymbol{B} \cup \boldsymbol{C}) =_{\mathcal{P}^{\mathbb{II}}(X \times Y)} (\boldsymbol{A} \times \boldsymbol{B}) \cup (\boldsymbol{A} \times \boldsymbol{C})$.
(ii) $\boldsymbol{A} \times (\boldsymbol{B} \cap \boldsymbol{C}) =_{\mathcal{P}^{\mathbb{II}}(X \times Y)} (\boldsymbol{A} \times \boldsymbol{B}) \cap (\boldsymbol{A} \times \boldsymbol{C})$.

Proof. We prove only (i). We have that

$$\begin{aligned}\boldsymbol{A} \times (\boldsymbol{B} \cup \boldsymbol{C}) &:= (A^1, A^0) \times (B^1 \cup C^1, B^0 \cap C^0) \\ &:= \big(A^1 \times (B^1 \cup C^1), (A^0 \times Y) \cup [X \times (B^0 \cap C^0)]\big) \\ &=_{\mathcal{P}^{\rrbracket\llbracket}(X \times Y)} \big((A^1 \times B^1) \cup (A^1 \times C^1), [(A^0 \times Y) \cup (X \times B^0)] \cap \\ &\qquad \cap [(A^0 \times Y) \cup (X \times C^0)]\big) \\ &:= (\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{A} \times \boldsymbol{C}). \end{aligned}$$ □

Proposition 2.8.8. *Let the sets $(X, =_X, \neq_X^f)$ and $(Y, =_Y, \neq_Y)$, where $f : X \to Y$ (see Remark 2.3.3). Let also $\boldsymbol{A} := (A^1, A^0)$ and $\boldsymbol{B} := (B^1, B^0)$ in $\mathcal{P}^{\rrbracket\llbracket}(Y)$.*
(i) $f^{-1}(\boldsymbol{A}) := \big(f^{-1}(A^1), f^{-1}(A^0)\big) \in \mathcal{P}^{\rrbracket\llbracket}(X)$.
(ii) $f^{-1}(\boldsymbol{A} \cup \boldsymbol{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\boldsymbol{A}) \cup f^{-1}(\boldsymbol{B})$.
(iii) $f^{-1}(\boldsymbol{A} \cap \boldsymbol{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\boldsymbol{A}) \cap f^{-1}(\boldsymbol{B})$.
(iv) $f^{-1}(-\boldsymbol{A}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} -f^{-1}(\boldsymbol{A})$.
(v) $f^{-1}(\boldsymbol{A} - \boldsymbol{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\boldsymbol{A}) - f^{-1}(\boldsymbol{B})$.

Proof. (i) By Definition 2.6.9 we have that

$$f^{-1}(A^1) := \{(x, a_1) \in X \times A^1 \mid f(x) =_Y i_{A^1}^X(a_1)\}, \quad i_{f^{-1}(A^1)}^X(x, a_1) := x,$$

$$f^{-1}(A^0) := \big(\{(x, a_0) \in X \times A^0 \mid f(x) =_Y i_{A^0}^X(a_0)\}, \quad i_{f^{-1}(A^0)}^X(x, a_0) := x.$$

Let $(x, a_1) \in f^{-1}(A^1)$ and $(z, a_0) \in f^{-1}(A^0)$. By the extensionality of $\neq_Y$ we have that

$$i_{f^{-1}(A^1)}^X(x, a_1) \neq_X^f i_{f^{-1}(A^0)}^X(z, a_0) :\Leftrightarrow x \neq_X^f z :\Leftrightarrow f(x) \neq_Y f(z) \Leftrightarrow i_{A^1}^X(a_1) \neq_Y i_{A^0}^X(a_0),$$

and the last inequality holds by the hypothesis $\boldsymbol{A} \in \mathcal{P}^{\rrbracket\llbracket}(Y)$. Next we show only (ii):

$$\begin{aligned} f^{-1}(\boldsymbol{A} \cup \boldsymbol{B}) &:= f^{-1}\big(A^1 \cup B^1, A^0 \cap B^0\big) \\ &:= \big(f^{-1}(A^1 \cup B^1), f^{-1}(A^0 \cap B^0)\big) \\ &= \big(f^{-1}(A^1) \cup f^{-1}(B^1), f^{-1}(A^0) \cap f^{-1}(B^0)\big) \\ &:= f^{-1}(\boldsymbol{A}) \cup f^{-1}(\boldsymbol{B}). \end{aligned}$$ □

Alternatively, one can define the following operations between complemented subsets.

Definition 2.8.9. *If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}^{\rrbracket\llbracket}(X)$ and $\boldsymbol{C} \in \mathcal{P}^{\rrbracket\llbracket}(Y)$, let*

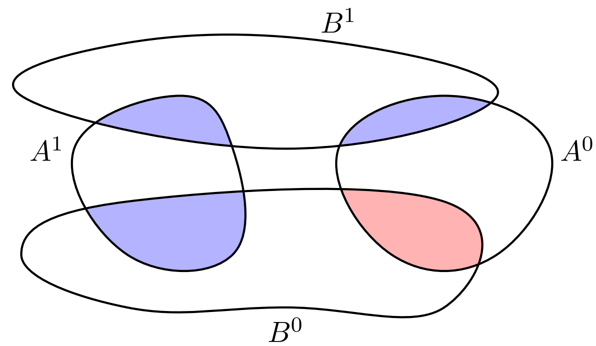
$$\boldsymbol{A} \vee \boldsymbol{B} := \big([A^1 \cap B^1] \cup [A^1 \cap B^0] \cup [A^0 \cap B^1],\ A^0 \cap B^0\big),$$

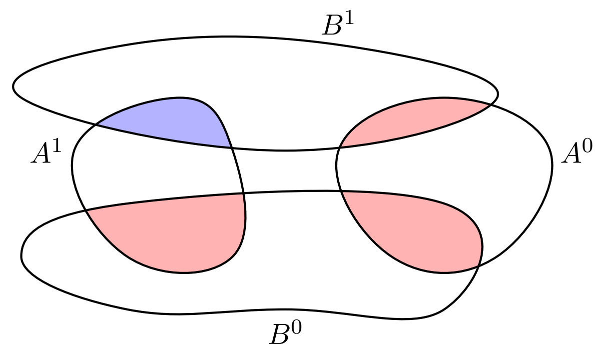
$$\boldsymbol{A} \wedge \boldsymbol{B} := \big(A^1 \cap B^1,\ [A^1 \cap B^0] \cup [A^0 \cap B^1] \cup [A^0 \cap B^0]\big),$$

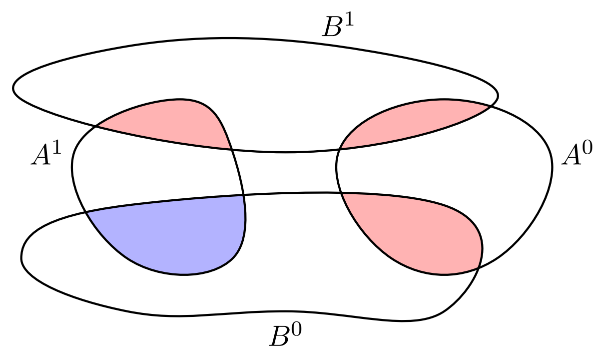
$$\boldsymbol{A} \ominus \boldsymbol{B} := \boldsymbol{A} \wedge (-\boldsymbol{B}),$$

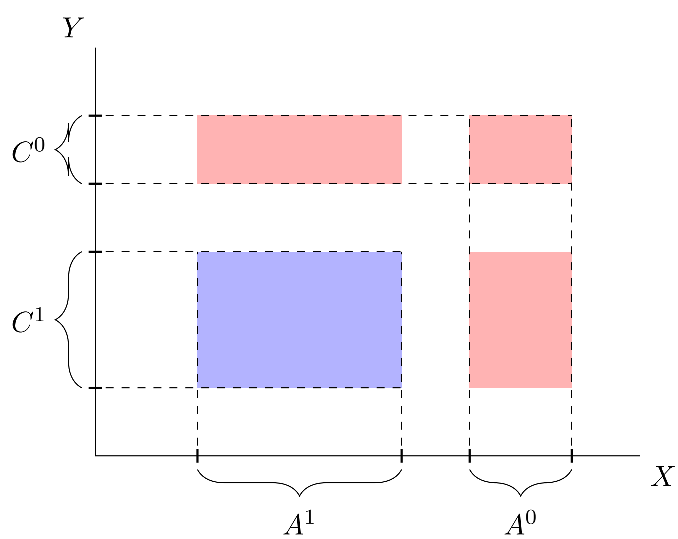
$$\boldsymbol{A} \otimes \boldsymbol{C} := \big(A^1 \times C^1,\ [A^1 \times C^0] \cup [A^0 \times C^1] \cup [A^0 \times C^0]\big),$$

The following diagrams depict $\boldsymbol{A} \vee \boldsymbol{B}, \boldsymbol{A} \wedge \boldsymbol{B}$, $\boldsymbol{A} \ominus \boldsymbol{B}$, and $\boldsymbol{A} \otimes \boldsymbol{C}$, respectively.


B^1
A^1
A^0
B^0


B^1
A^1
A^0
B^0


B^1
A^1
A^0
B^0


Y
C^0
C^1
X
A^1
A^0

With the previous definitions the corresponding characteristic functions are expressed through the characteristic functions of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Remark 2.8.10. *If $\boldsymbol{A}, \boldsymbol{B}$ are complemented subsets of X, then $\boldsymbol{A} \vee \boldsymbol{B}, \boldsymbol{A} \wedge \boldsymbol{B}, \boldsymbol{A} - \boldsymbol{B}$ and $-\boldsymbol{A}$ are complemented subsets of X with characteristic functions*

$$\chi_{\boldsymbol{A} \vee \boldsymbol{B}} =_{\mathfrak{F}(X,\mathbb{2})} \chi_{\boldsymbol{A}} \vee \chi_{\boldsymbol{B}}, \quad \chi_{\boldsymbol{A} \wedge \boldsymbol{B}} =_{\mathfrak{F}(X,\mathbb{2})} \chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}}, \quad \chi_{\boldsymbol{A} - \boldsymbol{B}} =_{\mathfrak{F}(X,\mathbb{2})} \chi_{\boldsymbol{A}}(1 - \chi_{\boldsymbol{B}}),$$

$$\chi_{\boldsymbol{A} \otimes \boldsymbol{B}}(x, y) =_{\mathfrak{F}(X \times X,\mathbb{2})} \chi_{\boldsymbol{A}}(x) \cdot \chi_{\boldsymbol{B}}(y), \quad \chi_{-\boldsymbol{A}} =_{\mathfrak{F}(X,\mathbb{2})} 1 - \chi_{\boldsymbol{A}}.$$

Proof. We show only the equality $\chi_{\boldsymbol{A} \wedge \boldsymbol{B}} =_{\mathfrak{F}(X,\mathbb{2})} \chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}}$. By Definition 2.7.4 the multiplication of the partial maps $\chi_{\boldsymbol{A}} \colon \mathtt{Dom}(\boldsymbol{A}) \to \mathbb{2}$ and $\chi_{\boldsymbol{B}} \colon \mathtt{Dom}(\boldsymbol{B}) \to \mathbb{2}$ is the partial function

$$\chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}} := \big(\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B}), i^X_{\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})}, (\chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}})^{\mathbb{2}}_{\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})}\big),$$

$$(\chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}})^{\mathbb{2}}_{\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})}(u, w) := \chi_{\boldsymbol{A}}(u) \cdot \chi_{\boldsymbol{B}}(w),$$

for every $(u, w) \in \mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})$. The partial function $\chi_{\boldsymbol{A} \wedge \boldsymbol{B}}$ is the triplet

$$\chi_{\boldsymbol{A} \wedge \boldsymbol{B}} := \big(\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B}), i^X_{\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B})}, (\chi_{\boldsymbol{A} \wedge \boldsymbol{B}})^{\mathbb{2}}_{\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B})}\big).$$

Since $\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})$, and if $(f, g) \colon \mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})$, it is straightforward to show that also the following outer diagram commutes

$$\begin{tikzcd}
\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B}) \arrow[rr, bend left, "f"] \arrow[dr, "i^X_{\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B})}"] \arrow[ddr, bend right, "(\chi_{\boldsymbol{A} \wedge \boldsymbol{B}})^{\mathbb{2}}_{\mathtt{Dom}(\boldsymbol{A} \wedge \boldsymbol{B})}"'] & & \mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B}) \arrow[ll, bend left, "g"] \arrow[dl, "i^X_{\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})}"'] \arrow[ddl, bend left, "(\chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}})^{\mathbb{2}}_{\mathtt{Dom}(\boldsymbol{A}) \cap \mathtt{Dom}(\boldsymbol{B})}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

and hence the two partial functions are equal in $\mathfrak{F}(X, \mathbb{2})$. $\square$

2.9 Notes

Note 2.9.1. In [55] Greenleaf introduced predicates on objects through the totality Ω of propositions and then he defined $\mathcal{P}(X)$ as $\mathbb{F}(X, \Omega)$. A similar treatment of the powerset $\mathcal{P}(X)$ is found in [113]. For us a predicate on a set X is a bounded formula $P(x)$ with x as a free variable. In order to define new objects from X through P we ask P to be extensional.

Note 2.9.2. In [27], pp. 114-5, Cantor described a set as follows:

> A manifold (a sum, a set) of elements belonging to some conceptual sphere is called well-defined if, on the basis of its definition and in accordance with the logical principle of the excluded third, it must be regarded as internally determined, both whether any object of that conceptual sphere belongs as an element to the mentioned set, and also whether two objects belonging to the set, in spite of formal differences in the mode of givenness, are equal to each other or not.

Bishop's intuitive notion of set is similar to Cantor's, except that he does not invoke the principle of the excluded middle (PEM). As it was pointed to me by W. Sieg, Dedekind's primitive notions in [44] were "systems" and "transformations of systems". Notice that here we study defined totalities that are not defined inductively. The inductively defined sets are expected to be studied in a future work within an extension BST^* of BST.

Note 2.9.3. Although $\mathbb{N}$ is the only primitive set considered in BST, one could, in principle, add more primitive sets. E.g., a primitive set of Booleans, of integers, and, more interestingly, a primitive continuous interval, or a primitive real line (see [23] for an axiomatic treatment of the set $\mathbb{R}$ of reals within BISH).

Note 2.9.4. In Martin-Löf type theory the definitional, or judgemental equality $a := b$, where a, b are terms of some type A, is never used in a formula. We permit the use of the definitional equality $:=$ for membership conditions only. In the membership condition for the product we use the primitive notion of a pair. The membership condition for an extensional subset X_P of X implies that an object x "has not unique typing", as it can be an element of more than one sets.

Note 2.9.5. The positively defined notion of discrete set used here comes from [76], p. 9. There it is also mentioned that a set without a specified inequality i.e., a pair $(X, =_X)$, is discrete, if $\forall_{x,y \in X}\big(x =_X y \ \vee \ \neg(x =_X y)\big)$. In [84] it is mentioned that the above discreteness of $\mathbb{F}(\mathbb{N}, \mathbb{N})$ implies the non-constructive principle "weak LPO"

$$\forall_{f \in \mathbb{F}(\mathbb{N},\mathbb{N})}\bigg(\forall_{n \in \mathbb{N}}\big(f(n) =_{\mathbb{N}} 0\big) \ \vee \ \neg\forall_{n \in \mathbb{N}}\big(f(n) =_{\mathbb{N}} 0\big)\bigg).$$

Because of a result of Bauer and Swan in [4], we cannot show in BISH the existence of an uncountable separable metric space, hence, using the discrete metric, the existence of an uncountable discrete set. Note that in [9], p. 66, a set S is called discrete, if the set $D := \{(s, t) \in S \times S \mid s =_S t\}$ is a free, or a detachable subset of $S \times S$. In Definition 2.2.4 we use the symbol $D(S)$ for D and we call it the diagonal of S. We employ here the diagonal of a set in the fundamental definition of a set-indexed family of sets (Definition 3.1.1).

Note 2.9.6. In [9] and [19], the negation $\neg\phi$ of a formula ϕ is not mentioned explicitly. E.g., the exact writing of condition (Ap_1) in Definition 2.2.5 is "if $x =_X y$ and $x \neq_X y$, then $0 =_{\mathbb{N}} 1$". Similarly, the condition of tightness in Definition 2.2.5 is written as follows: "if $x \neq_X y$ entails $0 = 1$, then $x =_X y$". hence, if $\neq_X$ is tight, the implication $x \neq_X y \Rightarrow 0 =_{\mathbb{N}} 1$ is logically equivalent to the (positively defined, if X is a defined totality) equality $x =_X y$. Within intuitionistic logic one defines $\neg\phi := \phi \Rightarrow \bot$.

Note 2.9.7. The definitions of (-2)-sets and (-1)-sets are proof-irrelevant translations of the corresponding notions in HoTT, which were introduced by Voevodsky (see [124]). The definition of a 0-*set* requires to determine a set $\mathtt{PrfEql}_0^X(x, y)$ of witnesses of the equality $x =_X y$. This is done in a universal way in MLTT, while in BST in a "local" way, and by definition (see Definition 5.6.3).

Note 2.9.8. In the literature of constructive mathematics (see e.g., [7], pp. 34–35) the term *preset* is used for a totality. Also, the term *operation* is used for a non-dependent assignment routine from a totality X to a totality Y (see [7], p. 44), while we use it only for a non-dependent assignment routine from a set X to a set Y.

Note 2.9.9. The notion of uniqueness associated to the definition of a function is *local*, in the following sense: if $f\colon X \to Y$, it is immediate to show that $\forall_{x\in X}\exists!_{y\in Y}\big(f(x) =_Y y\big)$. The converse is the local version of Myhill's axiom of non-choice (LANC). Let $P(x, y)$ be an extensional property on $X \times Y$ i.e., $\forall_{x,x',y,y'\in X}\big([x =_X x' \ \&\ y =_Y y' \ \&\ P(x, y)] \Rightarrow P(x', y')\big)$. The principle (LANC) is the formula

$$\forall_{x\in X}\exists!_{y\in Y}P(x,y) \Rightarrow \exists_{f\in\mathbb{F}(X,Y)}\forall_{x\in X}\big(P(x, f(x))\big).$$

Notice that LANC provides the existence of a function for which we only know how its outputs behave with respect to the equality of Y, and it gives no information on how f behaves definitionally. If we define $Q_x(y) := P(x, y)$, then if we suppose $Q_x(f(x))$ and $Q_x(g(x))$, for some $f, g \in \mathbb{F}(X, Y)$, we get $f(x) =_Y y =_Y g(x)$, and then (LANC) implies

$$\forall_{x\in X}\exists!_{y\in Y}P(x,y) \Rightarrow \exists!_{f\in\mathbb{F}(X,Y)}\forall_{x\in X}\big(P(x, f(x))\big).$$

We can use (LANC) to view an arbitrary subset (A, i_A^X) of X as an extensional subset of X. If $(A, i_A^X) \in \mathcal{P}(X)$, then the property P_A on X defined by $P_A(x) := \exists_{a\in A}\big(i_A^X(a) =_X x\big)$, is extensional, and $(i_A^X, j_A^X) : X_{P_A} =_{\mathcal{P}(X)} (A, i_A^X)$, for some function $j_A^X\colon X_{P_A} \to A$. To show this, let $x, y \in X$ such that $P_A(x)$ and $x =_X y$. By transitivity of $=_X$, if $i_A^X(a) =_X x$, then $i_A^X(a) =_X y$. If $x \in X$ and $a, b \in A$ such that $i_A^X(a) =_X x =_X i_A^X(b)$, then $a =_A b$ i.e., $\forall_{x\in X_{P_A}}\exists!_{a\in A}\big(i_A^X(a) =_X x\big)$, and since the property $Q(x, a) :\Leftrightarrow i_A^X(a) =_X x$ is extensional on $X_{P_A} \times A$, by (LANC) there is a (unique) function $j_A^X : X_{P_A} \to A$, such that for every $x \in X_P$ we have that $i_A^X(j_A^X(x)) =_X x$, and the required diagram commutes. The principle (LANC), which is also considered in [5], is included in Myhill's system CST (see [80]) as a principle of generating functions. This is in contrast to Bishop's algorithmic approach to the concept of function.

Note 2.9.10. In [19], p. 67, a function $f : A \to B$ is defined as a finite routine which, applied to *any* element of A, produces an element $b \equiv f(a)$ of B, such that $f(a) =_B f(a')$, whenever $a =_A a'$. In [19], p. 15, we read that f "affords an explicit, finite mechanical reduction of the procedure for constructing $f(a)$ to the procedure for constructing a". The pattern of defining a function $f\colon X \to Y$ by first defining an operation $f\colon X \rightsquigarrow Y$, and then proving that f is a function, is implicit in the more elementary parts of [9] and [19], and more explicit in the later parts of the books. E.g., in [19], p. 199, an inhabited subset U of $\mathbb{C}$ *has the maximal extent property*, if there is an operation μ from U to $\mathbb{R}^+$ satisfying certain properties. One can show *afterwords* that U is open and μ is a function on U. This property is used in Bishop's proof of the Riemann mapping theorem (see [19], pp. 209–210).

Note 2.9.11. Regarding the set-character of $\mathbb{F}(X, Y)$, Bishop, in [19], p. 67, writes:

> When X is not countable, the set $\mathbb{F}(X, Y)$ seems to have little practical interest, because to get a hold on its structure is too hard. For instance, it has been asserted by Brouwer that all functions in $\mathbb{F}(\mathbb{R}, \mathbb{R})$ are continuous, but no acceptable proof of this assertion is known.

Similar problems occur though, in function spaces where the domain of the functions is a countable set. E.g., we cannot accept constructively (i.e., in the sense of Bishop) that the Cantor space $\mathbb{F}(\mathbb{N}, 2)$ satisfies Markov's principle, but no one that we know of has doubted the set-character of $\mathbb{F}(\mathbb{N}, 2)$. The possibility of doubting the set-character of the Baire space $\mathbb{F}(\mathbb{N}, \mathbb{N})$ is discussed by Beeson in [7], p. 46.

Note 2.9.12. In intensional Martin-Löf Type Theory the type

$$\left(\prod_{x\colon X} f(x) = g(x)\right) \to f = g$$

is not provable (inhabited), and its inhabitance is known as the *axiom of function extensionality* (FunExt). In BST this axiom is part of the canonical definition of the function space $\mathbb{F}(X, Y)$. Because of this, many results in MLTT + FunExt are translatable in BST (see Chapter 5).

Note 2.9.13. The totality $\mathbb{V}_0$ is not mentioned by Bishop, although it is necessary, if we want to formulate the fundamental notion of a set-indexed family of sets. The defined equality on the universe $\mathbb{V}_0$ expresses that $\mathbb{V}_0$ is *univalent*, as isomorphic sets are equal in $\mathbb{V}_0$. In univalent type theory, which is MLTT extended with Voevodsky's *axiom of univalence* $\mathtt{UA}$ (see [124]), the existence of a pair of quasi-inverses between types A and B implies that they are equivalent in Voevodsky's sense, and by the univalence axiom, also propositionally equal. The axiom $\mathtt{UA}$ is partially translated in BST as the canonical definition of $\mathbb{V}_0$. Because of this, results in MLTT + $\mathtt{UA}$ that do not raise the level of the universe are translatable in BST. For example, Proposition 5.5.1 is lemma 4.9.2 in book HoTT [124], where $\mathtt{UA}$ is used in its proof: if $e\colon X \simeq Y$, then $Z \to X \simeq Z \to Y$, and by $\mathtt{UA}$ we get $e = \mathtt{idtoEqv}(p)$, for some $p\colon X =_{\mathcal{U}} Y$. Notice that in the formulation of this lemma the universe-level is not raised.

Note 2.9.14. The notion of a dependent operation is explicitly mentioned by Bishop in [9], p. 65, and repeated in [19], p. 70, in the definition of the intersection of a family of subsets of a set indexed by some set T:

> an element u of $\bigcap_{t\in T}\lambda(t)$ is a finite routine which associates an element x_t of $\lambda(t)$ with each element t of T, such that $i_t(x_t) = i_{t'}(x_{t'})$ whenever $t, t' \in T$.

This definition corresponds to Definition 4.3.1 in this Thesis.

Note 2.9.15. Bishop's definition of a subset of a set is related to the notion of a subobject in Category Theory (see [3], p. 89, and [54], p. 75). In practice the subsets of a set X are defined through an extensional property on X. In [20], p. 7, this approach to the notion of a subset is considered as its *definition*. Note that there the implication $x =_X y \Rightarrow (P(y) \Rightarrow P(x))$ is also included in the definition of an extensional property, something which follows though, from the symmetry of $=_X$. Such a form of separation axiom is used implicitly in [9] and in [19]. Myhill used in his system CST the axiom of bounded separation to implement the notion of an extensional subset of X. This axiom is also included in Aczel's system CZF (see [1], p. 26).

Note 2.9.16. One could have defined the equality $=_{A\cup B}$ without relying on the non-dependent assignment routine $i^X_{A\cup B}$. If we define first

$$z =_{A\cup B} w :\Leftrightarrow \begin{cases} i^X_A(z) =_X i_A(w) & , z, w \in A \\[1ex] i^X_A(z) =_X i_B(w) & , z \in A \ \&\ w \in B \\[1ex] i^X_B(z) =_X i_B(w) & , z, w \in B \\[1ex] i^X_B(z) =_X i_A(w) & , z \in B \ \&\ w \in A, \end{cases}$$

we can define afterwords the *operation* $i^X_{A\cup B} : A \cup B \to X$ as in Definition 2.6.5. In this way the non-dependent assignment routine $i^X_{A\cup B}$ is defined on a set, and it is an operation. Bishop avoids this definition, probably because this pattern cannot be extended to the definition of a union of a family of subsets (see Definition 4.2.1). In that case, we cannot write down the corresponding case distinction for $z =_{A\cup B} w$. Moreover, the proof of $\big(A \cup B, i^X_{A\cup B}\big) \subseteq X$ is immediate, if one uses Definition 2.6.5.

Note 2.9.17. The definition of the empty subset $\emptyset_X$ of a set X, given in [9], p. 65, can be formulated as follows. Let X be a set and $x_0 \in X$. The totality $\emptyset_X$ is defined by $z \in \emptyset_X :\Leftrightarrow x_0 \in X \ \&\ 0 =_{\mathbb{N}} 1$. Let $i^X_{\emptyset} \colon \emptyset_X \rightsquigarrow X$ be the non-dependent assignment routine, defined by $i(z) := x_0$, for every $z \in \emptyset_X$, and let $z =_{\emptyset_X} w :\Leftrightarrow i(z) =_X i(w) :\Leftrightarrow x_0 =_X x_0$. The pair $(\emptyset_X, i^X_{\emptyset})$ is the *empty subset* of X. One can show that $=_{\emptyset_X}$ is an equality on $\emptyset_X$, and hence $\emptyset_X$ can be considered to be a set. The assignment routine $i^X_{\emptyset}$ is an embedding of $\emptyset_X$ into X, and hence $(\emptyset_X, i^X_{\emptyset})$ is a subset of X. As Bishop himself writes in [9], p. 65, "the definition of $\emptyset$ is negativistic, and we prefer to mention the void set as seldom as possible". In [19], p. 69, Bishop and Bridges define two subsets A, B of X to be disjoint, when $A \cap B$ "is the void subset of X". Clearly, this "is" cannot be $A \cap B := \emptyset_X$. If we interpret it as $A \cap B =_{\mathcal{P}(X)} \emptyset_X$, we need the existence of certain functions from $\emptyset_X$ to $A \cap B$ and from $A \cap B$ to $\emptyset_X$. The latter approach is followed in MLTT for the empty type. Following Bishop, we refrain from elaborating this negatively defined notion.

Note 2.9.18. If $(A, i^X_A) \subseteq A$, $(B, i^Y_B) \subseteq Y$, and $f \colon X \to Y$, the *extensional image* $f[A]$ of A under f is defined through the extensional property $P(y) := \exists_{a\in A}\big(f(i_A(a)) =_Y y\big)$. Similarly, the *extensional pre-image* $f^{-1}[B]$ of B under f is defined through the extensional property $Q(x) := \exists_{b\in B}\big(f(x) =_Y i_B(b)\big)$. The subset $f(A)$ of Y contains exactly the outputs $f(i^X_A(a))$ of f, for every $a \in A$, while the subset $f[A]$ of Y contains all the elements of Y that are $=_Y$-equal to some output $f(i_A(a))$ of f, for every $a \in A$. It is useful to keep the "distinction" between the subsets $f(A)$, $f[A]$, and $f^{-1}(B)$, $f^{-1}[B]$. We need the equality in $\mathcal{P}(X)$ of a subset of X to its extensional version (see Note 2.9.9), hence the principle LANC, to get $f(A) =_{\mathcal{P}(Y)} f[A]$ and $f^{-1}(B) =_{\mathcal{P}(X)} f^{-1}[B]$.

Note 2.9.19. There are instances in Bishop's work indicating that the powerset of a set is treated as a set. In [9], p. 68, and in [19], p. 74, the following "function" is defined

$$j : \mathcal{P}^{\rrbracket\llbracket}(X) \to \mathcal{P}(X), \qquad (A^1, A^0) \mapsto A^1.$$

This is in complete contrast to our interpretation of a function as an operation between sets. Of course, such a rule is an exception in [9] and [19]. In the definition of an integration space, see [19], p. 216, the "set" $\mathfrak{F}(X, Y)$ of all strongly extensional partial functions from X to Y requires quantification over $\mathbb{V}_0$. Such a quantification is also implicit in the definition of a measure space given in [19], p. 282, and in the definition of a complete measure space in [19], p. 289. These definitions appeared first in [18], p. 47, and p. 55, respectively. The powerset is repeatedly used as a set in [20] and [76]. It is not known if the treatment of the powerset as a set implies some constructively unacceptable principle.

Note 2.9.20. There are instances in Bishop's work indicating that the powerset of a set is *not* treated as a set. See e.g., the definition of a set-indexed family of sets in [19], p. 78 (our Definition 3.1.1). Similarly, in the definition of a family of subsets of a set A indexed by some set T (see [19], p. 69), the notion of a finite routine that assigns a subset of A to an element of T is used, and not the notion of a function from T to $\mathcal{P}(A)$. In the definition of a measure space in [9], p. 183, a subfamily of a given family of complemented sets is considered in order to avoid quantification over the class of all complemented subsets in the formulations of the definitional clauses of a measure space (see Note 7.6.6). The powerset axiom is also avoided in Myhill's formalization [80] of BISH and in Aczel's subsequent system CZF of constructive set theory (see [1]). Although, as we said, it is not known if the use of the powerset as a set implies some constructively unacceptable principle, it is not accepted in any *predicative* development of constructive mathematics.

Note 2.9.21. The notion of a partial function was introduced by Bishop and Cheng in [18], p. 1, and this definition, together with the introduced term "partial function", was also included in Chapter 3 of [19], p. 71. The totality of partial functions $\mathfrak{F}(X)$ from a set X to $\mathbb{R}$ is crucial to the definition of an integration space in the new measure theory developed in [18], and seriously extended in [19]. Only the basic algebraic operations on $\mathfrak{F}(X)$ were defined in [19], p. 71. The composition of partial functions is mentioned in [39], pp. 66–67. A notion of a partial dependent operation can be defined as follows. If A, I are sets, a partial dependent operation is a triplet $(A, i_A^I, \Phi_A^{\lambda_0})$, where $(A, i_A) \subseteq I$, $\lambda_0 \colon A \rightsquigarrow \mathbb{V}_0$, and $\Phi_A^{\lambda_0} \colon \bigcurlywedge_{a \in A} \lambda_0(a)$. If $\lambda_0(a) := Y$, for every $a \in A$, then the corresponding partial dependent operation is reduced to a partial function in $\mathfrak{F}(I, Y)$.

Note 2.9.22. In the study of various subsets of a set X we avoided to define the complement of a subset, since this requires a negative definition. Recall that the negatively defined notion of empty subset of a set is not really used. In [9] Bishop introduced a positive notion of the complement of a subset of a set X, the notion of a complemented subset of X. For its definition we need a notion of a fixed inequality on X, which is compatible with the given equality of X. In this way we can express the disjointness of two subsets A, B of a set X in a positive way. Usually, A, B are called *disjoint*, if $A \cap B$ is not inhabited. It is computationally more informative though, if a positive way is found to express disjointness of subsets. In [25] a positive notion of *apartness* is used as a foundation of constructive topology.

Note 2.9.23. The definitions of $\boldsymbol{A} \cap \boldsymbol{B}$, $\boldsymbol{A} \cup \boldsymbol{B}$ and $\boldsymbol{A} - \boldsymbol{B}$ appear in [9], p. 66, where $\boldsymbol{A} \cup \boldsymbol{B}$ and $\boldsymbol{A} \cap \boldsymbol{B}$ are special cases of the complemented subsets $\bigcup_{i \in I} \boldsymbol{\lambda}_0(i)$ and $\bigcap_{i \in I} \boldsymbol{\lambda}_0(i)$, respectively (see Proposition 4.9.2). There the inequality on X is induced by an inhabited set of functions from X to $\mathbb{R}$. The definition of $\boldsymbol{A} \times \boldsymbol{C}$ appears in [9], p. 206, in the section of the product measures. One can motivate these definitions applying a "classical" thinking. If $x \in X$, recall the definitions

$$x \in \boldsymbol{A} :\Leftrightarrow x \in A^1 \quad \& \quad x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0.$$

Interpreting the connectives in a classical way, we get

$$x \in \boldsymbol{A} \cup \boldsymbol{B} \Leftrightarrow x \in \boldsymbol{A} \ \vee \ x \in \boldsymbol{B} :\Leftrightarrow x \in A^1 \ \vee \ x \in B^1 :\Leftrightarrow x \in A^1 \cup B^1,$$

$$x \notin \boldsymbol{A} \cup \boldsymbol{B} \Leftrightarrow x \notin \boldsymbol{A} \ \& \ x \notin \boldsymbol{B} :\Leftrightarrow x \in A^0 \ \& \ x \in B^1 :\Leftrightarrow x \in A^1 \cap B^1,$$

$$x \in \boldsymbol{A} \cap \boldsymbol{B} \Leftrightarrow x \in \boldsymbol{A} \ \& \ x \in \boldsymbol{B} :\Leftrightarrow x \in A^1 \ \& \ x \in B^1 :\Leftrightarrow x \in A^1 \cap B^1,$$

$$x \notin \boldsymbol{A} \cap \boldsymbol{B} \Leftrightarrow x \notin \boldsymbol{A} \ \vee \ x \notin \boldsymbol{B} :\Leftrightarrow x \in A^0 \ \vee \ x \in B^1 :\Leftrightarrow x \in A^1 \cup B^1,$$

$$x \in -\boldsymbol{A} \Leftrightarrow x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0 \quad \& \quad x \notin -\boldsymbol{A} \Leftrightarrow x \in \boldsymbol{A} :\Leftrightarrow x \in A^1,$$

$$(x, y) \in \boldsymbol{A} \times \boldsymbol{C} \Leftrightarrow x \in \boldsymbol{A} \ \& \ y \in \boldsymbol{C} :\Leftrightarrow x \in A^1 \ \& \ y \in B^1 :\Leftrightarrow (x, y) \in A^1 \times B^1,$$

$$(x, y) \notin \boldsymbol{A} \times \boldsymbol{C} \Leftrightarrow x \notin \boldsymbol{A} \ \vee \ y \notin \boldsymbol{C} :\Leftrightarrow x \in A^0 \ \vee \ y \in B^0 :\Leftrightarrow (x, y) \in (A^0 \times Y) \cup (X \times B^0).$$

Note 2.9.24. In [18], pp. 16–17, and in [19], p. 73, the operations between the complemented subsets of a set X follow Definition 2.8.9 in order to employ the good behaviour of the corresponding characteristic functions in the new measure theory. In the measure theory of [9], where the characteristic functions of complemented subsets are not crucial, the operations between complemented subsets are defined according to Definition 2.8.4. Bishop and Cheng use the notation $\boldsymbol{A} \times \boldsymbol{B}$ instead of $\boldsymbol{A} \otimes \boldsymbol{B}$. As it is evident from the previous figures, the 1- and 0-components of the complemented subsets in the Bishop-Cheng definition are subsets of the corresponding 1- and 0-components of the complemented subsets in the Bishop definition from [9]. Actually, the definitions of the operations of complemented subsets in [9] associate to the 1-component of the complemented subset a maximal complement. The two sets of operations though, share the same algebraic and set-theoretic properties. They only behave differently with respect to their characteristic functions. Based on the work [113] of Shulman, we can motivate the second set of operations in a way similar to the motivation provided for the first set of operations in Note 2.9.23. Keeping the definitions of $x \in \boldsymbol{A}$ and $x \notin \boldsymbol{B}$, we can apply a "linear" interpretation of the connectives $\vee$ and &. As it is mentioned in [113], p. 2, the multiplicative version P `par` Q of $P \vee Q$ in linear logic represents the pattern "if not P, then Q; and if not Q, then P". Let

$$x \in \boldsymbol{A} \vee \boldsymbol{B} :\Leftrightarrow [x \notin \boldsymbol{A} \Rightarrow x \in \boldsymbol{B}] \ \& \ [x \notin \boldsymbol{B} \Rightarrow x \in \boldsymbol{A}].$$

With the use of Ex falsum quodlibet the implication $x \notin \boldsymbol{A} \Rightarrow x \in \boldsymbol{B}$ holds if $x \in \boldsymbol{A} :\Leftrightarrow x \in A^1$, or if $x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0$ and $x \in \boldsymbol{B} :\Leftrightarrow x \in B^1$ i.e., if $x \in A^0 \cap B^1$. Hence, the first implication holds if $x \in A^1 \cup (A^0 \cap B^1)$. Similarly, the second holds if $x \in B^1 \cup (B^0 \cap A^1)$. Thus

$$x \in \boldsymbol{A} \vee \boldsymbol{B} \Leftrightarrow x \in [A^1 \cup (A^0 \cap B^1)] \cap [B^1 \cup (B^0 \cap A^1)],$$

and the last intersection is equal to $\texttt{Dom}(\boldsymbol{A} \vee \boldsymbol{B})$! One then can define $x \notin \boldsymbol{A} \vee \boldsymbol{B} :\Leftrightarrow x \notin \boldsymbol{A} \ \& \ x \notin \boldsymbol{B}$, and $x \in \boldsymbol{A} \wedge \boldsymbol{B} :\Leftrightarrow x \in \boldsymbol{A} \ \& \ x \in \boldsymbol{B}$, and $x \notin \boldsymbol{A} \wedge \boldsymbol{B} :\Leftrightarrow x \in (-\boldsymbol{A}) \vee (-\boldsymbol{B})$.

Chapter 3

Families of sets

We develop the basic theory of set-indexed families of sets and of family-maps between them. We study the exterior union of a family of sets Λ, or the $\sum$-set of Λ, and the set of dependent functions over Λ, or the $\prod$-set of Λ. We prove the distributivity of $\prod$ over $\sum$ for families of sets indexed by a product of sets, which is the translation of the type-theoretic axiom of choice into BST. Sets of sets are special set-indexed families of sets that allow "lifting" of functions on the index-set to functions on them. The direct families of sets and the set-relevant families of sets are introduced. The index-set of the former is a directed set, while the transport maps of the latter are more than one and appropriately indexed. With the use of the introduced universe $\mathbb{V}_0^{\mathrm{im}}$ of sets and impredicative sets we study families of families of sets.

3.1 Set-indexed families of sets

Roughly speaking, a family of sets indexed by some set I is an assignment routine $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ that behaves like a function i.e., if $i =_I j$, then $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$. Next follows an exact formulation of this description that reveals the witnesses of the equality $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$.

Definition 3.1.1. *If I is a set, a family of sets indexed by I, or an I-family of sets, is a pair $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 \colon I \rightsquigarrow \mathbb{V}_0$, and λ_1, a modulus of function-likeness for λ_0, is given by*

$$\lambda_1 \colon \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1(i,j) := \lambda_{ij}, \quad (i,j) \in D(I),$$

such that the transport maps λ_{ij} of Λ satisfy the following conditions:

(a) *For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.*

(b) *If $i =_I j$ and $j =_I k$, the following diagram commutes*

$$\begin{tikzcd} \lambda_0(i) \arrow[d, "\lambda_{ij}"'] \arrow[dr, "\lambda_{ik}"] & \\ \lambda_0(j) \arrow[r, "\lambda_{jk}"'] & \lambda_0(k). \end{tikzcd}$$

I is the index-set of the family Λ. If X is a set, the constant I-family of sets X is the pair $C^X := (\lambda_0^X, \lambda_1^X)$, where $\lambda_0(i) := X$, for every $i \in I$, and $\lambda_1(i,j) := \mathrm{id}_X$, for every $(i,j) \in D(I)$ (see the left diagram in Definition 3.1.2).

The dependent operation λ_1 should have been written as follows

$$\lambda_1\colon \bigcurlywedge_{z\in D(I)} \mathbb{F}\big(\lambda_0(\mathbf{pr}_1(z)), \lambda_0(\mathbf{pr}_2(z))\big),$$

but, for simplicity, we avoid the use of the primitive projections $\mathbf{pr}_1, \mathbf{pr}_2$. Condition (a) of Definition 3.1.1 could have been written as $\lambda_{ii} =_{\mathbb{F}(\lambda_0(i),\lambda_0(i))} \mathrm{id}_{\lambda_0(i)}$. If $i =_I j$, then by conditions (b) and (a) of Definition 3.1.1 we get $id_{\lambda_0(i)} := \lambda_{ii} = \lambda_{ji} \circ \lambda_{ij}$ and $\mathrm{id}_{\lambda_0(j)} := \lambda_{jj} = \lambda_{ij} \circ \lambda_{ji}$ i.e., $(\lambda_{ij}, \lambda_{ji})\colon \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$. In this sense λ_1 is a modulus of function-likeness for λ_0.

Definition 3.1.2. *The pair $\Lambda^{2} := (\lambda_0^{2}, \lambda_1^{2})$, where $\lambda_0^{2}\colon 2 \rightsquigarrow \mathbb{V}_0$ with $\lambda_0^{2}(0) := X$, $\lambda_0^{2}(1) := Y$, and $\lambda_1^{2}(0,0) := \mathrm{id}_X$ and $\lambda_1^{2}(1,1) := \mathrm{id}_Y$, is the 2-family of X and Y*

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d, "\mathrm{id}_Y"'] \arrow[dr, "\mathrm{id}_X"] & \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y. \end{tikzcd}$$

The $\mathbb{n}$-family $\Lambda^{\mathbb{n}}$ of the sets $X_1, \ldots X_n$, where $n \geq 1$, and the $\mathbb{N}$-family $\Lambda^{\mathbb{N}} := (\lambda_0^{\mathbb{N}}, \lambda_1^{\mathbb{N}})$ of the sets $(X_n)_{n\in\mathbb{N}}$ are defined similarly[1]*.*

Definition 3.1.3. *Let $\Lambda := (\lambda_0, \lambda_1)$ and $M := (\mu_0, \mu_1)$ be I-families of sets. A family-map from Λ to M, in symbols $\Psi\colon \Lambda \Rightarrow M$ is a dependent operation $\Psi\colon \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$ such that for every $(i,j) \in D(I)$ the following diagram commutes*

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j). \end{tikzcd}$$

Let $\mathtt{Map}_I(\Lambda, M)$ be the totality of family-maps from Λ to M, which is equipped with the equality

$$\Psi =_{\mathtt{Map}_I(\Lambda,M)} \Xi :\Leftrightarrow \forall_{i\in I}\big(\Psi_i =_{\mathbb{F}(\lambda_0(i),\mu_0(i))} \Xi_i\big).$$

If $\Xi : M \Rightarrow N$, the composition family-map $\Xi \circ \Psi\colon \Lambda \Rightarrow N$ is defined, for every $i \in I$, by $(\Xi \circ \Psi)_i := \Xi_i \circ \Psi_i$

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Psi_i"] \arrow[dd, bend right, "(\Xi \circ \Psi)_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"'] \arrow[dd, bend left, "(\Xi \circ \Psi)_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}"'] \arrow[d, "\Xi_i"] & \mu_0(j) \arrow[d, "\Xi_j"'] \\ \nu_0(i) \arrow[r, "\nu_{ij}"'] & \nu_0(j). \end{tikzcd}$$

[1]It is immediate to show that $\Lambda^{\mathbb{n}}$ is an $\mathbb{n}$-family, and $\Lambda^{\mathbb{N}}$ is an $\mathbb{N}$-family.

The identity family-map $Id_\Lambda \colon \bigwedge_{i\in I}\mathbb{F}\big(\lambda_0(i),\lambda_0(i)\big)$ on Λ, is defined by $\mathrm{Id}_\Lambda(i) := \mathrm{id}_{\lambda_0(i)}$, for every $i \in I$. Let $\mathtt{Fam}(I)$ be the totality of I-families, equipped with the canonical equality

$$\Lambda =_{\mathtt{Fam}(I)} M :\Leftrightarrow \exists_{\Phi\in\mathtt{Map}_I(\Lambda,M)}\exists_{\Xi\in\mathtt{Map}_I(M,\Lambda)}\big((\Phi,\Xi)\colon \Lambda =_{\mathtt{Fam}(I)} M\big),$$

$$(\Phi,\Xi)\colon \Lambda =_{\mathtt{Fam}(I)} M :\Leftrightarrow \big(\Phi\circ\Xi = \mathrm{id}_M \ \&\ \Xi\circ\Phi = \mathrm{id}_\Lambda\big).$$

It is straightforward to show that the composition family-map $\Xi \circ \Psi$ is a family-map from Λ to N, and that the equalities on $\mathtt{Map}_I(\Lambda, M)$ and $\mathtt{Fam}(I)$ satisfy the conditions of an equivalence relation. It is natural to accept the totality $\mathtt{Map}(\Lambda, M)$ as a set. If $\mathtt{Fam}(I)$ was a set though, the constant I-family with value $\mathtt{Fam}(I)$ would be defined though a totality in which it belongs to. From a predicative point of view, this cannot be accepted. The membership condition of the totality $\mathtt{Fam}(I)$ though, does not depend on the universe $\mathbb{V}_0$, therefore it is also natural not to consider $\mathtt{Fam}(I)$ to be a class. Hence, $\mathtt{Fam}(I)$ is a totality "between" a (predicative) set and a class. For this reason, we say that $\mathtt{Fam}(I)$ is an *impredicative set*. Next follows an obvious generalisation of a family-map.

Definition 3.1.4. *If $\Lambda, M \in \mathtt{Fam}(I)$, such that $\Lambda =_{\mathtt{Fam}(I)} M$, we define the set*

$$\mathtt{PrfEql}_0(\Lambda, M) := \big\{(\Phi,\Psi) \in \mathtt{Map}_I(\Lambda, M)\times\mathtt{Map}_I(M,\Lambda) \mid (\Phi,\Psi) : \Lambda =_{\mathtt{Fam}(I)} M\big\},$$

equipped with the equality of the product of sets. If $\Phi \in \mathtt{Map}_I(\Lambda, M), \Psi \in \mathtt{Map}_I(M,\Lambda), \Phi' \in \mathtt{Map}_I(M, N)$ and $\Psi' \in \mathtt{Map}_I(N, M)$, let $\mathtt{refl}(\Lambda) := \big(\mathrm{Id}_\Lambda, \mathrm{Id}_\Lambda\big)$ and $(\Phi,\Psi)^{-1} := (\Psi,\Phi)$ and $(\Phi,\Psi) * (\Phi',\Psi') := (\Phi'\circ\Phi, \Psi\circ\Psi')$.

As in the case of $\mathbb{V}_0$ and the corresponding set $\mathtt{PrfEql}_0(X, Y)$, in general, not all elements of $\mathtt{PrfEql}_0(\Lambda, M)$ are equal. If $I := \mathbb{1} := \{0\}$, and $\lambda_0(0) := \mathbb{2}$, and if $\Phi_0 := \mathrm{id}_{\mathbb{2}}$ and $\Psi_0 := \mathtt{sw}_{\mathbb{2}}$, then $(\Phi,\Phi) \in \mathtt{PrfEql}_0(\Lambda,\Lambda)$ and $(\Psi,\Psi) \in \mathtt{PrfEql}_0(\Lambda,\Lambda)$, while $\Phi \neq_{\mathtt{Map}_I(\Lambda,\Lambda)} \Psi$, since $\Phi_0 \neq_{\mathbb{F}(\mathbb{2},\mathbb{2})} \Psi_0$. It is immediate to show the groupoid-properties (i)-(iv) for the equality of the totality $\mathtt{Fam}(I)$.

Definition 3.1.5. *Let I, J, K be sets, $h \in \mathbb{F}(J, I), g \in \mathbb{F}(K, J)$, $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(J)$, and $N := (\nu_0, \nu_1) \in \mathtt{Fam}(K)$. A family-map from M to Λ over h is a dependent operation $\Psi\colon \bigwedge_{j\in J}\mathbb{F}\big(\mu_0(j), \lambda_0(h(j))\big)$, such that for every $(j, j') \in D(J)$ the following diagram commutes*

$$\begin{array}{ccc} \mu_0(j) & \xrightarrow{\ \mu_{jj'}\ } & \mu_0(j') \\ {\scriptstyle \Psi_j}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{j'}} \\ \lambda_0(h(j)) & \xrightarrow[\lambda_{h(j)h(j')}]{} & \lambda_0(h(j')), \end{array}$$

where $\Psi_j := \Psi(j)$ is the j-component of Ψ, for every $j \in J$. We write $\Psi\colon M \overset{h}{\Rightarrow} \Lambda$ for such a family-map. If $\Psi\colon M \overset{h}{\Rightarrow} \Lambda$ and $\Xi\colon N \overset{g}{\Rightarrow} M$, the composition family-map $\Psi\circ\Xi\colon N \overset{h\circ g}{\Longrightarrow} \Lambda$ over $h \circ g$ is defined, for every $k \in K$, by $(\Psi\circ\Xi)_k := \Psi_{g(k)} \circ \Xi_k$

$$\begin{tikzcd}
\nu_0(k) \arrow[r, "\nu_{kk'}"] \arrow[d, "\Xi_k"] \arrow[dd, bend right=60, "(\Psi\circ\Xi)_k"'] & \nu_0(k') \arrow[d, "\Xi_{k'}"'] \arrow[dd, bend left=60, "(\Psi\circ\Xi)_{k'}"] \\
\mu_0(g(k')) \arrow[r, "\mu_{g(k)g(k')}"'] \arrow[d, "\Psi_{g(k)}"] & \mu_0(g(k)) \arrow[d, "\Psi_{g(k')}"'] \\
\lambda_0(h(g(k))) \arrow[r, "\lambda_{h(g(k))h(g(k'))}"'] & \lambda_0(h(g(k'))).
\end{tikzcd}$$

Definition 3.1.6. *Let $\Lambda := (\lambda_0, \lambda_1), M := (\mu_0, \mu_1)$ be I-families of sets.*
(i) *The product family of Λ and M is the pair $\Lambda \times M := (\lambda_0 \times \mu_0, \lambda_1 \times \mu_1)$, where*

$$(\lambda_0 \times \mu_0)(i) := \lambda_0(i) \times \mu_0(i); \qquad i \in I,$$

$$\big(\lambda_1 \times \mu_1\big)_{ij} \colon \lambda_0(i) \times \mu_0(i) \to \lambda_0(j) \times \mu_0(j); \qquad (i,j) \in D(I),$$

$$\big(\lambda_1 \times \mu_1\big)_{ij}\big(x, y\big) := \big(\lambda_{ij}(x), \mu_{ij}(y)\big); \qquad x \in \lambda_0(i) \ \& \ y \in \mu_0(i).$$

(ii) *The function space family from Λ to M is the pair $\mathbb{F}(\Lambda, M) := \big(\mathbb{F}(\lambda_0, \mu_0), \mathbb{F}(\lambda_1, \mu_1)\big)$ where*

$$\big[\mathbb{F}(\lambda_0, \mu_0)\big](i) := \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big); \qquad i \in I,$$

$$\mathbb{F}(\lambda_1, \mu_1) \ : \ \bigcurlywedge_{(i,j)\in D(I)} \mathbb{F}\Big(\mathbb{F}\big(\lambda_0(i), \mu_0(i)\big), \mathbb{F}\big(\lambda_0(j), \mu_0(j)\big)\Big)$$

$$\mathbb{F}(\lambda_1, \mu_1)_{ij} := \mathbb{F}(\lambda_1, \mu_1)(i,j) \colon \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big) \to \mathbb{F}\big(\lambda_0(j), \mu_0(j)\big); \qquad (i,j) \in D(I),$$

$$\mathbb{F}(\lambda_1, \mu_1)_{ij}(f) := \mu_{ij} \circ f \circ \lambda_{ji}$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "f"] & \mu_0(i) \arrow[d, "\mu_{ij}"] \\
\lambda_0(j) \arrow[u, "\lambda_{ji}"] \arrow[r, "\mathbb{F}(\lambda_1,\mu_1)_{ij}(f)"'] & \mu_0(j).
\end{tikzcd}$$

(iii) *If K is a set, $\Sigma := (\sigma_0, \sigma_1)$ is a K-family of sets and $h : I \to K$, the composition family of Σ with h is the pair $\Sigma \circ h := (\sigma_0 \circ h, \sigma_1 \circ h)$, where*

$$(\sigma_0 \circ h)(i) := \sigma_0(h(i)); \qquad i \in I,$$

$$(\sigma_1 \circ h)_{ij} := (\sigma_1 \circ h)(i,j) \colon \sigma_0(h(i)) \to \sigma_0(h(j)); \qquad (i,j) \in D(I),$$

$$(\sigma_1 \circ h)_{ij} := \sigma_{h(i)h(j)}.$$

It is straightforward to show that $\Lambda \times M$, $\mathbb{F}(\Lambda, M)$, and $\Sigma \circ h$ are I-families. E.g., for $\mathbb{F}(\Lambda, M)$, and if $i, j, k \in I$ and $i =_I j =_I k$, we have that

$$\mathbb{F}(\lambda_1, \mu_1)_{ii}(f) := \mu_{ii} \circ f \circ \lambda_{ii} := \mathrm{id}_{\mu_0(i)} \circ f \circ \mathrm{id}_{\lambda_0(i)} := f,$$

$$\begin{aligned}\mathbb{F}(\lambda_1,\mu_1)_{jk}\Big(\mathbb{F}(\lambda_1,\mu_1)_{ij}(f)\Big) &:= \mu_{jk}\circ\big[\mu_{ij}\circ f\circ\lambda_{ji}\big]\circ\lambda_{kj}\\ &:= \big[\mu_{jk}\circ\mu_{ij}\big]\circ f\circ\big[\lambda_{ji}\circ\lambda_{kj}\big]\\ &= \mu_{ik}\circ f\circ\lambda_{ki}\\ &:= \mathbb{F}(\lambda_1,\mu_1)_{ik}(f).\end{aligned}$$

Proposition 3.1.7. *Let X, Y, I be sets and $C^X, C^Y, C^{X\times Y}, C^{\mathbb{F}(X,Y)}$ the constant I-families $X, Y, X\times Y$, and $\mathbb{F}(X,Y)$, respectively.*
(i) $C^X \times C^Y =_{\texttt{Fam}(I)} C^{X\times Y}$.
(ii) $\mathbb{F}(C^X, C^Y) =_{\texttt{Fam}(I)} C^{\mathbb{F}(X,Y)}$.

Proof. (i) Let $\Phi\colon C^X\times C^Y \Rightarrow C^{X\times Y}$ and $\Psi\colon C^{X\times Y}\Rightarrow C^X\times C^Y$ be defined by $\Phi_i := X\times Y := \Psi_i$, for every $i\in I$, then by the commutativity of the following left diagram

$$\begin{tikzcd} X\times Y \arrow[r, "(\lambda_1^X\times\mu_1^Y)_{ij}"] \arrow[d, "\mathrm{id}_{X\times Y}"'] & X\times Y \arrow[d, "\mathrm{id}_{X\times Y}"] \\ X\times Y \arrow[r, "\lambda_{ij}"'] & X\times Y \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}(X,Y) \arrow[r, "\mathbb{F}(\lambda_1^X,\mu_1^Y)_{ij}"] \arrow[d, "\mathrm{id}_{\mathbb{F}(X,Y)}"'] & \mathbb{F}(X,Y) \arrow[d, "\mathrm{id}_{\mathbb{F}(X,Y)}"] \\ \mathbb{F}(X,Y) \arrow[r, "\mu_{ij}"'] & \mathbb{F}(X,Y), \end{tikzcd}$$

Φ, Ψ are well-defined family-maps and $(\Phi,\Psi)\colon C^X\times C^Y =_{\texttt{Fam}(I)} C^{X\times Y}$.
(ii) Let $\Phi\colon \mathbb{F}(C^X, C^Y)\Rightarrow C^{\mathbb{F}(X,Y)}$ and $\Psi\colon C^{\mathbb{F}(X,Y)}\Rightarrow \mathbb{F}(C^X,C^Y)$ be defined by $\Phi_i := \mathbb{F}(X,Y) := \Psi_i$, for every $i\in I$, then by the commutativity of the above right diagram Φ,Ψ are well-defined family-maps and $(\Phi,\Psi)\colon \mathbb{F}(C^X,C^Y) =_{\texttt{Fam}(I)} C^{\mathbb{F}(X,Y)}$. $\square$

The operations on families of sets generate operations on family-maps.

Proposition 3.1.8. *Let $\Lambda := (\lambda_0,\lambda_1), M := (\mu_0,\mu_1), N := (\nu_0,\nu_1)$, $K := (\kappa_0,\kappa_1)\in\texttt{Fam}(I)$.*
(i) *If $\Phi\colon N\Rightarrow\Lambda$ and $\Psi\colon N\Rightarrow M$, then $\Phi\times\Psi\colon N\Rightarrow\Lambda\times M$ is the product family-map of Φ and Ψ, where, for every $i\in I$, the map $(\Phi\times\Psi)_i\colon \nu_0(i)\to\lambda_0(i)\times\mu_0(i)$ is defined by*

$$(\Phi\times\Psi)_i(z) := \big(\Phi_i(z),\Psi_i(z)\big);\qquad z\in\nu_0(i).$$

(ii) If $\Phi\colon N\Rightarrow\Lambda$ and $\Psi\colon K\Rightarrow M$, then $\Phi\times\Psi\colon N\times K\Rightarrow\Lambda\times M$ is the *product family-map* of Φ and Ψ, where, for every $i\in I$, the map $(\Phi\times\Psi)_i\colon \nu_0(i)\times\kappa_0(i)\to\lambda_0(i)\times\mu_0(i)$ is defined by

$$(\Phi\times\Psi)_i(x,y) := \big(\Phi_i(x),\Psi_i(y)\big);\qquad (x,y)\in\nu_0(i)\times\kappa_0(i).$$

(iii) *If $\Phi\colon N\Rightarrow\Lambda$, then $\mathbb{F}(\Phi)^c\colon\mathbb{F}(\Lambda,M)\Rightarrow\mathbb{F}(N,M)$, where, for every $i\in I$, the function $\mathbb{F}(\Phi)^c_i\colon\mathbb{F}\big(\lambda_0(i),\mu_0(i)\big)\to\mathbb{F}\big(\nu_0(i),\mu_0(i)\big)$ is defined by*

$$\mathbb{F}(\Phi)^c_i(f) := f\circ\Phi_i;\qquad f\in\mathbb{F}\big(\lambda_0(i),\mu_0(i)\big)$$

$$\begin{tikzcd} \nu_0(j) \arrow[r, "\Phi_i"] \arrow[rr, bend right, "\mathbb{F}(\Phi)^c_i(f)"'] & \lambda_0(i) \arrow[r, "f"] & \mu_0(i) \end{tikzcd} \qquad \begin{tikzcd} \mu_0(j) \arrow[r, "f"] \arrow[rr, bend right, "\mathbb{F}(\Phi)^d_i(f)"'] & \lambda_0(i) \arrow[r, "\Phi_i"] & \nu_0(i). \end{tikzcd}$$

If $\Phi\colon \Lambda \Rightarrow N$*, then* $\mathbb{F}(\Phi)^d \colon \mathbb{F}(M, \Lambda) \Rightarrow \mathbb{F}(M, N)$*, where, for every* $i \in I$ *and* $f \in \mathbb{F}\big(\mu_0(i), \lambda_0(i)\big)$*, the function* $\mathbb{F}(\Phi)^d_i \colon \mathbb{F}\big(\mu_0(i), \lambda_0(i)\big) \to \mathbb{F}\big(\mu_0(i), \nu_0(i)\big)$ *is defined by* $\mathbb{F}(\Phi)^d_i(f) := \Phi_i \circ f$*.*
(iv) *If* $\Phi\colon N \Rightarrow \Lambda$ *and* $\Psi\colon M \Rightarrow K$*, then* $\mathbb{F}(\Phi, \Psi)\colon \mathbb{F}(\Lambda, M) \Rightarrow \mathbb{F}(N, K)$*, where for every* $i \in I$*, the map* $\mathbb{F}(\Phi, \Psi)_i \colon \mathbb{F}\big(\lambda_0(i), \mu_0(i) \to \mathbb{F}\big(\nu_0(i), \kappa_0(i)$ *is defined by*

$$\mathbb{F}(\Phi, \Psi)_i(f) := \Psi_i \circ f \circ \Phi_i; \qquad f \in \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "f"] & \mu_0(i) \arrow[d, "\Psi_i"] \\
\nu_0(i) \arrow[u, "\Phi_i"] \arrow[r, "\mathbb{F}(\Phi,\Psi)_i(f)"'] & \kappa_0(i).
\end{tikzcd}$$

Proof. We prove (i) and (iii), as the proofs of (ii), (iv) are similar to that of (i), (iii), respectively.
(i) If $i =_I j$, the following diagram is commutative

$$\begin{tikzcd}
\nu_0(i) \arrow[r, "\nu_{ij}"] \arrow[d, "(\Phi \times \Psi)_i"'] & \nu_0(j) \arrow[d, "(\Phi \times \Psi)_j"] \\
\mu_0(i) \times \lambda_0(i) \arrow[r, "(\lambda_1 \times \mu_1)_{ij}"'] & \mu_0(j) \times \lambda_0(j),
\end{tikzcd}$$

since by the commutativity of the following two diagrams

$$\begin{tikzcd}
\nu_0(i) \arrow[r, "\nu_{ij}"] \arrow[d, "\Phi_i"'] & \nu_0(j) \arrow[d, "\Phi_j"] \\
\lambda_0(i) \arrow[r, "\lambda_{ij}"'] & \lambda_0(j)
\end{tikzcd}
\qquad
\begin{tikzcd}
\nu_0(i) \arrow[r, "\nu_{ij}"] \arrow[d, "\Psi_i"'] & \nu_0(j) \arrow[d, "\Psi_j"] \\
\mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j),
\end{tikzcd}$$

$$\begin{aligned}
(\Phi \times \Psi)_j\big(\nu_{ij}(z)\big) &:= \big(\Phi_j(\nu_{ij}(z)), \Psi_j(\nu_{ij}(z))\big) \\
&= \big(\lambda_{ij}(\Phi_i(z)), \mu_{ij}(\Psi_i(z))\big) \\
&:= (\lambda_1 \times \mu_1)_{ij}\big(\Phi_i(z), \Psi_i(z)\big) \\
&:= (\lambda_1 \times \mu_1)_{ij}\big((\Phi \times \Psi)_i(z)\big); \qquad z \in \nu_0(i).
\end{aligned}$$

(ii) If $i =_I j$, the following diagram is commutative

$$\begin{tikzcd}
\mathbb{F}\big(\lambda_0(i), \mu_0(i)\big) \arrow[r, "\mathbb{F}(\lambda_1,\mu_1)_{ij}"] \arrow[d, "\mathbb{F}(\Phi)^c_i"'] & \mathbb{F}\big(\lambda_0(j), \mu_0(j)\big) \arrow[d, "\mathbb{F}(\Phi)^c_j"] \\
\mathbb{F}\big(\nu_0(i), \mu_0(i)\big) \arrow[r, "\mathbb{F}(\nu_1,\mu_1)_{ij}"'] & \mathbb{F}\big(\nu_0(j), \mu_0(j)\big),
\end{tikzcd}$$

$$\begin{aligned}
\mathbb{F}(\Phi)^c_j\big(\mathbb{F}(\lambda_1,\mu_1)_{ij}(f)\big) &:= \mathbb{F}(\Phi)^c_j(\mu_{ij}\circ f\circ\lambda_{ji})\\
&:= (\mu_{ij}\circ f\circ\lambda_{ji})\circ\Phi_j\\
&:= \mu_{ij}\circ f\circ(\lambda_{ji}\circ\Phi_j)\\
&:= \mu_{ij}\circ f\circ(\Phi_i\circ\nu_{ji})\\
&:= \mu_{ij}\circ(f\circ\Phi_i)\circ\nu_{ji}\\
&:= \mathbb{F}(\nu_1,\mu_1)_{ij}(f\circ\Phi_i)\\
&:= \mathbb{F}(\nu_1,\mu_1)_{ij}\big(\mathbb{F}(\Phi)^c_i(f)\big);\qquad f\in\mathbb{F}\big(\lambda_0(i),\mu_0(i)\big).
\end{aligned}$$

The equality $\lambda_{ji}\circ\Phi_j = \Phi_i\circ\nu_{ji}$ used above follows from the definition of $\Phi\colon N\Rightarrow\Lambda$ on $(j,i)\in D(I)$. The proof of $\mathbb{F}(\Phi)^d\colon\mathbb{F}(M,\Lambda)\Rightarrow\mathbb{F}(M,N)$ is similar. $\square$

3.2 The exterior union of a family of sets

Definition 3.2.1. *Let $\Lambda := (\lambda_0,\lambda_1)$ be an I-family of sets. The exterior union, or disjoint union, or the $\sum$-set $\sum_{i\in I}\lambda_0(i)$ of Λ, and its canonical equality are defined by*

$$w\in\sum_{i\in I}\lambda_0(i) :\Leftrightarrow \exists_{i\in I}\exists_{x\in\lambda_0(i)}\big(w := (i,x)\big),$$

$$(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \&\ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

The $\sum$-set of the $\mathit{2}$-family Λ^{2} of the sets X and Y is the coproduct of X and Y, and we write

$$X+Y := \sum_{i\in\mathit{2}}\lambda_0^{\mathit{2}}(i).$$

Proposition 3.2.2. (i) *The equality on $\sum_{i\in I}\lambda_0(i)$ satisfies the conditions of an equivalence relation.*

(ii) *Let $(I,=_I,\neq_I)$ be a discrete set and $\neq_{\lambda_0(i)}$ an inequality on $\lambda_0(i)$, for every $i\in I$. If the transport map λ_{ij} is strongly extensional, for every $(i,j)\in D(I)$, then the relation*

$$(i,x)\neq_{\sum_{i\in I}\lambda_0(i)}(j,y) :\Leftrightarrow i\neq_I j \ \vee\ \big(i =_I j \ \&\ \lambda_{ij}(x)\neq_{\lambda_0(j)} y\big)$$

is an inequality on $\sum_{i\in I}\lambda_0(i)$. If $(\lambda_0(i),=_{\lambda_0(i)},\neq_{\lambda_0(i)})$ is a discrete set, for every $i\in I$, then $\big(\sum_{i\in I}\lambda_0(i),=_{\sum_{i\in I}\lambda_0(i)},\neq_{\sum_{i\in I}\lambda_0(i)}\big)$ is discrete. Moreover, if $\neq_I$ is tight, and if, for every $i\in I$, the inequality $\neq_{\lambda_0(i)}$ is tight, then the inequality $\neq_{\sum_{i\in I}\lambda_0(i)}$ is tight.

Proof. (i) Let $(i,x),(j,y),\ (k,z)\in\sum_{i\in I}\lambda_0(i)$. Since $i =_I i$ and $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$, we get $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$, then $j =_I i$ and $\lambda_{ji}(y) = \lambda_{ji}(\lambda_{ij}(x)) = \lambda_{ii}(x) := \mathrm{id}_{\lambda_0(i)}(x) := x$, hence $(j,y) =_{\sum_{i\in I}\lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$ and $(j,y) =_{\sum_{i\in I}\lambda_0(i)} (k,z)$, then $i =_I j \ \&\ j =_I k \Rightarrow i =_I k$, and

$$\lambda_{ik}(x) =_{\lambda_0(k)} (\lambda_{jk}\circ\lambda_{ij})(x) := \lambda_{jk}(\lambda_{ij}(x)) =_{\lambda_0(k)} \lambda_{jk}(y) =_{\lambda_0(k)} z.$$

(ii) The condition (Ap_1) of Definition 2.2.5 is trivially satisfied. To show condition (Ap_2), we suppose first that $i\neq_I j$, hence by the corresponding condition of $\neq_I$ we get $(j,y)\neq_{\sum_{i\in I}\lambda_0(i)} (i,x)$. If $i =_I j \ \&\ \lambda_{ij}(x)\neq_{\lambda_0(j)} y$, we show that $\lambda_{ji}(y)\neq_{\lambda_0(i)} x$. By the extensionality of $\neq_{\lambda_0(j)}$ (Remark 2.2.6) the inequality $\lambda_{ij}(x)\neq_{\lambda_0(j)} y$ implies the inequality $\lambda_{ij}(x)\neq_{\lambda_0(j)}\lambda_{ij}\big(\lambda_{ji}(y)\big)$, and since λ_{ij} is strongly extensional, we get $x \neq_{\lambda_0(i)} \lambda_{ji}(y)$. To show condition (Ap$_3$), let $(i,x) \neq_{\sum_{i\in I}\lambda_0(i)} (j,y)$, and let $(k,z) \in \sum_{i\in I}\lambda_0(i)$. If $i \neq_I j$, then by condition (Ap$_3$) of $\neq_I$ we get $k \neq_I i$, or $k \neq_I j$, hence $(k,z) \neq_{\sum_{i\in I}\lambda_0(i)} (i,x)$, or $(k,z) \neq_{\sum_{i\in I}\lambda_0(i)} (j,y)$. Suppose next $i =_I j$ & $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$. Since the set $(I, =_I, \neq_I)$ is discrete, $k \neq_I i$, or $k =_I i =_I j$. If $k \neq_I i$, then what we want to show follows immediately. If $k =_I i =_I j$, then by the extensionality of $\neq_{\lambda_0(j)}$ and the strong extensionality of the transport map λ_{kj} we have that

$$\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \lambda_{kj}\big(\lambda_{ik}(x)\big) \neq_{\lambda_0(j)} \lambda_{kj}\big(\lambda_{jk}(y)\big) \Rightarrow \lambda_{ik}(x) \neq_{\lambda_0(k)} \lambda_{jk}(y).$$

Hence, by condition (Ap$_3$) of $\neq_{\lambda_0(k)}$ we get $\lambda_{ik}(x) \neq_{\lambda_0(k)} z$, or $\lambda_{jk}(y) \neq_{\lambda_0(k)} z$, hence $(i,x) \neq_{\sum_{i\in I}\lambda_0(i)} (k,z)$, or $(j,y) \neq_{\sum_{i\in I}\lambda_0(i)} (k,z)$. Suppose next that $(\lambda_0(i), =_{\lambda_0(i)}, \neq_{\lambda_0(i)})$ is a discrete set, for every $i \in I$. We show that $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$ i.e., $i =_I j$ and $\lambda_{ij}(x) =_{\lambda_0(i)} y$, or $(i,x) \neq_{\sum_{i\in I}\lambda_0(i)} (j,y)$ i.e., $i \neq_I j$ or $i =_I j$ & $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$. Since $(I, =_I, \neq_I)$ is discrete, $i =_I j$, or $i \neq_I j$. In the first case, and since $(\lambda_0(j), =_{\lambda_0(j)}, \neq_{\lambda_0(j)})$ is discrete, we get $\lambda_{ij}(x) =_{\lambda_0(j)} y$ or $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$, and what we want follows immediately. If $i \neq_I j$, we get $(i,x) \neq_{\sum_{i\in I}\lambda_0(i)} (j,y)$. Finally, we suppose that $\neq_I$ is tight, and that $\neq_{\lambda_0(i)}$ is tight, for every $i \in I$. Let $\neg\big[(i,x) \neq_{\sum_{i\in I}\lambda_0(i)} (j,y)\big]$ i.e.,

$$\big[i \neq_I j \ \vee \ \big(i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y\big)\big] \Rightarrow \bot.$$

From this hypothesis we get the conjunction[2]

$$\big[i \neq_I j \Rightarrow \bot\big] \ \ \& \ \ \big[\big(i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y\big) \Rightarrow \bot\big].$$

By the tightness of $\neq_I$ we get $i =_I j$. The implication $\big(i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y\big) \Rightarrow \bot$ logically implies the implication $(i =_I j) \Rightarrow \big(\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \bot\big)$, and since its premiss $i =_I j$ is derived by the tightness of $\neq_I$, by Modus Ponens we get $\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \bot$. Since $\neq_{\lambda_0(i)}$ is tight, we conclude that $\lambda_{ij}(x) =_{\lambda_0(j)} y$, hence $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$. $\square$

The totality $\sum_{i\in I}\lambda_0(i)$ is considered to be a set. By the definition of $X + Y$

$$\begin{aligned} w \in X + Y &\Leftrightarrow \exists_{i\in 2}\exists_{x\in\lambda_0^{2}(i)}\big(w := (i,x)\big) \\ &\Leftrightarrow \exists_{x\in X}\big(w := (0,x)\big) \ \vee \ \exists_{y\in Y}\big(w := (1,y)\big), \end{aligned}$$

$$(i,x) =_{X+Y} (i',x') \Leftrightarrow (i =_2 i' =_2 0 \ \& \ x =_X x') \ \vee \ (i =_2 i' =_2 1 \ \& \ x =_Y x').$$

One could have defined $X + Y$ independently from Λ^{2}, and then prove $X + Y =_{\mathbb{V}_0} \sum_{i\in 2}\lambda_0^{2}(i)$.

Corollary 3.2.3. *If $(X, =_X, \neq_X)$, $(Y, =_Y, \neq_Y)$ are discrete, $(X+Y, =_{X+Y}, \neq_{X+Y})$ is discrete.*

Proof. Since $(2, =_2, \neq_2)$ is a discrete set, we use Proposition 3.2.2(ii). $\square$

Definition 3.2.4. *Let $\Lambda := (\lambda_0, \lambda_1), M := (\mu_0, \mu_1)$ be I-families of sets. The coproduct family of Λ and M is the pair $\Lambda + M := (\lambda_0 + \mu_0, \lambda_1 + \mu_1)$, where $(\lambda_0 + \mu_0)(i) := \lambda_0(i) + \mu_0(i)$, for every $i \in I$, and the map $\big(\lambda_1 + \mu_1\big)_{ij} \colon \lambda_0(i) + \mu_0(i) \to \lambda_0(j) + \mu_0(j)$ is defined by*

$$\big(\lambda_1 + \mu_1\big)_{ij}(w) := \begin{cases} \big(0, \lambda_{ij}(x)\big) & , w := (0,x) \\ \big(1, \mu_{ij}(y)\big) & , w := (1,y) \end{cases} ; \qquad w \in \lambda_0(i) + \mu_0(i).$$

[2]Here we use the logical implication $\big((\phi \vee \psi) \Rightarrow \bot\big) \Rightarrow [(\phi \Rightarrow \bot) \ \& \ (\psi \Rightarrow \bot)]$.

It is straightforward to show that $\Lambda + M$ is an I-family of sets.

Proposition 3.2.5. *Let $\Lambda := (\lambda_0, \lambda_1), M := (\mu_0, \mu_1)$, and $N := (\nu_0, \nu_1)$ be I-families of sets. If $\Phi \colon \Lambda \Rightarrow N$ and $\Psi \colon M \Rightarrow N$, then $\Phi + \Psi \colon \Lambda + M \Rightarrow N$ is the coproduct family-map of Φ and Ψ, where, for every $i \in I$, the map $(\Phi + \Psi)_i \colon \lambda_0(i) + \mu_0(i) \to \nu_0(i)$ is defined by*

$$(\Phi + \Psi)_i(w) := \begin{cases} \Phi_i(x) & , w := (0, x) \\ \Psi_i(y) & , w := (1, y) \end{cases} ; \qquad w \in \lambda_0(i) + \mu_0(i).$$

Proof. If $i =_I j$, the following diagram is commutative

$$\begin{tikzcd} \lambda_0(i)+\mu_0(i) \arrow[r, "(\lambda_1+\mu_1)_{ij}"] \arrow[d, "(\Phi+\Psi)_i"'] & \lambda_0(j)+\mu_0(j) \arrow[d, "(\Phi+\Psi)_j"] \\ \nu_0(i) \arrow[r, "\nu_{ij}"'] & \nu_0(j), \end{tikzcd}$$

since by the commutativity of the following left diagram

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Phi_i"'] & \lambda_0(j) \arrow[d, "\Phi_j"] \\ \nu_0(i) \arrow[r, "\nu_{ij}"'] & \nu_0(j) \end{tikzcd} \qquad \begin{tikzcd} \mu_0(i) \arrow[r, "\mu_{ij}"] \arrow[d, "\Psi_i"'] & \mu_0(j) \arrow[d, "\Psi_j"] \\ \nu_0(i) \arrow[r, "\nu_{ij}"'] & \nu_0(j), \end{tikzcd}$$

$$\begin{aligned} (\Phi + \Psi)_j\big((\lambda_1 + \mu_1)_{ij}(0, x)\big) &:= (\Phi + \Psi)_j\big(0, \lambda_{ij}(x)\big) \\ &= \Phi_j\big(\lambda_{ij}(x)\big) \\ &= \nu_{ij}\big(\Phi_i(x)\big) \\ &:= \nu_{ij}\big((\Phi + \Psi)_i(0, x)\big); \qquad x \in \lambda_0(i). \end{aligned}$$

By the commutativity of the right diagram, $(\Phi+\Psi)_j\big((\lambda_1+\mu_1)_{ij}(1, y)\big) = \nu_{ij}\big((\Phi+\Psi)_i(1, y)\big)$. $\square$

Proposition 3.2.6. *If $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, then*

$$\sum_{i \in I} \big(\lambda_0(i) + \mu_0(i)\big) =_{\mathbb{V}_0} \bigg(\sum_{i \in I} \lambda_0(i)\bigg) + \bigg(\sum_{i \in I} \mu_0(i)\bigg).$$

Proof. Let $f \colon \sum_{i \in I} \big(\lambda_0(i) + \mu_0(i)\big) \rightsquigarrow \sum_{i \in I} \lambda_0(i) + \sum_{i \in I} \mu_0(i)$ be defined by

$$f(i, w) := \begin{cases} \big(0, (i, x)\big) & , w := (0, x) \\ \big(1, (i, y)\big) & , w := (1, y) \end{cases} ; \qquad i \in I, \ w \in \lambda_0(i) + \mu_0(i).$$

Clearly, f is a well-defined operation. To show that f is a function, we suppose that

$$(i, w) =_{\sum_{i \in I}(\lambda_0(i)+\mu_0(i))} (j, u) :\Leftrightarrow i =_I j \ \& \ (\lambda_1 + \mu_1)_{ij}(w) =_{\lambda_0(j)+\mu_0(j)} u,$$

and we show that $f(i, w) = f(j, u)$. The equality $(\lambda_1 + \mu_1)_{ij}(w) =_{\lambda_0(j)+\mu_0(j)} u$ amounts to $\lambda_{ij}(x) =_{\lambda_0(j)} x'$, if $w := (0, x)$ and $u := (0, x')$, or to $\mu_{ij}(y) =_{\mu_0(j)} y'$, if $w := (1, y)$ and

$u := (1, y')$. With the use of these equalities and the definition of the canonical equality on the coproduct it is straightforward to show that $\big(0, (i, x)\big) = \big(0, (j, x')\big)$, or $\big(1, (i, y)\big) = \big(1, (j, y')\big)$, hence $f(i, w) = f(j, u)$. Let $g\colon \sum_{i\in I} \lambda_0(i) + \sum_{i\in I} \mu_0(i) \rightsquigarrow \sum_{i\in I} \big(\lambda_0(i) + \mu_0(i)\big)$ be defined by

$$g(U) := \begin{cases} \big(i, (0, x)\big) & , U := \big(0, (i, x)\big) \\ \big(i, (1, y)\big) & , U := \big(1, (i, y)\big) \end{cases} ; \qquad U \in \sum_{i\in I} \lambda_0(i) + \sum_{i\in I} \mu_0(i).$$

Proceeding similarly, we show that the operation g is a function. It is straightforward to show that $(f, g)\colon \sum_{i\in I} \big(\lambda_0(i) + \mu_0(i)\big) =_{\mathbb{V}_0} \sum_{i\in I} \lambda_0(i) + \sum_{i\in I} \mu_0(i)$. $\square$

Proposition 3.2.7. *Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, and $\Psi : \Lambda \Rightarrow M$.*
(i) *For every $i \in I$ the operation $e_i^{\Lambda} : \lambda_0(i) \rightsquigarrow \sum_{i\in I} \lambda_0(i)$, defined by $e_i^{\Lambda}(x) := (i, x)$, for every $x \in \lambda_0(i)$, is an embedding.*
(ii) *The operation $\Sigma\Psi : \sum_{i\in I} \lambda_0(i) \rightsquigarrow \sum_{i\in I} \mu_0(i)$, defined by*

$$\Sigma\Psi(i, x) := (i, \Psi_i(x)); \qquad (i, x) \in \sum_{i\in I} \lambda_0(i),$$

is a function, such that for every $i \in I$ the following diagram commutes

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "e_i^{\Lambda}"'] & \mu_0(i) \arrow[d, "e_i^{M}"] \\ \sum_{i\in I} \lambda_0(i) \arrow[r, "\Sigma\Psi"'] & \sum_{i\in I} \mu_0(i). \end{tikzcd}$$

(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Sigma\Psi$ is an embedding.*
(iv) *If Ψ_i is a surjection, for every $i \in I$, then $\Sigma\Psi$ is an surjection.*
(v) *If $\Phi : M \Rightarrow \Lambda$, where Φ_i is a modulus of surjectivity for Ψ_i, for every $i \in I$, then a modulus of surjectivity for $\Sigma\Psi$ is the operation $\sigma\Psi\colon \sum_{i\in I} \mu_0(i) \rightsquigarrow \sum_{i\in I} \lambda_0(i)$, defined by*

$$\sigma\Psi(i, y) := (i, \Phi_i(y)); \qquad (i, y) \in \sum_{i\in I} \mu_0(i).$$

Proof. (i) If $x, y \in \lambda_0(i)$, then $e_i^{\Lambda}(x) =_{\sum_{i\in I} \lambda_0(i)} e_i^{\Lambda}(y)$ if and only if $(i, x) =_{\sum_{i\in I} \lambda_0(i)} (i, y)$, which is equivalent to $\lambda_{ii}(x) =_{\lambda_0(i)} y :\Leftrightarrow x =_{\lambda_0(i)} y$.
(ii) If $(i, x) =_{\sum_{i\in I} \lambda_0(i)} (j, y)$ i.e., $i =_I j$ and $\lambda_{ij}(x) =_{\lambda_0(j)} y$, we show that $(i, \Psi_i(x)) =_{\sum_{i\in I} \mu_0(i)} (j, \Psi_j(y))$ i.e., $i =_I j$ and $\mu_{ij}(\Psi_i(x)) =_{\mu_0(j)} \Psi_j(y)$. Since $\Psi\colon \Lambda \Rightarrow M$, we get $\mu_{ij}(\Psi_i(x)) =_{\mu_0(j)} \Psi_j(\lambda_{ij}(x)) =_{\mu_0(j)} \Psi_j(y)$. The required commutativity of the diagram is immediate to show.
(iii) Since Ψ is a family-map from Λ to M, we have that

$$\begin{aligned} \Sigma\Psi(i, x) =_{\sum_{i\in I} \mu_0(i)} \Sigma\Psi(j, y) &:\Leftrightarrow (i, \Psi_i(x)) =_{\sum_{i\in I} \mu_0(i)} (j, \Psi_j(y)) \\ &:\Leftrightarrow i =_I j \ \&\ \mu_{ij}(\Psi(x)) =_{\mu_0(j)} \Psi_j(y) \\ &\Leftrightarrow i =_I j \ \&\ \Psi_j(\lambda_{ij}(x)) =_{\mu_0(j)} \Psi_j(y) \\ &\Rightarrow i =_I j \ \&\ \lambda_{ij}(x) =_{\lambda_0(j)} y \\ &:\Leftrightarrow (i, x) =_{\sum_{i\in I} \lambda_0(i)} (j, y). \end{aligned}$$

(iv) Let $(i,y) \in \sum_{i\in I}\mu_0(i)$. Since Ψ_i is a surjection, there is $x \in \lambda_0(i)$ such that $\Psi_i(x) = y$. Hence $\Sigma\Psi(i,x) := (i, \Psi_i(x)) =_{\sum_{i\in I}\mu_0(i)} (i,y)$, since $\mu_{ii}\big(\Psi_i(x)\big) := \Psi_i(x) =_{\mu_0(i)} y$.
(v) If $y \in \mu_0(i)$, then $\Psi_i(\Phi_i(y)) =_{\mu_0(i)} y$. To show that the operation $\sigma\Psi$ is a function, we suppose $(i,y) =_{\sum_{i\in I}\mu_0(i)} (j,z) :\Leftrightarrow i =_I j \ \&\ \mu_{ij}(y) =_{\mu_0(j)} z$ and we show that $(i,\sigma_i(y)) =_{\sum_{i\in I}\lambda_0(i)} (j,\sigma_j(z)) :\Leftrightarrow i =_I j \ \&\ \lambda_{ij}(\Phi_i(y)) =_{\lambda_0(j)} \Phi_j(z)$. Since $\Phi_j \colon \mu_0(j) \to \lambda_0(j)$, we have that $\mu_{ij}(y) =_{\mu_0(j)} z \Rightarrow \Phi_j\big(\mu_{ij}(y)\big) =_{\lambda_0(j)} \Phi_j(z)$. By the commutativity of the diagram

$$\begin{tikzcd}
\mu_0(i) \arrow[r, "\mu_{ij}"] \arrow[d, "\Phi_i"'] & \mu_0(j) \arrow[d, "\Phi_j"] \\
\lambda_0(i) \arrow[r, "\lambda_{ij}"'] & \lambda_0(j),
\end{tikzcd}$$

$\Phi_j(z) =_{\lambda_0(j)} \Phi_j\big(\mu_{ij}(y)\big) =_{\lambda_0(j)} \lambda_{ij}(\Phi_i(y))$. Since $\mu_{ii}\big(\Psi_i(\Phi_i(y))\big) := \Psi_i(\Phi_i(y))\big) =_{\mu_0(i)} y$,

$$\Sigma\Psi\big(\sigma\Psi(i,y)\big) := \Sigma\Psi\big((i,\Phi_i(y))\big) := \big(i, \Psi_i(\Phi_i(y))\big) =_{\sum_{i\in I}\mu_0(i)} (i,y). \qquad \square$$

Definition 3.2.8. *Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The first projection on $\sum_{i\in I}\lambda_0(i)$ is the operation $\mathtt{pr}_1^{\Lambda} \colon \sum_{i\in I}\lambda_0(i) \rightsquigarrow I$, defined by $pr_1^{\Lambda}(i,x) := \mathbf{pr}_1(i,x) := i$, for every $(i,x) \in \sum_{i\in I}\lambda_0(i)$. We may only write $\mathtt{pr}_1$, if Λ is clearly understood from the context.*

By the definition of the canonical equality on $\sum_{i\in I}\lambda_0(i)$ we get that $\mathtt{pr}_1^{\Lambda}$ is a function.

Definition 3.2.9. *Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The $\sum$-indexing of Λ is the pair $\Sigma^{\Lambda} := (\sigma_0^{\Lambda}, \sigma_1^{\Lambda})$, where $\sigma_0^{\Lambda} \colon \sum_{i\in I}\lambda_0(i) \rightsquigarrow \mathbb{V}_0$ is defined by $\sigma_0^{\Lambda}(i,x) := \lambda_0(i)$, for every $(i,x) \in \sum_{i\in I}\lambda_0(i)$, and $\sigma_1^{\Lambda}\big((i,x),(j,y)\big) := \lambda_{ij}$, for every $\big((i,x),(j,y)\big) \in D\big(\sum_{i\in I}\lambda_0(i)\big)$.*

Clearly, Σ^{Λ} is a family of sets over $\sum_{i\in I}\lambda_0(i)$, and $\Sigma \colon \Sigma^{\Lambda} \overset{\mathtt{pr}_1^{\Lambda}}{\Longrightarrow} \Lambda$ (see Definition 3.1.5), where, if $w := (i,x) \in \sum_{i\in I}\lambda_0(i)$, we define $\Sigma_w : \lambda_0(i) \to \lambda_0(\mathtt{pr}_1^{\Lambda}(w))$ to be the identity $\mathrm{id}_{\lambda_0(i)}$.

Definition 3.2.10. *Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The second projection on $\sum_{i\in I}\lambda_0(i)$ is the dependent operation $\mathtt{pr}_2^{\Lambda} \colon \bigcurlywedge_{(i,x)\in\sum_{i\in I}\lambda_0(i)}\lambda_0(i)$, defined by $\mathtt{pr}_2^{\Lambda}(i,x) := \mathbf{pr}_2(i,x) := x$, for every $(i,x) \in \sum_{i\in I}\lambda_0(i)$. We may only write $\mathtt{pr}_2$, when the family of sets Λ is clearly understood from the context.*

In Remark 3.3.2 we show that $\mathtt{pr}_2^{\Lambda}$ is a dependent function over the family Σ^{Λ}.

3.3 Dependent functions over a family of sets

Definition 3.3.1. *Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The totality $\prod_{i\in I}\lambda_0(i)$ of dependent functions over Λ, or the $\prod$-set of Λ, is defined by*

$$\Theta \in \prod_{i\in I}\lambda_0(i) :\Leftrightarrow \Theta \in \mathbb{A}(I,\lambda_0) \ \&\ \forall_{(i,j)\in D(I)}\big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}(\Theta_i)\big),$$

and it is equipped with the canonical equality and the canonical inequality of the set $\mathbb{A}(I,\lambda_0)$. If X is a set and Λ^X is the constant I-family X (see Definition 3.1.1), we use the notation

$$X^I := \prod_{i\in I} X.$$

Clearly, the property $P(\Phi) :\Leftrightarrow \forall_{(i,j)\in D(I)}\big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}(\Theta_i)\big)$ is extensional on $\mathbb{A}(I, \lambda_0)$, the equality on $\prod_{i\in I} \lambda_0(i)$ is an equivalence relation. $\prod_{i\in I} \lambda_0(i)$ is considered to be a set.

Remark 3.3.2. *If $\Lambda := (\lambda_0, \lambda_1)$ is an I-family of sets and $\Sigma^\Lambda := (\sigma_0^\Lambda, \sigma_1^\Lambda)$ is the $\sum$-indexing of Λ, then $\mathtt{pr}_2^\Lambda$ is a dependent function over Σ^Λ.*

Proof. By Definition 3.2.10 the second projection $\mathtt{pr}_2^\Lambda$ of Λ is the dependent assignment $\mathtt{pr}_2^\Lambda \colon \bigcurlywedge_{(i,x)\in\sum_{i\in I}\lambda_0(i)} \lambda_0(i)$, such that $\mathtt{pr}_2^\Lambda(i,x) := x$, for every $(i,x) \in \sum_{i\in I} \lambda_0(i)$. It suffices to show that if $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y$, then

$$\mathtt{pr}_2^\Lambda(j,y) := y =_{\lambda_0(j)} \lambda_{ij}(x) := \sigma_1^\Lambda\big((i,x),(j,y)\big)\big(\mathtt{pr}_2^\Lambda(i,x)\big). \qquad \square$$

Remark 3.3.3. (i) *If Λ^2 is the 2-family of the sets X and Y, then $\prod_{i\in 2} \lambda_0^2(i) =_{\mathbb{V}_0} X \times Y$.*
(ii) *If I, A are sets, and $\Lambda := (\lambda_0^A, \lambda_1)$ is the constant I-family A, then $A^I =_{\mathbb{V}_0} \mathbb{F}(I, A)$.*

Proof. (i) Let $f \colon \prod_{i\in 2} \lambda_0^2(i) \rightsquigarrow X\times Y$ be defined by $f(\Phi) := (\Phi_0, \Phi_1)$, for every $\Phi \in \prod_{i\in 2} \lambda_0^2(i)$. Let $g \colon X \times Y \rightsquigarrow \prod_{i\in 2} \lambda_0^2(i)$ be defined by $g(x,y) := \Phi_{(x,y)}$, for every $(x,y) \in X \times Y$. It is easy to show that f, g are well-defined functions and $(f,g) \colon \prod_{i\in 2} \lambda_0^2(i) =_{\mathbb{V}_0} X \times Y$.
(ii) Let $h \colon A^I \rightsquigarrow \mathbb{F}(I,A)$ be defined by $h(\Phi) := h_\Phi \colon I \to A$, where $h_\Phi(i) := \Phi_i$, for every $\Phi \in A^I$ and $i \in I$. Let $k \colon \mathbb{F}(I,A) \rightsquigarrow A^I$ be defined by $k(e) := \Phi_e$, where $[\Phi_e]_i := e(i)$, for every $e \in \mathbb{F}(I,A)$ and $i \in I$. Then h, k are well-defined functions and $(h,k) \colon A^I =_{\mathbb{V}_0} \mathbb{F}(I,A)$. $\square$

Corollary 3.3.4. *If $\Lambda, M \in \mathtt{Fam}(I)$ and $\Psi \colon \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$, the following are equivalent:*
(i) *$\Psi \colon \Lambda \Rightarrow M$.*
(ii) *$\Psi \in \prod_{i\in I} \big[\mathbb{F}(\lambda_0, \mu_0)\big](i)$.*

Proof. If $i =_I j$, the commutativity of the following left diagram

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Phi_i"'] & \lambda_0(j) \arrow[d, "\Phi_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j) \end{tikzcd} \qquad \begin{tikzcd} \lambda_0(i) \arrow[d, "\Phi_i"'] & \lambda_0(j) \arrow[l, "\lambda_{ji}"'] \arrow[d, "\Phi_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j), \end{tikzcd}$$

is equivalent to the commutativity of the above right one, hence the defining condition for $\Psi \in \mathtt{Map}(\Lambda, M)$ is equivalent to the defining condition $\Psi_j = \mathbb{F}(\lambda_1, \mu_1)_{ij}(\Psi_i) := \mu_{ij} \circ \Psi_i \circ \lambda_{ji}$ for $\Psi \in \prod_{i\in I} \big(\lambda_0(i) \times \mu_0(i)\big)$ (see Definition 3.1.6(ii)). $\square$

Proposition 3.3.5. *Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, and $\Psi : \Lambda \Rightarrow M$.*
(i) *If $i \in I$, the operation $\pi_i^\Lambda : \prod_{i\in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, defined by $\Theta \mapsto \Theta_i$, is a function.*
(ii) *The operation $\Pi\Psi : \prod_{i\in I} \lambda_0(i) \rightsquigarrow \prod_{i\in I} \mu_0(i)$, defined by*

$$[\Pi\Psi(\Theta)]_i := \Psi_i(\Theta_i); \qquad i \in I,$$

is a function, such that for every $i \in I$ the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\prod_{i \in I} \lambda_0(i) \arrow[u, "\pi_i^{\Lambda}"] \arrow[r, "\Pi\Psi"'] & \prod_{i \in I} \mu_0(i). \arrow[u, "\pi_i^{M}"']
\end{tikzcd}$$

(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Pi\Psi$ is an embedding.*

(iv) *If $\Phi : M \Rightarrow \Lambda$ such that Φ_i is a modulus of surjectivity for Ψ_i, for every $i \in I$, the operation $\pi\Psi \colon \prod_{i \in I} \mu_0(i) \rightsquigarrow \prod_{i \in I} \lambda_0(i)$ is a modulus of surjectivity for $\prod \Psi$, where*

$$\big[\pi\Psi(\Omega)\big]_i := \Phi_i(\Omega_i); \qquad \Omega \in \prod_{i \in I} \mu_0(i), \ i \in I.$$

Proof. (i) This follows immediately from the definition of equality on $\prod_{i \in I} \lambda_0(i)$.
(ii) First we show that $\Pi\Psi$ is well-defined i.e., $\Pi\Psi(\Theta) \in \prod_{i \in I} \mu_0(i)$. If $i =_I j$, then by the commutativity of the following left diagram from the definition of a family-map

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"] \\
\mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j)
\end{tikzcd} \qquad
\begin{tikzcd}
\mu_0(i) \arrow[r, "\mu_{ij}"] \arrow[d, "\Phi_i"'] & \mu_0(j) \arrow[d, "\Phi_j"] \\
\lambda_0(i) \arrow[r, "\lambda_{ij}"'] & \lambda_0(j),
\end{tikzcd}$$

$$[\Pi\Psi(\Theta)]_j := \Psi_j(\Theta_j) = \Psi_j\big(\lambda_{ij}(\Theta_i)\big) = \mu_{ij}\big(\Psi_i(\Theta_i)\big) := \mu_{ij}\big([\Pi\Psi(\Theta)]_i\big).$$

It is immediate to show that $\Pi\Psi$ is a function and that the required diagram commutes.
(iii) If $\Theta, \Theta' \in \prod_{i \in I} \lambda_0(i)$, then

$$\begin{aligned}
\Pi\Psi(\Theta) =_{\prod_{i \in I} \mu_0(i)} \Pi\Psi(\Theta') &:\Leftrightarrow \forall_{i \in I}\big(\Psi_i(\Theta_i) =_{\mu_0(i)} \Psi_i(\Theta'_i)\big) \\
&\Rightarrow \forall_{i \in I}\big(\Theta_i =_{\lambda_0(i)} \Theta'_i\big) \\
&:\Leftrightarrow \Theta =_{\prod_{i \in I} \lambda_0(i)} \Theta'.
\end{aligned}$$

(iv) First we show that $\pi\Psi$ is well-defined i.e., $\pi\Psi(\Omega) \in \prod_{i \in I} \lambda_0(i)$. If $i =_I j$, and since $\Phi : M \Rightarrow \Lambda$, by the commutativity of the above right diagram

$$\big[\pi\Psi(\Omega)\big]_j := \Phi_j(\Omega_j) =_{\lambda_0(j)} \Phi_j\big(\mu_{ij}(\Omega_i)\big) =_{\lambda_0(j)} \lambda_{ij}\big(\Phi_i(\Omega_i)\big) := \lambda_{ij}\big(\big[\pi\Psi(\Omega)\big]_i\big).$$

It is immediate to show that $\pi\Psi$ is a function. Finally we show that $\Pi\Psi\big(\pi\Psi(\Omega)\big) = \Omega$, for every $\Omega \in \prod_{i \in I} \mu_0(i)$. If $i \in I$, and since Φ_i is a modulus of surjectivity for Ψ_i, we get

$$\big[\Pi\Psi\big(\pi\Psi(\Omega)\big)\big]_i := \Psi_i\big(\big[\pi\Psi(\Omega)\big]_i\big) := \Psi_i\big(\Phi_i(\Omega_i)\big) = \Omega_i. \qquad \square$$

Proposition 3.3.6. *If $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, then*

$$\prod_{i \in I} \big(\lambda_0(i) \times \mu_0(i)\big) =_{\mathbb{V}_0} \bigg(\prod_{i \in I} \lambda_0(i)\bigg) \times \bigg(\prod_{i \in I} \mu_0(i)\bigg),$$

$$\mathtt{Map}_I(\Lambda, M) =_{\mathbb{V}_0} \prod_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big).$$

Proof. Let the operation $f\colon \prod_{i\in I}\big(\lambda_0(i)\times\mu_0(i)\big) \rightsquigarrow \prod_{i\in I}\lambda_0(i)\times\prod_{i\in I}\mu_0(i)$ be defined by $f(\Phi) := \big(\mathtt{pr}_1(\Phi), \mathtt{pr}_2(\Phi)\big)$, for every $\Phi\in\prod_{i\in I}\big(\lambda_0(i)\times\mu_0(i)\big)$, where $\mathtt{pr}_1(\Phi)_i := \mathtt{pr}_1(\Phi_i)$ and $\mathtt{pr}_2(\Phi)_i := \mathtt{pr}_2(\Phi_i)$, for every $i\in I$. Using Definition 3.1.6(i),

$$\mathtt{pr}_1(\Phi)_j := \mathtt{pr}_1(\Phi_j) = \mathtt{pr}_1\big((\lambda_1\times\mu_1)_{ij}(\Phi_i)\big) := \mathtt{pr}_1\big(\lambda_{ij}(\mathtt{pr}_1(\Phi)_i), \mu_{ij}(\mathtt{pr}_2(\Phi)_i\big) := \lambda_{ij}(\mathtt{pr}_1(\Phi)_i),$$

hence $\mathtt{pr}_1(\Phi)\in\prod_{i\in I}\lambda_0(i)$. Similarly, $\mathtt{pr}_2(\Phi)\in\prod_{i\in I}\mu_0(i)$. It is immediate to show that the operation f is a function. Let $g\colon \prod_{i\in I}\lambda_0(i)\times\prod_{i\in I}\mu_0(i)\rightsquigarrow\prod_{i\in I}\big(\lambda_0(i)\times\mu_0(i)\big)$ be defined by $g(\Psi,\Xi) := \Phi$, for every $\Psi\in\prod_{i\in I}\lambda_0(i)$ and $\Xi\in\prod_{i\in I}\mu_0(i)$, where $\Phi_i := (\Psi_i,\Xi_i)$, for every $i\in I$. We show that g is well-defined i.e., $\Phi\in\prod_{i\in I}\big(\lambda_0(i)\times\mu_0(i)\big)$. If $i =_I j$, then

$$(\lambda_1\times\mu_1)_{ij}(\Phi_i) := (\lambda_1\times\mu_1)_{ij}(\Psi_i,\Xi_i) := \big(\lambda_{ij}(\Psi_i),\mu_{ij}(\Xi_i)\big) = (\Psi_j,\Xi_j) := \Phi_j.$$

Clearly, f,g are inverse to each other. For the equality $\mathtt{Map}_I(\Lambda, M) =_{\mathbb{V}_0} \prod_{i\in I}\mathbb{F}\big(\lambda_0(i),\mu_0(i)\big)$, we use Coroallry 3.3.4 and the corresponding identity maps are its witnesses. □

3.4 Subfamilies of families of sets

Definition 3.4.1. *Let $\Lambda := (\lambda_0,\lambda_1)\in\mathtt{Fam}(I)$ and $h\colon J\to I$. The pair $\Lambda\circ h := (\lambda_0\circ h, \lambda_1\circ h)$, defined in Definition 3.1.6, is called the h-subfamily of Λ, and we write $(\Lambda\circ h)_J \leq \Lambda_I$. If $J := \mathbb{N}$, we call $\Lambda\circ h$ the h-subsequence of Λ.*

Remark 3.4.2. *If $\Lambda\in\mathtt{Set}(I)$, then $\Lambda\circ h\in\mathtt{Set}(J)$ if and only h is an embedding.*

Proof. Let $\Lambda\circ h\in\mathtt{Set}(J)$ and $h(j) =_I h(j')$, hence $(\lambda_{h(j)h(j')},\lambda_{h(j')h(j)})\colon \lambda_0(h(j)) =_{\mathbb{V}_0} \lambda_0(h(j'))$, and $j =_J j'$. If h is an embedding and $(\lambda_0\circ h)(j) =_{\mathbb{V}_0} (\lambda\circ h)(j') :\Leftrightarrow \lambda_0(j(j)) =_{\mathbb{V}_0} \lambda_0(h(j'))$, then $h(j) =_I h(j')$, since $\Lambda\in\mathtt{Set}(I)$, and hence $j =_J j'$. □

Remark 3.4.3. *Let $\Lambda, M\in\mathtt{Fam}(I)$, $h\in\mathbb{F}(J,I)$ and $g\in\mathbb{F}(I,K)$.*

(i) $\Lambda\circ\mathrm{id}_I := \Lambda$.

(ii) $(\Lambda\circ g)\circ h := \Lambda\circ(g\circ h)$.

(iii) *If $\Phi\colon\Lambda\Rightarrow M$, then $\Phi\circ h\colon\Lambda\circ h\Rightarrow M\circ h$, where*

$$(\Phi\circ h)_j\colon \lambda_0(h(j))\to\mu_0(h(j)), \quad (\Phi\circ h)_j := \Phi_{h(j)}; \quad j\in J.$$

(iv) *If $\Phi\colon\Lambda\Rightarrow M$, then $\Phi^h\colon\Lambda\circ h\overset{h}{\Rightarrow} M$, where*

$$\Phi^h_j\colon \lambda_0(h(j))\to\mu_0(h(j)), \quad \Phi^h_j := \Phi_{h(j)}; \quad j\in J.$$

(iv) $(\Lambda\circ h)\times(M\circ h) := (\Lambda\times M)\circ h$.

(v) $\mathbb{F}\big((\Lambda\circ h),(M\circ h)\big) := \mathbb{F}(\Lambda, M)\circ h$.

Proof. All cases are straightforward to show. □

Proposition 3.4.4. *Let $\Lambda\in\mathtt{Fam}(I)$, and $h\colon J\to I$.*

(i) *The operation $\sum_h\colon \sum_{j\in J}\lambda_0(h(j))\rightsquigarrow\sum_{i\in I}\lambda_0(i)$, defined by*

$$\sum_h(j,u) := (h(j),u); \quad (j,u)\in\sum_{j\in J}\lambda_0(h(j)),$$

is a function, and it is an embedding if h is an embedding.
(ii) *The operation $\prod_h \colon \prod_{i\in I} \lambda_0(i) \rightsquigarrow \prod_{j\in J} \lambda_0(h(j))$, defined by*

$$\Phi \mapsto \prod_h(\Phi), \qquad \Big(\prod_h \Phi\Big)_j := \Phi_{h(j)}; \qquad \Phi \in \prod_{i\in I} \lambda_0(i),\ j \in J,$$

is a function, and if h is an embedding, then $\prod_h$ is an embedding.

Proof. (i) By definition we have that

$$(j, u) =_{\sum_{j\in J}(\lambda_0\circ h)(j)} (j', u') :\Leftrightarrow j =_J j' \ \&\ \lambda_{h(j)h(j')}(u) =_{\lambda_0(h(j')} u',$$

$$(h(j), u) =_{\sum_{i\in I}\lambda_0(i)} (h(j'), u') :\Leftrightarrow h(j) =_I h(j') \ \&\ \lambda_{h(j)h(j')}(u) =_{\lambda_0(h(j')} u'.$$

Since h is a function, the operation $\sum_h$ is a function. If h is an embedding, it is immediate to show that $\sum_h$ is an embedding.
(ii) First we show that $\prod_h$ is well-defined. If $j =_J j'$, then

$$\Big(\prod_h \Phi\Big)_{j'} := \Phi_{h(j')} =_{\lambda_0(h(j'))} \lambda_{h(j)h(j')}\big(\Phi_{h(j)}\big) := (\lambda_1 \circ h)_{jj'}\bigg(\Big(\prod_h \Phi\Big)_j\bigg).$$

It is immediate to show that $\prod_h$ is a function. Let h be a surjection and let $\Phi, \Theta \in \prod_{i\in I} \lambda_0(i)$ such that $\prod_h(\Phi) =_{\prod_{j\in J}\lambda_0(h(j))} \prod_h(\Theta)$. If $i \in I$, let $j \in J$ with $h(j) =_I i$. As $\Phi_i =_{\lambda_0(i)} \lambda_{h(j)i}\Phi_{h(j)}$ and $\Theta_i =_{\lambda_0(i)} \lambda_{h(j)i}\Theta_{h(j)}$, and since $\Phi_{h(j)} =_{\lambda_0(h(j))} \Theta_{h(j)}$, we get $\Phi_i =_{\lambda_0(i)} \Theta_i$. $\square$

3.5 Families of sets over products

Proposition 3.5.1. *Let $\Lambda := (\lambda_0, \lambda_1), K := (k_0, k_1) \in \mathtt{Fam}(I)$ and $M := (\mu_0, \mu_1), N := (\nu_0, \nu_1) \in \mathtt{Fam}(J)$.*
(i) *$\Lambda \otimes M := (\lambda_0 \otimes \mu_0, \lambda_1 \otimes \mu_1) \in \mathtt{Fam}(I \times J)$, where $\lambda_0 \otimes \mu_0 \colon I \times J \rightsquigarrow \mathbb{V}_0$ is defined by*

$$(\lambda_0 \otimes \mu_0)(i, j) := \lambda_0(i) \times \mu_0(j); \qquad (i, j) \in I \times J,$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i'j')} \colon \lambda_0(i) \times \mu_0(j) \to \lambda_0(i') \times \mu_0(j'),$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i'j')}(u, w) := \big(\lambda_{ii'}(u), \mu_{jj'}(w)\big); \qquad (u, w) \in \lambda_0(i) \times \mu_0(j).$$

(ii) *If $\Phi \colon \Lambda \Rightarrow K$ and $\Psi \colon M \Rightarrow N$, then $\Phi \otimes \Psi \colon \Lambda \otimes M \Rightarrow K \otimes N$, where, for every $(i, j) \in I \times J$,*

$$(\Phi \otimes \Psi)_{(i,j)} \colon \lambda_0(i) \times \mu_0(j) \to k_0(i) \times \nu_0(j),$$

$$(\Phi \otimes \Psi)_{(i,j)}(u, w) := \big(\Phi_i(u), \Psi_j(w)\big); \qquad (u, w) \in \lambda_0(i) \times \mu_0(j).$$

(iii) *The following equalities hold*

$$\sum_{(i,j)\in I\times J} \big(\lambda_0(i) \times \mu_0(j)\big) =_{\mathbb{V}_0} \Big(\sum_{i\in I} \lambda_0(i)\Big) \times \Big(\sum_{j\in J} \mu_0(j)\Big),$$

$$\prod_{(i,j)\in I\times J} \big(\lambda_0(i) \times \mu_0(j)\big) =_{\mathbb{V}_0} \Big(\prod_{i\in I} \lambda_0(i)\Big) \times \Big(\prod_{j\in J} \mu_0(j)\Big).$$

Proof. (i) The proof is straightforward.
(ii) We show the required following commutativity by the following supposed ones by

$$\begin{tikzcd}
\lambda_0(i) \times \mu_0(j) \arrow[rr, "(\lambda_1 \otimes \mu_1)_{(i,j)(i',j')}"] \arrow[d, "(\Phi \otimes \Psi)_{(i,j)}"'] & & \lambda_0(i') \times \mu_0(j') \arrow[d, "(\Phi \otimes \Psi)_{(i',j')}"] \\
k_0(i) \times \nu_0(j) \arrow[rr, "(k_1 \otimes \nu_1)_{(i,j)(i',j')}"'] & & k_0(i') \times \nu_0(j')
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ii'}"] \arrow[d, "\Phi_i"'] & \lambda_0(i') \arrow[d, "\Phi_{i'}"] \\
k_0(i) \arrow[r, "k_{ii'}"'] & k_0(i')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu_0(j) \arrow[r, "\mu_{jj'}"] \arrow[d, "\Psi_j"'] & \mu_0(j') \arrow[d, "\Psi_{j'}"] \\
\mu_0(j) \arrow[r, "\nu_{jj'}"'] & \mu_0(j'),
\end{tikzcd}$$

$$\begin{aligned}
(\Phi \otimes \Psi)_{(i',j')}\big((\lambda_1 \otimes \mu_1)_{(i,j)(i',j')}\big) &:= (\Phi \otimes \Psi)_{(i',j')}\big(\lambda_{ii'}(u), \mu_{jj'}(w)\big) \\
&:= \big(\Phi_{i'}\big(\lambda_{ii'}(u)\big), \Psi_{j'}\big(\mu_{jj'}(w)\big)\big) \\
&= \big(k_{ii'}\big(\Phi_i(u)\big), \nu_{jj'}\big(\Psi_j(w)\big)\big) \\
&:= (k_1 \otimes \nu_1)_{(i,j)(i',j')}\big(\Phi_i(u), \Psi_j(w)\big) \\
&:= (k_1 \otimes \nu_1)_{(i,j)(i',j')}\big((\Phi \otimes \Psi)_{(i,j)}(u,w)\big).
\end{aligned}$$

(iii) For the equality on $\sum_{(i,j)\in I\times J}\big(\lambda_0(i) \times \mu_0(j)\big)$ we have that

$$\big((i,j),(u,w)\big) =_{\sum_{(i,j)\in I\times J}(\lambda_0(i)\times\mu_0(j))} \big((i',j'),(u',w')\big) :\Leftrightarrow i =_I i' \ \& \ j =_J j' \ \&$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i',j')}(u,w) =_{\lambda_0(i')\times\mu_0(j')} (u',w') :\Leftrightarrow \lambda_{ii'}(u) =_{\lambda_0(i')} u' \ \& \ \mu_{jj'}(w) =_{\mu_0(j')} w'.$$

For the equality on $\big(\sum_{i\in I}\lambda_0(i)\big) \times \big(\sum_{j\in J}\mu_0(j)\big)$ we have that

$$\big((i,u),(j,w)\big) =_{\big(\sum_{i\in I}\lambda_0(i)\big)\times\big(\sum_{j\in J}\mu_0(j)\big)} \big((i',u'),(j',w')\big) :\Leftrightarrow$$

$$(i,u) =_{\sum_{i\in I}\lambda_0(i)} (i',u') \ \& \ (j,w) =_{\sum_{j\in J}\mu_0(j)} (j',w'),$$

i.e., if $i =_I i'$ and $\lambda_{ii'}(u) =_{\lambda_0(i')} u'$, and $j =_J j'$ and $\mu_{jj'}(w) =_{\mu_0(j')} w'$. As the equality conditions for the two sets are equivalent, the operation $\phi\colon \big(\sum_{i\in I}\lambda_0(i)\big) \times \big(\sum_{j\in J}\mu_0(j)\big) \rightsquigarrow \sum_{(i,j)\in I\times J}\big(\lambda_0(i) \times \mu_0(j)\big)$, defined by the rule $\big((i,u),(j,w)\big) \mapsto \big((i,j),(u,w)\big)$, together with the operation $\theta\colon \sum_{(i,j)\in I\times J}\big(\lambda_0(i) \times \mu_0(j)\big) \rightsquigarrow \big(\sum_{i\in I}\lambda_0(i)\big) \times \big(\sum_{j\in J}\mu_0(j)\big)$, defined by the inverse rule $\big((i,j),(u,w)\big) \mapsto \big((i,u),(j,w)\big)$, are well-defined functions that witness the required equality of the two sets in $\mathbb{V}_0$.
(iv) We proceed similarly to the proof of Proposition 3.3.6. $\square$

Next we define new families of sets generated by a given family of sets indexed by the product $X \times Y$ of X and Y. These families will also be used in section 5.1.

Definition 3.5.2. *Let X, Y be sets, and let $R := (\rho_0, \rho_1)$ be an $(X \times Y)$-family of sets.*
(i) *If $x \in X$, the x-component of R is the pair $R^x := (\rho_0^x, \rho_1^x)$, where the assignment routines $\rho_0^x \colon Y \rightsquigarrow \mathbb{V}_0$ and $\rho_1^x \colon \curlywedge_{(y,y') \in D(Y)} \mathbb{F}\big(\rho_0^x(y), \rho_0^x(y')\big)$ are defined by $\rho_0^x(y) := \rho_0(x, y)$, for every $y \in Y$, and $\rho_1^x(y, y') := \rho_{yy'}^x := \rho_{(x,y)(x,y')}$, for every $(y, y') \in D(Y)$.*
(ii) *If $y \in Y$, the y-component of R is the pair $R^y := (\rho_0^y, \rho_1^y)$, where the assignment routines $\rho_0^y \colon Y \rightsquigarrow \mathbb{V}_0$ and $\rho_1^y \colon \curlywedge_{(x,x') \in D(X)} \mathbb{F}\big(\rho_0^y(x), \rho_0^y(x')\big)$ are defined by $\rho_0^y(x) := \rho_0(x, y)$, for every $x \in X$, and $\rho_1^y(x, x') := \rho_{xx'}^y := \rho_{(x,y)(x',y)}$, for every $(x, x') \in D(X)$.*
(iii) *Let $\sum^1 R := (\sum^1 \rho_0, \sum^1 \rho_1)$, where $\sum^1 \rho_0 : X \rightsquigarrow \mathbb{V}_0$ and*

$$\sum^1 \rho_1 \colon \curlywedge_{(x,x') \in D(X)} \mathbb{F}\Big(\big(\sum^1 \rho_0\big)(x), \big(\sum^1 \rho_0\big)(x')\Big) \quad \textit{are defined by}$$

$$\Big(\sum^1 \rho_0\Big)(x) := \sum_{y \in Y} \rho_0^x(y) := \sum_{y \in Y} \rho_0(x, y); \qquad x \in X,$$

$$\Big(\sum^1 \rho_1\Big)(x, x') := \Big(\sum^1 \rho_1\Big)_{xx'} \colon \sum_{y \in Y} \rho_0(x, y) \to \sum_{y \in Y} \rho_0(x', y); \qquad (x, x') \in D(X),$$

$$\Big(\sum^1 \rho_1\Big)_{xx'}(y, u) := \big(y, \rho_{(x,y)(x',y)}(u)\big); \qquad (y, u) \in \sum_{y \in Y} \rho_0(x, y).$$

(iv) *Let $\sum^2 R := (\sum^2 \rho_0, \sum^2 \rho_1)$, where $\sum^2 \rho_0 : Y \rightsquigarrow \mathbb{V}_0$ and*

$$\sum^2 \rho_1 \colon \curlywedge_{(y,y') \in D(X)} \mathbb{F}\Big(\big(\sum^2 \rho_0\big)(y), \big(\sum^2 \rho_0\big)(y')\Big) \quad \textit{are defined by}$$

$$\Big(\sum^2 \rho_0\Big)(y) := \sum_{x \in X} \rho_0^y(x) := \sum_{x \in X} \rho_0(x, y); \qquad y \in Y,$$

$$\Big(\sum^2 \rho_1\Big)(y, y') := \Big(\sum^2 \rho_1\Big)_{yy'} \colon \sum_{x \in X} \rho_0(x, y) \to \sum_{x \in X} \rho_0(x, y'); \qquad (y, y') \in D(Y),$$

$$\Big(\sum^2 \rho_1\Big)_{yy'}(x, w) := \big(x, \rho_{(x,y)(x,y')}(w)\big); \qquad (x, w) \in \sum_{x \in X} \rho_0(x, y).$$

(v) *Let $\prod^1 R := (\prod^1 \rho_0, \prod^1 \rho_1)$, where $\prod^1 \rho_0 : X \rightsquigarrow \mathbb{V}_0$ and*

$$\prod^1 \rho_1 \colon \curlywedge_{(x,x') \in D(X)} \mathbb{F}\Big(\big(\prod^1 \rho_0\big)(x), \big(\prod^1 \rho_0\big)(x')\Big) \quad \textit{are defined by}$$

$$\Big(\prod^1 \rho_0\Big)(x) := \prod_{y \in Y} \rho_0^x(y) := \prod_{y \in Y} \rho_0(x, y); \qquad x \in X,$$

$$\Big(\prod^1 \rho_1\Big)(x,x') := \Big(\prod^1 \rho_1\Big)_{xx'} : \prod_{y \in Y} \rho_0(x,y) \to \prod_{y \in Y} \rho_0(x',y); \qquad (x,x') \in D(X),$$

$$\bigg[\Big(\prod^1 \rho_1\Big)_{xx'}(\Theta)\bigg]_y := \rho_{(x,y)(x',y)}(\Theta_y)); \qquad \Theta \in \prod_{y \in Y} \rho_0(x,y),\ y \in Y.$$

(vi) *Let* $\prod^2 R := (\prod^2 \rho_0, \prod^2 \rho_1)$*, where* $\prod^2 \rho_0 : Y \rightsquigarrow \mathbb{V}_0$ *and*

$$\prod^2 \rho_1 \colon \bigcurlywedge_{(y,y') \in D(X)} \mathbb{F}\Big((\prod^2 \rho_0)(y), (\prod^2 \rho_0)(y') \Big) \quad \textit{are defined by}$$

$$\Big(\prod^2 \rho_0\Big)(y) := \prod_{x \in X} \rho_0^y(x) := \prod_{x \in X} \rho_0(x,y); \qquad y \in Y,$$

$$\Big(\prod^2 \rho_1\Big)(y,y') := \Big(\prod^2 \rho_1\Big)_{yy'} : \prod_{x \in X} \rho_0(x,y) \to \prod_{x \in X} \rho_0(x,y'); \qquad (y,y') \in D(Y),$$

$$\bigg[\Big(\prod^2 \rho_1\Big)_{yy'}(\Phi)\bigg]_x := \rho_{(x,y)(x,y')}(\Phi_x)); \qquad \Phi \in \prod_{x \in X} \rho_0(x,y),\ x \in X.$$

It is easy to show that $R^y, \sum^1 R, \prod^1 R \in \mathtt{Fam}(X)$ and $R^x, \sum^2 R, \prod^2 R \in \mathtt{Fam}(Y)$.

Proposition 3.5.3. *Let* $X, Y \in \mathbb{V}_0$*,* $R := (\rho_0, \rho_1), S := (\sigma_0, \sigma_1) \in \mathtt{Fam}(X \times Y)$*, and* $\Phi \colon R \Rightarrow S$*.*
(i) *Let* $\Phi^x \colon \bigcurlywedge_{y \in Y} \mathbb{F}\big(\rho_0^x(y), \sigma_0^x(y)\big)$*, where* $\Phi^x_y := \Phi_{(x,y)} \colon \rho_0^x(y) \to \sigma_0^x(y)$*.*
(ii) *Let* $\Phi^y \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(R^y(x), S^y(x)\big)$*, where* $\Phi^y_x := \Phi_{(x,y)} \colon \rho_0^y(x) \to \sigma_0^y(x)$*.*
(iii) *Let* $\sum^1 \Phi \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(\big(\sum^1 \rho_0\big)(x), \big(\sum^1 \sigma_0\big)(x)\big)$*, where, for every* $x \in X$*, we define*

$$\Big(\sum^1 \Phi\Big)_x : \sum_{y \in Y} \rho_0^y(x) \to \sum_{y \in Y} \sigma_0^x(y)$$

$$\Big(\sum^1 \Phi\Big)_x (y,u) := \big(y, \Phi_{(x,y)}(u)\big); \qquad (y,u) \in \sum_{y \in Y} \rho_0(x,y).$$

(iv) *If* $\sum^2 \Phi \colon \bigcurlywedge_{y \in Y} \mathbb{F}\big(\big(\sum^2 \rho_0\big)(y), \big(\sum^2 \sigma_0\big)(y)\big)$*, where, for every* $y \in Y$*, we define*

$$\Big(\sum^2 \Phi\Big)_y : \sum_{x \in X} \rho_0^x(y) \to \sum_{x \in X} \sigma_0^y(x)$$

$$\Big(\sum^2 \Phi\Big)_y (x,w) := \big(x, \Phi_{(x,y)}(w)\big); \qquad (x,w) \in \sum_{x \in X} \rho(x,y).$$

(v) *Let* $\prod^1 \Phi \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(\big(\prod^1 \rho_0\big)(x), \big(\prod^1 \sigma_0\big)(x)\big)$*, where, for every* $x \in X$*, we define*

$$\Big(\prod^1 \Phi\Big)_x : \prod_{y \in Y} \rho_0^y(x) \to \prod_{y \in Y} \sigma_0^x(y)$$

$$\left[\left(\prod^{1}\Phi\right)_x(\Theta)\right]_y := \Phi_{(x,y)}\big(\Theta_y)\big)); \qquad \Theta \in \prod_{y\in Y}\rho_0(x,y).$$

(vi) *Let* $\prod^2 \Phi \colon \curlywedge_{y\in Y} \mathbb{F}\big(\big(\prod^2 \rho_0\big)(y), \big(\prod^2 \sigma_0\big)(y)\big)$*, where, for every* $y \in Y$*, we define*

$$\left(\prod^{2}\Phi\right)_y \colon \prod_{x\in X}\rho_0^x(y) \to \prod_{x\in X}\sigma_0^y(x)$$

$$\left[\left(\prod^{2}\Phi\right)_y(\Theta)\right]_x := \Phi_{(x,y)}\big(\Theta_x)\big); \qquad \Theta \in \prod_{x\in X}\rho_0(x,y).$$

Then $\Phi^x \colon R^x \Rightarrow S^x$*,* $\Phi^y \colon R^y \Rightarrow S^y$*,* $\sum^1 \Phi \colon \big(\sum^1 R\big) \Rightarrow \big(\sum^1 S\big)$*,* $\sum^2 \Phi \colon \big(\sum^2 R\big) \Rightarrow \big(\sum^2 S\big)$*,* $\prod^1 \Phi \colon \big(\prod^1 R\big) \Rightarrow \big(\prod^1 S\big)$*, and* $\prod^2 \Phi \colon \big(\prod^2 R\big) \Rightarrow \big(\prod^2 S\big)$*.*

Proof. The proofs of (ii), (iv) and (vi) are like the proofs of (i), (iii), and (v), respectively. (i) It is immediate to show that the operation $\Phi_y^x \colon \rho_0^x(y) \rightsquigarrow \sigma_0^x(y)$ is a function. If $y =_Y y'$, the commutativity of the following left diagram from the hypothesis $\Phi \colon R \Rightarrow S$

$$\begin{tikzcd}
\rho_0(x,y) \arrow[r, "\rho_{(x,y)(x,y')}"] \arrow[d, "\Phi_{(x,y)}"'] & \rho_0(x,y') \arrow[d, "\Phi_{(x,y')}"] \\
\sigma_0(x,y) \arrow[r, "\sigma_{(x,y)(x,y')}"'] & \sigma_0(x,y')
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma_0^x(y) \arrow[r, "\rho_{yy'}^x"] \arrow[d, "\Phi_y^x"'] & \rho_0^x(y') \arrow[d, "\Phi_{y'}^x"] \\
\sigma_0^x(y) \arrow[r, "\sigma_{yy'}^x"'] & \sigma_0^x(y')
\end{tikzcd}$$

implies the required commutativity of the right above diagram, as these are the same diagrams. (iii) First we explain why the operation $\big(\sum^1 \Phi\big)_x$ is a function. If

$$(y,u) =_{\sum_{y\in Y}\rho_0^y(x)} (y',u') :\Leftrightarrow y =_Y y' \ \& \ \rho_{(x,y)(x,y')}(u) =_{\rho_0(x,y')} u',$$

$$\big(y, \Phi_{(x,y)}(u)\big) =_{\sum_{y\in Y}\sigma_0^y(x)} \big(y, \Phi_{(x,y)}(u)\big) :\Leftrightarrow y =_Y y' \ \& \ \sigma_{(x,y)(x,y')}\big(\Phi_{(x,y)}(u)\big) =_{\sigma_0(x,y')} \Phi_{(x,y')}(u').$$

From our hypothesis the second equality is equivalent to

$$\sigma_{(x,y)(x,y')}\big(\Phi_{(x,y)}(u)\big) =_{\sigma_0(x,y')} \Phi_{(x,y')}\big(\rho_{(x,y)(x,y')}(u)\big),$$

which is the commutativity of the above left diagram. If $x =_X x'$, and since $\Phi_{(x',y)} \circ \rho_{(x,y)(x',y)} = \sigma_{(x,y)(x',y)} \circ \Phi_{(x,y)}$ we get the commutativity of the following left diagram by

$$\begin{tikzcd}
\sum_{y\in Y}\rho_0(x,y) \arrow[r, "\left(\sum^1 \rho_1\right)_{xx'}"] \arrow[d, "\left(\sum^1 \Phi\right)_x"'] & \sum_{y\in Y}\rho_0(x',y) \arrow[d, "\left(\sum^1 \Phi\right)_{x'}"] \\
\sum_{y\in Y}\sigma_0(x,y) \arrow[r, "\left(\sum^1 \sigma_1\right)_{xx'}"'] & \sum_{y\in Y}\sigma_0(x',y)
\end{tikzcd}
\quad
\begin{tikzcd}
\prod_{y\in Y}\rho_0(x,y) \arrow[r, "\left(\prod^1 \rho_1\right)_{xx'}"] \arrow[d, "\left(\prod^1 \Phi\right)_x"'] & \prod_{y\in Y}\rho_0(x',y) \arrow[d, "\left(\prod^1 \Phi\right)_{x'}"] \\
\prod_{y\in Y}\sigma_0(x,y) \arrow[r, "\left(\prod^1 \sigma_1\right)_{xx'}"'] & \prod_{y\in Y}\sigma_0(x',y)
\end{tikzcd}$$

$$
\begin{aligned}
\Big(\sum^1 \Phi\Big)_{x'}\bigg(\Big(\sum^1 \rho_1\Big)_{xx'}(y,u)\bigg) &:= \Big(\sum^1 \Phi\Big)_{x'}\big(y, \rho_{(x,y)(x',y)}(u)\big)\\
&:= \big(y, \Phi_{(x',y)}\big(\rho_{(x,y)(x',y)}(u)\big)\big)\\
&= \big(y, \sigma_{(x,y)(x',y)}\big(\Phi_{(x,y)}(u)\big)\big)\\
&:= \Big(\sum^1 \sigma_1\Big)_{xx'}\big(y, \Phi_{(x,y)}(u)\big)\big)\\
&:= \Big(\sum^1 \sigma_1\Big)_{xx'}\bigg(\Big(\sum^1 \Phi\Big)_{x}(y,u)\bigg).
\end{aligned}
$$

(v) First we explain why the operation $\big(\prod^1 \Phi\big)_x$ is well-defined. If $y =_Y y'$ and $\Theta \in \prod_{y\in Y}\rho_0^x(y)$, then by the commutativity of the above left diagram we have that

$$
\begin{aligned}
\bigg[\Big(\prod^1 \Phi\Big)_x(\Theta)\bigg]_{y'} &:= \Phi_{(x,y')}\big(\Theta_{y'}\big)\\
&= \Phi_{(x,y')}\big(\rho_{(xy)(x,y')}(\Theta_y)\big)\\
&= \sigma_{(x,y)(x,y')}\big(\Phi_{(x,y)}(\Theta_y)\big)\\
&:= \sigma^x_{yy'}\bigg(\bigg[\Big(\prod^1 \Phi\Big)_x(\Theta)\bigg]_y\bigg).
\end{aligned}
$$

Clearly, the operation $\big(\prod^1 \Phi\big)_x$ is a function. If $x =_X x'$, and by the commutativity of the first diagram in the proof of (iii) we get the commutativity of the above right diagram

$$
\begin{aligned}
\bigg[\Big(\prod^1 \Phi\Big)_{x'}\bigg(\Big(\prod^1 \rho_1\Big)_{xx'}(\Theta)\bigg)\bigg]_y &:= \Phi_{(x',y)}\bigg[\Big(\prod^1 \rho_1\Big)_{xx'}(\Theta)\bigg]_y\bigg)\\
&:= \Phi_{(x',y)}\big(\rho_{(x,y)(x',y)}(\Theta_y)\big)\\
&= \sigma_{(x,y)(x',y)}\big(\Phi_{(x,y)}(\Theta_y)\big)\\
&:= \sigma_{(x,y)(x',y)}\bigg(\bigg[\Big(\prod^1 \Phi\Big)_x(\Theta)\bigg]_y\bigg)\\
&:= \bigg[\Big(\prod^1 \sigma_1\Big)_{xx'}\bigg(\Big(\prod^1 \Phi\Big)_x(\Theta)\bigg)\bigg]_y.
\end{aligned}
$$
□

Proposition 3.5.4. *If $R := (\rho_0, \rho_1) \in \mathtt{Fam}(X \times Y)$, the following equalities hold.*

$$\sum_{x\in X}\sum_{y\in Y}\rho_0(x,y) =_{\mathbb{V}_0} \sum_{y\in Y}\sum_{x\in X}\rho_0(x,y),$$

$$\prod_{x\in X}\prod_{y\in Y}\rho_0(x,y) =_{\mathbb{V}_0} \prod_{y\in Y}\prod_{x\in X}\rho_0(x,y).$$

Proof. The proof is straightforward. □

3.6 The distributivity of $\prod$ over $\sum$

We prove the translation of the type-theoretic axiom of choice in BST (Theorem 3.6.4).

Lemma 3.6.1. *Let $R := (\rho_0, \rho_1)$, $R^x := (\rho_0^x, \rho_1^x)$ and $\sum^1 R := (\sum^1 \rho_0, \sum^1 \rho_1)$ be the families of sets of Definition 3.5.2. If $\Phi \in \prod_{x\in X} \big(\sum^1 \rho_0\big)(x)$, the operation $f_\Phi \colon X \rightsquigarrow Y$, defined by $x \mapsto \mathtt{pr}_1^{R^x}(\Phi_x)$, for every $x \in X$, is a function from X to Y.*

Proof. If $x =_X x'$, then $\Phi_{x'} = \big(\sum^1 \rho_1\big)_{xx'}(\Phi_x)$. Since $\Phi_x \in \sum_{y\in Y} \rho_0(x, y)$, there are $y \in Y$ and $u \in \rho_0(x, y)$ such that $\Phi_x := (y, u)$. Hence $f_\Phi(x) := y$ and

$$f_\Phi(x') := \mathtt{pr}_1^{R^x}(\Phi_{x'}) =_Y \mathtt{pr}_1^{R^x}\bigg(\Big(\sum^1 \rho_1\Big)_{xx'}(\Phi_x)\bigg) := \mathtt{pr}_1^{R^x}\big(y, \rho_{(x,y)(x',y)}(u)\big) := y. \qquad \square$$

Lemma 3.6.2. *Let $R := (\rho_0, \rho_1)$, $R^x := (\rho_0^x, \rho_1^x)$ and $\sum^1 R := (\sum^1 \rho_0, \sum^1 \rho_1)$ be as above. If $f \colon X \to Y$, the pair $N^f := \big(\nu_0^f, \nu_1^f\big)$ is an X-family of sets, where the assignment routines $\nu_0^f : X \rightsquigarrow \mathbb{V}_0$ and $\nu_1^f \colon \curlywedge_{(x,x')\in D(X)} \mathbb{F}\big(\nu_0^f(x), \nu_0^f(x')\big)$ are given by $\nu_0^f(x) := \rho_0(x, f(x))$, for every $x \in X$, and $\nu_{xx'}^f := \rho_{(x,f(x))(x',f(x'))}$, for every $(x, x') \in D(X)$,*

Proof. The proof is straightforward (see also [95], p. 12). $\square$

Lemma 3.6.3. *If $R := (\rho_0, \rho_1)$ and $N^f := \big(\nu_0^f, \nu_1^f\big)$ are families of sets as above, then the pair $\Xi := (\xi_0, \xi_1)$ is an $\mathbb{F}(X, Y)$-family of sets, where the assignment routines $\xi_0 : \mathbb{F}(X, Y) \rightsquigarrow \mathbb{V}_0$ and $\xi_1 \colon \curlywedge_{(f,f')\in D(\mathbb{F}(X,Y))} \mathbb{F}\big(\xi_0(f), \xi_0(f')\big)$ are defined by*

$$\xi_0(f) := \prod_{x\in X} \nu_0^f(x) := \prod_{x\in X} \rho_0(x, f(x)); \qquad f \in \mathbb{F}(X, Y),$$

$$\xi_{ff'} : \prod_{x\in X} \rho_0(x, f(x)) \to \prod_{x\in X} \rho_0(x, f'(x)); \qquad (f, f') \in D(\mathbb{F}(X, Y)),$$

$$\big[\xi_{ff'}(H)\big]_x := \rho_{(x,f(x))(x,f'(x))}(H_x); \qquad H \in \prod_{x\in X} \rho_0(x, f(x)), \quad x \in X.$$

Proof. First we show that the operation $\xi_{ff'}$ is well-defined i.e., if

$$H \in \prod_{x\in X} \rho_0(x, f(x)) :\Leftrightarrow \forall_{(x,x')\in D(X)}\big(H_{x'} = \nu_{xx'}^f(H_x) := \rho_{(x,f(x))(x',f(x'))}(H_x)\big), \quad \text{then}$$

$$\xi_{ff'}(H) \in \prod_{x\in X} \rho_0(x, f'(x)) :\Leftrightarrow \forall_{(x,x')\in D(X)}\big(\big[\xi_{ff'}(H)\big]_{x'} = \nu_{xx'}^f\big(\big[\xi_{ff'}(H)\big]_x\big)$$

If $x =_X x'$, then $f(x) =_Y f(x') =_Y f'(x') =_Y f'(x)$, and

$$\begin{aligned}
\nu_{xx'}^f\big(\big[\xi_{ff'}(H)\big]_x\big) &:= \rho_{(x,f'(x))(x',f'(x'))}\big(\big[\xi_{ff'}(H)\big]_x\big) \\
&:= \rho_{(x,f'(x))(x',f'(x'))}\big(\rho_{(x,f(x))(x,f'(x))}(H_x)\big) \\
&= \rho_{(x,f(x))(x',f'(x'))}(H_x) \\
&= \rho_{(x',f(x'))(x',f'(x'))}\big(\rho_{(x,f(x))(x',f(x'))}(H_x)\big) \\
&= \rho_{(x',f(x'))(x',f'(x'))}(H_{x'}) \\
&:= \big[\xi_{ff'}(H)\big]_{x'}.
\end{aligned}$$

It is immediate to see to show that $\xi_{ff'}$ is a function. If $f \in \mathbb{F}(X,Y)$, then

$$\big[\xi_{ff}(H)\big]_x := \rho_{(x,f(x))(x,f(x))}(H_x) := \mathrm{id}_{\rho_0(x,f(x))}(H_x) := H_x.$$

Moreover, if $f =_{\mathbb{F}(X,Y)} f' =_{\mathbb{F}(X,Y)} f''$, the following diagram is commutative:

$$\begin{tikzcd}
\xi_0(f) \arrow[d, "\xi_{ff'}"'] \arrow[dr, "\xi_{ff''}"] & \\
\xi_0(f') \arrow[r, "\xi_{f'f''}"'] & \xi_0(f'')
\end{tikzcd}$$

$$\begin{aligned}
\big[(\xi_{f'f''} \circ \xi_{ff'})(H)\big]_x &:= \big[\xi_{f'f''}\big(\xi_{ff'}(H)\big)\big]_x \\
&:= \rho_{(x,f'(x))(x,f''(x))}\big(\big[\xi_{ff'}(H)\big]_x\big) \\
&:= \rho_{(x,f'(x))(x,f''(x))}\big(\rho_{(x,f(x))(x,f'(x))}(H_x)\big) \\
&= \rho_{(x,f(x))(x,f''(x))}(H_x) \\
&:= \big[\xi_{ff''}(H)\big]_x.
\end{aligned}$$

□

Theorem 3.6.4 (Distributivity of $\prod$ over $\sum$)**.** *Let X, Y be sets, $R := (\rho_0, \rho_1)$, $R^x := (\rho_0^x, \rho_1^x)$, and $\sum^1 R := (\sum^1 \rho_0, \sum^1 \rho_1)$ as above. If*

$$\Phi \in \prod_{x\in X} \big(\sum^1 \rho_0\big)(x) := \prod_{x\in X}\sum_{y\in Y} \rho_0(x,y), \quad \textit{there is}$$

$$\Theta_\Phi \in \prod_{x\in X} \nu_0^{f_\Phi}(x) := \prod_{x\in X} \rho_0(x, f_\Phi(x)),$$

where $f_\Phi \colon X \to Y$ is defined in Lemma 3.6.1. The following operation is a function:

$$\mathtt{ac} : \prod_{x\in X}\sum_{y\in Y} \rho_0(x,y) \rightsquigarrow \sum_{f\in\mathbb{F}(X,Y)} \prod_{x\in X} \rho_0(x, f(x))$$

$$\Phi \mapsto \big(f_\Phi, \Theta_\Phi\big); \qquad \Phi \in \prod_{x\in X}\sum_{y\in Y} \rho_0(x,y).$$

Proof. Since by Remark 3.3.2,

$$\mathtt{pr}_2^{R^x} \in \prod_{w\in\sum_{y\in Y}\rho_0^x(y)} \rho_0^x(\mathtt{pr}_1^{R^x}(w)) := \prod_{w\in\sum_{y\in Y}\rho_0(x,y)} \rho_0(x, \mathtt{pr}_1^{R^x}(w)),$$

$$\mathtt{pr}_2^{R^x}(\Phi_x) \in \rho_0(x, \mathtt{pr}_1^{R^x}(\Phi_x)) := \rho_0(x, f_\Phi(x)).$$

Hence, the dependent operation $\Theta_\Phi \in \curlywedge_{x\in X} \nu_0^{f_\Phi}(x) := \curlywedge_{x\in X} \rho_0(x, f_\Phi(x))$, defined by

$$\Theta_\Phi(x) := \mathtt{pr}_2^{R^x}(\Phi_x) := u; \qquad \Phi_x := (y,u), \ \ y := f_\Phi(x), \ \ x \in X,$$

is well-defined. To show that $\Theta_\Phi \in \prod_{x\in X} \nu_0^{f_\Phi}(x)$, let $x =_X x'$. Since,

$$\Theta_\Phi(x') := \mathtt{pr}_2^{R^{x'}}(\Phi_{x'}) := u'; \qquad \Phi_{x'} := (y', u'), \ \ y' := f_\Phi(x'),$$

we need to show that $u' =_{\rho_0(x',y')} \nu^{f_\Phi}_{xx'}(u)$. Since $\Phi \in \prod_{x\in X}\big(\sum^1 \rho_0\big)(x)$, we have that

$$(y',u') := (f_\Phi(x'),u') := \Phi_{x'} =_{\sum_{y\in Y}\rho_0(x',y)} \Big(\sum^1 \rho_1\Big)_{xx'}(\Phi_x) =_{\sum_{y\in Y}\rho_0(x',y)} \big(y, \rho_{(x,y)(x',y)}(u)\big).$$

By the last equality we get $y =_Y y'$ and

$$\rho^{x'}_{yy'}\big(\rho_{(x,y)(x',y)}(u)\big) =_{\rho^{x'}_0(y')} u' :\Leftrightarrow \rho_{(x',y)(x',y')}\big(\rho_{(x,y)(x',y)}(u)\big) =_{\rho_0(x',y')} u'$$
$$\Leftrightarrow \rho_{(x,y)(x',y')}(u) =_{\rho_0(x',y')} u'.$$

Hence,

$$\nu^{f_\Phi}_{xx'}(u) := \rho_{(x,f_\Phi(x))(x',f_\Phi(x'))}(u) := \rho_{(x,y)(x',y')}(u) =_{\rho_0(x',y')} u'.$$

To show that the operation $\mathtt{ac}$ is a function, we suppose that $\Phi =_{\prod_{x\in X}\mu_0(x)} \Phi'$, and we show that $\mathtt{ac}(\Phi) =_{\sum_{f\in\mathbb{F}(X,Y)}\xi_0(f)} \mathtt{ac}(\Phi')$ i.e.,

$$\big(f_\Phi,\Theta_\Phi\big) =_{\sum_{f\in\mathbb{F}(X,Y)}\xi_0(f)} \big(f_{\Phi'},\Theta_{\Phi'}\big) :\Leftrightarrow f_\Phi =_{\mathbb{F}(X,Y)} f_{\Phi'} \ \&\ \xi_{f_\Phi,f_{\Phi'}}(\Theta_\Phi) =_{\xi_0(f_{\Phi'})} \Theta_{\Phi'}.$$

By definition, $\Phi =_{\prod_{x\in X}\left(\sum^1\rho_0\right)(x)} \Phi'$ if and only if $\Phi_x =_{\left(\sum^1\rho_0\right)(x)} \Phi_x{}'$, for every $x \in X$. By Lemma 3.6.1

$$f_\Phi(x) := \mathtt{pr}^{R^x}_1(\Phi_x) := y; \qquad \Phi_x := (y,u),$$
$$f_{\Phi'}(x) := \mathtt{pr}^{\Lambda^x}_1(\Phi'{}_x) := y'; \qquad \Phi'{}_x := (y',u').$$

Since $\Phi_x =_{\left(\sum^1\rho_0\right)(x)} \Phi_x{}'$, we get $y =_Y y'$, and $\rho_{(x,y)(x,y')}(u) =_{\rho_0(x,y')} u'$. From the first equality we get and hence $f_\Phi(x) =_Y f_{\Phi'}(x)$, and from the second we conclude that

$$\big[\xi_{f_\Phi,f_{\Phi'}}(\Theta_\Phi)\big]_x := \rho_{(x,f_\Phi(x))(x,f_{\Phi'}(x))}\big([\Theta_f]_x\big) := \rho_{(x,y)(x,y')}(u) =_{\rho_0(x,y')} u' := \big[\Theta_{\Phi'}\big]_x. \qquad \square$$

3.7 Sets of sets

Definition 3.7.1. *If I is a set, a set of sets indexed by I, or an I-set of sets, is a pair $\Lambda := (\lambda_0,\lambda_1) \in \mathtt{Fam}(I)$ such that the following condition is satisfied:*

$$Q(\Lambda) :\Leftrightarrow \forall_{i,j\in I}\big(\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j) \Rightarrow i =_I j\big).$$

Let $\mathtt{Set}(I)$ be their totality, equipped with the canonical equality on $\mathtt{Fam}(I)$.

Remark 3.7.2. *If $\Lambda \in \mathtt{Set}(I)$ and $M \in \mathtt{Fam}(I)$ such that $\Lambda =_{\mathtt{Fam}(I)} M$, then $M \in \mathtt{Set}(I)$.*

Proof. Let $i,j \in I$, $f\colon \mu_0(i) \to \mu_0(j)$ and $g\colon \mu_0(j) \to \mu_0(i)$, such that $f \circ g = \mathrm{id}_{\mu_0(j)}$ and $g \circ f = \mathrm{id}_{\mu_0(i)}$. It suffices to show that $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$. Let $\Phi \in \mathtt{Map}_I(\Lambda, M)$ and $\Psi \in \mathtt{Map}_I(M,\Lambda)$ such that $\Phi\circ\Psi = \mathrm{id}_M$ and $\Psi\circ\Phi = \mathrm{id}_\Lambda$. We define $f'\colon \lambda_0(i) \to \lambda_0(j)$ and $g'\colon \lambda_0(j) \to \lambda_0(i)$ by

$$f' := \Psi_j \circ f \circ \Phi_i \quad \& \quad g' := \Psi_i \circ g \circ \Phi_j$$

$$\begin{tikzcd}
\mu_0(i) \arrow[r, "f"] & \mu_0(j) \arrow[d, "\Psi_j"] \\
\lambda_0(i) \arrow[u, "\Phi_i"] \arrow[r, "f'"'] & \lambda_0(j)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu_0(i) \arrow[d, "\Psi_i"'] & \mu_0(j) \arrow[l, "g"'] \\
\lambda_0(i) & \lambda_0(j). \arrow[l, "g'"] \arrow[u, "\Phi_j"']
\end{tikzcd}$$

It is straightforward to show that $(f', g') \colon \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$. □

By the previous remark $Q(\Lambda)$ is an extensional property on $\mathtt{Fam}(I)$. Since $\mathtt{Set}(I)$ is defined by separation on $\mathtt{Fam}(I)$, which is impredicative, $\mathtt{Set}(I)$ is also an impredicative set. We can also see that by an argument similar to the one used for the impredicativity of $\mathtt{Fam}(I)$.

If X, Y are not equal sets in $\mathbb{V}_0$, then with Ex falsum we get that the $\mathbb{2}$-family $\Lambda^{\mathbb{2}}$ of X and Y is a $\mathbb{2}$-set of sets. Similarly, if X_n and X_m are not equal in $\mathbb{V}_0$, for every $n \neq m$, then with Ex falsum we get that the $\mathbb{N}$-family $\Lambda^{\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ is an $\mathbb{N}$-set of sets. If I is a set with $(i, j) \in I \times I$ such that $\neg(i =_I j)$, then the constant I-family A, for some set A, is an I-family that is not an I-set of sets. We can easily turn an I-family of sets Λ into an I-set of sets.

Definition 3.7.3. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$. The equality $=_I^{\Lambda}$ on I induced by Λ is given by $i =_I^{\Lambda} j :\Leftrightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$, for every $i, j \in I$. The set $\lambda_0 I$ of sets generated by Λ is the totality I equipped with the equality $=_I^{\Lambda}$. For simplicity, we write $\lambda_0(i) \in \lambda_0 I$, instead of $i \in I$, when I is equipped with the equality $=_I^{\Lambda}$. The operation $\lambda_0^* : I \rightsquigarrow I$ from $(I, =_I)$ to $(I, =_I^{\Lambda})$, defined by $i \mapsto i$, for every $i \in I$, is denoted by $\lambda_0^* : I \rightsquigarrow \lambda_0 I$, and its definition is rewritten as $\lambda_0^*(i) := \lambda_0(i)$, for every $i \in I$.*

Clearly, λ_0^* is a function. In the next proof the hypothesis of a set of sets is crucial.

Proposition 3.7.4. *Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-set of sets, and let Y be a set. If $f \colon I \to Y$, there is a unique function $\lambda_0 f \colon \lambda_0 I \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\lambda_0"'] & Y. \\ \lambda_0 I \arrow[ur, dashed, "\lambda_0 f"'] & \end{tikzcd}$$

Conversely, if $f \colon I \rightsquigarrow Y$ and $f^* : \lambda_0 I \to Y$ such that the corresponding diagram commutes, then f is a function and f^* is equal to the function from $\lambda_0 I$ to Y generated by f.

Proof. The operation $\lambda_0 f$ from $\lambda_0 I$ to Y defined by $\lambda_0 f(\lambda_0(i)) := f(i)$, for every $\lambda_0(i) \in \lambda_0 I$, is a function, since, for every $i, j \in I$, we have that $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j) \Rightarrow i =_I j$, hence $f(i) =_Y f(j) :\Leftrightarrow \lambda_0 f(\lambda_0(i)) =_Y \lambda_0 f(\lambda_0(j))$. The commutativity of the diagram follows from the reflexivity of $=_Y$. If $g \colon \lambda_0 I \to Y$ makes the above diagram commutative, then for every $\lambda_0(i)$ we have that $g(\lambda_0(i)) =_Y f(i) =: \lambda_0 f(\lambda_0(i))$, hence $g =_{\mathbb{F}(\lambda_0 I, Y)} \lambda_0 f$. For the converse, if $i, j \in I$, then by the transitivity of $=_Y$ we have that $i =_I j \Rightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$, hence $\Rightarrow f^*(\lambda_0(i)) =_Y f^*(\lambda_0(j))$, and $f(i) =_Y f(j)$. The proof of the fact that f^* is the function from $\lambda_0 I$ to Y generated by f is immediate. □

Proposition 3.7.5. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$, and let Y be a set. If $f^* : \lambda_0 I \to Y$, there is a unique function $f : I \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, dashed, "f"] \arrow[d, "\lambda_0"'] & Y. \\ \lambda_0 I \arrow[ur, "f^*"'] & \end{tikzcd}$$

If $\Lambda \in \texttt{Set}(I)$, then f^* is equal to the function from $\lambda_0 I$ to Y generated by f.

Proof. Let $f^*\colon I \rightsquigarrow Y$, defined by $f(i) := f^*(\lambda_0(i))$, for every $i \in I$. Since $i =_I i' \Rightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(i') \Rightarrow f^*(\lambda_0(i)) =_Y f^*(\lambda_0(i')) \Leftrightarrow f(i) =_Y f(i')$, f is the required function. If $\Lambda \in \texttt{Set}(I)$, by Proposition 3.7.4 f^* is generated by f. The uniqueness of f follows immediately. $\square$

Remark 3.7.6. *Let $f \in \mathbb{F}(I, Y)$ and $g \in \mathbb{F}(Y, Z)$. If $\Lambda \in \texttt{Set}(I)$, then $\lambda_0(g \circ f) := g \circ \lambda_0 f$*

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "\lambda_0"'] & Y \arrow[r, "g"] & Z. \\
\lambda_0 I \arrow[ur, "\lambda_0 f"] \arrow[urr, "\lambda_0(g \circ f)"'] & &
\end{tikzcd}$$

Proof. If $i \in I$, then by Proposition 3.7.4 we have that $\lambda_0(g \circ f)(\lambda_0(i)) := (g \circ f)(i) := g(f(i)) =: g[\lambda_0 f(\lambda_0(i))] =: [g \circ \lambda_0 f](\lambda_0(i))$. $\square$

Proposition 3.7.7. *Let $\Lambda := (\lambda_0, \lambda_1) \in \texttt{Set}(I)$ and $M := (\mu_0, \mu_1) \in \texttt{Set}(J)$. If $f : I \to J$, there is a unique function $f^* : \lambda_0 I \to \mu_0 J$ such that the following diagram commutes*

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "\lambda_0"'] & J \arrow[d, "\mu_0"] \\
\lambda_0 I \arrow[r, dashed, "f^*"'] & \mu_0 J.
\end{tikzcd}$$

Conversely, if $f\colon I \rightsquigarrow J$, and $f^* : \lambda_0 I \to \mu_0 J$ such that the corresponding to the above diagram commutes, then $f \in \mathbb{F}(I, J)$ and f^* is equal to the function from $\lambda_0 I$ to $\mu_0 J$ generated by f.

Proof. Let $f^*\colon \lambda_0 I \rightsquigarrow \mu_0 J$ be defined by $f^*(\lambda_0(i)) := \mu_0(f(i))$, for every $\lambda_0(i) \in \lambda_0 I$. We show that $f^* \in \mathbb{F}(\lambda_0 I, \mu_0 J)$. If $i, j \in I$, such that $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$, then $i =_I j$, hence $f(i) =_J f(j)$, and consequently $\mu_0(f(i)) =_{\mathbb{V}_0} \mu_0(f(j))$ i.e., $f^*(\lambda_0(i)) =_{\mathbb{V}_0} f^*(\lambda_0(j))$. The uniqueness of f^* is trivial. For the converse, by the transitivity of $=_{\mathbb{V}_0}$, and since $M \in \texttt{Set}(J)$, we have that $i =_I j \Rightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j) \Rightarrow f^*(\lambda_0(i)) =_{\mathbb{V}_0} f^*(\lambda_0(j))$, hence $\mu_0(f(i)) =_{\mathbb{V}_0} \mu_0(f(j))$, which implies $f(i) =_J f(j)$. Clearly, f^* is equal to the function from $\lambda_0 I$ to $\mu_0 J$ generated by f. $\square$

Proposition 3.7.8. *Let $\Lambda := (\lambda_0, \lambda_1) \in \texttt{Fam}(I)$ and $M := (\mu_0, \mu_1) \in \texttt{Set}(J)$. If $f^*\colon \lambda_0 I \to \mu_0 J$, there is a unique function f from I to J, such that the following diagram commutes, and f^* is equal to the function from $\lambda_0 I$ to $\mu_0 J$ generated by f*

$$\begin{tikzcd}
I \arrow[r, dashed, "f"] \arrow[d, "\lambda_0"'] & J \arrow[d, "\mu_0"] \\
\lambda_0 I \arrow[r, "f^*"'] & \mu_0 J.
\end{tikzcd}$$

Proof. If $i \in I$, then $f^*(\lambda_0(i)) := \mu_0(j)$, for some $j \in J$. We define the routine $f(i) := j$ i.e., the output of f^* determines the output of f. Since $i =_I i' \Rightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(i') \Rightarrow f^*(\lambda_0(i)) =_{\mathbb{V}_0} f^*(\lambda_0(i'))$ we get $\mu_0(j) =_{\mathbb{V}_0} \mu_0(j') \Rightarrow j =_J j' \Leftrightarrow f(i) =_J f(i')$, hence f is a function. The required commutativity of the diagram follows immediately. If $g : I \to J$ such that the above diagram commutes, then $\mu_0(g(i)) =_{\mathbb{V}_0} f^*(\lambda_0(i)) := \mu_0(j) =: \mu_0(f(i))$, hence $g(i) =_J f(i)$. $\square$

3.8 Direct families of sets

Definition 3.8.1. *Let $(I, \preccurlyeq_I)$ be a directed set, and $D^{\preccurlyeq}(I) := \big\{(i,j) \in I \times I \mid i \preccurlyeq_I j\big\}$ the diagonal of $\preccurlyeq_I$. A direct family of sets $(I, \preccurlyeq_I)$, or an $(I, \preccurlyeq_I)$-family of sets, is a pair $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and $\lambda_1^{\preccurlyeq}$, a modulus of transport maps for λ_0, is defined by*

$$\lambda_1^{\preccurlyeq} : \bigcurlywedge_{(i,j)\in D^{\preccurlyeq}(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1^{\preccurlyeq}(i,j) := \lambda_{ij}^{\preccurlyeq}, \quad (i,j) \in D^{\preccurlyeq}(I),$$

such that the transport maps $\lambda_{ij}^{\preccurlyeq}$ of $\Lambda^{\preccurlyeq}$ satisfy the following conditions:

(a) *For every $i \in I$, we have that $\lambda_{ii}^{\preccurlyeq} := \mathrm{id}_{\lambda_0(i)}$.*

(b) *If $i \preccurlyeq_I j$ and $j \preccurlyeq_I k$, the following diagram commutes*

$$\begin{tikzcd} \lambda_0(i) \arrow[d, "\lambda_{ij}^{\preccurlyeq}"'] \arrow[dr, "\lambda_{ik}^{\preccurlyeq}"] & \\ \lambda_0(j) \arrow[r, "\lambda_{jk}^{\preccurlyeq}"'] & \lambda_0(k). \end{tikzcd}$$

If $X \in \mathbb{V}_0$, the constant $(I, \preccurlyeq_I)$-family X is the pair $C^{\preccurlyeq,X} := (\lambda_0^X, \lambda_1^{\preccurlyeq,X})$, where $\lambda_0^X(i) := X$, and $\lambda_1^{\preccurlyeq,X}(i,j) := \mathrm{id}_X$, for every $i \in I$ and $(i,j) \in D^{\preccurlyeq}(I)$.

Since in general $\preccurlyeq_I$ is not symmetric, the transport map $\lambda_{ij}^{\preccurlyeq}$ does not necessarily have an inverse. Hence $\lambda_1^{\preccurlyeq}$ is only a modulus of transport for λ_0, in the sense that determines the transport maps of $\Lambda^{\preccurlyeq}$, and not necessarily a modulus of function-likeness for λ_0.

Definition 3.8.2. *If $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$ and $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq})$ are $(I, \preccurlyeq_I)$-families of sets, a direct family-map Φ from $\Lambda^{\preccurlyeq}$ to $M^{\preccurlyeq}$, denoted by $\Phi \colon \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$, their set $\mathtt{Map}_{(I,\preccurlyeq_I)}(\Lambda^{\preccurlyeq}, M^{\preccurlyeq})$, and the totality $\mathtt{Fam}(I, \preccurlyeq_I)$ of $(I, \preccurlyeq_I)$-families are defined as in Definition 3.1.3. The direct sum $\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$ over $\Lambda^{\preccurlyeq}$ is the totality $\sum_{i\in I} \lambda_0(i)$ equipped with the equality*

$$(i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y) :\Leftrightarrow \exists_{k\in I}\big(i \preccurlyeq_I k \ \&\ j \preccurlyeq_I k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big).$$

The totality $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ of dependent functions over $\Lambda^{\preccurlyeq}$ is defined by

$$\Phi \in \prod_{i\in I}^{\preccurlyeq} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \&\ \forall_{(i,j)\in D^{\preccurlyeq}(I)}\big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}^{\preccurlyeq}(\Phi_i)\big),$$

and it is equipped with the equality of $\mathbb{A}(I, \lambda_0)$.

Clearly, the property $P(\Phi) :\Leftrightarrow \forall_{(i,j)\in D^{\preccurlyeq}(I)}\big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}^{\preccurlyeq}(\Phi_i)\big)$ is extensional on $\mathbb{A}(I, \lambda_0)$, the equality on $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ is an equivalence relation. $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ is considered to be a set.

Proposition 3.8.3. *The relation $(i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y)$ is an equivalence relation.*

Proof. If $i \in I$, and since $i \preccurlyeq_I i$, there is $k \in I$ such that $i \preccurlyeq_I k$, and by the reflexivity of the equality on $\lambda_0(k)$ we get $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{ik}^{\preccurlyeq}(x)$. The symmetry of $=_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)}$ follows from the symmetry of the equalities $=_{\lambda_0(k)}$. To prove transitivity, we suppose that

$$(i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y) :\Leftrightarrow \exists_{k\in I}\big(i \preccurlyeq_I k \ \& \ j \preccurlyeq_I k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big),$$

$$(j,y) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j',z) :\Leftrightarrow \exists_{k'\in I}\big(j \preccurlyeq_I k' \ \& \ j' \preccurlyeq_I k' \ \& \ \lambda_{jk'}^{\preccurlyeq}(y) =_{\lambda_0(k')} \lambda_{j'k'}^{\preccurlyeq}(z)\big),$$

and we show that

$$(i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j',z) :\Leftrightarrow \exists_{k''\in I}\big(i \preccurlyeq_I k'' \ \& \ j' \preccurlyeq_I k'' \ \& \ \lambda_{ik''}^{\preccurlyeq}(x) =_{\lambda_0(k'')} \lambda_{j'k''}^{\preccurlyeq}(z)\big).$$

By the definition of a directed set there is $k'' \in I$ such that $k \preccurlyeq_I k''$ and $k' \preccurlyeq_I k''$

$$\begin{tikzcd}
 & k \arrow[rr] & & k'' & & k' \arrow[ll] & \\
i \arrow[ur] & & & j \arrow[ull] \arrow[urr] & & & j', \arrow[ul]
\end{tikzcd}$$

hence by transitivity $i \preccurlyeq_I k''$ and $j' \preccurlyeq_I k''$. Moreover,

$$\begin{aligned}
\lambda_{ik''}^{\preccurlyeq}(x) &\overset{i\preccurlyeq_I k\preccurlyeq_I k''}{=} \lambda_{kk''}^{\preccurlyeq}\big(\lambda_{ik}^{\preccurlyeq}(x)\big) \\
&= \lambda_{kk''}^{\preccurlyeq}\big(\lambda_{jk}^{\preccurlyeq}(y)\big) \\
&\overset{j\preccurlyeq_I k\preccurlyeq_I k''}{=} \lambda_{jk''}^{\preccurlyeq}(y) \\
&\overset{j\preccurlyeq_I k'\preccurlyeq_I k''}{=} \lambda_{k'k''}^{\preccurlyeq}\big(\lambda_{jk'}^{\preccurlyeq}(y)\big) \\
&= \lambda_{k'k''}^{\preccurlyeq}\big(\lambda_{j'k'}^{\preccurlyeq}(z)\big) \\
&\overset{j'\preccurlyeq_I k'\preccurlyeq_I k''}{=} \lambda_{j'k''}^{\preccurlyeq}(z).
\end{aligned}$$

□

Notice that the projection operation from $\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$ to I is not a function.

Proposition 3.8.4. *If $(I, \preccurlyeq_I)$ is a directed set, $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$, $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq})$ are $(I, \preccurlyeq_I)$-families of sets, and $\Psi^{\preccurlyeq} : \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$, the following hold.*

(i) *For every $i \in I$ the operation $e_i^{\Lambda^{\preccurlyeq}} : \lambda_0(i) \rightsquigarrow \sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$, defined by $x \mapsto (i,x)$, for every $x \in \lambda_0(i)$, is a function from $\lambda_0(i)$ to $\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$.*

(ii) *The operation $\Sigma^{\preccurlyeq}\Psi : \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \rightsquigarrow \sum_{i\in I}^{\preccurlyeq} \mu_0(i)$, defined by $\big(\Sigma^{\preccurlyeq}\Psi\big)(i,x) := (i, \Psi_i(x))$, for every $(i,x) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$, is a function from $\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$ to $\sum_{i\in I}^{\preccurlyeq} \mu_0(i)$ such that, for every $i \in I$, the following left diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "e_i^{\Lambda^{\preccurlyeq}}"'] & \mu_0(i) \arrow[d, "e_i^{M^{\preccurlyeq}}"] \\
\sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \arrow[r, "\Sigma^{\preccurlyeq}\Psi"'] & \sum_{i\in I}^{\preccurlyeq} \mu_0(i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\prod_{i\in I}^{\preccurlyeq} \lambda_0(i) \arrow[r, "\Pi^{\preccurlyeq}\Psi"'] \arrow[u, "\pi_i^{\Lambda^{\preccurlyeq}}"] & \prod_{i\in I}^{\preccurlyeq} \mu_0(i). \arrow[u, "\pi_i^{M^{\preccurlyeq}}"']
\end{tikzcd}$$

(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Sigma^{\preccurlyeq}\Psi$ is an embedding.*

(iv) *For every $i \in I$ the operation $\pi_i^{\Lambda^{\preccurlyeq}} : \prod_{i\in I}^{\preccurlyeq} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, defined by $\Theta \mapsto \Theta_i$, for every $\Theta \in \prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$, is a function from $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ to $\lambda_0(i)$.*

(v) *The operation $\Pi^{\preccurlyeq}\Psi : \prod_{i\in I}^{\preccurlyeq} \lambda_0(i) \rightsquigarrow \prod_{i\in I}^{\preccurlyeq} \mu_0(i)$, defined by $[\Pi^{\preccurlyeq}\Psi(\Theta)]_i := \Psi_i(\Theta_i)$, for every $i \in I$ and $\Theta \in \prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$, is a function from $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ to $\prod_{i\in I}^{\preccurlyeq} \mu_0(i)$, such that, for every $i \in I$, the above right diagram commutes.*

(vi) *If Ψ_i is an embedding, for every $i \in I$, then $\Pi^{\preccurlyeq}\Psi$ is an embedding.*

Proof. (i) If $x, y \in \lambda_0(i)$ such that $x =_{\lambda_0(i)} y$, then, since $\preccurlyeq$ is reflexive, if we take $k := i$, we get $\lambda_{ii}^{\preccurlyeq}(x) := \mathrm{id}_{\lambda_0(i)}(x) := x =_{\lambda_0(i)} y := \mathrm{id}_{\lambda_0(i)}(y) := \lambda_{ii}^{\preccurlyeq}(y)$, hence $(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (i, y)$.

(ii) If $(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j, y)$, there is $k \in I$ such that $i \preccurlyeq_I k$, $j \preccurlyeq_I k$ and $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)$. We show the following equality:

$$\big(\Sigma^{\preccurlyeq}\Psi\big)(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \mu_0(i)} \big(\Sigma^{\preccurlyeq}\Psi\big)(j, y) :\Leftrightarrow (i, \Psi_i(x)) =_{\sum_{i\in I}^{\preccurlyeq} \mu_0(i)} (j, \Psi_j(y))$$
$$:\Leftrightarrow \exists_{k'\in I}\big(i, j \preccurlyeq_I k' \ \&\ \mu_{ik'}^{\preccurlyeq}(\Psi_i(x)) =_{\mu_0(k')} \mu_{jk'}^{\preccurlyeq}(\Psi_j(y))\big).$$

If we take $k' := k$, by the commutativity of the following diagrams, and since Ψ_k is a function,

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ik}^{\preccurlyeq}"] \arrow[d, "\Psi_i"'] & \lambda_0(k) \arrow[d, "\Psi_k"] \\ \mu_0(i) \arrow[r, "\mu_{ik}^{\preccurlyeq}"'] & \mu_0(k) \end{tikzcd} \qquad \begin{tikzcd} \lambda_0(j) \arrow[r, "\lambda_{jk}^{\preccurlyeq}"] \arrow[d, "\Psi_j"'] & \lambda_0(k) \arrow[d, "\Psi_k"] \\ \mu_0(j) \arrow[r, "\mu_{jk}^{\preccurlyeq}"'] & \mu_0(k) \end{tikzcd}$$

$$\mu_{ik}^{\preccurlyeq}\big(\Psi_i(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) =_{\mu_0(k)} \mu_{jk}^{\preccurlyeq}\big(\Psi_j(y)\big).$$

(iii) If we suppose $\big(\Sigma^{\preccurlyeq}\Psi\big)(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \mu_0(i)} \big(\Sigma^{\preccurlyeq}\Psi\big)(j, y)$ i.e., $\mu_{ik}^{\preccurlyeq}(\Psi_i(x)) =_{\mu_0(k)} \mu_{jk}^{\preccurlyeq}(\Psi_j(y))\big)$, for some $k \in I$ with $i, j \preccurlyeq_I k$, by the proof of case (ii) we get $\Psi_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big)$, and since Ψ_k is an embedding, we get $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)$ i.e., $(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j, y)$.

(iv)-(vi) Their proof is omitted, since a proof of their contravariant version (see Note 3.11.10) is given in the proof of Theorem 6.6.3. □

Since the transport functions $\lambda_{ik}^{\preccurlyeq}$ are not in general embeddings, we cannot show in general that $e_i^{\Lambda^{\preccurlyeq}}$ is an embedding, as it is the case for the map e_i^{Λ} in Proposition 3.2.7(i). The study of direct families of sets can be extended following the study of set-indexed families of sets.

3.9 Set-relevant families of sets

In general, we may want to have more than one transport maps from $\lambda_0(i)$ to $\lambda_0(j)$, if $i =_I j$. In this case, to each $(i, j) \in D(I)$ we associate a set of transport maps.

Definition 3.9.1. *If I is a set, a set-relevant family of sets indexed by I, is a triplet $\Lambda^* := \big(\lambda_0, \varepsilon_0^{\lambda}, \lambda_2\big)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\varepsilon_0^{\lambda} : D(I) \rightsquigarrow \mathbb{V}_0$, and*

$$\lambda_2 : \bigcurlywedge_{(i,j)\in D(I)} \ \bigcurlywedge_{p\in\varepsilon_0^{\lambda}(i,j)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_2\big((i, j), p\big) := \lambda_{ij}^{p}, \quad (i, j) \in D(I), \quad p \in \varepsilon_0^{\lambda}(i, j),$$

such that the following conditions hold:
(i) *For every* $i \in I$ *there is* $p \in \varepsilon_0^{\lambda}(i,i)$ *such that* $\lambda_{ii}^{p} =_{\mathbb{F}(\lambda_0(i),\lambda_0(i))} \mathrm{id}_{\lambda_0(i)}$.
(ii) *For every* $(i,j) \in D(I)$ *and every* $p \in \varepsilon_0^{\lambda}(i,j)$ *there is some* $q \in \varepsilon_0^{\lambda}(j,i)$ *such that such that the following left diagram commutes*

$$\begin{tikzcd} \lambda_0(i) \arrow[d, "\lambda_{ij}^{p}"'] \arrow[dr, "\mathrm{id}_{\lambda_0(i)}"] & \\ \lambda_0(j) \arrow[r, "\lambda_{ji}^{q}"'] & \lambda_0(i) \end{tikzcd} \qquad \begin{tikzcd} \lambda_0(i) \arrow[d, "\lambda_{ij}^{p}"'] \arrow[dr, "\lambda_{ik}^{r}"] & \\ \lambda_0(j) \arrow[r, "\lambda_{jk}^{q}"'] & \lambda_0(k). \end{tikzcd}$$

(iii) *If* $i =_I j =_I k$, *then for every* $p \in \varepsilon_0^{\lambda}(i,j)$ *and every* $q \in \varepsilon_0^{\lambda}(j,k)$ *there is* $r \in \varepsilon_0^{\lambda}(i,k)$ *such that the above right diagram commutes.*
We call Λ^* *function-like, if* $\forall_{(i,j)\in D(I)}\forall_{p,p'\in\varepsilon_0^{\lambda}(i,j)}\big(p =_{\varepsilon_0^{\lambda}(i,j)} p' \Rightarrow \lambda_{ij}^{p} =_{\mathbb{F}(\lambda_0(i),\lambda_0(j))} \lambda_{ij}^{p'}\big)$.

It is immediate to show that if $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$, then Λ generates a set-relevant family over I, where $\varepsilon_0^{\lambda}(i,j) := \mathbb{1}$, and $\lambda_2\big((i,j),p)\big) := \lambda_{ij}$, for every $(i,j) \in D(I)$.

Definition 3.9.2. *Let* $\Lambda^* := (\lambda_0, \varepsilon_0^{\lambda}, \lambda_2)$ *and* $M := (\mu_0, \varepsilon_0^{\mu}, \mu_2)$ *be set-relevamt families of sets over* I. *A covariant set-relevant family-map from* Λ^* *to* M^*, *in symbols* $\Psi \colon \Lambda^* \Rightarrow M^*$, *is a dependent operation* $\Psi \colon \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$ *such that for every* $(i,j) \in D(I)$ *and for every* $p \in \varepsilon_0^{\lambda}(i,j)$ *there is* $q \in \varepsilon_0^{\mu}(i,j)$ *such that the following diagram commutes*

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}^{p}"] \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}^{q}"'] & \mu_0(j). \end{tikzcd}$$

A contravariant set-relevant family-map is defined by the property: for every $q \in \varepsilon_0^{\mu}(i,j)$, *there is* $p \in \varepsilon_0^{\lambda}(i,j)$ *such that the above diagram commutes. Let* $\mathtt{Map}_I(\Lambda^*, M^*)$ *be the totality of covariant set-relevant family-maps from* Λ^* *to* M^*, *which is equipped with the pointwise equality. If* $\Xi : M^* \Rightarrow N^*$, *the composition set-relevant family-map* $\Xi \circ \Psi \colon \Lambda^* \Rightarrow N^*$ *is defined, for every* $i \in I$, *by* $(\Xi \circ \Psi)_i := \Xi_i \circ \Psi_i$

$$\begin{tikzcd} \lambda_0(i) \arrow[r, "\lambda_{ij}^{p}"] \arrow[d, "\Psi_i"] \arrow[dd, bend right, "(\Xi \circ \Psi)_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"'] \arrow[dd, bend left, "(\Xi \circ \Psi)_j"] \\ \mu_0(i) \arrow[r, "\mu_{ij}^{q}"'] \arrow[d, "\Xi_i"] & \mu_0(j) \arrow[d, "\Xi_j"'] \\ \nu_0(i) \arrow[r, "\nu_{ij}^{r}"'] & \nu_0(j). \end{tikzcd}$$

The composition of contravariant set-relevant family-maps is defined similarly The identity set-relevant family-map is defined by $\mathrm{Id}_{\Lambda^*}(i) := \mathrm{id}_{\lambda_0(i)}$*, for every* $i \in I$*. Let* $\mathtt{Fam}^*(I)$ *be the totality of set-relevant* I*-families, equipped with the obvious canonical equality.*

Id_{Λ^*} is both a covariant and a contravariant set-relevant family-map from Λ^* to itself.

Definition 3.9.3. *Let* $\Lambda^* := \big(\lambda_0, \varepsilon_0^{\lambda}, \lambda_2\big) \in \mathtt{Fam}^*(I)$*. The exterior union* $\sum_{i\in I}^* \lambda_0(i)$ *of* Λ^* *is the totality* $\sum_{i\in I} \lambda_0(i)$*, equipped with the following equality*

$$(i,x) =_{\sum_{i\in I}^* \lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \& \ \exists_{p\in\varepsilon_0^{\lambda}(i,j)}\big(\lambda_{ij}^p(x) =_{\lambda_0(j)} y\big).$$

The totality $\prod_{i\in I}^* \lambda_0(i)$ *of dependent functions over* Λ^* *is defined by*

$$\Theta \in \prod_{i\in I}^* \lambda_0(i) :\Leftrightarrow \Theta \in \mathbb{A}(I,\lambda_0) \ \& \ \forall_{(i,j)\in D(I)}\forall_{p\in\varepsilon_0^{\lambda}(i,j)}\big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}^p(\Theta_i)\big),$$

and it is equipped with the pointwise equality.

A motivation for the definitions of $\sum_{i\in I}^* \lambda_0(i)$ and $\prod_{i\in I}^* \lambda_0(i)$ is provided in Note 5.7.10.

Remark 3.9.4. *The equalities on* $\sum_{i\in I}^* \lambda_0(i)$ *and* $\prod_{i\in I}^* \lambda_0(i)$ *satisfy the conditions of an equivalence relation.*

Proof. Let $(i,x), (j,y)$ and $(k,z) \in \sum_{i\in I} \lambda_0(i)$. By definition there is $p \in \varepsilon_0^{\lambda}(i,i)$ such that $\lambda_{ii}^p = \mathrm{id}_{\lambda_0(i)}$, hence $(i,x) =_{\sum_{i\in I}^* \lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}^* \lambda_0(i)} (j,y)$, then $j =_I i$ and there is $q \in \varepsilon_0^{\lambda}(ji)$ such that $\lambda_{ji}^q(y) = \lambda_{ji}^q(\lambda_{ij}^p(x)) = \mathrm{id}_{\lambda_0(i)}(x) := x$, hence $(j,y) =_{\sum_{i\in I}^g {}^*\lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}^* \lambda_0(i)} (j,y)$ and $(j,y) =_{\sum_{i\in I}^* \lambda_0(i)} (k,z)$, then from the hypotheses $i =_I j$ and $j =_I k$, we get $i =_I k$. From the hypotheses $\exists_{p\in\varepsilon_0^{\lambda}(i,j)}\big(\lambda_{ij}^p(x) =_{\lambda_0(j)} y\big)$ and $\exists_{q\in\varepsilon_0^{\lambda}(j,k)}\big(\lambda_{jk}^q(y) =_{\lambda_0(k)} z\big)$, let $r \in \varepsilon_0^{\lambda}(i,k)$ such that $\lambda_{ik}^r = \lambda_{jk}^q \circ \lambda_{ij}^p$. Hence $\lambda_{ik}^r(x) =_{\lambda_0(k)} \lambda_{jk}^q(\lambda_{ij}^p(x)) =_{\lambda_0(k)} \lambda_{jk}^q(y) = z$. The proof for the equality on $\prod_{i\in I}^* \lambda_0(i)$ is trivial. $\square$

Proposition 3.9.5. *Let* $\Lambda := (\lambda_0, \varepsilon_0^{\lambda}, \lambda_2)$*,* $M := (\mu_0, \varepsilon_0^{\mu}, \mu_2) \in \mathtt{Fam}^*(I)$*, and* $\Psi : \Lambda^* \Rightarrow M^*$*.*

(i) *For every* $i \in I$ *the operation* $e_i^{\Lambda^*} : \lambda_0(i) \rightsquigarrow \sum_{i\in I}^* \lambda_0(i)$*, defined by* $e_i^{\Lambda^*}(x) := (i,x)$*, for every* $x \in \lambda_0(i)$*, is a function.*

(ii) *If* Ψ *is covariant, the operation* $\Sigma^*\Psi : \sum_{i\in I}^* \lambda_0(i) \rightsquigarrow \sum_{i\in I}^* \mu_0(i)$*, defined by* $\Sigma^*\Psi(i,x) := (i, \Psi_i(x))$*, for every* $(i,x) \in \sum_{i\in I}^* \lambda_0(i)$*, is a function, such that for every* $i \in I$ *the following left diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "e_i^{\Lambda^*}"'] & \mu_0(i) \arrow[d, "e_i^{M^*}"] \\
\sum_{i\in I}^* \lambda_0(i) \arrow[r, "\Sigma^*\Psi"'] & \sum_{i\in I}^* \mu_0(i).
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\prod_{i\in I}^* \lambda_0(i) \arrow[r, "\Pi^*\Psi"'] \arrow[u, "\pi_i^{\Lambda^*}"] & \prod_{i\in I}^* \mu_0(i). \arrow[u, "\pi_i^{M^*}"']
\end{tikzcd}$$

(iii) *If* $i \in I$*, the operation* $\pi_i^{\Lambda^*} : \prod_{i\in I}^* \lambda_0(i) \rightsquigarrow \lambda_0(i)$*, defined by* $\Theta \mapsto \Theta_i$*, is a function.*

(iv) *If* Ψ *is contravariant, the operation* $\Pi^*\Psi : \prod_{i\in I}^* \lambda_0(i) \rightsquigarrow \prod_{i\in I}^* \mu_0(i)$*, defined by* $[\Pi^*\Psi(\Theta)]_i := \Psi_i(\Theta_i)$*, for every* $i \in I$*, is a function, such that for every* $i \in I$ *the above right diagram commutes.*

(v) *If* Ψ_i *is an embedding, for every* $i \in I$*, then* $\Pi^*\Psi$ *is an embedding.*

Proof. We proceed as in the proofs of Propositions 3.2.7 and 3.3.5. $\square$

The definitions of operations on I-families of sets and their family-maps extend to operations on set-relevant I-families and their family-maps. An important example of a set-relevant family of sets is that of a family of sets over a set with a proof-relevant equality (see Definition 5.3.4). For reasons that are going to be clear in the study of these families of sets, the first definitional clause of a set-relevant family of sets over I involves the equality $=_{\mathbb{F}(\lambda_0(i),\lambda_0(i))}$ instead of the definitional one. Next follows the definition of the direct version of a set-relevant family of sets, the importance of which is explained in Note 6.10.4.

Definition 3.9.6. *If $(I, \preccurlyeq_I)$ is a directed set, a set-relevant direct family of sets indexed by I, is a quadruple $\Lambda^{*,\preccurlyeq} := \big(\lambda_0, \varepsilon_0^{\lambda^{\preccurlyeq}}, \Theta, \lambda_2^{\preccurlyeq}\big)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\varepsilon_0^{\lambda^{\preccurlyeq}} : D^{\preccurlyeq}(I) \rightsquigarrow \mathbb{V}_0$, $\Theta \in \bigcurlywedge_{(i,j)\in D^{\preccurlyeq}(I)} \varepsilon_0^{\lambda^{\preccurlyeq}}(i,j)$ is a modulus of inhabitedness for $\varepsilon_0^{\lambda^{\preccurlyeq}}$, and*

$$\lambda_2^{\preccurlyeq} : \bigcurlywedge_{(i,j)\in D^{\preccurlyeq}(I)} \bigcurlywedge_{p\in\varepsilon_0^{\lambda^{\preccurlyeq}}(i,j)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_2^{\preccurlyeq}\big((i,j),p\big) := \lambda_{ij}^{p,\preccurlyeq}, \quad (i,j) \in D(I), \quad p \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i,j),$$

such that the following conditions hold:

(i) *For every $i \in I$ there is $p \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i,i)$ such that $\lambda_{ii}^{p,\preccurlyeq} =_{\mathbb{F}(\lambda_0(i),\lambda_0(i))} \mathrm{id}_{\lambda_0(i)}$.*

(ii) *If $i \preccurlyeq_I j \preccurlyeq_I k$, then for every $p \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i,j)$ and every $q \in \varepsilon_0^{\lambda^{\preccurlyeq}}(j,k)$ there is $r \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i,k)$ such that $\lambda_{jk}^{q,\preccurlyeq} \circ \lambda_{ij}^{p,\preccurlyeq} = \lambda_{ik}^{r,\preccurlyeq}$.*

(iii) *For every $(i,j) \in D^{\preccurlyeq}(I)$ and every $p, p' \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i,j)$ and every $x \in \lambda_0(i)$ there is $k \in I$ such that $j \preccurlyeq_I k$ and there is $q \in \varepsilon_0^{\lambda^{\preccurlyeq}}(j,k)$ such that*

$$\lambda_{jk}^{q,\preccurlyeq}\big(\lambda_{ij}^{p,\preccurlyeq}(x)\big) =_{\lambda_0(k)} \lambda_{jk}^{q,\preccurlyeq}\big(\lambda_{ij}^{p',\preccurlyeq}(x)\big).$$

The modulus of inhabitedness Θ for $\varepsilon_0^{\lambda^{\preccurlyeq}}$ and the last condition in the previous definition guarantee that the equality on the corresponding $\sum$-set of a set-relevant direct family of sets satisfies the conditions of an equivalence relation.

Definition 3.9.7. *Let $\Lambda^{*,\preccurlyeq} := \big(\lambda_0, \varepsilon_0^{\lambda^{\preccurlyeq}}, \Theta, \lambda_2^{\preccurlyeq}\big)$ a set-relevant family of sets over a directed set $(I, \preccurlyeq)$. Its exterior union $\sum_{i\in I}^{*,\preccurlyeq} \lambda_0(i)$ is the totality $\sum_{i\in I} \lambda_0(i)$ equipped with the equality*

$$(i,x) =_{\sum_{i\in I}^{*,\preccurlyeq} \lambda_0(i)} (j,y) :\Leftrightarrow \exists_{k\in I}\bigg(i, j \preccurlyeq_I k \ \& \ \exists_{p\in\varepsilon_0^{\lambda^{\preccurlyeq}}(i,k)} \exists_{q\in\varepsilon_0^{\lambda^{\preccurlyeq}}(j,k)} \big(\lambda_{ik}^{p,\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{q,\preccurlyeq}(y)\big)\bigg).$$

The set $\prod_{i\in I}^{*,\preccurlyeq} \lambda_0(i)$ of dependent functions over $\Lambda^{*,\preccurlyeq}$ is defined by

$$\Phi \in \prod_{i\in I}^{*,\preccurlyeq} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \& \ \forall_{(i,j)\in D^{\preccurlyeq}(I)} \forall_{p\in\varepsilon_0^{\lambda,\preccurlyeq}(i,j)} \big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}^{p,\preccurlyeq}(\Phi_i)\big),$$

and it is equipped with the pointwise equality.

Proposition 3.9.8. *The equality on $\sum_{i\in I}^{*,\preccurlyeq} \lambda_0(i)$ satisfies the conditions of an equivalence relation.*

Proof. To show that $(i,x) =_{\sum_{i\in I}^{*,\preccurlyeq}\lambda_0(i)} (i,x)$ we use the first definitional clause of a set-relevant directed family of sets. The proof of the equality $(j,y) =_{\sum_{i\in I}^{*,\preccurlyeq}\lambda_0(i)} (i,x)$ from the equality $(i,x) =_{\sum_{i\in I}^{*,\preccurlyeq}\lambda_0(i)} (j,y)$ is trivial. For transitivity we suppose that $(i,x) =_{\sum_{i\in I}^{*,\preccurlyeq}\lambda_0(i)} (j,y)$ and $(j,y) =_{\sum_{i\in I}^{*,\preccurlyeq}\lambda_0(i)} (j',z)$ i.e.,

$$\exists_{k'\in I}\bigg(j \preccurlyeq_I k' \ \&\ j' \preccurlyeq_I k' \ \&\ \exists_{p'\in\varepsilon_0^{\lambda^{\preccurlyeq}}(j,k')}\exists_{q'\in\varepsilon_0^{\lambda^{\preccurlyeq}}(j',k')}\big(\lambda_{jk'}^{p',\preccurlyeq}(y) =_{\lambda_0(k')} \lambda_{j'k'}^{q',\preccurlyeq}(z)\big)\bigg).$$

There is $k''\in I$ such that $k \preccurlyeq_I k''$ and $k' \preccurlyeq_I k''$.

$$\begin{tikzcd}
& k \arrow[r] & k'' & k' \arrow[l] & \\
i \arrow[ur] & & j \arrow[ul] \arrow[ur] & & j' \arrow[ul]
\end{tikzcd}$$

Moreover, there are $r\in\varepsilon_0^{\lambda^{\preccurlyeq}}(i,k'')$ and $s\in\varepsilon_0^{\lambda^{\preccurlyeq}}(j,k'')$ such that

$$\lambda_{ik''}^{r,\preccurlyeq}(x) \overset{i\preccurlyeq_I k\preccurlyeq_I k''}{=} \lambda_{kk''}^{\Theta(k,k''),\preccurlyeq}\big(\lambda_{ik}^{p,\preccurlyeq}(x)\big) = \lambda_{kk''}^{\Theta(k,k''),\preccurlyeq}\big(\lambda_{jk}^{q,\preccurlyeq}(y)\big) \overset{j\preccurlyeq_I k\preccurlyeq_I k''}{=} \lambda_{jk''}^{s,\preccurlyeq}(y).$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[dr, "\lambda_{ik}^{p,\preccurlyeq}"] \arrow[ddrr, bend right, "\lambda_{ik''}^{r,\preccurlyeq}"'] \arrow[dddrr, bend right=60, "\lambda_{il}^{r''',\preccurlyeq}"'] & & \lambda_0(j) \arrow[dl, "\lambda_{jk}^{q,\preccurlyeq}"'] \arrow[dr, "\lambda_{jk'}^{p',\preccurlyeq}"] \arrow[dd, "\lambda_{jk''}^{s,\preccurlyeq}"] & & \lambda_0(j') \arrow[dl, "\lambda_{j'k'}^{q',\preccurlyeq}"'] \arrow[ddll, bend left, "\lambda_{j'k''}^{r',\preccurlyeq}"] \arrow[dddll, bend left=60, "\lambda_{j'l}^{r'',\preccurlyeq}"] \\
& \lambda_0(k) \arrow[dr, "\lambda_{kk'}^{\Theta(k,k''),\preccurlyeq}"'] & & \lambda_0(k') \arrow[dl, "\lambda_{k'k''}^{\Theta(k'k''),\preccurlyeq}"] & \\
& & \lambda_0(k'') \arrow[d, "\lambda_{k''l}^{t,\preccurlyeq}"] & & \\
& & \lambda_0(l) & &
\end{tikzcd}$$

If we apply condition (iii) of Definition 3.9.7 to $j \preccurlyeq_I k''$, $y\in\lambda_0(j)$ and the transport maps $\lambda_{jk''}^{s,\preccurlyeq}$ and $\lambda_{k'k''}^{\Theta(k',k''),\preccurlyeq}\circ\lambda_{jk'}^{p',\preccurlyeq}$ from $\lambda_0(j)$ to $\lambda_0(k'')$, then there is $l\in I$, such that $k''\preccurlyeq_I l$, and some $t\in\varepsilon_0^{\lambda^{\preccurlyeq}}(k'',l)$ such that

$$\begin{aligned}
\lambda_{k''l}^{t,\preccurlyeq}\big(\lambda_{jk''}^{s,\preccurlyeq}(y)\big) &= \lambda_{k''l}^{t,\preccurlyeq}\Big(\lambda_{k'k''}^{\Theta(k',k''),\preccurlyeq}\big(\lambda_{jk'}^{p',\preccurlyeq}(y)\big)\Big)\\
&= \lambda_{k''l}^{t,\preccurlyeq}\Big(\lambda_{k'k''}^{\Theta(k',k''),\preccurlyeq}\big(\lambda_{j'k'}^{q',\preccurlyeq}(z)\big)\Big)\\
&= \lambda_{k''l}^{t,\preccurlyeq}\big(\lambda_{j'k''}^{r',\preccurlyeq}(z)\Big)\\
&= \lambda_{j'l}^{r'',\preccurlyeq}(z),
\end{aligned}$$

for some $r' \in \varepsilon_0^{\lambda^{\preccurlyeq}}(j', k'')$ and some $r'' \in \varepsilon_0^{\lambda^{\preccurlyeq}}(j', l)$. From $\lambda_{ik''}^{r,\preccurlyeq}(x) =_{\lambda_0(k'')} \lambda_{jk''}^{s,\preccurlyeq}(y)$ we get

$$\lambda_{k''l}^{t,\preccurlyeq}\big(\lambda_{jk''}^{s,\preccurlyeq}(y)\big) = \lambda_{k''l}^{t,\preccurlyeq}\big(\lambda_{ik''}^{r,\preccurlyeq}(x)\big) = \lambda_{il}^{r''',\preccurlyeq}(x),$$

for some $r''' \in \varepsilon_0^{\lambda^{\preccurlyeq}}(i, l)$. Hence, $\lambda_{il}^{r''i,\preccurlyeq}(x) =_{\lambda_0(l)} \lambda_{j'l}^{r'',\preccurlyeq}(z)$ i.e., $(i, x) =_{\sum_{i\in I}^{*,\preccurlyeq} \lambda_0(i)} (j', z)$. $\square$

3.10 Families of families of sets, an impredicative interlude

We define the notion of a family of families of sets $(\Lambda^i)_{i\in I}$, where each family of sets Λ^i is indexed by some set $\mu_0(i)$, and $i \in I$. As expected, the index-sets are given by some family $M \in \mathtt{Fam}(I)$, and $(\Lambda^i)_{i\in I}$ must must be a function-like object i.e., if $i =_I j$, the family Λ^i of sets over the index-set $\mu_0(i)$ is "equal" to the family Λ^j of sets over the index-set $\mu_0(j)$. This equality can be expressed through the notion of a family map from Λ^i to Λ^j over μ_{ij} (see Definition 3.1.5). As in the case of the definition of a family of sets we provide the a priori given transport maps of $(\Lambda^i)_{i\in I}$ with certain properties that guarantee the existence of these family-maps. As $\mathtt{Fam}(I)$ is an impredicative set, to define a family of families of sets, we need to introduce, in complete analogy to the introduction of $\mathbb{V}_0$, the class $\mathbb{V}_0^{\mathrm{im}}$ *of sets and impredicative sets*. All notions of assignment routines defined in Chapter 2 are defined in a similar way when the class $\mathbb{V}_0^{\mathrm{im}}$ is used instead of $\mathbb{V}_0$. We add the superscript $^{\mathrm{im}}$ to a symbol in order to denote the version of the corresponding notion that requires the use of $\mathbb{V}_0^{\mathrm{im}}$.

Definition 3.10.1. *Let $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$ and, if $(i, j) \in D(I)$ let the set*

$$T_{ij}(M) := \{(m, n) \in \mu_0(i) \times \mu_0(j) \mid \mu_{ij}(m) =_{\mu_0(j)} n\}.$$

A family of families of sets over I and M, or an (I, M)-family of families of sets, is a pair $(\Lambda^i)_{i\in I} := \big(\Lambda^{0,M}, \Lambda^{1,M}\big)$, where

$$\Lambda^{0,M} \colon \overset{\mathrm{im}}{\curlywedge} \mathtt{Fam}(\mu_0(i)), \qquad \Lambda_i^{0,M} := \big(\lambda_0^i, \lambda_1^1\big); \qquad i \in I,$$

$$\Lambda^{1,M} \colon \curlywedge_{(i,j)\in D(I)} \curlywedge_{(m,n)\in T_{ij}(M)} \mathbb{F}\big(\lambda_0^i(m), \lambda_0^j(n)\big),$$

$$\big(\Lambda_{(i,j)}^{1,M}\big)_{(m,n)} := \lambda_{mn}^{ij} \colon \lambda_0^i(m) \to \lambda_0^j(n); \qquad (i, j) \in D(I),\ (m, n) \in T_{ij}(M),$$

such that the transport maps λ_{mn}^{ij} of $(\Lambda^i)_{i\in I}$, satisfy the following conditions:

(i) *For every $i \in I$ and $(m, m') \in T_{ij}(M)$, we have that $\lambda_{mm'}^{ii} := \lambda_{mm'}^{i}$.*

(ii) *If $i =_I j =_I k$, for every $(m, n) \in T_{ij}(M)$ and $(n, l) \in T_{jk}(M)$, the following diagram commutes*

$$\begin{tikzcd}
\lambda_0^i(m) \arrow[d, "\lambda_{mn}^{ij}"'] \arrow[dr, "\lambda_{ml}^{ik}"] & \\
\lambda_0^j(n) \arrow[r, "\lambda_{nl}^{jk}"'] & \lambda_0^k(l).
\end{tikzcd}$$

Let $\mathtt{Fam}(I, M)$ be the totality of (I, M)-families of families of sets.

For condition (i) above, we have that $\mu_{ii}(m) = m' :\Leftrightarrow m =_{\mu(i)} m'$ and for condition (ii), from the hypotheses $(m, n) \in T_{ij}(M) :\Leftrightarrow \mu_{ij}(m) = n$ and $(n, l) \in T_{jk}(M) :\Leftrightarrow \mu_{jk}(n) = l$ we get $(m, l) \in T_{ik}(M) :\Leftrightarrow \mu_{ik}(m) = l$, as $\mu_{ik}(m) = \mu_{jk}(\mu_{ij}(m)) = l$. The main intuition behind this defintion is that if $i =_I j$, then $\mu_0(i) =_{\mathbb{V}_0} \mu_0(j)$, hence, if $m \in \mu_0(i)$ and $n \in \mu_0(j)$, there is a transport map λ_{mn}^{ij} from $\lambda_0^i(m)$ to $\lambda_0^j(n)$. It is easy to see that if $\Lambda \in \mathtt{Fam}(J)$, then by taking $I := \mathbb{1}$ and M the constant family J over $\mathbb{1}$, then Λ can be viewed as an $(\mathbb{1}, M)$-family of families of sets.

Lemma 3.10.2. *If $(\Lambda^i)_{i\in I} \in \mathtt{Fam}(I, M)$, for its transport maps λ_{mn}^{ij} the following hold:*
(i) $\lambda_{mm}^{ii} := \mathrm{id}_{\lambda_0^i(m)}$.
(ii) $\lambda_{mn}^{ij} \circ \lambda_{nm}^{ji} = \mathrm{id}_{\lambda_0^j(n)}$.
(iii) *If $\mu_{ij}(m) = n$, then* $\lambda_{m\mu_{ij}(m)}^{ij} = \lambda_{n\mu_{ij}(m)}^{j} \circ \lambda_{mn}^{ij}$

$$\begin{tikzcd}
\lambda_0^i(m) \arrow[r, "\lambda_{mm}^i"] \arrow[d, "\lambda_{m\mu_{ij}(m)}^{ij}"'] & \lambda_0^i(m) \arrow[d, "\lambda_{mn}^{ij}"] \\
\lambda_0^j(\mu_{ij}(m)) & \lambda_0^j(n). \arrow[l, "\lambda_{n\mu_{ij}(m)}^{j}"]
\end{tikzcd}$$

(iv) *If $m =_{\mu_0(i)} m'$, then* $\lambda_{\mu_{ij}(m)\mu_{ij}(m')}^{j} \circ \lambda_{m\mu_{ij}(m)}^{ij} = \lambda_{m'\mu_{ij}(m')}^{ij} \circ \lambda_{mm'}^{i}$

$$\begin{tikzcd}
\lambda_0^i(m) \arrow[r, "\lambda_{mm'}^i"] \arrow[d, "\lambda_{m\mu_{ij}(m)}^{ij}"'] & \lambda_0^i(m') \arrow[d, "\lambda_{m'\mu_{ij}(m')}^{ij}"] \\
\lambda_0^j(\mu_{ij}(m)) \arrow[r, "\lambda_{\mu_{ij}(m)\mu_{ij}(m')}^{j}"'] & \lambda_0^j(\mu_{ij}(m')).
\end{tikzcd}$$

(v) *If $i =_I j =_I k$, then* $\lambda_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))}^{k} \circ \lambda_{m\mu_{ik}(m)}^{ik} = \lambda_{\mu_{ij}(m)\mu_{jk}(\mu_{ij}(m))}^{jk} \circ \lambda_{m\mu_{ij}(m)}^{ij}$

$$\begin{tikzcd}
\lambda_0^i(m) \arrow[r, "\lambda_{m\mu_{ik}(m)}^{ik}"] \arrow[d, "\lambda_{m\mu_{ij}(m)}^{ij}"'] & \lambda_0^k(\mu_{ik}(m)) \arrow[d, "\lambda_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))}^{ik}"] \\
\lambda_0^j(\mu_{ij}(m)) \arrow[r, "\lambda_{\mu_{ij}(m)\mu_{jk}(\mu_{ij}(m)}^{jk}"'] & \lambda_0^k(\mu_{jk}(\mu_{ij}(m)).
\end{tikzcd}$$

Proof. (i) By Definition 3.10.1 $\lambda_{mm}^{ii} := \lambda_{mm}^{i} := \mathrm{id}_{\lambda_0^i(m)}$.
(ii) By the composition-rule and case (i) we get $\lambda_{mn}^{ij} \circ \lambda_{nm}^{ji} = \lambda_{nn}^{jj} = \mathrm{id}_{\lambda_0^j(n)}$.
(iii) By the composition-rule we get $\lambda_{m\mu_{ij}(m)}^{ij} = \lambda_{n\mu_{ij}(m)}^{jj} \circ \lambda_{mn}^{ij} := \lambda_{n\mu_{ij}(m)}^{j} \circ \lambda_{mn}^{ij}$.

(iv) By Definition 3.10.1 we have that

$$\begin{aligned}\lambda^{j}_{\mu_{ij}(m)\mu_{ij}(m')} \circ \lambda^{ij}_{m\mu_{ij}(m)} &:= \lambda^{jj}_{\mu_{ij}(m)\mu_{ij}(m')} \circ \lambda^{ij}_{m\mu_{ij}(m)} \\ &= \lambda^{ij}_{m\mu_{ij}(m')} \\ &= \lambda^{ij}_{m'\mu_{ij}(m')} \circ \lambda^{ii}_{mm'} \\ &:= \lambda^{ij}_{m'\mu_{ij}(m')} \circ \lambda^{i}_{mm'}.\end{aligned}$$

(v) If $l := \mu_{jk}(\mu_{ij}(m))$, then by case (iii) $\lambda^{ik}_{m\mu_{ik}(m)} = \lambda^{k}_{\mu_{jk}(\mu_{ij}(m))\mu_{ik}(m)} \circ \lambda^{ik}_{m\mu_{jk}(\mu_{ij}(m))}$, hence

$$\begin{aligned}\lambda^{k}_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))} \circ \lambda^{ik}_{m\mu_{ik}(m)} &= \lambda^{k}_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))} \circ \big[\lambda^{k}_{\mu_{jk}(\mu_{ij}(m))\mu_{ik}(m)} \circ \lambda^{ik}_{m\mu_{jk}(\mu_{ij}(m))}\big] \\ &:= \big[\lambda^{k}_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))} \circ \lambda^{k}_{\mu_{jk}(\mu_{ij}(m))\mu_{ik}(m)}\big] \circ \lambda^{ik}_{m\mu_{jk}(\mu_{ij}(m))} \\ &= \lambda^{k}_{\mu_{jk}(\mu_{ij}(m))\mu_{jk}(\mu_{ij}(m))} \circ \lambda^{ik}_{m\mu_{jk}(\mu_{ij}(m))} \\ &:= \lambda^{ik}_{m\mu_{jk}(\mu_{ij}(m))} \\ &= \lambda^{jk}_{\mu_{ij}(m)\mu_{jk}(\mu_{ij}(m))} \circ \lambda^{ij}_{m\mu_{ij}(m)}.\end{aligned}$$ □

Definition 3.10.3. *If $(\Lambda^i)_{i\in I} \in \mathtt{Fam}(I, M)$, and based on Definition 3.1.5, its transport family-maps Φ^{Λ}_{ij} are the family-maps $\Phi^{\Lambda}_{ij} \colon \Lambda^{0,M}_i \overset{\mu_{ij}}{\Longrightarrow} \Lambda^{0,M}_j$, defined by the rule*

$$\big[\Phi^{\Lambda}_{ij}\big]_m := \lambda^{ij}_{m\mu_{ij}(m)}; \qquad m \in \mu_0(i), (i,j) \in D(I).$$

The fact that $\lambda^{ij}_{m\mu_{ij}(m)} \colon \Lambda^{0,M}_i \overset{\mu_{ij}}{\Longrightarrow} \Lambda^{0,M}_j$ is shown by the commutativity of the diagram in case (iv) of Lemma 3.10.2. In analogy to the transport maps λ_{ij} of an I-family of sets Λ, the transport family-maps Φ^{Λ}_{ij} witness the equality between the $\mu_0(i)$-family of sets $\Lambda^{0,M}_i$ and the $\mu_0(j)$-family of sets $\Lambda^{0,M}_j$.

Definition 3.10.4. *If $(\Lambda^i)_{i\in I} \in \mathtt{Fam}(I, M)$, its exterior union $\sum_{i\in I}\sum_{m\in\mu_0(i)} \lambda^i_0(m)$ is defined by*

$$w \in \sum_{i\in I}\sum_{m\in\mu_0(i)} \lambda^i_0(m) :\Leftrightarrow \exists_{i\in I}\exists_{m\in\mu_0(i)}\exists_{x\in\lambda^i_0(m)}\big(w := (i,m,x)\big),$$

$$(i,m,x) =_{\sum_{i\in I}\sum_{m\in\mu_0(i)} \lambda^i_0(m)} (j,n,y) :\Leftrightarrow i =_I j \ \&\ \mu_{ij}(m) =_{\mu_0(j)} n \ \&\ \lambda^{ij}_{mn}(x) =_{\lambda^j_0(n)} y.$$

Remark 3.10.5. *The equality on $\sum_{i\in I}\sum_{m\in\mu_0(i)} \lambda^i_0(m)$ satisfies the conditions of an equivalence relation.*

Proof. To show $(i,m,x) = (i,m,x) :\Leftrightarrow i =_I i \ \&\ \mu_{ii}(m) = m \ \&\ \lambda^{ii}_{mm}(x) = x$, we use Lemma 3.10.2(i). If $(i,m,x) = (j,n,y) :\Leftrightarrow i = j \ \&\ \mu_{ij}(m) = n \ \&\ \lambda^{ij}_{mn}(x) = y$, then $j = i$ and $\mu_{ji}(n) = m$, and, using Lemma 3.10.2(ii), $\lambda^{ji}_{nm}(y) = \lambda^{ji}_{nm}\big(\lambda^{ij}_{mn}(x)\big) = \lambda^{ii}_{mm}(x) = x$ i.e., $(j,n,y) = (i,m,x)$. If $(i,m,x) = (j,n,y)$ and $(j,n,xy = (k,l,z) :\Leftrightarrow j = k \ \&\ \mu_{jk}(n) = l \ \&\ \lambda^{jk}_{nl}(y) = z$, then $i = k$ and $\mu_{ik}(m) = \mu_{jk}(\mu_{ij}(m)) = \mu_{jk}(n) = l$, and $\lambda^{ik}_{ml}(x) = \lambda^{jk}_{nl}\big(\lambda^{ij}_{mn}(x)\big) = \lambda^{jk}_{nl}(y) = z$ i.e., $(i,m,x) = (k,l,z)$. □

Proposition 3.10.6. *If $(\Lambda^i)_{i\in I} \in \texttt{Fam}(I, M)$, then $\Sigma := (\sigma_0, \sigma_1) \in \texttt{Fam}(I)$, where*

$$\sigma_0(i) := \sum_{m\in\mu_0(i)} \lambda_0^i(m); \qquad i \in I,$$

$$\sigma_1(i,j) := \sigma_{ij} \colon \Big(\sum_{m\in\mu_0(i)} \lambda_0^i(m)\Big) \to \sum_{n\in\mu_0(j)} \lambda_0^j(n); \qquad (i,j) \in D(I),$$

$$\sigma_{ij}(m,x) := \big(\mu_{ij}(m), \lambda^{ij}_{m\mu_{ij}(m)}(x)\big); \qquad m \in \mu_0(i),\ x \in \lambda_0^i(m).$$

Proof. First we show that the operation σ_{ij} is a function. We suppose that $(m,x) =_{\sum_{m\in\mu_0(i)}\lambda_0^i(m)} (m',x') :\Leftrightarrow m =_{\mu_0(i)} m'$ & $\lambda^i_{mm'}(x) =_{\lambda_0^i(m')} x$, and we show that

$$\big(\mu_{ij}(m), \lambda^{ij}_{m\mu_{ij}(m)}(x)\big) =_{\sum_{n\in\mu_0(j)}\lambda_0^j(n)} \big(\mu_{ij}(m'), \lambda^{ij}_{m\mu_{ij}(m')}(x')\big) \Leftrightarrow$$

$$\mu_{ij}(m) =_{\mu_0(j)} \mu_{ij}(m') \ \&\ \lambda^j_{\mu_{ij}(m)\mu_{ij}(m')}\big(\lambda^{ij}_{m\mu_{ij}(m)}(x)\big) =_{\lambda_0^j(\mu_{ij}(m'))} \lambda^{ij}_{m'\mu_{ij}(m')}(x').$$

The first conjunct follows from $m =_{\mu_0(i)} m'$, and the second is Lemma 3.10.2(iv). Since

$$\sigma_{ii}(m,x) := \big(\mu_{ii}(m), \lambda^{ii}_{m\mu_{ii}(m)}(x)\big) := (m, \lambda^{ii}_{mm}(x)) := (m, \mathrm{id}_{\lambda_0^i(m)}(x)) := (m,x),$$

we get $\sigma_{ii} := \mathrm{id}_{\sum_{m\in\mu_0(i)}\lambda_0^i(m)}$. For the commutativity of the diagram

$$\begin{tikzcd}
\sum_{m\in\mu_0(i)} \lambda_0^i(m) \arrow[d, "\sigma_{ij}"'] \arrow[dr, "\sigma_{ik}"] & \\
\sum_{n\in\mu_0(j)} \lambda_0^j(n) \arrow[r, "\sigma_{jk}"'] & \sum_{l\in\mu_0(k)} \lambda_0^k(l)
\end{tikzcd}$$

we have that by definition $\sigma_{ik}(m,x) := \big(\mu_{ik}(m), \lambda^{ik}_{m\mu_{ik}(m)}(x)\big)$, and

$$\begin{aligned}
\sigma_{jk}\big(\sigma_{ij}(m,x)\big) &:= \sigma_{jk}\big(\mu_{ij}(m), \lambda^{ij}_{m\mu_{ij}(m)}(x)\big) \\
&:= \Big(\mu_{jk}(\mu_{ij}(m)),\ \lambda^{jk}_{\mu_{ij}(m)\mu_{jk}(\mu_{ij}(m))}\big(\lambda^{ij}_{m\mu_{ij}(m)}(x)\big)\Big).
\end{aligned}$$

Hence, $\sigma_{ik}(m,x) =_{\sum_{l\in\mu_0(k)}\lambda_0^k(l)} \sigma_{jk}\big(\sigma_{ij}(m,x)\big) :\Leftrightarrow \mu_{ik}(m) =_{\mu_0(k)} \mu_{jk}(\mu_{ij}(m))$ and

$$\lambda^k_{\mu_{ik}(m)\mu_{jk}(\mu_{ij}(m))}\big(\lambda^{ik}_{m\mu_{ik}(m)}(x)\big) = \lambda^{jk}_{\mu_{ij}(m)\mu_{jk}(\mu_{ij}(m))}\big(\lambda^{ij}_{m\mu_{ij}(m)}(x)\big).$$

The first conjunct is immediate to show, and the second is exactly Lemma 3.10.2(v). □

Clearly, for the exterior union of $(\Lambda^i)_{i\in I}$ we have that

$$\sum_{i\in I}\sum_{m\in\mu_0(i)} \lambda_0^i(m) =_{\mathbb{V}_0} \sum_{i\in I} \sigma_0(i) := \sum_{i\in I}\Big(\sum_{m\in\mu_0(i)} \lambda_0^i(m)\Big).$$

If $(\Lambda^i)_{i\in I}$ and $(M^i)_{i\in I}$ are (I,M)-families of families, a map from $(\Lambda^i)_{i\in I}$ to $(M^i)_{i\in I}$ is an appropriate dependent function $(\Psi^i)_{i\in I}$ such that Ψ^i is a family map from Λ^i to M^i, for every $i \in I$. Before giving this definition we show a fact of independent interest.

Proposition 3.10.7. *Let* $(K^i)_{i\in I} := (K^{0,M}, K^{1,M}), (\Lambda^i)_{i\in I} := (\Lambda^{0,M}, \Lambda^{1,M}) \in \mathtt{Fam}(I, M)$, *and let* $i =_I j$.

(i) *The operation* $\gamma_{ij} : \mathtt{Map}\big(K_i^{0,M}, \Lambda_i^{0,M}\big) \rightsquigarrow \mathtt{Map}\big(K_j^{0,M}, \Lambda_j^{0,M}\big)$, *defined by the rule* $\Psi^i \mapsto \big[\gamma_{ij}(\Psi^i)\big]^j$, *is a function, where, for every* $n \in \mu_0(j)$, *the map* $\big[\gamma_{ij}(\Psi^i)\big]^j_n \colon \kappa_0^j(n) \to \lambda_0^j(n)$ *is defined by* $\big[\gamma_{ij}(\Psi^i)\big]^j_n := \lambda^{ij}_{\mu_{ji}(n)n} \circ \Psi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)}$

$$\begin{tikzcd}
\kappa_0^j(n) \arrow[r, "{[\gamma_{ij}(\Psi^i)]^j_n}"] \arrow[d, "\kappa^{ji}_{n\mu_{ji}(n)}"'] & \lambda_0^j(n) \\
\kappa_0^i(\mu_{ji}(n)) \arrow[r, "\Psi^i_{\mu_{ji}(n)}"'] & \lambda_0^i(\mu_{ji}(n)). \arrow[u, "\lambda^{ij}_{\mu_{ji}(n)n}"']
\end{tikzcd}$$

(ii) *The pair* $\Gamma := (\gamma_0, \gamma_1) \in \mathtt{Fam}(I)$, *where* $\gamma_0(i) := \mathtt{Map}\big(K_i^{0,M}, \Lambda_i^{0,M}\big)$, *for every* $i \in I$, *and* $\gamma_1(i,j) := \gamma_{ij}$, *for every* $(i,j) \in D(I)$.

Proof. (i) First we show that γ_{ij} is well-defined i.e., $\big[\gamma_{ij}(\Psi^i)\big]^j \in \mathtt{Map}(K^j, \Lambda^j)$. If $n, n' \in \mu_0(j)$, we show that the following diagram commutes

$$\begin{tikzcd}
\kappa_0^j(n) \arrow[r, "\kappa^j_{nn'}"] \arrow[d, "{[\gamma_{ij}(\Psi^i)]^j_n}"'] & \kappa_0^j(n') \arrow[d, "{[\gamma_{ij}(\Psi^i)]^j_{n'}}"] \\
\lambda_0^j(n) \arrow[r, "\lambda^j_{nn'}"'] & \lambda_0^j(n').
\end{tikzcd}$$

By definition we have that $\big[\gamma_{ij}(\Psi^i)\big]^j_{n'} \circ \kappa^j_{nn'} := \big[\lambda^{ij}_{\mu_{ji}(n')n'} \circ \Psi^i_{\mu_{ji}(n')} \circ \kappa^{ji}_{n'\mu_{ji}(n')}\big] \circ \kappa^j_{nn'}$. Since Ψ^i is in $\mathtt{Map}(K_i^{0,M}, \Lambda_i^{0,M})$, by the commutativity of the following diagram

$$\begin{tikzcd}
\kappa_0^i(\mu_{ji}(n)) \arrow[r, "\kappa^i_{\mu_{ji}(n)\mu_{ji}(n')}"] \arrow[d, "\Psi^i_{\mu_{ji}(n)}"'] & \kappa_0^i(\mu_{ji}(n')) \arrow[d, "\Psi^i_{\mu_{ji}(n')}"] \\
\lambda_0^i(\mu_{ji}(n)) \arrow[r, "\lambda^i_{\mu_{ji}(n)\mu_{ji}(n')}"'] & \lambda_0^i(\mu_{ji}(n')),
\end{tikzcd}$$

we get $\Psi^i_{\mu_{ji}(n')} \circ \kappa^i_{\mu_{ji}(n)\mu_{ji}(n')} = \lambda^i_{\mu_{ji}(n)\mu_{ji}(n')} \circ \Psi^i_{\mu_{ji}(n)}$, and hence

$$\begin{aligned}
\Psi^i_{\mu_{ji}(n')} \circ \kappa^{ji}_{n'\mu_{ji}(n')} &= \Psi^i_{\mu_{ji}(n')} \circ \big(\kappa^i_{\mu_{ji}(n)\mu_{ji}(n')} \circ \kappa^{ji}_{n'\mu_{ji}(n)}\big) \\
&:= \big(\Psi^i_{\mu_{ji}(n')} \circ \kappa^i_{\mu_{ji}(n)\mu_{ji}(n')}\big) \circ \kappa^{ji}_{n'\mu_{ji}(n)} \\
&= \big(\lambda^i_{\mu_{ji}(n)\mu_{ji}(n')} \circ \Psi^i_{\mu_{ji}(n)}\big) \circ \kappa^{ji}_{n'\mu_{ji}(n)},
\end{aligned}$$

$$
\begin{aligned}
\big[\gamma_{ij}(\Psi^i)\big]_{n'} \circ \kappa^j_{nn'} &:= \big[\lambda^{ij}_{\mu_{ji}(n')n'} \circ \big(\Psi^i_{\mu_{ji}(n')} \circ \kappa^{ji}_{n'\mu_{ji}(n')}\big)\big] \circ \kappa^j_{nn'} \\
&= \big[\lambda^{ij}_{\mu_{ji}(n')n'} \circ \big(\big(\lambda^i_{\mu_{ji}(n)\mu_{ji}(n')} \circ \Psi^i_{\mu_{ji}(n)}\big) \circ \kappa^{ji}_{n'\mu_{ji}(n)}\big)\big] \circ \kappa^j_{nn'} \\
&:= \big(\lambda^{ij}_{\mu_{ji}(n')n'} \circ \lambda^i_{\mu_{ji}(n)\mu_{ji}(n')}\big) \circ \Psi^i_{\mu_{ji}(n)} \circ \big(\kappa^{ji}_{n'\mu_{ji}(n)} \circ \kappa^j_{nn'}\big) \\
&= \lambda^{ij}_{\mu_{ji}(n)n'} \circ \Psi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)} \\
&= \lambda^j_{nn'} \circ \big(\lambda^{ij}_{\mu_{ji}(n)n} \circ \Psi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)}\big) \\
&:= \lambda^j_{nn'} \circ \big[\gamma_{ij}(\Psi^i)\big]^j_n.
\end{aligned}
$$

If $\Psi^i = \Phi^i$, we show that $\big[\gamma_{ij}(\Psi^i)\big]^j = \big[\gamma_{ij}(\Phi^i)\big]^j$. As

$$
[\gamma_{ij}(\Psi^i)]^j_n := \lambda^{ij}_{\mu_{ji}(n)n} \circ \Psi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)} \quad \& \quad [\gamma_{ij}(\Phi^i)]^j_n := \lambda^{ij}_{\mu_{ji}(n)n} \circ \Phi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)},
$$

and since $\Psi^i = \Phi^i$, we get $\Psi^i_{\mu_{ji}(n)} = \Phi^i_{\mu_{ji}(n)}$, and hence the following diagram commutes

$$
\begin{tikzcd}
\kappa^j_0(n) \arrow[r, "\lambda^{ij}_{\mu_{ji}(n)n} \circ \Psi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)}"] \arrow[d, "\kappa^j_{nn}"'] & \lambda^j_0(n) \arrow[d, "\lambda^j_{nn}"] \\
\kappa^j_0(n) \arrow[r, "\lambda^{ij}_{\mu_{ji}(n)n} \circ \Phi^i_{\mu_{ji}(n)} \circ \kappa^{ji}_{n\mu_{ji}(n)}"'] & \lambda^j_0(n).
\end{tikzcd}
$$

(ii) If $m \in \mu_0(i)$, then $[\gamma_{ii}(\Psi^i)]^i_m := \lambda^{ii}_{\mu_{ii}(m)m} \circ \Psi^i_{\mu_{ii}(m)} \circ \kappa^{ii}_{m\mu_{ii}(m)} := \lambda^{ii}_{mm} \circ \Psi^i_m \circ \kappa^{ii}_{mm} :=$ $\mathrm{id}_{\lambda^i_0(m)} \circ \Psi^i_m \circ \mathrm{id}_{\kappa^i_0(m)} := \Psi^i_m$, hence $[\gamma_{ii}(\Psi^i)]^i_m := \Psi^i_m$, and consequently $\big[\gamma_{ii}(\Psi^i)\big]^i := \Psi^i$. For the commutativity of the diagram

$$
\begin{tikzcd}
\mathtt{Map}\big(K^{0,M}_i, \Lambda^{0,M}_i\big) \arrow[d, "\gamma_{ij}"'] \arrow[dr, "\gamma_{ik}"] & \\
\mathtt{Map}\big(K^{0,M}_j, \Lambda^{0,M}_j\big) \arrow[r, "\gamma_{jk}"'] & \mathtt{Map}\big(K^{0,M}_k, \Lambda^{0,M}_k\big)
\end{tikzcd}
$$

we need to show the equality between the maps

$$
\chi^k_l := \lambda^{jk}_{\mu_{kj}(l)l} \circ \big[\gamma_{ij}(\Psi^i)\big]^j_{\mu_{kj}(l)} \circ \kappa^{kj}_{l\mu_{kj}(l)} \quad \& \quad \upsilon^k_l := \lambda^{ik}_{\mu_{ki}(l)l} \circ \Psi^j_{\mu_{ki}(l)} \circ \kappa^{ki}_{l\mu_{ki}(l)}.
$$

By the definition of $\big[\gamma_{ij}(\Psi^i)\big]^j_{\mu_{kj}(l)}$ we get

$$
\begin{aligned}
\chi^k_l &:= \lambda^{jk}_{\mu_{kj}(l)l} \circ \big(\lambda^{ij}_{\mu_{ji}(\mu_{kj}(l))\mu_{kj}(l)} \circ \Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \kappa^{ji}_{\mu_{kj}(l)\mu_{ji}(\mu_{kj}(l))}\big) \circ \kappa^{kj}_{l\mu_{kj}(l)} \\
&= \big(\lambda^{jk}_{\mu_{kj}(l)l} \circ \lambda^{ij}_{\mu_{ji}(\mu_{kj}(l))\mu_{kj}(l)}\big) \circ \Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \big(\kappa^{ji}_{\mu_{kj}(l)\mu_{ji}(\mu_{kj}(l))} \circ \kappa^{kj}_{l\mu_{kj}(l)}\big) \\
&= \lambda^{ik}_{\mu_{ji}(\mu_{kj}(l))l} \circ \Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \kappa^{ki}_{l\mu_{ji}(\mu_{kj}(l))}.
\end{aligned}
$$

By the supposed commutativity of the following diagram

$$\begin{tikzcd}
\kappa_0^i(\mu_{ki}(l)) \arrow[rr, "\kappa^i_{\mu_{ki}(l)\mu_{ji}(\mu_{kj}(l))}"] \arrow[d, "\Psi^i_{\mu_{ki}(l)}"'] & & \kappa_0^i(\mu_{ji}(\mu_{kj}(l))) \arrow[d, "\Psi^i_{\mu_{ji}(\mu_{kj}(l))}"] \\
\lambda_0^i(\mu_{ki}(l)) \arrow[rr, "\lambda^i_{\mu_{ki}(l)\mu_{ji}(\mu_{kj}(l))}"'] & & \lambda_0^i(\mu_{ji}(\mu_{kj}(l))),
\end{tikzcd}$$

$$\begin{aligned}
\upsilon_l^k &:= \lambda^{ik}_{\mu_{ki}(l)l} \circ \Psi^j_{\mu_{ki}(l)} \circ \kappa^{ki}_{l\mu_{ki}(l)} \\
&= \lambda^{ik}_{\mu_{ji}(\mu_{kj}(l))l} \circ \big(\lambda^i_{\mu_{ki}(l)\mu_{ji}(\mu_{kj}(l))} \circ \Psi^j_{\mu_{ki}(l)}\big) \circ \kappa^{ki}_{l\mu_{ki}(l)} \\
&= \lambda^{ik}_{\mu_{ji}(\mu_{kj}(l))l} \circ \big(\Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \kappa^i_{\mu_{ki}(l)\mu_{ji}(\mu_{kj}(l))}\big) \circ \kappa^{ki}_{l\mu_{ki}(l)} \\
&= \lambda^{ik}_{\mu_{ji}(\mu_{kj}(l))l} \circ \Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \big(\kappa^i_{\mu_{ki}(l)\mu_{ji}(\mu_{kj}(l))} \circ \kappa^{ki}_{l\mu_{ki}(l)}\big) \\
&= \lambda^{ik}_{\mu_{ji}(\mu_{kj}(l))l} \circ \Psi^i_{\mu_{ji}(\mu_{kj}(l))} \circ \kappa^{ki}_{l\mu_{ji}(\mu_{kj}(l))} \\
&= \chi_l^k.
\end{aligned}$$ ☐

Definition 3.10.8. *If $(K^i)_{i\in I}, (\Lambda^i)_{i\in I} \in \texttt{Fam}(I, M)$, a family of families-map from $(K^i)_{i\in I}$ to $(\Lambda^i)_{i\in I}$, in symbols $\Psi\colon (K^i)_{i\in I} \Rightarrow (\Lambda^i)_{i\in I}$, is a dependent operation $\Psi\colon \bigcurlywedge_{i\in I} \texttt{Map}\big(K_i^{0,M}, \Lambda_i^{0,M}\big)$ such that for every $(i,j) \in D(I)$ the following diagram commutes*

$$\begin{tikzcd}
K_i^{0,M} \arrow[r, Rightarrow, "\Phi^K_{ij}"] \arrow[d, Rightarrow, "\Psi^i"'] & K_j^{0,M} \arrow[d, Rightarrow, "\Psi^j"] \\
\Lambda_i^{0,M} \arrow[r, Rightarrow, "\Phi^\Lambda_{ij}"'] & \Lambda_j^{0,M},
\end{tikzcd}$$

where Φ^K_{ij} and Φ^Λ_{ij} are the transport family-maps of $(K^i)_{i\in I}$ and $(\Lambda^i)_{i\in I}$, respectively, according to Definition 3.10.3. If $\Xi\colon (\Lambda^i)_{i\in I} \Rightarrow (N^i)_{i\in I}$, the composition $\Xi \circ \Psi\colon (K^i)_{i\in I} \Rightarrow (N^i)_{i\in I}$ is defined, for every $i \in I$, by $(\Xi \circ \Psi)^i := \Xi^i \circ \Psi^i$

$$\begin{tikzcd}
K_i^{0,M} \arrow[r, Rightarrow, "\Phi^K_{ij}"] \arrow[d, Rightarrow, "\Psi^i"] \arrow[dd, Rightarrow, bend right=60, "(\Xi\circ\Psi)^i"'] & K_j^{0,M} \arrow[d, Rightarrow, "\Psi^j"'] \arrow[dd, Rightarrow, bend left=60, "(\Xi\circ\Psi)^j"] \\
\Lambda_i^{0,M} \arrow[r, Rightarrow, "\Phi^\Lambda_{ij}"'] \arrow[d, Rightarrow, "\Xi^i"] & \Lambda_j^{0,M} \arrow[d, Rightarrow, "\Xi^j"'] \\
N_i^{0,M} \arrow[r, Rightarrow, "\Phi^N_{ij}"'] & N_j^{0,M}.
\end{tikzcd}$$

The identity family of families-map $\mathrm{Id}_{(\Lambda^i)_{i\in I}}$ is defined by the rule $\big[\mathrm{Id}_{(\Lambda^i)_{i\in I}}\big]^i := \mathrm{Id}_{\Lambda_i^{0,M}}$, for every $i \in I$. The totality of family of families-maps from $(K^i)_{i\in I}$ to $(\Lambda^i)_{i\in I}$, and the canonical equality on $\texttt{Fam}(I, M)$ is defined in analogy to Definition 3.1.3.

If $\Psi\colon (K^i)_{i\in I} \Rightarrow (\Lambda^i)_{i\in I}$, the commutativity of the diagram in Definition 3.10.8 is unfolded as follows. If $i =_I j$ and $m \in \mu_0(i)$, then

$$\big[\Psi^j \circ \Phi^K_{ij}\big]_m = \big[\Phi^\Lambda_{ij} \circ \Psi^i\big]_m :\Leftrightarrow \Psi^j_{\mu_{ij}(m)} \circ \big[\Phi^K_{ij}\big]_m = \big[\Phi^\Lambda_{ij}\big]_m \circ \Psi^i_m$$
$$:\Leftrightarrow \Psi^j_{\mu_{ij}(m)} \circ \kappa^{ij}_{m\mu_{ij}(m)} = \lambda^{ij}_{m\mu_{ij}(m)} \circ \Psi^i_m$$

i.e., the following diagram commutes

$$\begin{tikzcd}
\kappa^i_0(m) \arrow[r, "\kappa^{ij}_{m\mu_{ij}(m)}"] \arrow[d, "\Psi^i_m"'] & \kappa^j_0(\mu_{ij}(m)) \arrow[d, "\Psi^j_{\mu_{ij}(m)}"] \\
\lambda^i_0(m) \arrow[r, "\lambda^{ij}_{m\mu_{ij}(m)}"'] & \lambda^j_0(\mu_{ij}(m)).
\end{tikzcd}$$

In analogy to Corollary 3.3.4 we have the following.

Corollary 3.10.9. *If $(K^i)_{i\in I}, (\Lambda^i)_{i\in I} \in \mathtt{Fam}(I, M)$ and $\Psi\colon \curlywedge_{i\in I} \mathtt{Map}\big(K^{0M}_i, \Lambda^{0,M}_i\big)$, the following are equivalent:*
(i) $\Psi\colon (K^i)_{i\in I} \Rightarrow (\Lambda^i)_{i\in I}$.
(ii) $\Psi \in \prod_{i\in I} \mathtt{Map}\big(K^{0M}_i, \Lambda^{0,M}_i\big)$.

Proof. If $i =_I j$, the commutativity of the diagram in the definition of a family of families-map $\Psi\colon (K^i)_{i\in I} \Rightarrow (\Lambda^i)_{i\in I}$ is equivalent to the membership condition $\Psi \in \prod_{i\in I} \mathtt{Map}\big(K^{0M}_i, \Lambda^{0,M}_i\big)$ using the above unfolding of the equality $\big[\Psi^j \circ \Phi^K_{ij}\big]_m = \big[\Phi^\Lambda_{ij} \circ \Psi^i\big]_m$. $\square$

Definition 3.10.10. *The totality $\prod_{i\in I}\prod_{m\in\mu_0(i)} \lambda^i_0(m)$ of dependent functions over a family of families of set $(\Lambda^i)_{i\in I} \in \mathtt{Fam}(I, M)$ is defined by*

$$\Theta \in \prod_{i\in I}\prod_{m\in\mu_0(i)} \lambda^i_0(m) :\Leftrightarrow \Theta\colon \curlywedge_{i\in I}\curlywedge_{m\in\mu_0(i)} \lambda^i_0(m) \ \&\ \forall_{(i,j)\in D(I)}\forall_{(m,n)\in T_{ij}(M)}\big(\Theta^j_n =_{\lambda^j_0(n)} \lambda^{ij}_{mn}(\Theta^i_m)\big),$$

$$\Theta =_{\prod_{i\in I}\prod_{m\in\mu_0(i)} \lambda^i_0(m)} \Phi :\Leftrightarrow \forall_{i\in I}\forall_{m\in\mu_0(i)}\big(\Theta^i_m =_{\lambda^i_0(m)} \Phi^i_m\big).$$

The theory of families of families of sets over (I, M) within $\mathbb{V}^{\mathrm{im}}_0$ can be developed further along the lines of the theory of families of sets over I within $\mathbb{V}_0$.

3.11 Notes

Note 3.11.1. The concept of a family of sets indexed by a (discrete) set was asked to be defined in [9], Exercise 2, p. 72, and the required definition, given by Richman, is included in [19], Exercise 2, p. 78, where the discreteness hypothesis is omitted. The definition has a strong type-theoretic flavour, although, Richman's motivation had categorical origin, rather than type-theoretic. In a personal communication regarding this definition, Richman referred to the definition of a set-indexed family of objects of a category, given in [76], p. 18, as the source of the definition attributed to him in [19], p. 78. Given the categorical flavour of

Bishop's notion of a subset, it might be that Bishop was also thinking in categorical terms, although Bishop, to our knowledge, neither used a purely categorical language to describe his concepts, nor he used general category theory as a foundational framework for BISH.

Specifically, in [76] Richman presented a set I as a category with objects its elements and

$$\mathrm{Hom}_{=_I}(i,j) := \{x \in \{0\} \mid i =_I j\},$$

for every $i, j \in I$. If we view $\mathbb{V}_0$ as a category with objects its elements and

$$\mathrm{Hom}_{=_{\mathbb{V}_0}}(X,Y) := \big\{(f,f') : \mathbb{F}(X,Y) \times \mathbb{F}(Y,X) \mid (f,f') : X =_{\mathbb{V}_0} Y\big\},$$

for every $X, Y \in \mathbb{V}_0$, then an I-family of sets is a functor from the category I to the category $\mathbb{V}_0$. Notice that in the definitions of $\mathrm{Hom}_{=_I}(i,j)$ and of $\mathrm{Hom}_{=_{\mathbb{V}_0}}(X,Y)$ the properties $P(x) := i =_I j$ and $Q(f,f') := (f,f') : X =_{\mathbb{V}_0} Y$ are extensional. In [95] we reformulated Richman's definition using the universe $\mathbb{V}_0$ of sets and the universe $\mathbb{V}_1$ of triplets (A,B,f), where $A, B \in \mathbb{V}_0$ and $f\colon A \to B$. Definition 3.1.1 rests on the notion of dependent operation, in order to be absolutely faithful to Bishop's account of sets and functions in [9] and [19]. For the definition of the concept of a family of sets in ZF, or CZF, see [82], p. 35, and Note 1.3.4.

The term "transport map" in Definition 3.1.1 is drawn from MLTT. Actually, Definition 3.1.1 is a "definitional form" of the type-theoretic transport i.e., the existence of the transport map $p_*\colon P(x) \to P(y)$, where $p\colon x =_A y$ and $P\colon A \to \mathcal{U}$ is a type-family over $A\colon \mathcal{U}$ in the universe of types $\mathcal{U}$. In MLTT the existence of p_* follows from Martin-Löf's J-rule, the induction principle that accommodates the indentity type-family $=_A\colon A \to A \to \mathcal{U}$, for every type $A\colon \mathcal{U}$. In Definition 3.1.1 we describe in a proof-irrelevant way i.e., using only the fact that $i =_I j$ and not referring to witnesses of this equality, a structure of transport maps. This structure in BST is defined, and not generated from the equality type family of MLTT.

Note 3.11.2. In the categorical setting of Richman (see Note 3.11.1), a family map $\Psi \in \mathtt{Map}_I(\Lambda, M)$ is a natural transformation from the functor Λ to the functor M. The fact that the most fundamental concepts of category theory, that of a functor and of a natural transformation, are formulated in a natural way in BST through the notion of a dependent operation explains why category theory is so closely connected to BST. For more on the connections between BST, dependent type theory and category theory see section 8.1.

Note 3.11.3. The exterior union, is necessary to the definition of the infinite product of a sequence of sets. In [19], p. 125, the following is noted:

> Within the main body of this text, we have only defined the product of a family of subsets of a given set. However, with the aid of Problem 2 of Chapter 3 we can define the product of an arbitrary sequence of sets. Definition (1.7) then applies to such a product[3].

Note 3.11.4. If $\Lambda^{\mathbb{N}}$ is the sequence of sets defined in Definition 3.1.2, the definitional clauses of the corresponding exterior union can be written as follows:

$$\sum_{n \in \mathbb{N}} X_n =: \{(n,x) \mid n \in \mathbb{N} \ \&\ x \in X_n\},$$

$$(n,x) =_{\sum_{n \in \mathbb{N}} X_n} (m,y) :\Leftrightarrow n =_{\mathbb{N}} m \ \&\ x =_{X_n} y.$$

[3]This is the definition of the countable product of metric spaces.

Traditionally, the countable product of this sequence of sets is defined by

$$\prod_{n\in\mathbb{N}} X_n := \bigg\{\phi \colon \mathbb{N} \to \sum_{n\in\mathbb{N}} X_n \mid \forall_{n\in\mathbb{N}}\big(\phi(n) \in X_n\big)\bigg\},$$

which is a rough writing of the following

$$\prod_{n\in\mathbb{N}} X_n := \bigg\{\phi \colon \mathbb{N} \to \sum_{n\in\mathbb{N}} X_n \mid \forall_{n\in\mathbb{N}}\big(\mathbf{pr}_1(\phi(n)) =_{\mathbb{N}} n\big)\bigg\}.$$

In the second writing $\mathbf{pr}_1(\phi(n)) =_{\mathbb{N}} n$ implies that $\mathbf{pr}_1(\phi(n) := n$, hence, if $\phi(n) := (m, y)$, then $m = n$ and $y \in X_n$. When the equality of I though, is not like that of $\mathbb{N}$, we cannot solve this problem in a satisfying way. Although Bishop did not consider products other than countable ones, in more abstract areas of mathematics, like e.g., the general topology of Bishop spaces, arbitrary products are considered (see [88]). One could have defined

$$\Phi \in \prod_{i\in I} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{F}\bigg(I, \sum_{i\in I} \lambda_0(i)\bigg) \ \& \ \forall_{i\in I}\big(\mathbf{pr}_1(\Phi(i)) := i\big).$$

This approach has the problem that the property

$$Q(\Phi) :\Leftrightarrow \forall_{i\in I}\big(\mathbf{pr}_1(\Phi(i)) := i\big)$$

is not necessarily extensional; let $\Phi =_{\mathbb{F}(I, \sum_{i\in I} \lambda_0(i))} \Theta$ i.e., $\forall_{i\in I}\big(\Phi(i) =_{\sum_{i\in I} \lambda_0(i)} \Theta(i)\big)$, and suppose that $Q(\Phi)$. If we fix some $i \in I$, and $\Phi(i) := (i, x)$ and $\Theta(i) := (j, y)$, we only get that $j =_I i$. The use of dependent operations allows us to define the right analogue to the $\prod$-type of MLTT and being at the same time compatible with the use of dependent operations by Bishop in [9], p. 65.

Note 3.11.5. A precise formulation of the definition in [19], p. 85, of the countable product of a sequence $\big(X_n, \rho_n\big)_{n\in\mathbb{N}}$ of metric spaces, where ρ_n is bounded by 1, for every $n \in \mathbb{N}$, is the following. Let $\Lambda^{\mathbb{N}} := (\lambda_0^{\mathbb{N}}, \lambda_1^{\mathbb{N}})$ be the $\mathbb{N}$-family of the sets $(X_n)_{n\in\mathbb{N}}$ (see Definition 3.1.2). Notice that the dependent operation $\lambda_1^{\mathbb{N}}$ is compatible to the corresponding metric structures in the sense that each transport map $\lambda_{nn}^{\mathbb{N}} := \mathrm{id}_{X_n}$ is a morphism in any category of metric spaces considered. This is an example of a *spectrum of metric spaces* over $\Lambda^{\mathbb{N}}$ (see also the introduction to section 6.1). The *countable product metric* ρ_∞ on $\prod_{n\in\mathbb{N}} X_n$, for every $\Phi, \Theta \in \prod_{n\in\mathbb{N}} X_n$, is defined by

$$\rho_\infty(\Phi, \Theta) := \sum_{n=0}^{\infty} \frac{\rho_n\big(\Phi_n, \Theta_n\big)}{2^n}.$$

Note 3.11.6. The equality $\Phi_j =_{\lambda_0(j)} \lambda_{ij}(\Phi_i)$ in Definition 3.3.1 is the proof-irrelevant version of dependent application of a dependent function in MLTT (see also Note 5.7.10).

Note 3.11.7. As it is mentioned in [84], the axiom of choice is "freely used in Bishop constructivism". In Theorem 3.6.4 we show only the formal version of the type-theoretic axiom choice within BST i.e., the the distributivity of $\prod$ over $\sum$. This term was suggested to us by M. Maietti. In [95] a proof of this result is also given, where dependecy is formulated with the help of the universe $\mathbb{V}_1$ of triplets (A, B, f) (see Note 3.11.1). As it was first noted to us by E. Palmgren, this distributivity holds in every locally cartesian closed category. In [128] it is mentioned that this fact is attributed to Martin-Löf and his work [73]. For a proof see [2].

Note 3.11.8. The notion of an I-set of sets is in accordance with Bishop's predicative spirit, and his need to avoid the treatment of the universe $\mathbb{V}_0$ as a set. This notion was not defined by Bishop, only its "internal" version, the notion of an I-set of subsets, was defined similarly by him in [9], p. 65. The use of the term "set of subsets" was a source of misreading of [9] from the side of Myhill in [80] (see also Notes 7.6.5 and 7.6.7). The definition of the set $\lambda_0 I$ is in the spirit of the definition of the quotient group G/H of the group G by its normal subgroup H, given in [76], p. 38. If I is equipped with the equality $=_I^{\Lambda}$, then Λ does not become necessarily an I-set of sets. The reason for this is that the transport maps of Λ are given beforehand, and if we equip I with $=_I^{\Lambda}$ we need to add a transport map λ_{ij} for every pair (i,j) for which $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$ and $(i,j) \notin D(I)$, where $D(I)$ is understood here as the diagonal $D(I, =_I)$ with respect to the equality $=_I$. So, λ_1 has to be extended, and define a new family of sets over $(I, =_I^{\Lambda})$, which is going to be an $(I, =_I^{\Lambda})$-set of sets.

Note 3.11.9. A direct family of sets is a useful variation of the notion of a set-indexed family of sets (see Chapter 6). A directed set $(I, \preccurlyeq_I)$ can also be seen as a category with objects the elements of I, and $\mathrm{Hom}_{\preccurlyeq_I}(i,j) := \{x \in \{0\} \mid i \preccurlyeq_I j\}$. If the universe $\mathbb{V}_0$ is seen as a category with objects its elements and $\mathrm{Hom}_{\preccurlyeq_{\mathbb{V}_0}}(X,Y) := \mathbb{F}(X,Y)$, an $(I, \preccurlyeq_I)$-family of sets is a functor from the category $(I, \preccurlyeq_I)$ to this new category $\mathbb{V}_0$.

Note 3.11.10. A generalisation of the notion of a direct family of sets is that of a preorder family of sets. If $(I, \preccurlyeq_I)$ is a preorder (see Definition 9.2.1), a *covariant preorder family of sets* over $(I, \preccurlyeq_I)$ is defined as a direct family of sets. One needs though the property of a directed set to define an interesting equality on the exterior union of the corresponding family. A *contravariant preorder family of sets* over $(I, \preccurlyeq_I)$, or an $(I, \succcurlyeq_I)$-family of sets, is a pair $M^{\succcurlyeq} := (\mu_0, \mu_1^{\succcurlyeq})$, where if $(j,i) \in D^{\succcurlyeq}(I)$, the transport maps $\mu_1^{\succcurlyeq}(j,i) \colon \mu_0(j) \to \mu_0(i)$ behave in a dual way i.e., for every $i,j,k \in I$ with $k \succcurlyeq_I j \succcurlyeq_I i$, the following diagram commutes

$$\begin{tikzcd}
\mu_0(i) & \\
\mu_0(j) \arrow[u, "\mu_{ij}^{\succcurlyeq}"] & \mu_0(k). \arrow[l, "\mu_{jk}^{\succcurlyeq}"] \arrow[ul, "\mu_{ik}^{\succcurlyeq}"']
\end{tikzcd}$$

If $(I, \preccurlyeq)$ is an inverse-directed set (see Definition 9.2.1) and $M^{\succcurlyeq}$ is an $(I, \succcurlyeq_I)$-contravariant direct family of sets, defined in the obvious way, the *inverse-direct sum* $\sum_{i\in I}^{\succcurlyeq} \mu_0(i)$ of $M^{\succcurlyeq}$ is the totality $\sum_{i \in I} \mu_0(i)$, equipped with the equality

$$(i,x) =_{\sum_{i\in I}^{\succcurlyeq} \mu_0(i)} (j,y) :\Leftrightarrow \exists_{k \in I}\big(i \succcurlyeq_I k \ \&\ j \succcurlyeq_I k \ \&\ \mu_{ik}^{\succcurlyeq}(x) =_{\mu_0(k)} \mu_{jk}^{\succcurlyeq}(y)\big).$$

The set $\prod_{i\in I}^{\succcurlyeq} \mu_0(i)$ is defined in the expected way. Thinking classically, a topology T of open sets on a set X, equipped with the subset order $\subseteq$, is an inverse-directed set, and the notion of a *presheaf of sets* on (X,T) is an example of a $(T, \supseteq)$-contravariant direct family of sets. In the language of presheaves (see [65], p. 72) the transport maps $\mu_{ij}^{\succcurlyeq}$ are called *restriction maps*, and a family-map $\Phi \colon \Lambda^{\succcurlyeq} \Rightarrow M^{\succcurlyeq}$ is called a morphism of presheaves. It is natural to use also the term *extension map* for the transport map $\lambda_{ij}^{\preccurlyeq}$ of a covariant (direct) preorder family of sets. The notion of a family of sets over a partial order is also used in the definition of a Kripke model for intuitionistic predicate logic. For that see [125], p. 85, where the transport maps $\lambda_{ij}^{\preccurlyeq}$ are called there *transition functions*.

Note 3.11.11. If a set-relevant family-map $\Psi \colon \Lambda^* \Rightarrow M^*$ was defined by the stronger condition: for every $(i,j) \in D(I)$, every $p \in \varepsilon_0^{\lambda}(i,j)$ and *every* $q \in \varepsilon_0^{\mu}(i,j)$ the diagram in Definition 3.9.2 commutes, then the expected fact $\mathrm{id}_{\Lambda^*} \colon \Lambda^* \Rightarrow \Lambda^*$ implies that $\lambda_{ij}^{p} = \lambda_{ij}^{q}$, for every $p, q \in \varepsilon_0^{\lambda}(i,j)$. This property is called proof-irrelevance in Definition 5.3.4.

Note 3.11.12. The theory of families of families of sets over (I, M) within $\mathbb{V}_0^{\mathrm{im}}$ is the third rung of the ladder of set-like objects in $\mathbb{V}_0^{\mathrm{im}}$. The first three rungs can be described as follows:

$$X, Y \in \mathbb{V}_0, \qquad f \colon X \to Y,$$

$$\Lambda, M \in \mathtt{Fam}(I), \quad \Psi \colon \Lambda \Rightarrow M \Leftrightarrow \Psi \in \prod_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big),$$

$$(K^i)_{i\in I}, (\Lambda^i)_{i \in I} \in \mathtt{Fam}(I, M), \quad \Psi \colon (K^i)_{i \in I} \Rightarrow (\Lambda^i)_{i \in I} \Leftrightarrow \Psi \in \prod_{i \in I} \mathtt{Map}\big(K_i^{0,M}, \Lambda_i^{0,M}\big).$$

This hierarchy of universes and families can be extended further, if necessary.

Note 3.11.13 (Small categories within BST)**.** As it is mentioned in the introduction to Chapter 9 of [124], where category theory is developed within HoTT, categories do not fit well with set-based mathematics. Quit earlier, see e.g, in [61], it is mentioned that "type theory is adequate to represent faithfully categorical reasoning". In [61] the objects are modelled as types and the Hom-sets as Hom-setoids of arrows, within the Calculus of Inductive Constructions. In [87] there are elements of such a development of category theory within type theory, where both the algebraic and the hom-definition are given. In [86] are included interesting remarks on the formulation of category theory in [124]. For relations between category theory and Explicit Mathematics see [64]. In this note we briefly explain why small categories fit well with BST.

As we have already explained in Note 3.11.1, Richman used the notion of a functor to define the fundamental notion of a set-indexed family of sets, as a special case of a set-indexed family of objects in some category $\mathcal{C}$. Here we do the opposite. The notion of a set-indexed family of sets is fundamental and comes first. We use the basic theory of set-indexed families of sets to describe the basic notions of category theory within BST. In what follows we consider the objects of a category to be a set, although that could also be a class. The totality of arrows is always a set i.e., we could study locally small categories, but here we only present small categories. A set is not necessarily in the homotopy sense of the book-HoTT (see the corresponding notion of a strict category in [124], section 9.6). At this point we do not equip $\mathrm{Ob}_{\mathcal{C}}$ with equality with evidence (EwE) that makes possible the formulation of precategory and category in the sense of the book-HoTT (see section 5.3). For a general discussion on the relations between categories and sets in BST see section 8.1.

Definition 3.11.14. *A (small) category is a structure* $\mathcal{C} := \big(\mathrm{Ob}_{\mathcal{C}}, \mathrm{mor}_0^{\mathcal{C}}, \mathrm{mor}_1^{\mathcal{C}}, \mathrm{Comp}^{\mathcal{C}}, \mathrm{Id}^{\mathcal{C}}\big)$*, where* $\mathrm{Ob}_{\mathcal{C}}$ *is a set,* $\big(\mathrm{mor}_0^{\mathcal{C}}, \mathrm{mor}_1^{\mathcal{C}}\big) \in \mathtt{Fam}(\mathrm{Ob}_{\mathcal{C}} \times \mathrm{Ob}_{\mathcal{C}})$*,*

$$\mathrm{Comp}^{\mathcal{C}} : \bigcurlywedge_{x,y,z \in \mathrm{Ob}_{\mathcal{C}}} \mathbb{F}\big(\mathrm{mor}_0^{\mathcal{C}}(y,z) \times \mathrm{mor}_0^{\mathcal{C}}(x,y), \mathrm{mor}_0^{\mathcal{C}}(x,z)\big),$$

$$\mathrm{Comp}_{xyz}^{\mathcal{C}} := \mathrm{Comp}^{\mathcal{C}}(x,y,z) : \mathrm{mor}_0^{\mathcal{C}}(y,z) \times \mathrm{mor}_0^{\mathcal{C}}(x,y) \to \mathrm{mor}_0^{\mathcal{C}}(x,z), \quad \mathrm{Comp}_{xyz}^{\mathcal{C}}(\phi, \psi) := \phi \circ \psi,$$

$$\mathrm{Id}^{\mathcal{C}} : \bigcurlywedge_{x \in \mathrm{Ob}_{\mathcal{C}}} \mathrm{mor}_0^{\mathcal{C}}(x,x), \quad \mathrm{Id}_x^{\mathcal{C}} := \mathrm{Id}^{\mathcal{C}}(x); \quad x \in \mathrm{Ob}_{\mathcal{C}},$$

such that the following conditions are satisfied:

(Cat$_1$) *For every* $x, y, z, w \in \mathrm{Ob}_{\mathcal{C}}$, $\chi \in \mathrm{mor}_0^{\mathcal{C}}(z, w), \phi \in \mathrm{mor}_0^{\mathcal{C}}(y, z)$, $\psi \in \mathrm{mor}_0^{\mathcal{C}}(x, y)$,

$$\begin{aligned}\chi \circ (\phi \circ \psi) &:= \mathrm{Comp}_{xzw}^{\mathcal{C}}\big(\chi, \mathrm{Comp}_{xyz}^{\mathcal{C}}(\phi, \psi)\big)\\ &=_{\mathrm{mor}_0^{\mathcal{C}}(x,w)} \mathrm{Comp}_{xyw}^{\mathcal{C}}\big(\mathrm{Comp}_{yzw}^{\mathcal{C}}(\chi, \phi), \psi\big)\\ &:= (\chi \circ \phi) \circ \psi.\end{aligned}$$

(Cat$_2$) *For every* $x, y \in \mathrm{Ob}_{\mathcal{C}}$, *and for every* $\psi \in \mathrm{mor}_0^{\mathcal{C}}(x, y)$,

$$\mathrm{Id}_y^{\mathcal{C}} \circ \psi := \mathrm{Comp}_{xyy}^{\mathcal{C}}(\mathrm{Id}_y^{\mathcal{C}}, \psi) =_{\mathrm{Mor}_0^{\mathcal{C}}(x,y)} \psi \quad \& \quad \psi \circ \mathrm{Id}_x^{\mathcal{C}} := \mathrm{Comp}_{xxy}^{\mathcal{C}}(\psi, \mathrm{Id}_x^{\mathcal{C}}, \psi) =_{\mathrm{mor}_0^{\mathcal{C}}(x,y)} \psi.$$

(Cat$_3$) *For every* $x, y, z, x', y', z' \in \mathrm{Ob}_{\mathcal{C}}$, *with* $x =_{\mathrm{Ob}_{\mathcal{C}}} x'$, $y =_{\mathrm{Ob}_{\mathcal{C}}} y'$ *and* $z =_{\mathrm{Ob}_{\mathcal{C}}} z'$, *for every* $\psi \in \mathrm{mor}_0^{\mathcal{C}}(x, y)$, $\phi \in \mathrm{mor}_0^{\mathcal{C}}(y, z)$, $\mathrm{mor}_{(x,z)(x',z')}^{\mathcal{C}}(\phi \circ \psi) =_{\mathrm{mor}_0^{\mathcal{C}}(x',z')} \mathrm{mor}_{(y,z)(y',z')}^{\mathcal{C}}(\phi) \circ \mathrm{mor}_{(x,y)(x',y')}^{\mathcal{C}}(\psi)$

$$\begin{tikzcd}
x \arrow[r, "\psi"] \arrow[d, equal] & y \arrow[r, "\phi"] \arrow[d, equal] & z \arrow[d, equal] \\
x' \arrow[r, "\mathrm{mor}_{(x,y)(x',y')}(\psi)"] \arrow[rr, bend right=20, "\mathrm{mor}_{(y,z)(y',z')}^{\mathcal{C}}(\phi)\ \circ\ \mathrm{mor}_{(x,y)(x',y')}^{\mathcal{C}}(\psi)"] \arrow[rr, bend right=60, "\mathrm{mor}_{(x,z)(x',z')}^{\mathcal{C}}(\phi\circ\psi)"'] & y' \arrow[r, "\mathrm{mor}_{(y,z)(y',z')}(\phi)"] & z'.
\end{tikzcd}$$

(Cat$_4$) *For every* $x, x' \in \mathrm{Ob}_{\mathcal{C}}$, *with* $x =_{\mathrm{Ob}_{\mathcal{C}}} x'$, $\mathrm{mor}_{(x,x)(x',x')}^{\mathcal{C}}\big(\mathrm{Id}_x^{\mathcal{C}}\big) =_{\mathrm{mor}_0^{\mathcal{C}}(x',x')} \mathrm{Id}_{x'}^{\mathcal{C}}$

$$\begin{tikzcd}
x \arrow[r, "\mathrm{Id}_x"] \arrow[d, equal] & x \arrow[d, equal] \\
x' \arrow[r, "\mathrm{mor}_{(x,x)(x',x')}(\mathrm{Id}_x)"] \arrow[r, bend right=40, "\mathrm{Id}_{x'}"'] & x'.
\end{tikzcd}$$

The last two conditions, which reflect a functorial behaviour of the transport maps of $\mathrm{mor}_1^{\mathcal{C}}$ and are not found in the standard definition of a category, are necessary compatibility conditions between these transport maps and the $\big(\mathrm{Comp}^{\mathcal{C}}, \mathrm{Id}^{\mathcal{C}}\big)$-structure of the category $\mathcal{C}$. While in intensional MLTT these conditions follow from the transport, hence the J-rule, here we need to include them in our definition.

As a characteristic example of a category in the above sense, we consider the constructive analogue to the category of posets. Classically, the category of posets has objects the collection of all posets and arrows the monotone functions. In order to formulate this constructively, we need to generalise Definition 3.11.14 to categories with objects an abstract totality $\mathrm{Ob}_{\mathcal{C}}$. In Definition 3.11.15 we define the category generated by a spectrum of posets. We can define similarly the category generated by a spectrum of groups, rings, modules etc. (for the notion of an S-spectrum, where S is a structure on a set X, see the introduction to section 6.1).

Definition 3.11.15. *A spectrum of posets over a set I is an I-family of sets $\Lambda := (\lambda_0, \lambda_1)$ such that $(\lambda_0(i), \leq_i)$ is a poset for every $i \in I$, and for every $(i,j) \in D(I)$ the transport map $\lambda_{ij} : \lambda_0(i) \to \lambda_0(j)$ is a monotone function. If $\mathbb{F}^{\mathrm{mn}}\big(\lambda_0(i), \lambda_0(j)\big)$ is the set of monotone functions from $\lambda_0(i)$ to $\lambda_0(j)$, the category $\mathcal{C}_\Lambda$ generated by the I-spectrum Λ is the structure $\mathcal{C}_\Lambda := \big(\mathrm{Ob}^{\mathcal{C}_\Lambda}, \mathrm{mor}_0^{\mathcal{C}_\Lambda}, \mathrm{mor}_1^{\mathcal{C}_\Lambda}, \mathrm{Comp}^{\mathcal{C}_\Lambda}, \mathrm{Id}^{\mathcal{C}_\Lambda}\big)$, where $\mathrm{Ob}^{\mathcal{C}_\Lambda} := \lambda_0 I$, and $\mathrm{mor}_0^{\mathcal{C}_\Lambda} := \mathbb{F}^{\mathrm{mn}}\big(\lambda_0(i), \lambda_0(j)\big)$. If $i =_I i'$ and $j =_I j'$, and since the composition of monotone functions is monotone, let $\mathrm{mor}_{(i,j)(i'j')}^{\mathcal{C}_\Lambda} : \mathbb{F}^{\mathrm{mn}}\big(\lambda_0(i), \lambda_0(j)\big) \to \mathbb{F}^{\mathrm{mn}}\big(\lambda_0(i'), \lambda_0(j')\big)$, defined by*

$$f \mapsto \mathrm{mor}_{(i,j)(i',j')}^{\mathcal{C}_\Lambda}(f), \qquad \mathrm{mor}_{(i,j)(i',j')}^{\mathcal{C}_\Lambda}(f) := \lambda_{jj'} \circ f \circ \lambda_{i'i}; \qquad f \in \mathbb{F}^{\mathrm{mn}}\big(\lambda_0(i), \lambda_0(j)\big),$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "f"] & \lambda_0(j) \arrow[d, "\lambda_{jj'}"] \\
\lambda_0(i') \arrow[u, "\lambda_{i'i}"] \arrow[r, "\mathrm{mor}_{(i,j)(i'j')}^{\mathcal{C}_\Lambda}(f)"'] & \lambda_0(j').
\end{tikzcd}$$

The dependent operations $\mathrm{Id}^{\mathcal{C}_\Lambda}\big(\lambda_0(i)\big) := \mathrm{Id}_{\lambda_0(i)}$ and $\mathrm{Comp}^{\mathcal{C}_\Lambda}$ are defined as expected.

Next we only show (Cat_3) and (Cat_4) for $\mathcal{C}_\Lambda$. If $i =_I i'$, $f =_I j'$ and $k =_I k'$, we have that

$$\begin{aligned}
\mathrm{mor}_{(k,i)(k',i')}^{\mathcal{C}_\Lambda}(\phi) \circ \mathrm{mor}_{(j,k)(j',k')}^{\mathcal{C}_\Lambda}(\psi) &:= \big(\lambda_{ii'} \circ \phi \circ \lambda_{k'k}\big) \circ \big(\lambda_{kk'} \circ \psi \circ \lambda_{j'j}\big) \\
&= \lambda_{ii'} \circ \phi \circ \big(\lambda_{k'k} \circ \lambda_{kk'}\big) \circ \psi \circ \lambda_{j'j} \\
&= \lambda_{ii'} \circ (\phi \circ \psi) \circ \lambda_{j'j} \\
&:= \mathrm{mor}_{(j,i)(j',i')}^{\mathcal{C}_\Lambda}(\phi \circ \psi),
\end{aligned}$$

$$\mathrm{mor}_{(i,i)(i',i')}^{\mathcal{C}_\Lambda}\big(\mathrm{Id}^{\mathcal{C}_\Lambda}\big(\lambda_0(i)\big)\big) := \mathrm{mor}_{(i,i)(i',i')}^{\mathcal{C}_\Lambda}\big(\mathrm{Id}_{\lambda_0(i)}\big) := \lambda_{ii'} \circ \mathrm{Id}_{\lambda_0(i)} \circ \lambda_{i'i} = \mathrm{Id}_{\lambda_0(j)} := \mathrm{Id}^{\mathcal{C}_\Lambda}\big(\lambda_0(j)\big).$$

Definition 3.11.16. *A functor $\boldsymbol{F} : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C} := \big(\mathrm{Ob}_{\mathcal{C}}, \mathrm{mor}_0^{\mathcal{C}}, \mathrm{mor}_1^{\mathcal{C}}, \mathrm{Comp}^{\mathcal{C}}, \mathrm{Id}^{\mathcal{C}}\big)$ to $\mathcal{D} := \big(\mathrm{Ob}_{\mathcal{D}}, \mathrm{mor}_0^{\mathcal{D}}, \mathrm{mor}_1^{\mathcal{D}}, \mathrm{Comp}^{\mathcal{D}}, \mathrm{Id}^{\mathcal{D}}\big)$ is a pair $\boldsymbol{F} := (F_0, F_1)$, where $F_0 : \mathrm{Ob}_{\mathcal{C}} \to \mathrm{Ob}_{\mathcal{D}}$ and*

$$F_1 : \bigcurlywedge_{x,y \in \mathrm{Ob}_{\mathcal{C}}} \mathbb{F}\big(\mathrm{mor}_0^{\mathcal{C}}(x,y), \mathrm{mor}_0^{\mathcal{D}}(F_0(x), F_0(y))\big),$$

$$F_{xy} := F_1(x,y) : \mathrm{mor}_0^{\mathcal{C}}(x,y) \to \mathrm{mor}_0^{\mathcal{D}}(F_0(x), F_0(y)),$$

such that the following conditions are satisfied:

(Funct_1) *For every $x, y, z \in \mathrm{Ob}_{\mathcal{C}}$, and for every $\psi \in \mathrm{mor}_0^{\mathcal{C}}(x,y)$, $\phi \in \mathrm{mor}_0^{\mathcal{C}}(y,z)$ we have that*

$$F_{xz}(\phi \circ \psi) =_{\mathrm{mor}_0^{\mathcal{D}}(F_0(x), F_0(z))} F_{yz}(\phi) \circ F_{xy}(\psi).$$

(Funct_2) *For every $x \in \mathrm{Ob}_{\mathcal{C}}$ we have that $F_{xx}\big(\mathrm{Id}_x^{\mathcal{C}}\big) =_{\mathrm{mor}_0^{\mathcal{D}}(F_0(x), F_0(x))} \mathrm{Id}_{F_0(x)}^{\mathcal{D}}$.*

(Funct_3) *For every $x, y, x', y' \in \mathrm{Ob}_{\mathcal{C}}$, such that $x =_{\mathrm{Ob}_{\mathcal{C}}} x'$ and $y =_{\mathrm{Ob}_{\mathcal{C}}} y'$, hence $(x,y) = (x',y')$ and $\big(F_0(x), F_0(y)\big) = \big(F_0(x'), F_0(y')\big)$, the following diagram commutes*

$$\begin{tikzcd}[column sep=huge]
\mathrm{mor}_0^{\mathcal{C}}(x,y) \arrow[r, "\mathrm{mor}^{\mathcal{C}}_{(x,y)(x',y')}"] \arrow[d, "F_{xy}"'] & \mathrm{mor}_0^{\mathcal{C}}(x',y') \arrow[d, "F_{x'y'}"] \\
\mathrm{mor}_0^{\mathcal{D}}(F_0(x),F_0(y)) \arrow[r, "\mathrm{mor}^{\mathcal{D}}_{\left(F_0(x),F_0(y)\right)\left(F_0(x'),F_0(y')\right)}"'] & \mathrm{mor}_0^{\mathcal{D}}(F_0(x'),F_0(y')).
\end{tikzcd}$$

The last condition, which is not found in the standard definition of a functor, is a compatibility condition between the F_1-part of a functor $\boldsymbol{F} : \mathcal{C} \to \mathcal{D}$, the transport maps $\mathrm{mor}_1^{\mathcal{C}}$ of $\mathcal{C}$ and the transport maps $\mathrm{mor}_1^{\mathcal{D}}$ of $\mathcal{D}$. As an example of a standard categorical construction in this framework, we formulate the notion of slice category. If $\mathcal{C} := \big(\mathrm{Ob}_{\mathcal{C}}, \mathrm{mor}_0^{\mathcal{C}}, \mathrm{mor}_1^{\mathcal{C}}, \mathrm{Comp}^{\mathcal{C}}, \mathrm{Id}^{\mathcal{C}}\big)$ is a category and $x \in \mathrm{Ob}_{\mathcal{C}}$, then $\Lambda^x := \big(\lambda_0^x, \lambda_1^x\big) \in \mathtt{Fam}(\mathrm{Ob}_{\mathcal{C}})$, where

$$\lambda_0^x : \mathrm{Ob}_{\mathcal{C}} \rightsquigarrow \mathbb{V}_0, \quad \lambda_0^x(y) := \mathrm{mor}_0^{\mathcal{C}}(y,x); \quad y \in \mathrm{Ob}_{\mathcal{C}},$$

$$\lambda_1^x : \bigcurlywedge_{(y,y') \in D(\mathrm{Ob}_{\mathcal{C}})} \mathbb{F}\big(\mathrm{mor}_0^{\mathcal{C}}(y,x), \mathrm{mor}_0^{\mathcal{C}}(y',x)\big),$$

$$\lambda_{yy'}^x := \lambda_1^x(y,y') := \mathrm{mor}^{\mathcal{C}}_{(y,x)(y',x)} : \mathrm{mor}_0^{\mathcal{C}}(y,x) \to \mathrm{mor}_0^{\mathcal{C}}(y',x); \quad (y,y') \in (\mathrm{Ob}_{CC}).$$

Then we can prove the following fact.

Proposition 3.11.17. *Let* $\mathcal{C} := \big(\mathrm{Ob}_{\mathcal{C}}, \mathrm{mor}_0^{\mathcal{C}}, \mathrm{mor}_1^{\mathcal{C}}, \mathrm{Comp}^{\mathcal{C}}, \mathrm{Id}^{\mathcal{C}}\big)$ *be a category and* $x, z \in \mathrm{Ob}_{\mathcal{C}}$. *Let the structure* $\mathcal{C}/x := \big(\mathrm{Ob}_{\mathcal{C}/x}, \mathrm{mor}_0^{\mathcal{C}/x}, \mathrm{mor}_1^{\mathcal{C}/x}, \mathrm{Comp}^{\mathcal{C}/x}, \mathrm{Id}^{\mathcal{C}/x}\big)$, *where*

$$\mathrm{Ob}_{\mathcal{C}/x} := \sum_{y \in \mathrm{Ob}_{\mathcal{C}}} \lambda_0^x(y) := \sum_{y \in \mathrm{Ob}_{\mathcal{C}}} \mathrm{mor}_0^{\mathcal{C}}(y,x),$$

$$\mathrm{mor}_0^{\mathcal{C}/x}\big((y,f),(z,g)\big) := \big\{h \in \mathrm{mor}_0^{\mathcal{C}}(y,z) \mid g \circ h = f\big\}.$$

If $(y,f) =_{\mathrm{Ob}_{\mathcal{C}/x}} (y',f')$ *and* $(z,g) =_{\mathrm{Ob}_{\mathcal{C}/x}} (z',g')$, *the function*

$$\mathrm{mor}^{\mathcal{C}/x}_{\big((y,f),(z,g)\big),\big((y',f'),(z',g')\big)} : \mathrm{mor}_0^{\mathcal{C}/x}\big((y,f),(z,g)\big) \to \mathrm{mor}_0^{\mathcal{C}/x}\big((y',f'),(z',g')\big),$$

$$h \mapsto \mathrm{mor}^{\mathcal{C}}_{(y,z)(y',z')}(h); \quad h \in \mathrm{mor}_0^{\mathcal{C}/x}\big((y,f),(z,g)\big),$$

is well-defined. If $\mathrm{Comp}^{\mathcal{C}/x}$ *is defined in the expected compositional way, and if* $\mathrm{Id}^{\mathcal{C}/x}\big((y,f)\big) := \mathrm{Id}^{\mathcal{C}}(y)$, *for every* $(y,f) \in \mathrm{Ob}_{\mathcal{C}/x}$, *then* $\mathcal{C}/x$ *is a category. Moreover, if* $h \in \mathrm{mor}_0^{\mathcal{C}}(x,z)$, *then* $\boldsymbol{H} := (H_0, H_1) \colon \mathcal{C}/x \to \mathcal{C}/z$, *where*

$$H_0 \colon \Big(\sum_{y \in \mathrm{Ob}_{\mathcal{C}}} \mathrm{mor}_0^{\mathcal{C}}(y,x)\Big) \to \Big(\sum_{y \in \mathrm{Ob}_{\mathcal{C}}} \mathrm{mor}_0^{\mathcal{C}}(y,z)\Big),$$

$$(y,f) \mapsto (y, h \circ f); \quad (y,f) \in \sum_{y \in \mathrm{Ob}_{\mathcal{C}}} \mathrm{mor}_0^{\mathcal{C}}(y,x),$$

$$H_1 \colon \bigcurlywedge_{(y,f),(y',f') \in \mathcal{C}/x} \mathbb{F}\Big(\mathrm{mor}_0^{\mathcal{C}/x}\big((y,f)(y',f')\big), \mathrm{mor}_0^{\mathcal{C}/z}\big((y,h\circ f),(y',h\circ f')\big)\Big),$$

$$H_{(y,f)(y',f')} \colon \mathrm{mor}_0^{\mathcal{C}/x}\big((y,f)(y',f')\big) \to \mathrm{mor}_0^{\mathcal{C}/z}\big((y,h\circ f),(y',h\circ f')\big),$$

$$g \mapsto g; \quad g \in \mathrm{mor}_0^{\mathcal{C}/x}((y,f)(y',f')).$$

Chapter 4

Families of subsets

We develop the basic theory of set-indexed families of subsets and of the corresponding family-maps between them. In contrast to set-indexed families of sets, the properties of which are determined "externally" through their transport maps, the properties of a set-indexed family $\Lambda(X)$ of subsets of a given set X are determined "internally" through the embeddings of the subsets of $\Lambda(X)$ to X. The interior union of $\Lambda(X)$ is the internal analogue to the $\sum$-set of a set-indexed family of sets Λ, and the intersection of $\Lambda(X)$ is the internal analogue to the $\prod$-set of Λ. Families of sets over products, sets of subsets, and direct families of subsets are the internal analogue to the corresponding notions for families of sets. Set-indexed families of partial functions and set-indexed families of complemented subsets, together with their corresponding family-maps, are studied.

4.1 Set-indexed families of subsets

Roughly speaking, a family of subsets of a set X indexed by some set I is an assignment routine $\lambda_0 : I \rightsquigarrow \mathcal{P}(X)$ that behaves like a function i.e., if $i =_I j$, then $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. The following definition is a formulation of this rough description that reveals the witnesses of the equality $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. This is done "internally", through the embeddings of the subsets into X. The equality $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$, which in the previous chapter is defined "externally" through the transport maps, follows, and a family of subsets is also a family of sets.

Definition 4.1.1. *Let X and I be sets. A family of subsets of X indexed by I, or an I-family of subsets of X, is a triplet $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$,*

$$\mathcal{E}^X : \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), X\big), \qquad \mathcal{E}^X(i) := \mathcal{E}^X_i; \qquad i \in I,$$

$$\lambda_1 : \bigcurlywedge_{(i,j)\in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \qquad \lambda_1(i,j) := \lambda_{ij}; \qquad (i,j) \in D(I),$$

such that the following conditions hold:

(a) *For every $i \in I$, the function $\mathcal{E}^X_i : \lambda_0(i) \to X$ is an embedding.*

(b) *For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.*

(c) *For every $(i,j) \in D(I)$ we have that $\mathcal{E}^X_i = \mathcal{E}^X_j \circ \lambda_{ij}$ and $\mathcal{E}^X_j = \mathcal{E}^X_i \circ \lambda_{ji}$*

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, bend left, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & & \lambda_0(j) \arrow[ll, hook, bend left, "\lambda_{ji}"] \arrow[dl, hook', "\mathcal{E}_j^X"] \\
& X. &
\end{tikzcd}$$

$\mathcal{E}^X$ is a modulus of embeddings for λ_0, and λ_1 a modulus of transport maps for λ_0. Let $\Lambda := (\lambda_0, \lambda_1)$ be the I-family of sets that corresponds to $\Lambda(X)$. If $(A, i_A) \in \mathcal{P}(X)$, the constant I-family of subsets A is the pair $C^A(X) := (\lambda_0^A, \mathcal{E}^{X,A}, \lambda_1^A)$, where $\lambda_0(i) := A$, $\mathcal{E}_i^{X,A} := i_A$, and $\lambda_1(i,j) := \mathrm{id}_A$, for every $i \in I$ and $(i,j) \in D(I)$ (see the left diagram in Definition 4.1.3).

Proposition 4.1.2. *Let X and I be sets, $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\mathcal{E}^X$ a modulus of embeddings for λ_0, and λ_1 a modulus of transport maps for λ_0. The following are equivalent.*
(i) *$\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$ is an I-family of subsets of X.*
(ii) *$\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$ and $\mathcal{E}^X : \Lambda \Rightarrow C^X$, where C^X is the constant I-family X.*

Proof. (i)⇒(ii) First we show that $\Lambda \in \mathtt{Fam}(I)$. If $i =_I j =_I k$, then $\mathcal{E}_k^X \circ (\lambda_{jk} \circ \lambda_{ij}) = (\mathcal{E}_k^X \circ \lambda_{jk}) \circ \lambda_{ij} = \mathcal{E}_j^X \circ \lambda_{ij} = \mathcal{E}_i^X$ and $\mathcal{E}_k^X \circ \lambda_{ik} = \mathcal{E}_i^X$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\lambda_{ij}"'] \arrow[rr, hook, bend left, "\lambda_{ik}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & \lambda_0(j) \arrow[r, hook, "\lambda_{jk}"'] \arrow[d, hook, "\mathcal{E}_j^X"'] & \lambda_0(k) \arrow[dl, hook', "\mathcal{E}_k^X"] \\
& X, &
\end{tikzcd}$$

hence $\mathcal{E}_k^X \circ (\lambda_{jk} \circ \lambda_{ij}) = \mathcal{E}_k^X \circ \lambda_{ik}$, and since $\mathcal{E}_k^X$ is an embedding, we get $\lambda_{jk} \circ \lambda_{ij} = \lambda_{ik}$. If $\mathcal{E}^X : \Lambda \Rightarrow C^X$, the following squares are commutative

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, hook, "\mathcal{E}_i^X"'] & \lambda_0(j) \arrow[d, hook', "\mathcal{E}_j^X"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(j) \arrow[r, "\lambda_{ji}"] \arrow[d, hook, "\mathcal{E}_j^X"'] & \lambda_0(i) \arrow[d, hook', "\mathcal{E}_i^X"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & & \lambda_0(j) \arrow[dl, hook', "\mathcal{E}_j^X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(j) \arrow[rr, hook, "\lambda_{ji}"] \arrow[dr, hook, "\mathcal{E}_j^X"'] & & \lambda_0(i) \arrow[dl, hook', "\mathcal{E}_i^X"] \\
& X &
\end{tikzcd}$$

if and only if the above triangles are commutative. The implication (ii)⇒(i) follows immediately from the equivalence between the commutativity of the above pairs of diagrams. □

Definition 4.1.3. *Let X be a set and $(A, i_A^X), (B, i_B^X) \subseteq X$. The triplet $\Lambda^2(X) := (\lambda_0^2, \mathcal{E}^X, \lambda_1^2)$, where $\Lambda^2 := \lambda_0^2, \lambda_1^2$ is the $\mathbb{2}$-family of A, B, $\mathcal{E}_0^X := i_A^X$, and $\mathcal{E}_1^X := i_B^X$*

$$\begin{tikzcd}
A \arrow[rr, hook, "\mathrm{id}_A"] \arrow[dr, hook, "i_A^X"'] & & A \arrow[dl, hook', "i_A^X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, hook, "\mathrm{id}_B"] \arrow[dr, hook, "i_B^X"'] & & B \arrow[dl, hook', "i_B^X"] \\
& X &
\end{tikzcd}$$

is the $\mathbb{2}$-family of subsets A and B of X. The $\mathbb{n}$-family $\Lambda^{\mathbb{n}}(X)$ of the subsets $(A_1, i_1), \ldots, (A_n, i_n)$ of X, and the $\mathbb{N}$-family of subsets $(A_n, i_n)_{n\in\mathbb{N}}$ of X are defined similarly.

Definition 4.1.4. *If $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1), M(X) := (\mu_0, \mathcal{Z}^X, \mu_1)$ and $N(X) := (\nu_0, \mathcal{H}^X, \nu_1)$ are I-families of subsets of X, a family of subsets-map $\Psi\colon \Lambda(X) \Rightarrow M(X)$ from $\Lambda(X)$ to $M(X)$ is a dependent operation $\Psi : \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$, where $\Psi(i) := \Psi_i$, for every $i \in I$, such that, for every $i \in I$, the following diagram commutes*[1]

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\Psi_i"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & & \mu_0(i) \arrow[dl, hook', "\mathcal{Z}_i^X"] \\
& X. &
\end{tikzcd}$$

The totality $\mathtt{Map}_I(\Lambda(X), M(X))$ of family of subsets-maps from $\Lambda(X)$ to $M(X)$ is equipped with the pointwise equality. If $\Psi\colon \Lambda(X) \Rightarrow M(X)$ and $\Xi\colon M(X) \Rightarrow N(X)$, the composition family of subsets-map $\Xi \circ \Psi\colon \Lambda(X) \Rightarrow N(X)$ is defined by $(\Xi \circ \Psi)(i) := \Xi_i \circ \Psi_i$,

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[rr, bend left, "(\Xi\circ\Psi)_i"] \arrow[dr, hook, "\mathcal{E}_i"'] & \mu_0(i) \arrow[r, "\Xi_i"] \arrow[d, hook, "\mathcal{Z}_i"] & \nu_0(i) \arrow[dl, hook', "\mathcal{H}_i"] \\
& X, &
\end{tikzcd}$$

for every $i \in I$. The identity family of subsets-map $\mathrm{Id}_{\Lambda(X)}\colon \Lambda(X) \Rightarrow \Lambda(X)$ and the equality on the totality $\mathtt{Fam}(I, X)$ of I-families of subsets of X are defined as in Definition 3.1.3.

We see no obvious reason, like the one for $\mathtt{Fam}(I)$, not to consider $\mathtt{Fam}(I, X)$ to be a set. In the case of $\mathtt{Fam}(I)$ the constant I-family $\mathtt{Fam}(I)$ would be in $\mathtt{Fam}(I)$, while the constant I-family $\mathtt{Fam}(I, X)$ is not clear how could be seen as a family of subsets of X. If $\nu_0(i) := \mathtt{Fam}(I, X)$, for every $i \in I$, we need to define a modulus of embeddings $\mathcal{N}_i^X\colon \mathtt{Fam}(I, X) \hookrightarrow X$, for every $i \in I$. From the given data one could define the assignment routine $\mathcal{N}_i^X$ by the rule $\mathcal{N}_i^X\big(\Lambda(X)\big) := \mathcal{E}_i^X(u_i)$, if it is known that $u_i \in \lambda_0(i)$. Even in that case, the assignment routine $\mathcal{N}_i^X$ cannot be shown to satisfy the expected properties. Clearly, if $\mathcal{N}_i^X$ was defined by the rule $\mathcal{N}_i^X\big(\Lambda(X)\big) := x_0 \in X$, then it cannot be an embedding.

[1] Trivially, for every $i \in I$ the map $\Psi_i\colon \lambda_0(i) \to \mu_0(i)$ is an embedding.

Definition 4.1.5. *If $\Lambda(X) M(X) \in \mathtt{Fam}(I, X)$, let*

$$\Lambda(X) \leq M(X) :\Leftrightarrow \exists_{\Phi \in \mathtt{Map}_I(\Lambda(X), M(X))} \big(\Phi \colon \Lambda(X) \Rightarrow M(X)\big),$$

If $\Phi \in \mathtt{Map}_I(\Lambda(X), M(X)), \Psi \in \mathtt{Map}_I(M(X), \Lambda(X)), \Phi' \in \mathtt{Map}_I(M(X), N(X))$ and $\Psi' \in \mathtt{Map}_I(N(X), M(X))$, let the following set and operations

$$\mathtt{PrfEql}_0(\Lambda(X), M(X)) := \mathtt{Map}_I(\Lambda(X), M(X)) \times \mathtt{Map}_I(M(X), \Lambda(X)),$$

$$\mathtt{refl}(\Lambda) := \big(\mathrm{Id}_{\Lambda_X}, \mathrm{Id}_{\Lambda_X}\big) \;\;\&\;\; (\Phi, \Psi)^{-1} := (\Psi, \Phi) \;\;\&\;\; (\Phi, \Psi) * (\Phi', \Psi') := (\Phi' \circ \Phi, \Psi \circ \Psi').$$

Proposition 4.1.6. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1), M(X) := (\mu_0, \mathcal{Z}^X, \mu_1) \in \mathtt{Fam}(I, X)$.*
(i) *If $\Psi \colon \Lambda(X) \Rightarrow M(X)$, then $\Psi \colon \Lambda \Rightarrow M$.*
(ii) *If $\Psi \colon \Lambda(X) \Rightarrow M(X)$ and $\Phi \colon \Lambda(X) \Rightarrow M(X)$, then $\Phi =_{\mathtt{Map}_I(\Lambda(X), M(X))} \Psi$.*

Proof. (i) By the commutativity of the following inner diagrams

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\Psi_i"] \arrow[dd, "\lambda_{ij}"'] \arrow[dr, hook, "\mathcal{E}_i^X"] & & \mu_0(i) \arrow[dd, "\mu_{ij}"] \arrow[dl, hook', "\mathcal{Z}_i^X"'] \\
& X & \\
\lambda_0(j) \arrow[rr, hook, "\Psi_j"'] \arrow[ur, hook', "\mathcal{E}_j^X"'] & & \mu_0(j), \arrow[ul, hook, "\mathcal{Z}_j^X"]
\end{tikzcd}$$

we get the required commutativity of the above outer diagram. If $x \in \lambda_0(i)$, then

$$(\mathcal{Z}_j^X \circ \Psi_j)\big(\lambda_{ij}(x)\big) = \mathcal{E}_j^X\big(\lambda_{ij}(x)\big) = \mathcal{E}_i^X(x) = (\mathcal{Z}_i^X \circ \Psi_i)(x) = \mathcal{Z}_j^X\big(\mu_{ij}(\Psi_i(x))\big).$$

Since $\mathcal{Z}_j^X\big(\Psi_j(\lambda_{ij}(x))\big) = \mathcal{Z}_j^X\big(\mu_{ij}(\Psi_i(x))\big)$, we get $\Psi_j(\lambda_{ij}(x) = \mu_{ij}(\Psi_i(x))$.
(ii) If $i \in I$, then $\Psi_i \colon \lambda_0(i) \subseteq \mu_0(i)$, $\Phi_i \colon \lambda_0(i) \subseteq \mu_0(i)$

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, bend left, "\Phi_i"] \arrow[rr, hook, bend right, "\Psi_i"'] \arrow[dr, hook, "\mathcal{E}_i"'] & & \mu_0(i) \arrow[dl, hook', "E_i"] \\
& X, &
\end{tikzcd}$$

hence by Proposition 2.6.2 we get $\Psi_i =_{\mathbb{F}(\lambda_0(i), \mu_0(i))} \Phi_i$. $\square$

Because of Proposition 4.1.6(ii) all the elements of $\mathtt{PrfEql}_0(\Lambda(X), M(X))$ are equal to each other, hence the groupoid- properties (i)-(iv) for $\mathtt{PrfEql}_0(\Lambda(X), M(X))$ hold trivially. Of course, $\Lambda(X) =_{\mathtt{Fam}(I,X)} M(X) :\Leftrightarrow \Lambda(X) \leq M(X) \;\&\; M(X) \leq \Lambda(X)$. The characterisation of a family of subsets given in Proposition 4.1.2 together with the operations on family-maps help us define new families of subsets from given ones.

Proposition 4.1.7. *If* $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ *and* $M(Y) := (\mu_0, \mathcal{E}^Y, \mu_1) \in \mathtt{Fam}(I, Y)$*, then*

$$(\Lambda \times M)(X \times Y) := \Lambda(X) \times M(Y) := \big(\lambda_0 \times \mu_0, \mathcal{E}^X \times \mathcal{E}^Y, \lambda_1, \mu_1\big) \in \mathtt{Fam}(I, X \times Y),$$

where the I*-family* $\Lambda \times M := (\lambda_0 \times \mu_0, \lambda_1 \times \mu_1)$ *is defined in Definition 3.1.6 and the family-map* $\mathcal{E}^X \times \mathcal{E}^Y : \Lambda \times M \Rightarrow C^X \times C^Y$ *is defined in Proposition 3.1.8*(ii).

Proof. By Proposition 3.1.8(ii) $\mathcal{E}^X \times \mathcal{E}^Y : \Lambda \times M \Rightarrow C^X \times C^Y$, where $(\mathcal{E}^X \times \mathcal{E}^Y)_i : \lambda_0(i) \times \mu_0(i) \hookrightarrow X \times Y$ is defined by the rule $(u, w) \mapsto \big(\mathcal{E}^X(u), \mathcal{E}^Y(w)\big)$, for every $(u, w) \in \lambda_0(i) \times \mu_0(i)$. By Proposition 2.6.11 this is a well-defined subset of $X \times Y$. By Proposition 3.1.7(i) $\mathcal{E}^X \times \mathcal{E}^Y : \Lambda \times M \Rightarrow C^{X \times Y}$, and we use Proposition 4.1.2. $\square$

The operations on subsets induce operations on families of subsets.

Proposition 4.1.8. *Let* $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$ *and* $M(X) := (\mu_0, \mathcal{Z}^X, \mu_1) \in \mathtt{Fam}(I, X)$.
(i) $(\Lambda \cap M)(X) := (\lambda_0 \cap \mu_0, \mathcal{E}^X \cap \mathcal{Z}^X, \lambda_1 \cap \mu_1) \in \mathtt{Fam}(I, X)$*, where* $\lambda_0 \cap \mu_0 \colon I \rightsquigarrow \mathbb{V}_0$ *is defined by* $(\lambda_0 \cap \mu_0)(i) := \lambda_0(i) \cap \mu_0(i)$*, for every* $i \in I$*, and the dependent operations* $\mathcal{E}^X \cap \mathcal{Z}^X \colon \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i) \cap \mu_0(i), X\big)$*,* $\lambda_1 \cap \mu_1 \colon \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i) \cap \mu_0(i), \lambda_0(j) \cap \mu_0(j)\big)$ *are defined by*

$$\big(\mathcal{E}^X \cap \mathcal{Z}^X\big)_i := i^X_{\lambda_0(i) \cap \mu_0(i)}; \qquad i \in I,$$

$$\big[\big(\lambda_1 \cap \mu_1\big)_1(i, j)\big](u, w) := \big(\lambda_1 \cap \mu_1\big)_{ij}(u, w) := \big(\lambda_{ij}(u), \mu_{ij}(w)\big); \qquad (u, w) \in \lambda_0(i) \cap \mu_0(i).$$

(ii) $(\Lambda \cup M)(X) := (\lambda_0 \cup \mu_0, \mathcal{E}^X \cup \mathcal{Z}^X, \lambda_1 \cup \mu_1) \in \mathtt{Fam}(I, X)$*, where* $\lambda_0 \cup \mu_0 \colon I \rightsquigarrow \mathbb{V}_0$ *is defined by* $(\lambda_0 \cup \mu_0)(i) := \lambda_0(i) \cup \mu_0(i)$*, for every* $i \in I$*, and the dependent operations* $\mathcal{E}^X \cup \mathcal{Z}^X \colon \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i) \cup \mu_0(i), X\big)$*,* $\lambda_1 \cup \mu_1 \colon \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i) \cup \mu_0(i), \lambda_0(j) \cup \mu_0(j)\big)$ *are defined by*

$$\big(\mathcal{E}^X \cup \mathcal{Z}^X\big)_i(z) := \left\{ \begin{array}{ll} \mathcal{E}^X_i(z) & , z \in \lambda_0(i) \\ \mathcal{Z}^X_i(z) & , z \in \mu_0(i), \end{array} \right. ; \qquad i \in I,\ z \in \lambda_0(i) \cup \mu_0(i)$$

$$(\lambda_1 \cup \mu_1)_{ij}(z) := \left\{ \begin{array}{ll} \lambda_{ij}(z) & , z \in \lambda_0(i) \\ \mu_{ij}(z) & , z \in \mu_0(i), \end{array} \right. ; \qquad (i, j) \in D(I),\ z \in \lambda_0(i) \cup \mu_0(i).$$

Proof. (i) By Definition 2.6.9 we have that

$$\lambda_0(i) \cap \mu_0(i) := \{(u, w) \in \lambda_0(i) \times \mu_0(i) \mid \mathcal{E}^X_i(u) =_X \mathcal{Z}^X_i(w)\}, \quad i^X_{\lambda_0(i) \cap \mu_0(i)}(u, w) := \mathcal{E}^X_i(u),$$

$$\lambda_0(j) \cap \mu_0(j) := \{(u', w') \in \lambda_0(j) \times \mu_0(j) \mid \mathcal{E}^X_j(u') =_X \mathcal{Z}^X_j(w')\},\ i^X_{\lambda_0(j) \cap \mu_0(j)}(u', w') := \mathcal{E}^X_j(u').$$

Since $\mathcal{E}^X_j\big(\lambda_{ij}(u)\big) =_X \mathcal{E}^X_i(u) =_X \mathcal{Z}^X_i(w) =_X \mathcal{Z}^X_j\big(\mu_{ij}(w)\big)$, we get $\big(\lambda_1 \cap \mu_1\big)_{ij}(u, w) \in \lambda_0(j) \cap \mu_0(j)$. Clearly, $\big(\lambda_1 \cap \mu_1\big)_{ij}$ is a function. The commutativity of the following left inner diagrams

$$\begin{tikzcd}
\lambda_0(i) \cap \mu_0(i) \arrow[rr, bend left, "(\lambda_1 \cap \mu_1)_{ij}"] \arrow[dr, "(\mathcal{E}^X \cap \mathcal{Z}^X)_i"'] & & \lambda_0(j) \cap \mu_0(j) \arrow[ll, bend left, "(\lambda_1 \cap \mu_1)_{ji}"] \arrow[dl, "(\mathcal{E}^X \cap \mathcal{Z}^X)_j"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(i) \cup \mu_0(i) \arrow[rr, bend left, "(\lambda_1 \cup \mu_1)_{ij}"] \arrow[dr, "(\mathcal{E}^X \cup \mathcal{Z}^X)_i"'] & & \lambda_0(i) \cup \mu_0(i) \arrow[ll, bend left, "(\lambda_1 \cup \mu_1)_{ji}"] \arrow[dl, "(\mathcal{E}^X \cup \mathcal{Z}^X)_j"] \\
& X &
\end{tikzcd}$$

follows by the equalities $\big(\mathcal{E}^X \cap \mathcal{X}^X\big)_j\big((\lambda_1 \cap \mu_1)_{ij}(u,w)\big) := i^X_{\lambda_0(i)\cap\mu_0(i)}\big(\lambda_{ij}(u), \mu_{ij}(w)\big) := \mathcal{E}^X_j\big(\lambda_{ij}(u)\big) =_X \mathcal{E}^X_i(u) := \big(\mathcal{E}^X \cap \mathcal{X}^X\big)_i(u,w)$.
(ii) First we show that $(\lambda_1 \cup \mu_1)_{ij}$ is a function. The more interesting case is $z \in \lambda_0(i)$, $w \in \mu_0(i)$ and $\mathcal{E}^X_i(z) =_X \mathcal{Z}^X_i(w)$. Hence $\mathcal{E}^X_j\big(\lambda_{ij}(u)\big) =_X =_X \mathcal{Z}^X_j\big(\mu_{ij}(w)\big)$, and $\lambda_{ij}(z) =_{\lambda_0(i)\cup\mu_0(i)} \mu_{ij}(w)$. The commutativity of the above right inner diagrams is straightforward to show. $\square$

Proposition 4.1.9. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I,X)$ and $M(Y) := (\mu_0, \mathcal{E}^Y, \mu_1) \in \mathtt{Fam}(J,Y)$. If $f \colon X \to Y$, let $[f(\Lambda)](Y) := \big(f(\lambda_0), f(\mathcal{E}^X)^Y, f(\lambda_1)\big)$, where the non-dependent assignment routine $f(\lambda_0) \colon I \rightsquigarrow \mathbb{V}_0$, and the dependent operations $f(\mathcal{E}^X)^Y \colon \bigcurlywedge_{i\in I} \mathbb{F}(f(\lambda_0)(i), Y)$ and $f(\lambda_1) \colon \bigcurlywedge_{(i,i')\in D(I)} \mathbb{F}\big(f(\lambda_0)(i), f(\lambda_0)(i')\big)$ are defined by*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\mathcal{E}^X_i"] \arrow[rr, bend right, "f^Y_i"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$[f(\lambda_0)](i) := f(\lambda_0(i)) := (\lambda_0(i), f_i), \qquad f^Y_i := f \circ \mathcal{E}^X_i; \qquad i \in I,$$
$$\big[f(\mathcal{E}^X)^Y\big](i) := f^Y_i, \qquad f(\lambda_1)_{ii'} := \lambda_{ii'}; \qquad i \in I,\ (i,i') \in D(I).$$

We call $[f(\Lambda)](Y)$ the image of Λ under f. The pre-image of M under f is the triplet $[f^{-1}(M)](X) := \big(f^{-1}(\mu_0), f^{-1}(\mathcal{E}^Y)^X, f^{-1}(\mu_1)\big)$, where the non-dependent assignment routine $f^{-1}(\mu_0) \colon J \rightsquigarrow \mathbb{V}_0$, and the dependent operations $f^{-1}(\mathcal{E}^Y)^X \colon \bigcurlywedge_{j\in J} \mathbb{F}(f^{-1}(\mu_0)(j), X)$ and $f^{-1}(\mu_1) \colon \bigcurlywedge_{(j,j')\in D(J)} \mathbb{F}\big(f^{-1}(\mu_0)(j), f^{-1}(\mu_0)(j')\big)$ are defined by

$$[f^{-1}(\mu_0)](j) := f^{-1}(\mu_0(j)) := \big\{(x,y) \in X \times \mu_0(j) \mid f(x) =_Y \mathcal{E}^Y_j(y)\big\}; \qquad j \in J$$

$$e_j \colon f^{-1}(\mu_0(j)) \hookrightarrow X \qquad e_j(x,y) := x; \qquad x \in X,\ y \in \mu_0(j),\ j \in J,$$
$$\big[f^{-1}(\mathcal{E}^Y)^X\big](j) := e_j; \qquad j \in J,$$
$$f^{-1}(\mu_1)_{jj'} \colon f^{-1}(\mu_0)(j) \to f^{-1}(\mu_0)(j') \qquad f^{-1}(\mu_1)_{jj'}(x,y) := (x, \mu_{jj'}(y)); \qquad (j,j') \in D(J).$$

Then $[f(\Lambda)](Y) \in \mathtt{Fam}(I,Y)$ and $[f^{-1}(M)](X) \in \mathtt{Fam}(J,X)$.

Proof. It suffices to show the commutativity of the following diagrams

$$\begin{tikzcd}
 & \lambda_0(i) \arrow[rr, bend left, "\lambda_{ii'}"] \arrow[dr, hook, "\mathcal{E}^X_i"'] & & \lambda_0(i') \arrow[ll, bend left, "\lambda_{i'i}"] \arrow[dl, hook', "\mathcal{E}^X_{i'}"] \\
Y & & X \arrow[ll, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f^{-1}(\mu_0)(j) \arrow[rr, bend left, "f^{-1}(\mu_1)_{jj'}"] \arrow[dr, hook, "{[f^{-1}(\mathcal{E}^Y)^X](j)}"'] & & f^{-1}(\mu_0)(j') \arrow[ll, bend left, "f^{-1}(\mu_1)_{j'j}"] \arrow[dl, hook', "{[f^{-1}(\mathcal{E}^Y)^X](j')}"] \\
 & X. &
\end{tikzcd}$$

For the left, we use the supposed commutativity of the two diagrams without the arrow $f \colon X \to Y$. For the above right outer diagram we have that $\big[f^{-1}(\mathcal{E}^Y)^X\big](j')\big(f^{-1}(\mu_1)_{jj'}(x,y)\big) := \big[f^{-1}(\mathcal{E}^Y)^X\big](j')(x, \mu_{jj'}(y)) := e_{j'}((x, \mu_{jj'}(y)) := x := e_j(x,y) := \big[f^{-1}(\mathcal{E}^Y)^X\big](j)(x,y)$. For the commutativity of the above right inner diagram we proceed similarly. $\square$

The operations on families of subsets generate operations on family of subsets-maps.

Proposition 4.1.10. *Let $\Lambda(X), K(X), M(X), N(X) \in \mathtt{Fam}(I, X)$, $P(Y), Q(Y) \in \mathtt{Fam}(J, Y)$, and $f\colon X \to Y$. Let also $\Phi\colon \Lambda(X) \Rightarrow K(X)$, $\Psi\colon M(X) \Rightarrow N(X)$, and $\Xi\colon P(Y) \Rightarrow Q(Y)$.*
(i) $\Phi \cap \Psi\colon (\Lambda \cap M)(X) \Rightarrow (K \cap N)(X)$, *where, for every $i \in I$ and $(u, w) \in \lambda_0(i) \cap \mu_0(i)$,*

$$(\Phi \cap \Psi)_i\colon \lambda_0(i) \cap \mu_0(i) \to k_0(i) \cap \nu_0(i), \qquad (\Phi \cap \Psi)_i(u, w) := \big(\Phi_i(u), \Psi_i(w)\big).$$

(ii) $\Phi \cup \Psi\colon (\Lambda \cup M)(X) \Rightarrow (K \cup N)(X)$, *where, for every $i \in I$,*

$$(\Phi \cup \Psi)_i\colon \lambda_0(i) \cup \mu_0(i) \to k_0(i) \cup \nu_0(i),$$

$$(\Phi \cup \Psi)_i(z) := \begin{cases} \Phi_i(z) & , z \in \lambda_0(i) \\ \Psi_i(z) & , z \in \mu_0(i) \end{cases}$$

(iii) $\Phi \times \Xi\colon (\Lambda \times P)(X \times Y) \Rightarrow (K \times Q)(X \times Y)$, *where, for every $i \in I$ and $(u, w) \in \lambda_0(i) \times p_0(i)$,*

$$(\Phi \times \Xi)_i\colon \lambda_0(i) \times p_0(i) \to k_0(i) \times q_0(i), \qquad (\Phi \times \Xi)_i(u, w) := \big(\Phi_i(u), \Xi_i(w)\big).$$

(iv) $f(\Phi)\colon [f(\Lambda)](Y) \Rightarrow [f(K)](Y)$, *where, for every $i \in I$ and $u \in f(\lambda_0(i))$,*

$$[f(\Phi)]_i\colon f(\lambda_0(i)) \to f(k_0(i)), \qquad [f(\Phi)]_i(u) := \Phi_i(u).$$

(v) $f^{-1}(\Xi)\colon [f^{-1}(P)](X) \Rightarrow [f^{-1}(Q)](X)$, *where, for every $j \in J$ and $(x, y) \in f^{-1}(p_0(j))$,*

$$[f^{-1}(\Xi)]_i\colon f^{-1}(p_0(j)) \to f^{-1}(q_0(j)), \qquad [f^{-1}(\Xi)]_j(x, y) := \big(x, \Xi_j(y)\big).$$

Proof. It is straightforward to show that all family of subsets-maps above are well-defined. $\square$

Definition 4.1.11. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $h\colon J \to I$. The triplet $\Lambda(X) \circ h := (\lambda_0 \circ h, \mathcal{E}^X \circ h, \lambda_1 \circ h)$, where $\Lambda \circ h := (\lambda_0 \circ h, \lambda_1 \circ h)$ is the h-subfamily of Λ, and the dependent operation $\mathcal{E}^X \circ h\colon \bigcurlywedge_{j \in J} \mathbb{F}\big(\lambda_0(h(j)), X\big)$ is defined by $(\mathcal{E}^X \circ h)_j := \mathcal{E}^X_{h(j)}$, for every $j \in J$, is called the h-subfamily of $\Lambda(X)$. If $J := \mathbb{N}$, we call $\Lambda(X) \circ h$ the h-subsequence of $\Lambda(X)$.*

It is immediate to show that $\Lambda(X) \circ h \in \mathtt{Fam}(J, X)$, and if $\Lambda(X) \circ h \in \mathtt{Set}(J, X)$, then h is an embedding. All notions and results of section 3.4 on subfamilies of families of sets extend naturally to subfamilies of families of subsets.

4.2 The interior union of a family of subsets

Definition 4.2.1. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$ be an I-family of subsets of X. The interior union, or simply the union of $\Lambda(X)$ is the totality $\sum_{i \in I} \lambda_0(i)$, which we denote in this case by $\bigcup_{i \in I} \lambda_0(i)$. Let the non-dependent assignment routine $e^{\Lambda(X)}_{\bigcup}\colon \bigcup_{i \in I} \lambda_0(i) \rightsquigarrow X$ defined by $(i, x) \mapsto \mathcal{E}^X_i(x)$, for every $(i, x) \in \bigcup_{i \in I} \lambda_0(i)$, and let*

$$(i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow e^{\Lambda(X)}_{\bigcup}(i, x) =_X e^{\Lambda(X)}_{\bigcup}(j, y) :\Leftrightarrow \mathcal{E}^X_i(x) =_X \mathcal{E}^X_j(y).$$

If $\neq_X$ is an inequality on X, let $(i, x) \neq_{\bigcup_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow \mathcal{E}^X_i(x) \neq_X \mathcal{E}^X_j(y)$. The family $\Lambda(X)$ is called a covering of X, or $\Lambda(X)$ covers X, if

$$\bigcup_{i \in I} \lambda_0(i) =_{\mathcal{P}(X)} X.$$

If $\neq_I$ is an inequality on I, and $\neq_X$ an inequality on X, we say that $\Lambda(X)$ is a family of disjoint subsets of X (with respect to $\neq_I$), if

$$\forall_{i,j\in I}\big(i \neq_I j \Rightarrow \lambda_0(i)]\![\![\lambda_0(j)\big),$$

where by Definition 2.8.1 $\lambda_0(i)]\!][\![\lambda_0(j) :\Leftrightarrow \forall_{u\in\lambda_0(i)}\forall_{w\in\lambda_0(j)}\big(\mathcal{E}_i^X(u) \neq_X \mathcal{E}_j^X(w)\big)$. $\Lambda(X)$ is called a partition of X, if it covers X and it is a family of disjoint subsets of X.

Clearly, $=_{\bigcup_{i\in I}\lambda_0(i)}$ is an equality on $\bigcup_{i\in I}\lambda_0(i)$, which is considered to be a set, and the operation $e_{\bigcup}^{\Lambda(X)}$ is an embedding of $\bigcup_{i\in I}\lambda_0(i)$ into X, hence $\big(\bigcup_{i\in I}\lambda_0(i), e_{\bigcup}^{\Lambda(X)}\big) \subseteq X$. The inequality $\neq_{\bigcup_{i\in I}\lambda_0(i)}$ is the canonical inequality of the subset $\bigcup_{i\in I}\lambda_0(i)$ of X (see Corollary 2.6.3). Hence, if $(X, =_X, \neq_X)$ is discrete, then $\big(\bigcup_{i\in I}\lambda_0(i), =_{\bigcup_{i\in I}\lambda_0(i)}, \neq_{\bigcup_{i\in I}\lambda_0(i)}\big)$ is discrete, and if $\neq_X$ is tight, then $=_{\bigcup_{i\in I}\lambda_0(i)}$ is tight. As the following left diagram commutes, $\Lambda(X)$ covers X, if and only of the following right diagram commutes i.e., if and only if $X \subseteq \bigcup_{i\in I}\lambda_0(i)$

$$\begin{tikzcd}
\bigcup_{i\in I}\lambda_0(i) \arrow[rr, hook, bend left, "e_{\bigcup}^{\Lambda(X)}"] \arrow[dr, hook, "e_{\bigcup}^{\Lambda(X)}"'] & & X \arrow[dl, hook, "\mathrm{id}_X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigcup_{i\in I}\lambda_0(i) \arrow[dr, hook, "e_{\bigcup}^{\Lambda(X)}"'] & & X \arrow[ll, hook, bend right, "g"'] \arrow[dl, hook, "\mathrm{id}_X"] \\
& X. &
\end{tikzcd}$$

If $(i,x) =_{\bigcup_{i\in I}\lambda_0(i)} (j,y)$, it is not necessary that $i =_I j$, hence it is not necessary that $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$ (as we show in the next proposition, the converse implication holds). Consequently, the first projection operation $\mathtt{pr}_1^{\Lambda(X)} := \mathtt{pr}_1^{\Lambda}$, where Λ is the I-family of sets induced by $\Lambda(X)$, is not necessarily a function! The second projection map on $\Lambda(X)$ is defined by $\mathtt{pr}_2^{\Lambda(X)} := \mathtt{pr}_2^{\Lambda}$. Notice that $\neq_{\bigcup_{i\in I}\lambda_0(i)}$ is an inequality on $\bigcup_{i\in I}\lambda_0(i)$, without supposing neither an inequality on I, nor an inequality on the sets $\lambda_0(i)$'s, as we did in Proposition 3.2.2(ii). Moreover, $\neq_{\bigcup_{i\in I}}$ is tight, if $\neq_X$ is tight. Cases (ii) and (iii) of the next proposition are due to M. Zeuner.

Proposition 4.2.2. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$.*
(i) *If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$, then $(i,x) =_{\bigcup_{i\in I}\lambda_0(i)} (j,y)$.*
(ii) *If $e_{\bigcup}^{\Lambda(X)} \colon \sum_{i\in I}\lambda_0(i) \rightsquigarrow X$ is an embedding, $\big(\sum_{i\in I}\lambda_0(i), e_{\bigcup}^{\Lambda(X)}\big) =_{\mathcal{P}(X)} \big(\bigcup_{i\in I}\lambda_0(i), e_{\bigcup}^{\Lambda(X)}\big)$.*
(iii) *If $\neq_I$ is a tight inequality on X, and $\Lambda(X)$ is a family of disjoint subsets of X with respect to $\neq_I$, then $e_{\bigcup}^{\Lambda(X)} \colon \sum_{i\in I}\lambda_0(i) \hookrightarrow X$.*

Proof. (i) If $i =_I j$, and since $\mathcal{E}_j^X$ is a function, we get $\mathcal{E}_i^X(x) = \mathcal{E}_j^X(\lambda_{ij}(x)) = \mathcal{E}_j^X(y)$.
(ii) Let $\mathcal{E}_i^X(x) =_X \mathcal{E}_j^X(y) \Leftrightarrow (i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$. We define the operations $\mathrm{id}_1 \colon \sum_{i\in I}\lambda_0(i) \rightsquigarrow \bigcup_{i\in I}\lambda_0(i)$ and $\mathrm{id}_2 \colon \bigcup_{i\in I}\lambda_0(i) \rightsquigarrow \sum_{i\in I}\lambda_0(i)$, both defined by the identity map-rule. That id_1 is a function, follows from (i). That id_2 is a function, follows from the hypothesis on $e_{\bigcup}^{\Lambda(X)}$.
(iii) We suppose that $e_{\bigcup}^{\Lambda(X)}(i,x) := \mathcal{E}_i^X(x) =_X \mathcal{E}_j^X(y) := e_{\bigcup}^{\Lambda(X)}(j,y)$ and we show that $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$. The converse implication follows from (i). If $\neg(i \neq_I j)$, then $\lambda_0(i)]\!][\![\lambda_0(j)$, hence $\mathcal{E}_i^X(x) \neq_X \mathcal{E}_j^X(y)$, which contradicts our hypothesis. By the tightness of $\neq_I$ we get $i =_I j$,

and it remains to show that $\lambda_{ij}(x) =_{\lambda_0(j)} y$. By the equalities $\mathcal{E}_j^X\big(\lambda_{ij}(x)\big) =_X \mathcal{E}_i^X(y) =_X \mathcal{E}_j^X(y)$, and as $\mathcal{E}_j^X$ is an embedding, we get $\lambda_{ij}(x) =_{\lambda_0(j)} y$. $\square$

Remark 4.2.3. *Let $i_0 \in I$, $(A, i_A^X) \subseteq X$, and $C^A(X) := (\lambda_0^A, \mathcal{E}^{A,X}, \lambda_1^A) \in \mathtt{Fam}(I, X)$ the constant family A of subsets of X. Then*

$$\bigcup_{i \in I} A := \bigcup_{i \in I} \lambda_0^A(i) =_{\mathcal{P}(X)} A.$$

Proof. By definition $(i,a) =_{\bigcup_{i\in I} A} (j,b) :\Leftrightarrow \mathcal{E}_i^{A,X}(a) =_X \mathcal{E}_j^{A,X}(b) :\Leftrightarrow i_A^X(a) =_X i_A^X(b) \Leftrightarrow a =_A b$. Let the operation $\phi\colon \bigcup_{i\in I} A \rightsquigarrow A$, defined by $\phi(i,a) := a$, for every $(i,a) \in \bigcup_{i\in I} A$, and let the operation $\theta\colon A \rightsquigarrow \bigcup_{i\in I} A$, defined by $\theta(a) := (i_0, a)$, for every $a \in A$. Clearly, ϕ and θ are functions. The required equality of these subsets follows from the following equalities: $i_A^X(a) =_X e_{\bigcup}^{C^A(X)}(i,a) := \mathcal{E}_i^{A,X}(a) := i_A^X(a)$, and $e_{\bigcup}^{C^A(X)}(i_0,a) := \mathcal{E}_{i_0}^{A,X}(a) := i_A^X(a)$. $\square$

The interior union of a family of subsets generalises the union of two subsets.

Proposition 4.2.4. *If $\Lambda^2(X)$ is the 2-family of subsets A, B of X, $\bigcup_{i\in 2} \lambda_0^2(i) =_{\mathcal{P}(X)} A \cup B$.*

Proof. The operation $g\colon \bigcup_{i\in 2} \lambda_0^2(i) \rightsquigarrow A \cup B$, defined by $g(i,x) := \mathbf{pr}_2(i,x) := x$, for every $(i,x) \in \bigcup_{i\in 2} \lambda_0^2(i)$, is well-defined, and it is an embedding, since $g(i,x) =_{A\cup B} g(j,y) :\Leftrightarrow x =_{A\cup B} y :\Leftrightarrow i_A^X(x) =_X i_B^X(y) :\Leftrightarrow (i,x) =_{\bigcup_{i\in 2}\lambda_0(i)} (j,y)$. The operation $f : A\cup B \rightsquigarrow \bigcup_{i\in 2} \lambda_0^2(i)$, defined by $f(z) := (0,z)$, if $z \in A$, and $f(z) := (1,z)$, if $z \in B$, is easily seen to be a function. For the commutativity of the following inner diagrams

$$\begin{tikzcd}
A \cup B \arrow[rr, bend left, "f"] \arrow[dr, "i_{A\cup B}^X"'] & & \bigcup_{i\in 2} \lambda_0^2(i) \arrow[ll, bend left, "g"] \arrow[dl, "e_{\bigcup}^{\Lambda^2(X)}"] \\
 & X &
\end{tikzcd}$$

we use the equalities $i_{A\cup B}^X(g(i,x)) := i_{A\cup B}^X(x) := i_A^X(x) := e_{\bigcup}^{\Lambda^2(X)}(i,x)$ and

$$e_{\bigcup}^{\Lambda^2(X)} f(z)) := \begin{cases} e_{\bigcup}^{\Lambda^2(X)}(0,z) & , z \in A \\ \\ e_{\bigcup}^{\Lambda^2(X)}(1,z) & , z \in B \end{cases} := \begin{cases} i_A^X(z) & , z \in A \\ \\ i_B^X(z) & , z \in B \end{cases} := i_{A\cup B}^X(z). \qquad \square$$

Proposition 4.2.5. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $M(Y) := (\mu_0, \mathcal{E}^Y, \mu_1) \in \mathtt{Fam}(I, Y)$. If $f\colon X \to Y$, the following hold:*

(i) $f\Big(\bigcup_{i\in I} \lambda_0(i)\Big) =_{\mathcal{P}(Y)} \bigcup_{i\in I} f\big(\lambda_0(i)\big)$.

(ii) $f^{-1}\Big(\bigcup_{i\in I} \mu_0(i)\Big) =_{\mathcal{P}(X)} \bigcup_{i\in I} f^{-1}\big(\mu_0(i)\big)$.

Proof. (i) By Definition 2.6.9 we have that $f\big(\bigcup_{i\in I}\lambda_0(i)\big) := \big(\bigcup_{i\in I}\lambda_0(i), f_{\bigcup}^{\Lambda(X)}\big)$, where $f_{\bigcup}^{\Lambda(X)} := f \circ e_{\bigcup}^{\Lambda(X)}$, and by Proposition 4.1.9 we have that

$$\begin{aligned}
(i,x) =_{f\left(\bigcup_{i\in I}\lambda_0(i)\right)} (j,y) &:\Leftrightarrow f_{\bigcup}^{\Lambda(X)}(i,x) =_Y f_{\bigcup}^{\Lambda(X)}(j,y) \\
&:\Leftrightarrow \big(f \circ e_{\bigcup}^{\Lambda(X)}\big)(i,x) =_Y \big(f \circ e_{\bigcup}^{\Lambda(X)}\big)(j,y) \\
&:\Leftrightarrow f\big(\mathcal{E}_i^X(x)\big) =_Y f\big(\mathcal{E}_j^X(y)\big) \\
&:\Leftrightarrow f_i^Y(x) =_Y f_j^Y(y)
\end{aligned}$$

By Proposition 4.1.9 and Definition 4.2.1 for the subset $\big(\bigcup_{i\in I} f\big(\lambda_0(i)\big), e_{\bigcup}^{[f(\Lambda)](Y)}\big)$ of Y we have that $f(\lambda_0(i)) := \big(\lambda_0(i), f_i^Y\big)$, where $f_i^Y := f \circ \mathcal{E}_i^X$, for every $i \in I$, and $x =_{f(\lambda_0(i))} x' :\Leftrightarrow f_i^Y(x) =_Y f_o^Y(x')$. Moreover, $e_{\bigcup}^{[f(\Lambda)](Y)}(i,x) := \big[f(\mathcal{E}^X)^Y\big]_i(x) := f_i^Y(x)$. Let the operations $g\colon f\big(\bigcup_{i\in I}\lambda_0(i)\big) \rightsquigarrow \bigcup_{i\in I} f\big(\lambda_0(i)\big)$ and $h\colon \bigcup_{i\in I} f\big(\lambda_0(i)\big) \rightsquigarrow f\big(\bigcup_{i\in I}\lambda_0(i)\big)$, defined by the same rule $(i,x) \mapsto (i,x)$.

$$\begin{tikzcd}
f\big(\bigcup_{i\in I}\lambda_0(i)\big) \arrow[rr, bend left, "g"] \arrow[dr, hook, "f_{\bigcup}^{\Lambda(X)}"'] & & \bigcup_{i\in I} f\big(\lambda_0(i)\big) \arrow[ll, bend left, "h"] \arrow[dl, hook, "{\left[f^{-1}(\mathcal{E}^Y)^X\right](j')}"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
f^{-1}\big(\bigcup_{i\in I}\mu_0(i)\big) \arrow[rr, bend left, "g'"] \arrow[dr, hook, "e_{\bigcup}^{[f^{-1}(M)](X)}"'] & & \bigcup_{i\in I} f^{-1}\big(\mu_0(i)\big) \arrow[ll, bend left, "h'"] \arrow[dl, hook, "{\left[f^{-1}(\mathcal{E}^Y)^X\right]_i}"] \\
& X &
\end{tikzcd}$$

It is immediate by the previous equalities that the above left diagrams commute.
(ii) By Definitions 4.2.1 and 2.6.9 for the subset $\big(\bigcup_{i\in I}\mu_0(i), e_{\bigcup}^{M(Y)}\big)$ of Y we have that the embedding $e_{\bigcup}^{M(Y)}\colon \bigcup_{i\in I}\mu_0(i) \hookrightarrow Y$ is given by the rule $(i,y) \mapsto \mathcal{E}_i^Y(y)$, and

$$f^{-1}\bigg(\bigcup_{i\in I}\mu_0(i)\bigg) := \bigg\{\big(x,(i,y)\big) \in X \times \bigcup_{i\in I}\mu_0(i) \mid f(x) =_Y e_{\bigcup}^{M(Y)}(i,y)\bigg\},$$

with embedding into X the mapping e^X, defined by the rule $e^X\big(x,(i,y)\big) := x$. Moreover,

$$\big(x,(i,y)\big) =_{f^{-1}(\bigcup_{i\in I}\mu_0(i))} \big(x',(i',y')\big) :\Leftrightarrow x =_X x' \ \&\ \mathcal{E}_i^Y(y) =_Y \mathcal{E}_{i'}^Y(y').$$

The subset $f^{-1}(\mu_0(i)) := \{(x,y) \in X \times \mu_0(i) \mid f(x) =_Y \mathcal{E}_i^Y(y)\}$ of X is equipped with the embedding $e_{f^{-1}(\mu_0(i))}^X\colon f^{-1}(\mu_0(i)) \hookrightarrow X$, which is defined by $e_{f^{-1}(\mu_0(i))}^X(x,y) := x$, for every $(x,y) \in f^{-1}(\mu_0(i))$. Moreover, we have that

$$\begin{aligned}
\big(x,(i,y)\big) =_{\bigcup_{i\in I} f^{-1}\left(\mu_0(i)\right)} \big(x',(i',y')\big) &:\Leftrightarrow \big[f^{-1}(\mathcal{E}^Y)^X\big]_i(x,y) =_X \big[f^{-1}(\mathcal{E}^Y)^X\big]_{i'}(x',y') \\
&:\Leftrightarrow e_{f^{-1}(\mu_0(i))}^X(x,y) =_X f^{-1}(\mu_0(i'))(x',y') \\
&:\Leftrightarrow x =_X x'.
\end{aligned}$$

If the operation $g'\colon f^{-1}\big(\bigcup_{i\in I}\mu_0(i)\big) \rightsquigarrow \bigcup_{i\in I} f^{-1}\big(\mu_0(i)\big)$ is defined by the rule $\big(x,(i,y)\big) \mapsto \big(i,(x,y)\big)$ and the operation $h'\colon \bigcup_{i\in I} f^{-1}\big(\mu_0(i)\big) \rightsquigarrow f^{-1}\big(\bigcup_{i\in I}\mu_0(i)\big)$ is defined by the rule inverse rule, then it is immediate to show that g' is a function. To show that h' is a function, we suppose that $x =_X x'$, hence $f(x) =_Y f(x')$, and by the definition of $f^{-1}\big(\bigcup_{i \in I} \mu_0(i)\big)$ we get $e_{\bigcup}^{M(Y)}(i, y) =_Y e_{\bigcup}^{M(Y)}(i', y') :\Leftrightarrow \mathcal{E}_i^Y(y) =_Y \mathcal{E}_{i'}(y')$, hence $\big(x, (i, y)\big) =_{f^{-1}\left(\bigcup_{i \in I} \mu_0(i)\right)} \big(x', (i', y')\big)$. It is immediate to show the commutativity of the above right diagrams. $\square$

Theorem 4.2.6 (Extension theorem for coverings)**.** *Let X, Y be sets, and let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ be a covering of X. If $f_i \colon \lambda_0(i) \to Y$, for every $i \in I$, such that*

$$f_{i|\lambda_0(i) \cap \lambda_0(j)} =_{\mathbb{F}(\lambda_0(i) \cap \lambda_0(j), Y)} f_{j|\lambda_0(i) \cap \lambda_0(j)},$$

for every $i, j \in I$, there is a unique $f \colon X \to Y$ such that $f_{|\lambda_0(i)} =_{\mathbb{F}(\lambda_0(i), Y)} f_i$, for every $i \in I$.

Proof. Let $e \colon X \hookrightarrow \bigcup_{i \in I} \lambda_0(i)$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, hook, bend left, "e"] \arrow[dr, "\mathrm{id}_X"'] & & \bigcup_{i \in I} \lambda_0(i) \arrow[dl, "e_{\bigcup}^{\Lambda(X)}"] \\
& X. &
\end{tikzcd}$$

Let the operation $f \colon X \rightsquigarrow Y$ defined by

$$f(x) := f_{\mathtt{pr}_1^{\Lambda(X)}(e(x))}\big(\mathtt{pr}_2^{\Lambda(X)}(e(x))\big),$$

for every $x \in X$. Hence, if $x \in X$, and $e(x) := (i, u)$, for some $i \in I$ and $u \in \lambda_0(i)$, then $f(x) := f_i(u)$. We show that f is a function. Recall that $\lambda_0(i) \cap \lambda_0(j) := \big\{(u, w) \in \lambda_0(i) \times \lambda_0(j) \mid \mathcal{E}_i^X(u) =_X \mathcal{E}_j^X(w)\big\}$. If $x, x' \in X$, let $e(x) := (i, u)$ and $e(x') := (j, w)$. If $x =_X x'$, then

$$e(x) =_{\bigcup_{i \in I} \lambda_0(i)} e(x') :\Leftrightarrow (i, u) =_{\bigcup_{i \in I} \lambda_0(i)} (j, w) :\Leftrightarrow \mathcal{E}_i^X(u) =_X \mathcal{E}_j^X(w) :\Leftrightarrow (u, w) \in \lambda_0(i) \cap \lambda_0(j).$$

By the definition of f we have that $f(x) := f_i(u)$ and $f(x') := f_j(w)$. We show that $f_i(u) =_Y f_j(w)$. Since $\lambda_0(i) \cap \lambda_0(j) \subseteq \lambda_0(i)$ and $\lambda_0(i) \cap \lambda_0(j) \subseteq \lambda_0(j)$, and as we have explained right before Proposition 2.6.8, by Definition 2.6.9 we have that

$$f_{i|\lambda_0(i) \cap \lambda_0(j)} := f_i \circ \mathtt{pr}_{\lambda_0(i)} \ \ \& \ \ f_{j|\lambda_0(i) \cap \lambda_0(j)} := f_j \circ \mathtt{pr}_{\lambda_0(j)}.$$

Since $(u, w) \in \lambda_0(i) \cap \lambda_0(j)$, by the equality of the restrictions of f_i and f_j to $\lambda_0(i) \cap \lambda_0(j)$

$$f_i(u) := \big(f_i \circ \mathtt{pr}_{\lambda_0(i)}\big)(u, w) =_Y \big(f_j \circ \mathtt{pr}_{\lambda_0(j)}\big)(u, w) := f_j(w).$$

Next we show that, if $i \in I$, then $f_{|\lambda_0(i)} = f_i$. Since $\mathcal{E}_i^X \colon (\lambda_0(i), \mathcal{E}_i^X) \subseteq (X, \mathrm{id}_X)$

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, bend left, "\mathcal{E}_i^X"] \arrow[dr, "\mathcal{E}_i^X"'] & & X \arrow[dl, "\mathrm{id}_X"] \\
& X, &
\end{tikzcd}$$

by Definition 2.6.9 we have that $f_{|\lambda_0(i)} := f \circ \mathcal{E}_i^X$. If $u \in \lambda_0(i)$, let $e\big(\mathcal{E}_i^X(u)\big) := (j, w)$, for some $j \in I$ and $w \in \lambda_0(j)$. Hence, by the definition of f we get

$$f_{|\lambda_0(i)}(u) := f\big(\mathcal{E}_i^X(u)\big) := f_j(w).$$

By the commutativity of the first diagram in this proof we get for $\mathcal{E}_i^X(u) \in X$

$$\mathcal{E}_j^X(w) := e_{\bigcup}^{\Lambda(X)}\big(e\big(\mathcal{E}_i^X(u)\big)\big) =_X \mathrm{id}_X\big(\mathcal{E}_i^X(u)\big) := \mathcal{E}_i^X(u)$$

i.e., $(u, w) \in \lambda_0(i) \cap \lambda_0(j)$. Hence, $f_{|\lambda_0(i)}(u) := f_j(w) =_Y f_i(u)$. Finally, let $f^* \colon X \to Y$ such that $f^*_{|\lambda_0(i)} := f^* \circ \mathcal{E}_i^X =_{\mathbb{F}(\lambda_0(i),Y)} f_i$, for every $i \in I$. If $x \in X$ let $e(x) := (i, u)$, for some $i \in I$ and $u \in \lambda_0(i)$. By the commutativity of the first diagram, and since f^* is a function, we get

$$\begin{aligned}
f^*(x) &=_Y f^*\big(e_{\bigcup}^{\Lambda(X)}(e(x))\big) \\
&:= f^*\big(e_{\bigcup}^{\Lambda(X)}(i, u)\big) \\
&:= f^*\big(\mathcal{E}_i^X(u)\big) \\
&=_Y f_i(u) \\
&:= f_{\mathtt{pr}_1^{\Lambda(X)}(e(x))}\big(\mathtt{pr}_2^{\Lambda(X)}(e(x))\big) \\
&:= f(x).
\end{aligned}$$ ☐

Corollary 4.2.7. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ be a partition of X. If $f_i \colon \lambda_0(i) \to Y$, for every $i \in I$, there is a unique $f \colon X \to Y$ with $f_{|\lambda_0(i)} =_{\mathbb{F}(\lambda_0(i),Y)} f_i$, for every $i \in I$.*

Proof. The condition $f_{i|\lambda_0(i)\cap\lambda_0(j)} =_{\mathbb{F}(\lambda_0(i)\cap\lambda_0(j),Y)} f_{j|\lambda_0(i)\cap\lambda_0(j)}$ of Theorem 4.2.6 is trivially satisfied using the logical principle Ex falso quodlibet. If we suppose that $(u, w) \in \lambda_0(i) \cap \lambda_0(j)$, which is impossible as $\lambda_0(i)]\!][\![\lambda_0(j)$, the equality $(f_i \circ \mathtt{pr}_{\lambda_0(i)})(u, w) =_Y (f_j \circ \mathtt{pr}_{\lambda_0(j)})(u, w)$, where $i, j \in I$, follows immediately. ☐

Proposition 4.2.8. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$, $M(X) := (\mu_0, \mathcal{Z}^X, \mu_1) \in \mathtt{Fam}(I, X)$, $\Psi : \Lambda(X) \Rightarrow M(X)$, and $(B, i_B^X) \subseteq X$.*

(i) *For every $i \in I$ the operation $e_i^{\Lambda(X)} : \lambda_0(i) \rightsquigarrow \bigcup_{i\in I} \lambda_0(i)$, defined by $x \mapsto (i, x)$, is an embedding, and $e_i^{\Lambda(X)} \colon \lambda_0(i) \subseteq \bigcup_{i\in I} \lambda_0(i)$.*

(ii) *If $\lambda_0(i) \subseteq B$, for every $i \in I$, then $\bigcup_{i\in I} \lambda_0(i) \subseteq B$.*

(iii) *The operation $\bigcup \Psi : \bigcup_{i\in I} \lambda_0(i) \rightsquigarrow \bigcup_{i\in I} \mu_0(i)$, defined by $\bigcup \Psi(i, x) := (i, \Psi_i(x))$, is an embedding, such that for every $i \in I$ the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\Psi_i"] \arrow[d, hook, "e_i^{\Lambda(X)}"'] & \mu_0(i) \arrow[d, hook, "e_i^{M(X)}"] \\
\bigcup_{i\in I} \lambda_0(i) \arrow[r, hook, "\bigcup \Psi"'] & \bigcup_{i\in I} \mu_0(i).
\end{tikzcd}$$

Proof. (i) If $x, x' \in \lambda_0(i)$, and since $\mathcal{E}_i^X$ is an embedding, we have that

$$e_i^{\Lambda(X)}(x) =_{\bigcup_{i\in I} \lambda_0(i)} e_i^{\Lambda(X)}(x') :\Leftrightarrow (i, x) =_{\bigcup_{i\in I} \lambda_0(i)} (i, x') :\Leftrightarrow \mathcal{E}_i^X(x) = \mathcal{E}_i^X(x') \Leftrightarrow x =_{\lambda_0(i)} x'.$$

Moreover, $e^X_{\bigcup}\big(e_i^{\Lambda(X)}(x)\big) := e^X_{\bigcup}(i,x) := \mathcal{E}_i^X(x)$, hence $e_i^{\Lambda(X)} \colon \lambda_0(i) \subseteq \bigcup_{i\in I}\lambda_0(i)$.
(ii) If $i \in I$ and $e_i^B \colon \lambda_0(i) \subseteq B$, then $i_B^X(e_i^B(x)) =_X \mathcal{E}_i^X(x)$, for every $x \in \lambda_0(i)$. Let the operation $e^B \colon \bigcup_{i\in I}\lambda_0(i) \rightsquigarrow B$, defined by $e^B(i,x) := e_i^B(x)$, for every $(i,x) \in \bigcup_{i\in I}\lambda_0(i)$. The operation e^B is a function:

$$\begin{aligned}(i,x) =_{\bigcup_{i\in I}\lambda_0(i)} (j,y) &:\Leftrightarrow \mathcal{E}_i^X(x) =_X \mathcal{E}_j^X(y)\\ &\Rightarrow i_B^X(e_i^B(x)) =_X i_B^X(e_j^B(y))\\ &\Rightarrow e_i^B(x) =_X e_j^B(y)\\ &:\Leftrightarrow e^B(i,x) =_X e^B(j,y).\end{aligned}$$

Moroever, $i_B^X\big(e^B(i,x)\big) := i_B^X(e_i^B(x)) =_X \mathcal{E}_i^X(x) := e_{\bigcup}^{\Lambda(X)}(i,x)$, hence $e^B \colon \bigcup_{i\in I}\lambda_0(i) \subseteq B$.
(iii) The required commutativity of the diagram is immediate, and $\bigcup\Psi$ is an embedding, since

$$\begin{tikzcd} \lambda_0(i) \arrow[rr, hook, "\Psi_i"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & & \mu_0(i) \arrow[dl, hook', "\mathcal{Z}_i^X"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \lambda_0(j) \arrow[rr, hook, "\Psi_j"] \arrow[dr, hook, "\mathcal{E}_j^X"'] & & \mu_0(j) \arrow[dl, hook', "\mathcal{Z}_j^X"] \\ & X, & \end{tikzcd}$$

$$\begin{aligned}(i,x) =_{\bigcup_{i\in I}\lambda_0(i)} (j,y) &:\Leftrightarrow \mathcal{E}_i^X(x) =_X \mathcal{E}_j^X(y)\\ &\Leftrightarrow \mathcal{Z}_i^X(\Psi_i(x)) =_X \mathcal{Z}_j^X(\Psi_j(y))\\ &:\Leftrightarrow (i,\Psi_i(x)) =_{\bigcup_{i\in I}\mu_0(i)} (j,\Psi_j(y))\\ &:\Leftrightarrow \bigcup\Psi(i,x) =_{\bigcup_{i\in I}\mu_0(i)} \bigcup\Psi(j,y).\end{aligned}$$

□

4.3 The intersection of a family of subsets

Definition 4.3.1. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I,X)$, and $i_0 \in I$. The intersection $\bigcap_{i\in I}\lambda_0(i)$ of $\Lambda(X)$ is the totality defined by*

$$\Phi \in \bigcap_{i\in I}\lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I,\lambda_0) \ \&\ \forall_{i,j\in I}\big(\mathcal{E}_i^X(\Phi_i) =_X \mathcal{E}_j^X(\Phi_j)\big).$$

Let $e_{\bigcap}^{\Lambda(X)} : \bigcap_{i\in I}\lambda_0(i) \rightsquigarrow X$ be defined by $e_{\bigcap}^{\Lambda(X)}(\Phi) := \mathcal{E}_{i_0}^X\big(\Phi_{i_0}\big)$, for every $\Phi \in \bigcap_{i\in I}\lambda_0(i)$, and

$$\Phi =_{\bigcap_{i\in I}\lambda_0(i)} \Theta :\Leftrightarrow e_{\bigcap}^{\Lambda(X)}(\Phi) =_X e_{\bigcap}^{\Lambda(X)}(\Theta) :\Leftrightarrow \mathcal{E}_{i_0}^X\big(\Phi_{i_0}\big) =_X \mathcal{E}_{i_0}^X\big(\Theta_{i_0}\big),$$

If $\neq_X$ is a given inequality on X, let $\Phi \neq_{\bigcap_{i\in I}\lambda_0(i)} \Theta :\Leftrightarrow \mathcal{E}_{i_0}^X\big(\Phi_{i_0}\big) \neq_X \mathcal{E}_{i_0}^X\big(\Theta_{i_0}\big)$.

Clearly, $=_{\bigcap_{i\in I}\lambda_0(i)}$ is an equality on $\bigcap_{i\in I}\lambda_0(i)$, which is considered to be a set, and $e_{\bigcap}^{\Lambda(X)}$ is an embedding, hence $\big(\bigcap_{i\in I}\lambda_0(i), e_{\bigcap}^{\Lambda(X)}\big) \subseteq X$. Moreover, the inequality $\neq_{\bigcap_{i\in I}\lambda_0(i)}$ is the canonical inequality of the subset $\bigcap_{i\in I}\lambda_0(i)$ of X (see Corollary 2.6.3).

Proposition 4.3.2. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I,X)$.*
(i) $\Phi =_{\bigcap_{i\in I}\lambda_0(i)} \Theta \Leftrightarrow \Phi =_{\mathbb{A}(I,\lambda_0)} \Theta$.
(ii) *If $\Phi \in \bigcap_{i\in I}\lambda_0(i)$, then $\Phi \in \prod_{i\in I}\lambda_0(i)$.*
(iii) *If $(X, =_X, \neq_X)$ is discrete, the set $\big(\bigcap_{i\in I}\lambda_0(i), =_{\bigcap_{i\in I}\lambda_0(i)}, \neq_{\bigcap_{i\in I}\lambda_0(i)}\big)$ is discrete.*

Proof. (i) To show the implication $(\Rightarrow)$, if $i \in I$, then $\mathcal{E}_i^X(\Phi_i) =_X \mathcal{E}_{i_0}^X(\Phi_{i_0}) =_X \mathcal{E}_{i_0}^X(\Theta_{i_0}) =_X \mathcal{E}_i^X(\Theta_i)$, and since $\mathcal{E}_i^X$ is an embedding, $\Phi_i =_{\lambda_0(i)} \Theta_i$. For the converse implication, the pointwise equality of Φ and Θ implies that $\Phi_{i_0} =_{\lambda_0(i_0)} \Theta_{i_0}$, hence $\mathcal{E}_{i_0}^X(\Phi_{i_0}) =_X \mathcal{E}_{i_0}^X(\Theta_{i_0})$.
(ii) If $i =_I j$, then $\mathcal{E}_j^X\big(\lambda_{ij}(\Phi_i)\big) =_X \mathcal{E}_i^X(\Phi_i) =_X \mathcal{E}_j(\Phi_j)$, and as $\mathcal{E}_j^X$ is an embedding, we get the required equality $\lambda_{ij}(\Phi_i) =_{\lambda_0(j)} \Phi_j$. The proof of (iii) is immediate. □

Since the equality of $\prod_{i\in I} \lambda_0(i)$ is the pointwise equality of $\mathbb{A}(I, \lambda_0)$, then, as we explained above, the equality of $\prod_{i\in I} \lambda_0(i)$ is the equality of $\bigcap_{i\in I} \lambda_0(i)$.

Remark 4.3.3. *Let $i_0 \in I$, $(A, i_A^X) \subseteq X$, and $C^A(X) := (\lambda_0^A, \mathcal{E}^{A,X}, \lambda_1^A) \in \mathtt{Fam}(I, X)$ the constant family A of subsets of X. Then*

$$\bigcap_{i\in I} A := \bigcap_{i\in I} \lambda_0^A(i) =_{\mathcal{P}(X)} A.$$

Proof. We proceed similarly to the proof of Remark 4.2.3. □

Proposition 4.3.4. *If $\Lambda^{\mathbb{2}}(X)$ is the $\mathbb{2}$-family of subsets A, B of X, $\bigcap_{i\in\mathbb{2}} \lambda_0^{\mathbb{2}}(i) =_{\mathcal{P}(X)} A \cap B$.*

Proof. By definition $\Phi \in \bigcap_{i\in I} \lambda_0^{\mathbb{2}}(i) :\Leftrightarrow \Phi\colon \bigcurlywedge_{i\in I} \lambda_0^{\mathbb{2}}(i)$ and for every $i, j \in \mathbb{2}$ we have that $\mathcal{E}_i^X(\Phi_i) =_X \mathcal{E}_j^X(\Phi_j)$, where $\mathcal{E}_0^X := i_A^X$ and $\mathcal{E}_1^X := i_B^X$. Moreover, $e_{\bigcap}^{\Lambda^{\mathbb{2}}(X)} : \bigcap_{i\in I} \lambda_0^{\mathbb{2}}(i) \rightsquigarrow X$ is given by $e_{\bigcap}^{\Lambda^{\mathbb{2}}(X)}(\Phi) := \mathcal{E}_0^X\big(\Phi_0\big)$, for every $\Phi \in \bigcap_{i\in I} \lambda_0^{\mathbb{2}}(i)$, and $\Phi =_{\bigcap_{i\in I} \lambda_0^{\mathbb{2}}(i)} \Theta :\Leftrightarrow \mathcal{E}_0^X\big(\Phi_0\big) =_X \mathcal{E}_0^X\big(\Theta_0\big)$. Let $f\colon A \cap B \rightsquigarrow \bigcap_{i\in\mathbb{2}} \lambda_0^{\mathbb{2}}(i)$ be defined by $f(a, b) := \Phi_{(a,b)}$, for every $(a, b) \in A \cap B$, where $\Phi_{(a,b)}\colon \bigcurlywedge_{i\in I} \lambda_0^{\mathbb{2}}(i)$, such that $\Phi_{(a,b)}(0) := a$ and $\Phi_{(a,b)}(1) := b$. Since $\mathcal{E}_0^X\big(\Phi_{(a,b)}(0)\big) =_X \mathcal{E}_1^X\big(\Phi_{(a,b)}(1)\big) \Leftrightarrow \mathcal{E}_0^X(a) =_X \mathcal{E}_1(b) :\Leftrightarrow i_A^X(a) =_X i_B^X(b)$, where the last equality holds by the definition of $A\cap B$ (see Definition 2.6.6), the operation f is well-defined. It is straightforward to show that f is a function. Let the operation $g\colon \bigcap_{i\in\mathbb{2}} \lambda_0^{\mathbb{2}}(i) \rightsquigarrow A\cap B$, defined by $g(\Phi) := (\Phi_0, \Phi_1)$, for every $\Phi \in \bigcap_{i\in\mathbb{2}} \lambda_0^{\mathbb{2}}(i)$. Since $i_Z^X(\Phi_0) := \mathcal{E}_0^X(\Phi_0) =_X \mathcal{E}_1^X(\Phi_1) := i_B^X(\Phi_1)$, we have that g is well-defined. It is easy to show that g is a function,

$$\begin{tikzcd}
A \cap B \arrow[rr, bend left, "f"] \arrow[dr, "i^X_{A\cap B}"'] & & \bigcap_{i\in\mathbb{2}} \lambda_0^{\mathbb{2}}(i) \arrow[ll, bend left, "g"] \arrow[dl, "e_{\bigcap}^{\Lambda^{\mathbb{2}}(X)}"] \\
& X. &
\end{tikzcd}$$

and the above inner diagrams commute. □

Proposition 4.3.5. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $M(Y) := (\mu_0, \mathcal{E}^Y, \mu_1) \in \mathtt{Fam}(I, Y)$. If $f\colon X \to Y$, the following hold:*

(i) $f\bigg(\bigcap_{i\in I} \lambda_0(i)\bigg) \subseteq \bigcap_{i\in I} f\big(\lambda_0(i)\big)$.

(ii) $f^{-1}\bigg(\bigcap_{i\in I} \mu_0(i)\bigg) =_{\mathcal{P}(X)} \bigcap_{i\in I} f^{-1}\big(\mu_0(i)\big)$.

Proof. We proceed similarly to the proof of Proposition 4.2.5. □

Proposition 4.3.6. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1), M_X := (\mu_0, \mathcal{Z}^X, \mu_1) \in \mathtt{Fam}(I, X)$, let $i_0 \in I$, $\Psi : \Lambda(X) \Rightarrow M(X)$, and $(B, i_B^X) \subseteq X$.*

(i) *The operation $\pi_i^{\Lambda(X)} : \bigcap_{i \in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, defined by $\Theta \mapsto \Theta_i$, is a function, and $\pi_i^{\Lambda(X)} \colon \bigcap_{i \in I} \lambda_0(i) \subseteq \lambda_0(i)$, for every $i \in I$.*

(ii) *If $B \subseteq \lambda_0(i)$, for every $i \in I$, then $B \subseteq \bigcap_{i \in I} \lambda_0(i)$.*

(iii) *The operation $\bigcap \Psi : \bigcap_{i \in I} \lambda_0(i) \rightsquigarrow \bigcap_{i \in I} \mu_0(i)$, defined by $[\bigcap \Psi(\Theta)]_i := \Psi_i(\Theta_i)$, for every $i \in I$, is an embedding, such that for every $i \in I$ the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\Psi_i"] & \mu_0(i) \\
\bigcap_{i \in I} \lambda_0(i) \arrow[u, "\pi_i^{\Lambda(X)}"] \arrow[r, hook, "\bigcap \Psi"'] & \bigcap_{i \in I} \mu_0(i). \arrow[u, "\pi_i^{M(X)}"']
\end{tikzcd}$$

Proof. (i) Since $\Phi =_{\bigcap_{i \in I} \lambda_0(i)} \Theta \Rightarrow \Phi =_{\mathbb{A}(I, \lambda_0)} \Theta$, we get $\Phi_i = \Theta_i$, for every $i \in I$. Since $\mathcal{E}_i^X\big(\pi_i^{\Lambda(X)}(\Theta)\big) := \mathcal{E}_i^X(\Theta_i) =_X \mathcal{E}_{i_0}^X(\Theta_{i_0}) := e_{\bigcap}^X(\Theta)$, we get $\pi_i^{\Lambda(X)} \colon \bigcap_{i \in I} \lambda_0(i) \subseteq \lambda_0(i)$.

(ii) If $i \in I$, let $e_B^i \colon B \subseteq \lambda_0(i)$, hence $\mathcal{E}_i^X(e_B^i(b)) =_X i_B^X(b)$, for every $b \in B$. Let the operation $e_B \colon B \rightsquigarrow \bigcap_{i \in I} \lambda_0(i)$, defined by the rule $b \mapsto e_B(b)$, where $[e_B(b)]_i := e_B^i(b)$, for every $b \in B$ and $i \in I$. First we show that e_B is well defined. If $i, j \in I$, then

$$\mathcal{E}_i^X\big([e_B(b)]_i\big) =_X \mathcal{E}_j^X\big([e_B(b)]_j\big) :\Leftrightarrow \mathcal{E}_i^X(e_B^i(b)) =_X \mathcal{E}_j^X(e_B^j(b)) \Leftrightarrow i_B^X(b) =_X i_B^X(b).$$

Clearly, e_B is a function. Moreover, $e_B \colon B \subseteq \bigcap_{i \in I} \lambda_0(i)$, since, for every $b \in B$,

$$e_{\bigcap}(e_B(b)) := \mathcal{E}_{i_0}^X\big([e_B(b)]_{i_0}\big) := \mathcal{E}_{i_0}^X(e_B^{i_0}(b)) =_X i_B^X(b).$$

(iii) It suffices to show that $\bigcap \Psi$ is an embedding. If $\Phi, \Theta \in \bigcap_{i \in I} \lambda_0(i)$, then

$$\begin{aligned}
\Phi =_{\bigcap_{i \in I} \lambda_0(i)} \Theta &:\Leftrightarrow \mathcal{E}_{i_0}^X\big(\Phi_{i_0}\big) =_X \mathcal{E}_{i_0}^X\big(\Theta_{i_0}\big) \\
&\Leftrightarrow \mathcal{Z}_{i_0}^X(\Psi_{i_0}(\Phi_{i_0})) =_X \mathcal{Z}_{i_0}^X(\Psi_{i_0}(\Theta_{i_0})) \\
&:\Leftrightarrow \mathcal{Z}_{i_0}^X\bigg(\Big[\bigcap \Psi(\Phi)\Big]_{i_0}\bigg) =_X \mathcal{Z}_{i_0}^X\bigg(\Big[\bigcap \Psi(\Theta)\Big]_{i_0}\bigg) \\
&:\Leftrightarrow \Big(\bigcap \Psi\Big)(\Phi) =_{\bigcap_{i \in I} \mu_0(i)} \Big(\bigcap \Psi\Big)(\Theta).
\end{aligned}$$

□

The above notions and results can be generalised as follows.

Definition 4.3.7. *Let X and Y be sets, and $h : X \to Y$. If $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$, and $M(Y) := (\mu_0, \mathcal{Z}^Y, \mu_1) \in \mathtt{Fam}(I, Y)$, a family of subsets-map from $\Lambda(X)$ to $M(Y)$ is a dependent operation $\Psi : \bigwedge_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$, where if $\Psi(i) := \Psi_i$, for every $i \in I$, then, for every $i \in I$, the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, hook, "\mathcal{E}_i^X"'] & \mu_0(i) \arrow[d, hook, "\mathcal{Z}_i^Y"] \\
X \arrow[r, "h"'] & Y.
\end{tikzcd}$$

The totality $\mathtt{Map}_{I,h}(\Lambda(X), M(Y))$ *of family of subsets-maps from* $\Lambda(X)$ *to* $M(Y)$ *is equipped with the pointwise equality, and we write* $\Psi\colon \Lambda(X) \overset{h}{\Rightarrow} M(Y)$*, if* $\Psi \in \mathtt{Map}(\Lambda(X), M(X))$*. If* $\Xi\colon M(Y) \overset{g}{\Rightarrow} N(Z)$*, where* $g\colon Y \to Z$*, the composition family of subsets-map* $\Xi \circ \Psi : \Lambda(X) \overset{g\circ h}{\Longrightarrow} N(Z)$ *is defined by* $(\Xi \circ \Psi)(i) := \Xi_i \circ \Psi_i$*, for every* $i \in I$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, hook, "\mathcal{E}_i^X"'] & \mu_0(i) \arrow[r, "\Xi_i"] \arrow[d, hook, "\mathcal{Z}_i^Y"] & \nu_0(i) \arrow[d, hook, "\mathcal{H}_i^Z"] \\
X \arrow[r, "h"'] & Y \arrow[r, "g"'] & Z.
\end{tikzcd}$$

If $Y := X$, and $h := \mathrm{id}_X$, and if $\Psi\colon \Lambda(X) \overset{\mathrm{id}_X}{\Longrightarrow} M(X)$, then $\Psi\colon \Lambda(X) \Rightarrow M(X)$. In the general case, if $\Psi\colon \Lambda(X) \overset{h}{\Rightarrow} M(Y)$, then Ψ_i is an embedding, if h is an embedding.

Proposition 4.3.8. *Let* X *and* Y *be sets, and* $h : X \to Y$*. Let also* $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$*,* $M(Y) := (\mu_0, \mathcal{Z}^Y, \mu_1) \in \mathtt{Fam}(I, Y)$*, and* $\Psi\colon \Lambda(X) \overset{h}{\Rightarrow} M(Y)$*.*

(i) *The operation* $\bigcup_h \Psi : \bigcup_{i\in I} \lambda_0(i) \rightsquigarrow \bigcup_{i\in I} \mu_0(i)$*, defined by* $\big(\bigcup_h \Psi\big)(i, u) := (i, \Psi_i(u))$*, for every* $(i, u) \in \bigcup_{i\in I} \lambda_0(i)$*, is a function, and for every* $i \in I$ *the following left diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, hook, "e_i^{\Lambda(X)}"'] & \mu_0(i) \arrow[d, hook, "e_i^{M(Y)}"] \\
\bigcup_{i\in I} \lambda_0(i) \arrow[r, "\bigcup_h \Psi"'] & \bigcup_{i\in I} \mu_0(i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\bigcap_{i\in I} \lambda_0(i) \arrow[r, "\bigcap_h \Psi"'] \arrow[u, "\pi_i^{\Lambda(X)}"] & \bigcap_{i\in I} \mu_0(i). \arrow[u, "\pi_i^{M(Y)}"']
\end{tikzcd}$$

(ii) *If* $i_0 \in I$*, the operation* $\bigcap_h \Psi : \bigcap_{i\in I} \lambda_0(i) \rightsquigarrow \bigcap_{i\in I} \mu_0(i)$*, defined by* $[\bigcap_h \Psi(\Theta)]_i := \Psi_i(\Theta_i)$*, for every* $i \in I$*, is a function, such that for every* $i \in I$ *the above right diagram commutes.*

Proof. (i) The commutativity of the diagram is trivial, and we show that $\bigcup_h \Psi$ is a function:

$$\begin{aligned}
(i, u) =_{\bigcup_{i\in I} \lambda_0(i)} (j, w) &:\Leftrightarrow \mathcal{E}_i(u) =_X \mathcal{E}_j(w) \\
&\Rightarrow h(\mathcal{E}_i(u)) =_Y h(\mathcal{E}_j(w)) \\
&\Leftrightarrow E_i(\Psi_i(u)) =_Y E_j(\Psi_j(w)) \\
&:\Leftrightarrow (i, \Psi_i(u)) =_{\bigcup_{i\in I} \mu_0(i)} (j, \Psi_j(w)) \\
&:\Leftrightarrow \bigg(\bigcup_h \Psi\bigg)(i, u) =_{\bigcup_{i\in I} \mu_0(i)} \bigg(\bigcup_h \Psi\bigg)(j, w).
\end{aligned}$$

(ii) The commutativity of the diagram is trivial, and we show that $\bigcap_h \Psi$ is a function:

$$\begin{aligned}
\Phi =_{\bigcap_{i\in I} \lambda_0(i)} \Theta &\Leftrightarrow \mathcal{E}_{i_0}(\Phi_{i_0}) =_X \mathcal{E}_{i_0}(\Theta_{i_0}) \\
&\Rightarrow h\big(\mathcal{E}_{i_0}(\Phi_{i_0})\big) =_Y h\big(\mathcal{E}_{i_0}(\Theta_{i_0})\big) \\
&\Leftrightarrow E_{i_0}\big(\Psi_{i_0}(\Phi_{i_0})\big) =_Y E_{i_0}\big(\Psi_{i_0}(\Theta_{i_0})\big) \\
&:\Leftrightarrow E_{i_0}\bigg(\bigg[\bigcap_h \Psi(\Phi)\bigg]_{i_0}\bigg) =_Y E_{i_0}\bigg(\bigg[\bigcap_h \Psi(\Theta)\bigg]_{i_0}\bigg) \\
&:\Leftrightarrow \bigg(\bigcap_h \Psi\bigg)(\Phi) =_{\bigcap_{i\in I} \mu_0(i)} \bigg(\bigcap_h \Psi\bigg)(\Theta).
\end{aligned}$$

$\square$

4.4 Families of subsets over products

Proposition 4.4.1. *Let* $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1), K(X) := (k_0, \mathcal{H}^X, k_1) \in \mathtt{Fam}(I, X)$, $M(Y) := (\mu_0, \mathcal{E}^Y, \mu_1)$, *and* $N(Y) := (\nu_0, \mathcal{H}^Y, \nu_1) \in \mathtt{Fam}(J, Y)$.
(i) $(\Lambda \otimes M)(X \times Y) := (\lambda_0 \otimes \mu_0, \mathcal{E}^X \otimes \mathcal{E}^Y, \lambda_1 \otimes \mu_1) \in \mathtt{Fam}(I \times J, X \times Y)$, *where*

$$(\lambda_0 \otimes \mu_0)(i,j) := \lambda_0(i) \times \mu_0(j); \qquad (i,j) \in I \times J,$$

$$\big(\mathcal{E}^X \otimes \mathcal{E}^Y\big)_{(i,j)} \colon \lambda_0(i) \times \mu_0(j) \hookrightarrow X \times Y,$$

$$\big(\mathcal{E}^X \otimes \mathcal{E}^Y\big)_{(i,j)}(u,w) := \big(\mathcal{E}_i^X(u), \mathcal{E}_j^Y(w)\big); \qquad (u,w) \in \lambda_0(i) \times \mu_0(j),\ (i,j) \in I \times J,$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i'j')} \colon \lambda_0(i) \times \mu_0(j) \to \lambda_0(i') \times \mu_0(j'),$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i'j')}(u,w) := \big(\lambda_{ii'}(u), \mu_{jj'}(w)\big); \qquad (u,w) \in \lambda_0(i) \times \mu_0(j).$$

(ii) *If* $\Phi \colon \Lambda(X) \Rightarrow K(X)$ *and* $\Psi \colon M(X) \Rightarrow N(X)$, *then* $\Phi \otimes \Psi \colon (\Lambda \otimes M)(X \times Y) \Rightarrow (K \otimes N)(X \times Y)$, *where, for every* $(i,j) \in I \times J$,

$$(\Phi \otimes \Psi)_{(i,j)} \colon \lambda_0(i) \times \mu_0(j) \to k_0(i) \times \nu_0(j),$$

$$(\Phi \otimes \Psi)_{(i,j)}(u,w) := \big(\Phi_i(u), \Psi_j(w)\big); \qquad (u,w) \in \lambda_0(i) \times \mu_0(j).$$

(iii) *The following equality holds*

$$\bigcup_{(i,j) \in I \times J} \big(\lambda_0(i) \times \mu_0(j)\big) =_{\mathcal{P}(X \times Y)} \bigg(\bigcup_{i \in I} \lambda_0(i)\bigg) \times \bigg(\bigcup_{j \in J} \mu_0(j)\bigg).$$

(iv) *If* $i_0 \in I$ *and* $j_0 \in J$, *the following equality holds*

$$\bigcap_{(i,j) \in I \times J} \big(\lambda_0(i) \times \mu_0(j)\big) =_{\mathcal{P}(X \times Y)} \bigg(\bigcap_{i \in I} \lambda_0(i)\bigg) \times \bigg(\bigcap_{j \in J} \mu_0(j)\bigg).$$

(v) *If* $\Lambda(X)$ *covers* X *and* $M(Y)$ *covers* Y, *then* $(\Lambda \otimes M)(X \times Y)$ *covers* $X \times Y$.
(vi) *Let the inequalities* $\neq_I, \neq_J, \neq_X$ *and* $\neq_Y$ *on* I, J, X *and* Y, *respectively. If* $\Lambda(X)$ *is a partition of* X *and* $M(Y)$ *is a partition of* Y, *then* $(\Lambda \otimes M)(X \times Y)$ *is a partition of* $X \times Y$.

Proof. The proofs of (i)-(iv) are the internal analogue to the proofs of Proposition 3.5.1(i)-(iii).
(v) Since $X =_{\mathcal{P}(X)} \bigcup_{i \in I} \lambda_0(i)$ and $Y =_{\mathcal{P}(Y)} \bigcup_{j \in J} \mu_0(j)$, by case (iii) and by Proposition 2.6.11(iv) we have that

$$X \times Y =_{\mathcal{P}(X \times Y)} \bigg(\bigcup_{i \in I} \lambda_0(i)\bigg) \times \bigg(\bigcup_{j \in J} \mu_0(j)\bigg) =_{\mathcal{P}(X \times Y)} \bigcup_{(i,j) \in I \times J} \big(\lambda_0(i) \times \mu_0(j)\big).$$

(vi) By Definition 4.2.1 we have that

$$i \neq_I i' \Rightarrow \mathcal{E}_i^X(u) \neq_X \mathcal{E}_{i'}^X(u'); \qquad i, i' \in I,\ u \in \lambda_0(i),\ u' \in \lambda_0(i'),$$

$$j \neq_J j' \Rightarrow \mathcal{E}_i^Y(w) \neq_Y \mathcal{E}_{j'}^Y(w'); \qquad j, j' \in J,\ w \in \mu_0(j),\ w' \in \mu_0(j').$$

Let $(i,j) \neq_{I \times J} (i',j') :\Leftrightarrow i \neq_I i' \vee j \neq_J j'$. If $i \neq_I i'$ is the case, then $\mathcal{E}_i^X(u) \neq_X \mathcal{E}_{i'}^X(u')$, hence $\big(\mathcal{E}_i^X(u), \mathcal{E}_j^Y(w)\big) \neq_{X \times Y} \big(\mathcal{E}_i^X(u'), \mathcal{E}_j^Y(w')\big)$. If $j \neq_J j'$, we proceed similarly. $\square$

Proposition 4.4.2. *Let* $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1), K(X) := (k_0, \mathcal{H}^X, k_1) \in \mathtt{Fam}(I, X)$, $M(X) := (\mu_0, \mathcal{Z}^X, \mu_1)$, *and* $N(X) := (\nu_0, \mathcal{F}^X, \nu_1) \in \mathtt{Fam}(J, X)$.

(i) $(\Lambda \wedge M)(X) := (\lambda_0 \wedge \mu_0, \mathcal{E}^X \wedge \mathcal{Z}^X, \lambda_1 \wedge \mu_1) \in \mathtt{Fam}(I \times J, X)$, *where*

$$(\lambda_0 \wedge \mu_0)(i,j) := \lambda_0(i) \cap \mu_0(j); \qquad (i,j) \in I \times J,$$

$$\big(\mathcal{E}^X \wedge \mathcal{Z}^X\big)_{(i,j)} \colon \lambda_0(i) \cap \mu_0(j) \hookrightarrow X,$$

$$\big(\mathcal{E}^X \wedge \mathcal{Z}^X\big)_{(i,j)}(u,w) := \mathcal{E}_i^X(u); \qquad (u,w) \in \lambda_0(i) \cap \mu_0(j),\ (i,j) \in I \times J,$$

$$(\lambda_1 \wedge \mu_1)_{(i,j)(i'j')} \colon \lambda_0(i) \cap \mu_0(j) \to \lambda_0(i') \cap \mu_0(j'),$$

$$(\lambda_1 \otimes \mu_1)_{(i,j)(i'j')}(u,w) := \big(\lambda_{ii'}(u), \mu_{jj'}(w)\big); \qquad (u,w) \in \lambda_0(i) \cap \mu_0(j).$$

(ii) $(\Lambda \vee M)(X) := (\lambda_0 \vee \mu_0, \mathcal{E}^X \vee \mathcal{Z}^X, \lambda_1 \vee \mu_1) \in \mathtt{Fam}(I \times J, X)$, *where*

$$(\lambda_0 \vee \mu_0)(i,j) := \lambda_0(i) \cup \mu_0(j); \qquad (i,j) \in I \times J,$$

$$\big(\mathcal{E}^X \vee \mathcal{Z}^X\big)_{(i,j)} \colon \lambda_0(i) \cup \mu_0(j) \hookrightarrow X,$$

$$\big(\mathcal{E}^X \vee \mathcal{Z}^X\big)_{(i,j)}(z) := \begin{cases} \mathcal{E}_i^X(z) & , z \in \lambda_0(i) \\ \mathcal{Z}_j^Y(z) & , z \in \mu_0(j) \end{cases}; \qquad i \in I,\ z \in \lambda_0(i) \cup \mu_0(i)$$

$$(\lambda_1 \vee \mu_1)_{(i,j)(i'j')}(z) := \begin{cases} \lambda_{ii'}(z) & , z \in \lambda_0(i) \\ \mu_{jj'}(z) & , z \in \mu_0(j) \end{cases}; \qquad \big((i,j),(i'j')\big) \in D(I \times J).$$

(iii) *If* $\Phi \colon \Lambda(X) \Rightarrow K(X)$ *and* $\Psi \colon M(X) \Rightarrow N(X)$, *then* $\Phi \wedge \Psi \colon (\Lambda \wedge M)(X) \Rightarrow (K \wedge N)(X)$, *where, for every* $(i,j) \in I \times J$,

$$(\Phi \wedge \Psi)_{(i,j)} \colon \lambda_0(i) \cap \mu_0(j) \to k_0(i) \cap \nu_0(j),$$

$$(\Phi \wedge \Psi)_{(i,j)}(u,w) := \big(\Phi_i(u), \Psi_j(w)\big); \qquad (u,w) \in \lambda_0(i) \cap \mu_0(j).$$

(iv) *If* $\Phi \colon \Lambda(X) \Rightarrow K(X)$ *and* $\Psi \colon M(X) \Rightarrow N(X)$, *then* $\Phi \vee \Psi \colon (\Lambda \vee M)(X) \Rightarrow (K \vee N)(X)$, *where, for every* $(i,j) \in I \times J$,

$$(\Phi \wedge \Psi)_{(i,j)} \colon \lambda_0(i) \cup \mu_0(j) \to k_0(i) \cup \nu_0(j),$$

$$(\Phi \vee \Psi)_{(i,j)}(z) := \begin{cases} \Phi_i(z) & , z \in \lambda_0(i) \\ \Psi_j(z) & , z \in \mu_0(j). \end{cases}$$

(v) *The following equality holds*

$$\bigcup_{(i,j) \in I \times J} \big(\lambda_0(i) \cap \mu_0(j)\big) =_{\mathcal{P}(X)} \bigg(\bigcup_{i \in I} \lambda_0(i)\bigg) \cap \bigg(\bigcup_{j \in J} \mu_0(j)\bigg).$$

(vi) *If* $(i_0, j_0) \in I \times J$, *the following equality holds*

$$\bigcap_{(i,j) \in I \times J} \big(\lambda_0(i) \cup \mu_0(j)\big) =_{\mathcal{P}(X)} \bigg(\bigcap_{i \in I} \lambda_0(i)\bigg) \cup \bigg(\bigcap_{j \in J} \mu_0(j)\bigg).$$

(vii) *If* $\Lambda(X)$ *covers* X *and* $M(Y)$ *covers* Y, *then* $(\Lambda \wedge M)(X)$ *covers* X.

(viii) *Let the inequalities* $\neq_I, \neq_J, \neq_X$ *and* $\neq_Y$ *on* I, J, X *and* Y, *respectively. If* $\Lambda(X)$ *is a partition of* X *and* $M(Y)$ *is a partition of* Y, *then* $(\Lambda \wedge M)(X)$ *is a partition of* X.

Proof. We proceed as in the proof of Proposition 4.4.1. $\square$

Let $M(Y) := (\mu_0, \mathcal{Z}^Y, \mu_1) \in \mathtt{Fam}(J, Y)$, $(A, i_A^X) \subseteq X$, $(B, i_B^Y) \subseteq Y$, and let $\Lambda^A(X) := (\lambda_0^A, \mathcal{E}^{A,X}, \lambda_1^X) \in \mathtt{Fam}(\mathbb{1}, X)$ the constant family A of subsets of X, and $\Lambda^B := (\lambda_0^B, \mathcal{E}^{B,Y}, \lambda_1^B)$ the constant family B of subsets of Y. By Propositions 4.4.1 and 4.4.2 we have that

$$\begin{aligned}
\bigcup_{j\in J}(A \times \mu_0(j)) &:= \bigcup_{(i,j)\in\mathbb{1}\times J} A \times \mu_0(j) \\
&:= \bigcup_{(i,j)\in\mathbb{1}\times J} \big(\lambda_0^A(i) \times \mu_0(j)\big) \\
&=_{\mathcal{P}(X\times Y)} \bigg(\bigcup_{i\in\mathbb{1}} \lambda_0^A(i)\bigg) \times \bigg(\bigcup_{j\in J} \mu_0(j)\bigg) \\
&=_{\mathcal{P}(X\times Y)} A \times \bigg(\bigcup_{j\in J} \mu_0(j)\bigg),
\end{aligned}$$

$$\begin{aligned}
\bigcap_{j\in J}(A \times \mu_0(j)) &:= \bigcap_{(i,j)\in\mathbb{1}\times J} A \times \mu_0(j) \\
&:= \bigcap_{(i,j)\in\mathbb{1}\times J} \big(\lambda_0^A(i) \times \mu_0(j)\big) \\
&=_{\mathcal{P}(X\times Y)} \bigg(\bigcap_{i\in\mathbb{1}} \lambda_0^A(i)\bigg) \times \bigg(\bigcap_{j\in J} \mu_0(j)\bigg) \\
&=_{\mathcal{P}(X\times Y)} A \times \bigg(\bigcap_{j\in J} \mu_0(j)\bigg),
\end{aligned}$$

$$\begin{aligned}
\bigcup_{j\in J}(B \cap \mu_0(j)) &:= \bigcup_{(i,j)\in\mathbb{1}\times J} B \cap \mu_0(j) \\
&:= \bigcup_{(i,j)\in\mathbb{1}\times J} \big(\lambda_0^B(i) \cap \mu_0(j)\big) \\
&=_{\mathcal{P}(Y)} \bigg(\bigcup_{i\in\mathbb{1}} \lambda_0^B(i)\bigg) \cap \bigg(\bigcup_{j\in J} \mu_0(j)\bigg) \\
&=_{\mathcal{P}(Y)} B \cap \bigg(\bigcup_{j\in J} \mu_0(j)\bigg),
\end{aligned}$$

$$\begin{aligned}
\bigcap_{j\in J}(B \cup \mu_0(j)) &:= \bigcap_{(i,j)\in\mathbb{1}\times J} B \cup \mu_0(j) \\
&:= \bigcap_{(i,j)\in\mathbb{1}\times J} \big(\lambda_0^B(i) \cup \mu_0(j)\big) \\
&=_{\mathcal{P}(Y)} \bigg(\bigcap_{i\in\mathbb{1}} \lambda_0^B(i)\bigg) \cup \bigg(\bigcap_{j\in J} \mu_0(j)\bigg) \\
&=_{\mathcal{P}(Y)} B \cup \bigg(\bigcap_{j\in J} \mu_0(j)\bigg).
\end{aligned}$$

Definition 4.4.3. *Let $X, Y, Z \in \mathbb{V}_0$, $x_0 \in X$, $y_0 \in Y$, and $R(Z) := (\rho_0, \mathcal{E}^Z \rho_1) \in \mathtt{Fam}(X \times Y, Z)$.*
(i) *If $x \in X$, the x-component of R is the triplet $R^x(Z) := (\rho_0^x, \mathcal{E}^{x,Z}, \rho_1^x)$, where the assignment routines ρ_0^x, ρ_1^x are as in Definition 3.5.2, and the dependent operation $\mathcal{E}^{x,Z} \colon \bigcurlywedge_{y \in Y} \mathbb{F}\big(\rho_0^x(y), Z\big)$ is defined by $\mathcal{E}_y^{x,Z} := \mathcal{E}_{(x,y)}^Z$, for every $y \in Y$.*
(ii) *If $y \in Y$, the y-component of R is the triplet $R^y(Z) := (\rho_0^y, \mathcal{E}^{y,Z}, \rho_1^y)$, where the assignment routines ρ_0^y, ρ_1^y are as in Definition 3.5.2, and the dependent operation $\mathcal{E}^{y,Z} \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(\rho_0^y(x), Z\big)$ is defined by $\mathcal{E}_x^{y,Z} := \mathcal{E}_{(x,y)}^Z$, for every $x \in Y$.*
(iii) *Let $\bigcup^1 R := (\bigcup^1 \rho_0, \big(\bigcup^1 \mathcal{E}\big)^Z, \bigcup^1 \rho_1)$, where $\bigcup^1 \rho_0 : X \rightsquigarrow \mathbb{V}_0$,*
$\overset{1}{\bigcup}\rho_1 \colon \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\Big((\overset{1}{\bigcup}\rho_0)(x), (\overset{1}{\bigcup}\rho_0)(x')\Big), (\overset{1}{\bigcup}\mathcal{E})^Z \colon \bigcurlywedge_{x \in X} \mathbb{F}\big((\overset{1}{\bigcup}\rho_0)(x), Z\big)$ *are defined by*

$$\Big(\overset{1}{\bigcup}\rho_0\Big)(x) := \bigcup_{y \in Y} \rho_0^x(y) := \bigcup_{y \in Y} \rho_0(x,y); \qquad x \in X,$$

$$\Big(\overset{1}{\bigcup}\rho_1\Big)(x,x') := \Big(\overset{1}{\bigcup}\rho_1\Big)_{xx'} : \bigcup_{y \in Y} \rho_0(x,y) \to \bigcup_{y \in Y} \rho_0(x',y); \qquad (x,x') \in D(X),$$

$$\Big(\overset{1}{\bigcup}\rho_1\Big)_{xx'}(y,u) := \big(y, \rho_{(x,y)(x',y)}(u)\big); \qquad (y,u) \in \bigcup_{y \in Y} \rho_0(x,y),$$

$$(\overset{1}{\bigcup}\mathcal{E})_x^Z(y,u) := \mathcal{E}_{(x,y)}^Z(u); \qquad x \in X,\ (y,u) \in \bigcup_{y \in Y} \rho_0(x,y).$$

(iv) Let $\bigcup^2 R := (\bigcup^2 \rho_0, \big(\bigcup^2 \mathcal{E}\big)^Z, \bigcup^2 \rho_1)$, where $\bigcup^2 \rho_0 : Y \rightsquigarrow \mathbb{V}_0$,
$\overset{2}{\bigcup}\rho_1 \colon \bigcurlywedge_{(y,y') \in D(Y)} \mathbb{F}\Big((\overset{2}{\bigcup}\rho_0)(x), (\overset{2}{\bigcup}\rho_0)(x')\Big), (\overset{2}{\bigcup}\mathcal{E})^Z \colon \bigcurlywedge_{y \in Y} \mathbb{F}\big((\overset{2}{\bigcup}\rho_0)(y), Z\big)$ are defined by

$$\Big(\overset{2}{\bigcup}\rho_0\Big)(y) := \bigcup_{x \in X} \rho_0^y(x) := \bigcup_{x \in X} \rho_0(x,y); \qquad y \in Y,$$

$$\Big(\overset{2}{\bigcup}\rho_1\Big)(y,y') := \Big(\overset{2}{\bigcup}\rho_1\Big)_{yy'} : \bigcup_{x \in X} \rho_0(x,y) \to \bigcup_{x \in X} \rho_0(x',y); \qquad (y,y') \in D(Y),$$

$$\Big(\overset{2}{\bigcup}\rho_1\Big)_{yy'}(x,w) := \big(x, \rho_{(x,y)(x',y)}(w)\big); \qquad (x,w) \in \bigcup_{x \in X} \rho_0(x,y),$$

$$(\overset{2}{\bigcup}\mathcal{E})_y^Z(x,w) := \mathcal{E}_{(x,y)}^Z(w); \qquad y \in Z,\ (x,w) \in \bigcup_{x \in X} \rho_0(x,y).$$

(v) *Let $\bigcap^1 R := (\bigcap^1 \rho_0, \big(\bigcap^1 \mathcal{E}\big)^Z, \bigcap^1 \rho_1)$, where $\bigcap^1 \rho_0 : X \rightsquigarrow \mathbb{V}_0$,*
$\overset{1}{\bigcap}\rho_1 \colon \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\Big((\overset{1}{\bigcap}\rho_0)(x), (\overset{1}{\bigcap}\rho_0)(x')\Big), (\overset{1}{\bigcap}\mathcal{E})^Z \colon \bigcurlywedge_{x \in X} \mathbb{F}\big((\overset{1}{\bigcap}\rho_0)(x), Z\big)$ *are defined by*

$$\Big(\bigcap^1 \rho_0\Big)(x) := \bigcap_{y \in Y} \rho_0^x(y) := \bigcap_{y \in Y} \rho_0(x, y); \qquad x \in X,$$

$$\Big(\bigcap^1 \rho_1\Big)(x, x') := \Big(\bigcap^1 \rho_1\Big)_{xx'} : \bigcap_{y \in Y} \rho_0(x, y) \to \bigcap_{y \in Y} \rho_0(x', y); \qquad (x, x') \in D(X),$$

$$\Big[\Big(\bigcap^1 \rho_1\Big)_{xx'}(\Phi)\Big]_y := \rho_{(x,y)(x',y)}(\Phi_y); \qquad \Phi \in \bigcap_{y \in Y} \rho_0(x, y),$$

$$(\bigcap^1 \mathcal{E})_x^Z(\Phi) := \mathcal{E}^Z_{(x,y_0)}(\Phi_{y_0}); \quad \Phi \in \bigcap_{y \in Y} \rho_0(x, y).$$

(vi) *Let* $\bigcap^2 R := (\bigcap^2 \rho_0, \big(\bigcap^2 \mathcal{E}\big)^Z, \bigcap^2 \rho_1)$, *where* $\bigcap^2 \rho_0 : X \rightsquigarrow \mathbb{V}_0$,

$\bigcap^2 \rho_1 \colon \bigcurlywedge_{(y,y') \in D(Y)} \mathbb{F}\Big((\bigcap^2 \rho_0)(y), (\bigcap^2 \rho_0)(y')\Big), (\bigcap^2 \mathcal{E})^Z \colon \bigcurlywedge_{y \in Z} \mathbb{F}\big((\bigcap^2 \rho_0)(y), Z\big)$ *are defined by*

$$\Big(\bigcap^2 \rho_0\Big)(y) := \bigcap_{x \in X} \rho_0^y(x) := \bigcap_{x \in X} \rho_0(x, y); \qquad y \in Y,$$

$$\Big(\bigcap^2 \rho_1\Big)(y, y') := \Big(\bigcap^2 \rho_1\Big)_{yy'} : \bigcap_{x \in X} \rho_0(x, y) \to \bigcap_{x \in X} \rho_0(x, y'); \qquad (y, y') \in D(Y),$$

$$\Big[\Big(\bigcap^2 \rho_1\Big)_{yy'}(\Phi)\Big]_x := \rho_{(x,y)(x,y')}(\Phi_x); \qquad \Phi \in \bigcap_{x \in X} \rho_0(x, y),$$

$$(\bigcap^2 \mathcal{E})_y^Z(\Phi) := \mathcal{E}^Z_{(x_0,y)}(\Phi_{x_0}); \quad \Phi \in \bigcap_{x \in X} \rho_0(x, y).$$

Clearly, $R^y(Z), \bigcup^1 R(Z), \bigcap^1 R(Z) \in \mathtt{Fam}(X, Z)$ and $R^x(Z), \bigcup^2 R(Z), \bigcap^2 R(Z) \in \mathtt{Fam}(Y, Z)$.

Proposition 4.4.4. *Let* $X, Y, Z \in \mathbb{V}_0$, $R(Z) := (\rho_0, \mathcal{E}^Z \rho_1), S(Z) := (\sigma_0, \mathcal{A}^Z, \sigma_1) \in \mathtt{Fam}(X \times Y, Z)$, *and* $\Phi \colon R(Z) \Rightarrow S(Z)$.

(i) *Let* $\Phi^x \colon \bigcurlywedge_{y \in Y} \mathbb{F}\big(\rho_0^x(y), \sigma_0^x(y)\big)$, *where* $\Phi^x_y := \Phi_{(x,y)} \colon \rho_0^x(y) \to \sigma_0^x(y)$.

(ii) *Let* $\Phi^y \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(R^y(x), S^y(x)\big)$, *where* $\Phi^y_x := \Phi_{(x,y)} \colon \rho_0^y(x) \to \sigma_0^y(x)$.

(iii) *Let* $\bigcup^1 \Phi \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(\big(\bigcup^1 \rho_0\big)(x), \big(\bigcup^1 \sigma_0\big)(x)\big)$, *where, for every* $x \in X$, *we define*

$$\Big(\bigcup^1 \Phi\Big)_x : \sum_{y \in Y} \rho_0^y(x) \to \bigcup_{y \in Y} \sigma_0^x(y)$$

$$\Big(\bigcup^1 \Phi\Big)_x (y, u) := \big(y, \Phi_{(x,y)}(u)\big); \qquad (y, u) \in \bigcup_{y \in Y} \rho_0(x, y).$$

(iv) *Let* $\bigcup^2 \Phi \colon \curlywedge_{y \in Y} \mathbb{F}\big(\big(\bigcup^2 \rho_0\big)(y), \big(\bigcup^2 \sigma_0\big)(y)\big)$*, where, for every* $y \in Y$*, we define*

$$\bigg(\bigcup^2 \Phi\bigg)_y \colon \bigcup_{x \in X} \rho_0^x(y) \to \bigcup_{x \in X} \sigma_0^y(x)$$

$$\bigg(\bigcup^2 \Phi\bigg)_y (x, w) := \big(x, \Phi_{(x,y)}(w)\big); \qquad (x, w) \in \bigcup_{x \in X} \rho(x, y).$$

(v) *Let* $\bigcap^1 \Phi \colon \curlywedge_{x \in X} \mathbb{F}\big(\big(\bigcap^1 \rho_0\big)(x), \big(\bigcap^1 \sigma_0\big)(x)\big)$*, where, for every* $x \in X$*, we define*

$$\bigg(\bigcap^1 \Phi\bigg)_x \colon \bigcap_{y \in Y} \rho_0^y(x) \to \bigcap_{y \in Y} \sigma_0^x(y)$$

$$\bigg[\bigg(\bigcap^1 \Phi\bigg)_x (\Theta)\bigg]_y := \Phi_{(x,y)}\big(\Theta_y)\big)\big); \qquad \Theta \in \bigcap_{y \in Y} \rho_0(x, y).$$

(vi) *Let* $\bigcap^2 \Phi \colon \curlywedge_{y \in Y} \mathbb{F}\big(\big(\bigcap^2 \rho_0\big)(y), \big(\bigcap^2 \sigma_0\big)(y)\big)$*, where, for every* $y \in Y$*, we define*

$$\bigg(\bigcap^2 \Phi\bigg)_y \colon \bigcap_{x \in X} \rho_0^x(y) \to \bigcap_{x \in X} \sigma_0^y(x)$$

$$\bigg[\bigg(\bigcap^2 \Phi\bigg)_y (\Theta)\bigg]_x := \Phi_{(x,y)}\big(\Theta_x\big)\big); \qquad \Theta \in \bigcap_{x \in X} \rho_0(x, y).$$

Then we have that $\Phi^x \colon R^x(Z) \Rightarrow S^x(Z)$ *and* $\Phi^y \colon R^y(Z) \Rightarrow S^y(Z)$ *and* $\bigcup^1 \Phi \colon \big(\bigcup^1 R\big)(Z) \Rightarrow \big(\bigcup^1 S\big)(Z)$ *and* $\bigcup^2 \Phi \colon \big(\bigcup^2 R\big)(Z) \Rightarrow \big(\bigcup^2 S\big)(Z)$ *and* $\bigcap^1 \Phi \colon \big(\bigcap^1 R\big)(Z) \Rightarrow \big(\bigcap^1 S\big)(Z)$ *and* $\bigcap^2 \Phi \colon \big(\bigcap^2 R\big)(Z) \Rightarrow \big(\bigcap^2 S\big)(Z)$.

Proof. We proceed similarly to the proof of Proposition 3.5.3. $\square$

Proposition 4.4.5. *If* $R := (\rho_0, \rho_1) \in \mathtt{Fam}(X \times Y, Z)$*, the following equalities hold.*

$$\bigcup_{x \in X} \bigcup_{y \in Y} \rho_0(x, y) =_{\mathcal{P}(Z)} \bigcup_{y \in Y} \bigcup_{x \in X} \rho_0(x, y),$$

$$\bigcap_{x \in X} \bigcap_{y \in Y} \rho_0(x, y) =_{\mathcal{P}(Z)} \bigcap_{y \in Y} \bigcap_{x \in X} \rho_0(x, y).$$

Proof. The proof is straightforward. $\square$

4.5 The semi-distributivity of $\bigcap$ over $\bigcup$

Section 4.4 is the "internal" analogue to section 3.5, as the presentation of the families of subsets over products follows the presentation of the families of sets over products. The distributivity of $\bigcap$ over $\bigcup$ though, cannot be approached as the distributivity of Σ over $\prod$, as the crucial Lemma 3.6.1 depends on the fact that the operation $\mathtt{pr}_1^{R^x}$ is a function, something which is not the case, as we have already explained in section 4.2, when the totality of the exterior union is equipped with the equality of the interior union.

Definition 4.5.1. *If $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $h\colon J \to I$, the composition family of $\Lambda(X)$ with h is the triplet $\Lambda(X) \circ h := (\lambda_0 \circ h, \mathcal{E}^X \circ h, \lambda_1 \circ h)$, where $\lambda_0 \circ h\colon J \rightsquigarrow \mathbb{V}_0$ and $\lambda_1 \circ h\colon \bigcurlywedge_{(j,j') \in D(J)} \mathbb{F}\big(\lambda_0(h(j)), \lambda_0(h(j'))\big)$ are given in Definition 3.1.6(iii), and the dependent operation $\mathcal{E}^X \circ h\colon \bigcurlywedge_{j \in J} \mathbb{F}\big(\lambda_0(h(j)), X\big)$ is defined by $(\mathcal{E}^X \circ h)_j := \mathcal{E}^X_{h(j)}$, for every $j \in J$.*

Clearly, $\Lambda(X) \circ h \in \mathtt{Fam}(J, X)$. To formulate the distributivity of $\bigcap$ over $\bigcup$ in the language of BST we need to introduce a family of subsets $P(I)$ of the index-set I of a given family of subsets of a set X. Throughout this section let the following data:

(a) $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$.

(b) $(K, =_K, \neq_K)$ is a set, and $k_0 \in K$.

(c) $P(I) := (p_0, \mathcal{Z}^I, p_1) \in \mathtt{Fam}(K, I)$.

(d) $\Lambda(X) \circ \mathcal{Z}^I_k := (\lambda_0 \circ \mathcal{Z}^I_k, \mathcal{E}^X \circ \mathcal{Z}^I_k, \lambda_1 \circ \mathcal{Z}^I_k) \in \mathtt{Fam}(p_0(k), X)$, for every $k \in K$.

(e) $T := \bigcap_{k \in K} p_0(k)$.

Proposition 4.5.2. *$N(K) := (\nu_0, \mathcal{N}^X, \nu_1) \in \mathtt{Fam}(K, X)$, where $\nu_0\colon K \rightsquigarrow \mathbb{V}_0$ is defined by*

$$\nu_0(k) := \bigcup_{j \in p_0(k)} (\lambda_0 \circ \mathcal{Z}^I_k)(j) := \bigcup_{j \in p_0(k)} \lambda_0\big(\mathcal{Z}^I_k(j)\big); \qquad k \in K,$$

and $\mathcal{N}^X\colon \bigcurlywedge_{k \in K} \mathbb{F}(\nu_0(k), X)$, $\nu_1\colon \bigcurlywedge_{(k,k') \in D(K)} \mathbb{F}\big(\nu_0(k), \nu_0(k')\big)$ are defined by

$$\mathcal{N}^X_k\colon \Bigg(\bigcup_{j \in p_0(k)} \lambda_0\big(Z^I_k(j)\big)\Bigg) \hookrightarrow X, \;\; \mathcal{N}^X_k(j, u) := \mathcal{E}^X_{\mathcal{Z}^I_k(j)}(u); \qquad j \in p_0(k),\; u \in \lambda_0\big(Z^I_k(j)\big),$$

$$\nu_1(k, k') := \nu_{kk'}\colon \bigcup_{j \in p_0(k)} \lambda_0\big(Z^I_k(j)\big) \to \bigcup_{j \in p_0(k')} \lambda_0\big(Z^I_{k'}(j)\big),$$

$$\nu_{kk'}(j, u) := \big(p_{kk'}(j), \lambda_{Z^I_k(j) Z^I_{k'}(p_{kk'}(j))}(u)\big); \qquad j \in p_0(k),\; u \in \lambda_0\big(Z^I_k(j)\big).$$

Proof. The operation $\mathcal{N}^X_k$ is an embedding, since by Definition 4.5.1

$$(j, u) =_{\bigcup_{j \in p_0(k)} \lambda_0(Z^I_k(j))} (j', u') :\Leftrightarrow \mathcal{E}^X_{\mathcal{Z}^I_k(j)}(u) =_X \mathcal{E}^X_{\mathcal{Z}^I_k(j')}(u') :\Leftrightarrow \mathcal{N}^X_k(j, u) =_X \mathcal{N}^X_k(j', u').$$

Let $k =_K k'$, $j \in p_0(k)$ and $u \in \lambda_0\big(Z^I_k(j)\big)$. By the commutativity of the left inner diagrams

$$\begin{tikzcd}
p_0(k) \arrow[rr, hook, bend left, "p_{kk'}"] \arrow[dr, "\mathcal{Z}^I_k"'] & & p_0(k') \arrow[ll, hook, bend left, "p_{k'k}"] \arrow[dl, "\mathcal{Z}^I_{k'}"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigcup_{j \in p_0(k)} \lambda_0\big(Z^I_k(j)\big) \arrow[rr, hook, bend left, "\nu_{kk'}"] \arrow[dr, "\mathcal{N}^X_k"'] & & \bigcup_{j' \in p_0(k')} \lambda_0\big(Z^I_{k'}(j')\big) \arrow[ll, hook, bend left, "\nu_{k'k}"] \arrow[dl, "\mathcal{N}^X_{k'}"] \\
& X &
\end{tikzcd}$$

we have that $\mathcal{Z}^I_{k'}\big(p_{kk'}(j)\big) =_I \mathcal{Z}^I_k(j)$. Hence $\lambda_{Z^I_k(j) Z^I_{k'}(p_{kk'}(j))}\colon \lambda_0\big(\mathcal{Z}^I_k(j)\big) \to \lambda_0\big(\mathcal{Z}^I_{k'}(p_{kk'}(j))\big)$ and $\nu_{kk'}(j, u)$ is well defined. Next we show that the above right inner diagrams commute. If

$$j' := p_{kk'}(j) \;\;\&\;\; i := Z^I_k(j) \;\;\&\;\; i' := \mathcal{Z}^I_{k'}(j'), \qquad \text{then}$$

$$\mathcal{N}^X_{k'}\big(\nu_{kk'}(j, u)\big) := \mathcal{N}^X_{k'}\big(j', \lambda_{ii'}(u)\big) := \mathcal{E}^X_{i'}\big(\lambda_{ii'}(u)\big) =_X \mathcal{E}^X_i(u) := \mathcal{N}^X_k(j, u),$$

using the commutativity of the following diagram

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, bend left, hook, "\lambda_{ii'}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] & & \lambda_0(i') \arrow[ll, bend left, hook, "\lambda_{i'i}"] \arrow[dl, hook', "\mathcal{E}_{i'}^X"] \\
& X. &
\end{tikzcd}$$

For the other above right inner diagram we proceed similarly. Clearly, $\nu_{kk}(j,u) := (j,u)$. $\square$

Proposition 4.5.3. *If $\tau \in T$, then $\tau(X) := (\tau_0, \mathcal{T}^X, \tau_1) \in \mathtt{Fam}(K,X)$, where $\tau_0 \colon K \rightsquigarrow \mathbb{V}_0$ is defined by $\tau_0(k) := \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big)$, for every $k \in K$, and the dependent operations $\mathcal{T}^X \colon \bigcurlywedge_{k\in K} \mathbb{F}(\tau_0(k), X)$, $\tau_1 \colon \bigcurlywedge_{(k,k')\in D(K)} \mathbb{F}\big(\tau_0(k), \tau_0(k')\big)$ are defined by*

$$\mathcal{T}_k^X \colon \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big) \hookrightarrow X, \quad \mathcal{T}_k^X := \mathcal{E}^X_{\mathcal{Z}_k^I(\tau_k)},$$

$$\tau_1(k,k') := \tau_{kk'} \colon \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big) \to \lambda_0\big(\mathcal{Z}_{k'}^I(\tau_{k'})\big), \qquad \tau_{kk'} := \lambda_{Z_k^I(\tau_k) Z_{k'}^I(\tau_{k'})}.$$

Proof. What we want follows in a straightforward way from the fact that $\Lambda(X) \in \mathtt{Fam}(I,X)$. $\square$

Proposition 4.5.4. *$\Xi(X) := (\xi_0, \mathcal{H}^X, \xi_1) \in \mathtt{Fam}(T,X)$, where $\xi_0 \colon T \rightsquigarrow \mathbb{V}_0$ is defined by*

$$\xi_0(\tau) := \bigcap_{k\in K} \tau_0(k) := \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big); \qquad \tau \in T,$$

and the dependent operations $\mathcal{H}^X \colon \bigcurlywedge_{\tau\in T} \mathbb{F}(\xi_0(\tau), X)$, $\xi_1 \colon \bigcurlywedge_{(\tau,\tau')\in D(T)} \mathbb{F}\big(\xi_0(\tau), \xi_0(\tau')\big)$ are defined, respectively, by $\mathcal{H}_\tau^X \colon \left(\bigcap_{k\in K} \tau_0(k)\right) \hookrightarrow X$, where $\mathcal{H}_\tau^X := e_{\bigcap}^{\tau(X)}$, for every $\tau \in T$,

$$\xi_1(\tau,\tau') := \xi_{\tau\tau'} \colon \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big) \to \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I({\tau'}_k)\big),$$

$$\Phi \mapsto \xi_{\tau\tau'}(\Phi); \qquad \big[\xi_{\tau\tau'}(\Phi)\big]_k := \lambda_{\mathcal{Z}_k^I(\tau_k)\mathcal{Z}_k^I({\tau'}_k)}(\Phi_k); \qquad \Phi \colon \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big), \ k \in K.$$

Proof. If $\tau \in T$, then by the definition of the embedding $e_{\bigcap}^{\tau(X)}$ we get

$$\mathcal{H}_\tau^X(\Phi) := \mathcal{T}_{k_0}^X(\Phi_{k_0}) := \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi_{k_0}); \qquad \Phi \colon \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big).$$

$\mathcal{H}_\tau^X$ is an embedding. Next we show that $\xi_{\tau\tau'}(\Phi) \in \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I({\tau'}_k)\big)$. As $\Phi \colon \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big)$,

$$\begin{aligned}
\mathcal{T}_k^X\big(\big[\xi_{\tau\tau'}(\Phi)\big]_k\big) &:= \mathcal{E}^X_{\mathcal{Z}_k^I({\tau'}_k)}\Big(\big[\xi_{\tau\tau'}(\Phi)\big]_k\Big) \\
&:= \mathcal{E}^X_{\mathcal{Z}_k^I({\tau'}_k)}\Big(\lambda_{\mathcal{Z}_k^I(\tau_k)\mathcal{Z}_k^I({\tau'}_k)}(\Phi_k)\Big) \\
&=_X \mathcal{E}^X_{\mathcal{Z}_k^I(\tau_k)}(\Phi_k) \\
&=_X \mathcal{E}^X_{\mathcal{Z}_l^I(\tau_l)}(\Phi_l) \\
&=_X \mathcal{E}^X_{\mathcal{Z}_l^I({\tau'}_l)}\Big(\lambda_{\mathcal{Z}_k^I(\tau_l)\mathcal{Z}_k^I({\tau'}_l)}(\Phi_l)\Big) \\
&:= \mathcal{T}_l^X\big(\big[\xi_{\tau\tau'}(\Phi)\big]_l\big),
\end{aligned}$$

for every $k, l \in K$. Similarly we show that $\xi_{\tau\tau'}$ is a function. If $\tau =_T \tau'$, then

$$\begin{tikzcd}
\bigcap_{k\in K} \lambda_0(\mathcal{Z}^I_k(\tau_k))) \arrow[rr, hook, bend left, "\xi_{\tau\tau'}"] \arrow[dr, hook, "\mathcal{H}^X_\tau"'] & & \bigcap_{k\in K} \lambda_0(\mathcal{Z}^I_k(\tau'_k)) \arrow[ll, hook, bend left, "\xi_{\tau'\tau}"] \arrow[dl, hook, "\mathcal{H}^X_{\tau'}"] \\
& X &
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{H}^X_{\tau'}\big(\xi_{\tau\tau'}(\Phi)\big) &:= \mathcal{E}^X_{\mathcal{Z}^I_{k_0}(\tau'_{k_0})}\bigg(\big[\xi_{\tau\tau'}(\Phi)\big]_{k_0}\bigg)\\
&:= \mathcal{E}^X_{\mathcal{Z}^I_{k_0}(\tau'_{k_0})}\bigg(\lambda_{\mathcal{Z}^I_{k_0}(\tau_{k_0})\mathcal{Z}^I_{k_0}(\tau'_{k_0})}(\Phi_{k_0})\bigg)\\
&=_X \mathcal{E}^X_{\mathcal{Z}^I_{k_0}(\tau_{k_0})}(\Phi_{k_0})\\
&:= \mathcal{H}^X_\tau(\Phi).
\end{aligned}$$ □

The set

$$W := \bigcap_{k\in K} \nu_0(k) := \bigcap_{k\in K}\bigg[\bigcup_{j\in p_0(k)} \lambda_0\big(\mathcal{Z}^I_k(j)\big)\bigg]$$

is embedded into X through the map $e^{N(X)}_{\bigcap}$, where $e^{N(X)}_{\bigcap}(A) := \mathcal{N}^X_{k_0}(A_{k_0})$, for every $A \in \bigcap_{k\in K}\nu_0(k)$. By definition, if $A\colon \bigcurlywedge_{k\in K}\nu_0(k)$, then

$$A \in \bigcap_{k\in K}\nu_0(k) \Leftrightarrow \forall_{k,l\in K}\big(\mathcal{N}^X_k(A_k) =_X \mathcal{N}^X_l(A_l)\big),$$

$$A_k \in \bigcup_{j\in p_0(k)} \lambda_0\big(\mathcal{Z}^I_k(j)\big), \quad \text{ i.e., } \quad A_k := (j,u),\ j\in p_0(k),\ u \in \lambda_0\big(\mathcal{Z}^I_k(j)\big),$$

$$\mathcal{N}^X_k(A_k) := \mathcal{N}^X_k(j,u) := \mathcal{E}^X_{\mathcal{Z}^I_k(j)}(u),$$

$$A =_{\bigcap_{k\in K}\nu_0(k)} B :\Leftrightarrow \mathcal{N}^X_{k_0}(A_{k_0}) =_X \mathcal{N}^X_{k_0}(B_{k_0}).$$

The set

$$V := \bigcup_{\tau\in T}\xi_0(\tau) := \bigcup_{\tau\in T}\bigg[\bigcap_{k\in K}\lambda_0\big(\mathcal{Z}^I_k(\tau_k)\big)\bigg]$$

is embedded into X through the map $e^{\Xi(X)}_{\bigcup}$, where

$$e^{\Xi(X)}_{\bigcup}(\tau,\Phi) := \mathcal{H}^X_\tau(\Phi) := \mathcal{E}^X_{\mathcal{Z}^I_{k_0}(\tau_{k_0})}(\Phi_{k_0}); \quad (\tau,\Phi)\in\bigcup_{\tau\in T}\xi_0(\tau),$$

$$(\tau,\Phi) =_{\bigcup_{\tau\in T}\xi_0(\tau)} (\tau',\Phi') :\Leftrightarrow \mathcal{H}^X_\tau(\Phi) =_X \mathcal{H}^X_{\tau'}(\Phi').$$

Proposition 4.5.5 (Semi-distributivity of $\bigcap$ over $\bigcup$)**.** $\big(V, e^{\Xi(X)}_{\bigcup}\big) \subseteq \big(W, e^{N(X)}_{\bigcap}\big)$.

Proof. Let the operation $\theta\colon V \rightsquigarrow W$, defined by

$$\theta(\tau, \Phi) := A^{(\tau,\phi)}, \qquad (\tau, \Phi) \in V,$$

$$A_k^{(\tau,\Phi)} := (\tau_k, \Phi_k); \qquad k \in K.$$

By definition $\tau_k \in p_0(k)$ and $\Phi_k \in \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big)$. We show that θ is well-defined i.e., $A^{(\tau,\Phi)} \in \bigcap_{k\in K} \nu_0(k)$. If $k, l \in K$, by the above unfolding of $A \in W$ we need to show that

$$\mathcal{E}^X_{\mathcal{Z}_k^I(\tau_k)}(\Phi_k) =_X \mathcal{E}^X_{\mathcal{Z}_k^I(\tau_l)}(\Phi_l),$$

which follows immediately from the unfolding of the membership $\Phi \in \bigcap_{k\in K} \lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big)$. If

$$(\tau, \Phi) =_{\bigcup_{\tau\in T} \xi_0(\tau)} (\tau', \Phi') :\Leftrightarrow \mathcal{H}_\tau^X(\Phi) =_X \mathcal{H}_{\tau'}^X(\Phi') :\Leftrightarrow \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi_{k_0}) =_X \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi'_{k_0}),$$

$$A^{(\tau,\Phi)} =_{\bigcap_{k\in K} \nu_0(k)} A^{(\tau',\Phi')} :\Leftrightarrow \mathcal{N}_{k_0}^X(A_{k_0}^{(\tau,\Phi)}) =_X \mathcal{N}_{k_0}^X(A_{k_0}^{(\tau',\Phi')})$$
$$:\Leftrightarrow \mathcal{N}_{k_0}^X(A_{k_0}^{(\tau_k,\Phi_k)}) =_X \mathcal{N}_{k_0}^X(A_{k_0}^{(\tau'_k,\Phi'_k)}) :\Leftrightarrow \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi_{k_0}) =_X \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi'_{k_0}),$$

hence θ is a function. The commutativity of the following diagram is shown by the equalities

$$\begin{tikzcd}
\bigcup_{\tau\in T}\Big[\bigcap_{k\in K}\lambda_0\big(\mathcal{Z}_k^I(\tau_k)\big)\Big] \arrow[rr, hook, bend left, "\theta"] \arrow[dr, hook, "e_{\bigcup}^{\Xi(X)}"'] & & \bigcap_{k\in K}\Big[\bigcup_{j\in p_0(k)}\lambda_0\big(\mathcal{Z}_k^I(j)\big)\Big] \arrow[dl, hook', "e_{\bigcap}^{N(X)}"] \\
& X &
\end{tikzcd}$$

$$e_{\bigcap}^{N(X)}\big(\theta(\tau, \Phi)\big) := \mathcal{N}_{k_0}^X\big(\tau_{k_0}, \Phi_{k_0}\big) := \mathcal{E}^X_{\mathcal{Z}_{k_0}^I(\tau_{k_0})}(\Phi_{k_0}) := e_{\bigcup}^{\Xi(X)}(\tau, \Phi). \qquad \square$$

For the converse inclusion see Note 4.11.6.

4.6 Sets of subsets

Definition 4.6.1. *If $I, X \in \mathbb{V}_0$, a set of subsets of X indexed by I, or an I-set of subsets of X, is triplet $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ such that the following condition is satisfied:*

$$Q(\Lambda(X)) :\Leftrightarrow \forall_{i,j\in I}\big(\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j) \Rightarrow i =_I j\big).$$

Let $\mathtt{Set}(I, X)$ be their totality, equipped with the canonical equality on $\mathtt{Fam}(I, X)$.

Remark 4.6.2. *If $\Lambda(X) \in \mathtt{Set}(I, X)$ and $M(X) \in \mathtt{Fam}(I, X)$ such that $\Lambda(X) =_{\mathtt{Fam}(I,X)} M(X)$, then $M(X) \in \mathtt{Set}(I, X)$.*

Proof. Let $\Phi\colon \Lambda(X) \Rightarrow M(X)$ and $\Psi\colon M(X) \Rightarrow \Lambda(X)$ such that $(\Phi, \Psi)\colon \Lambda(X) =_{\mathtt{Fam}(I,X)} M(X)$. Let also $(f, g)\colon \mu_0(i) =_{\mathcal{P}(X)} \mu_0(j)$. It suffices to show that $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$.

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, bend left, hook, "\Phi_i"] \arrow[drr, bend right, "\mathcal{E}_i^X"'] & \mu_0(i) \arrow[l, bend left, hook, "\Psi_i"] \arrow[rr, bend left, hook, "f"] \arrow[dr, "\mathcal{Z}_i^X"'] & & \mu_0(j) \arrow[ll, bend left, hook, "g"] \arrow[r, bend left, hook, "\Psi_j"] \arrow[dl, "\mathcal{Z}_j^X"] & \lambda_0(j) \arrow[l, bend left, hook, "\Phi_j"] \arrow[dll, bend left, "\mathcal{E}_j^X"] \\
& & X & &
\end{tikzcd}$$

If we define $f' := \Psi_j \circ f \circ \Phi_i$ and $g' := \Psi_i \circ g \circ \Phi_j$, it is straightforward to show that $(f', g') \colon \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$, hence $i =_I j$. $\square$

By the previous remark $Q(\Lambda(X))$ is an extensional property on $\texttt{Fam}(I, X)$. Since $\texttt{Set}(I, X)$ is defined by separation on $\texttt{Fam}(I, X)$, and since we see no objection to consider $\texttt{Fam}(I, X)$ to be a set, we also see no objection to consider $\texttt{Set}(I, X)$ to be a set.

Definition 4.6.3. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X \lambda_1) \in \texttt{Fam}(I, X)$. Let the equality $=_I^{\Lambda(X)}$ on I given by $i =_I^{\Lambda(X)} j :\Leftrightarrow \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$, for every $i, j \in I$. The set $\lambda_0 I(X)$ of subsets of X generated by $\Lambda(X)$ is the totality I equipped with the equality $=_I^{\Lambda(X)}$. We write $\lambda_0(i) \in \lambda_0 I(X)$, instead of $i \in I$, when I is equipped with the equality $=_I^{\Lambda(X)}$. The operation $\lambda_0^* : I \rightsquigarrow I$ from $(I, =_I)$ to $(I, =_I^{\Lambda(X)})$ is defined as in Definition 3.7.3.*

Clearly, λ_0^* is a function. All results in section 3.7 are shown similarly for sets of subsets, and for convenience we include them here without proof.

Proposition 4.6.4. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X \lambda_1) \in \texttt{Set}(I, X)$, and let Y be a set. If $f \colon I \to Y$, there is a unique function $\lambda_0 f \colon \lambda_0 I(X) \to Y$ such that the following diagram commutes*

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "\lambda_0"'] & Y. \\
\lambda_0 I(X) \arrow[ur, dashed, "\lambda_0 f"'] &
\end{tikzcd}$$

Conversely, if $f \colon I \rightsquigarrow Y$ and $f^* \colon \lambda_0 I(X) \to Y$ such that the corresponding diagram commutes, then f is a function and f^* is equal to the function from $\lambda_0 I(X)$ to Y generated by f.

Proposition 4.6.5. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \texttt{Fam}(I, X)$, and let Y be a set. If $f^* : \lambda_0 I(X) \to Y$, there is a unique function $f \colon I \to Y$ such that the following diagram commutes*

$$\begin{tikzcd}
I \arrow[r, dashed, "f"] \arrow[d, "\lambda_0"'] & Y. \\
\lambda_0 I(X) \arrow[ur, "f^*"'] &
\end{tikzcd}$$

If $\Lambda \in \texttt{Set}(I, X)$, then f^* is equal to the function from $\lambda_0 I(X)$ to Y generated by f.

Definition 4.6.6. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X \lambda_1) \in \mathtt{Set}(I, X)$, and let Y be a set. If $f^*\colon \lambda_0 I(X) \to Y$, we denote the unique function $f\colon I \to Y$ generated by f^* by $f^* \circ \lambda_0$.*

Corollary 4.6.7. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X \lambda_1) \in \mathtt{Fam}(I, X)$, and let Y be a set.*

(i) *The operation $\Phi\colon \mathbb{F}(\lambda_0 I, Y) \rightsquigarrow \mathbb{F}(I, Y)$, defined by $\Phi(f^*) := f^* \circ \lambda_0$, for every $f^* \in \mathbb{F}(\lambda_0 I, Y)$, is an embedding.*

(ii) *If $\Lambda(X) \in \mathtt{Set}(I, X)$, then Φ is a surjection, the operation $\Theta\colon \mathbb{F}(I, Y) \rightsquigarrow \mathbb{F}(\lambda_0 I, Y)$, defined by $\Theta(f) := \lambda_0 f$, for every $f \in \mathbb{F}(I, Y)$, is an embedding, and $(\Theta, \Phi)\colon (\mathbb{F}(I, Y) =_{\mathbb{V}_0} \mathbb{F}(\lambda_0 I, Y)$.*

Proof. (i) By definition of the corresponding equalities we have that

$$\begin{aligned} f^* =_{\mathbb{F}(\lambda_0 I(X), Y)} g^* &\Leftrightarrow \forall_{i \in I}\big(f^*(\lambda_0(i)) =_Y g^*(\lambda_0(i))\big) \\ &\Leftrightarrow \forall_{i \in I}\big((f^* \circ \lambda_0)(i)) =_Y (g^* \circ \lambda_0)(i))\big) \\ &\Leftrightarrow f^* \circ \lambda_0 =_{\mathbb{F}(I,Y)} g^* \circ \lambda_0. \end{aligned}$$

(ii) If $f \in \mathbb{F}(I, Y)$, then by Proposition 4.6.4 there is unique $\lambda_0 f \in \mathbb{F}(\lambda_0 I, Y)$ such that $\Phi(\lambda_0 f) := \lambda_0 f \circ \lambda_0 =_{\mathbb{F}(I,Y)} f$. By definition of the corresponding equalities we have that

$$\begin{aligned} f =_{\mathbb{F}(I,Y)} g &\Leftrightarrow \forall_{i \in I}\big(f(i)) =_Y g(i)\big) \\ &\Leftrightarrow \forall_{i \in I}\big(\lambda_0 f(\lambda_0(i)) =_Y \lambda_0 g(\lambda_0(i))\big) \\ &\Leftrightarrow \lambda_0 f =_{\mathbb{F}(\lambda_0 I, Y)} \lambda_0 g. \end{aligned}$$

Moreover, we have that $(\Theta \circ \Phi)(f^*) := \Theta(f^* \circ \lambda_0) := \lambda_0(f^* \circ \lambda_0) =_{\mathbb{F}(\lambda_0 I(X), Y)} f^*$, and $(\Phi \circ \Theta)(f) := \Phi(\lambda_0 f) := (\lambda_0 f) \circ \lambda_0 =_{\mathbb{F}(I,Y)} f$. $\square$

Proposition 4.6.8. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Set}(I, X)$ and $M(X) := (\mu_0, \mathcal{Z}^X \mu_1) \in \mathtt{Set}(J, Y)$. If $f\colon I \to J$, there is a unique function $f^*\colon \lambda_0 I(X) \to \mu_0 J(Y)$ such that the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\lambda_0"'] & J \arrow[d, "\mu_0"] \\ \lambda_0 I(X) \arrow[r, dashed, "f^*"'] & \mu_0 J(Y). \end{tikzcd}$$

If $f\colon I \rightsquigarrow J$, and $f^* : \lambda_0 I(X) \to \mu_0 J(Y)$ such that the corresponding to the above diagram commutes, then $f \in \mathbb{F}(I, J)$ and f^* is equal to the map in $\mathbb{F}\big(\lambda_0 I(X), \mu_0 J(Y)\big)$ generated by f.

Remark 4.6.9. *Let the set $\big(X, =_X, \neq_X^{\mathbb{F}(X,2)}\big)$, and $\Delta^1(X) := \big(\delta_0^1, \mathcal{E}^{1,X}, \delta_1^1\big)$, where the non-dependent assignment routine $\delta_0^1\colon \mathbb{F}(X, 2) \rightsquigarrow \mathbb{V}_0$ is defined by the rule $f \mapsto \delta_0^1(f)$, for every $f \in \mathbb{F}(X, 2)$ (see Definition 2.8.3), and the dependent operations $\mathcal{E}^{1,X}\colon \bigwedge_{f \in DF(X,2)} \mathbb{F}(\delta_0^1(f), X)$ and $\delta_1^1\colon \bigwedge_{(f,g) \in D(\mathbb{F}(X,2))} \mathbb{F}(\delta_0^1(f), \delta_0^1(g))$ are defined, respectively, by*

$$\mathcal{E}_f^{1,X}\colon \delta_0^1(f) \hookrightarrow X \qquad x \mapsto x; \qquad x \in \delta_0^1(f),$$

$$\delta_1^1(f, g) := \delta_{fg}^1\colon \delta_0^1(f) \to \delta_0^1(g) \qquad x \mapsto x; \qquad x \in \delta_0^1(f).$$

If $\Delta^0(X) := \big(\delta_0^0, \mathcal{E}^{0,X}, \delta_1^0\big)$, where $\delta_0^0\colon \mathbb{F}(X, 2) \rightsquigarrow \mathbb{V}_0$ is defined by the rule $f \mapsto \delta_0^0(f)$, for every $f \in \mathbb{F}(X, 2)$, and the dependent operations $\mathcal{E}^{0,X}, \delta_1^0$ are defined similarly, then $\Delta^1(X), \Delta^0(X) \in \mathtt{Set}(\mathbb{F}(X, 2), X)$, and they are called the $\mathbb{F}(X, 2)$-sets of detachable subsets of X.

Proof. We give the proof only for $\Delta^1(X)$. It is easy to show that $\Delta^1(X) \in \mathtt{Fam}(\mathbb{F}(X,2), X)$. Let $f, g \in \mathbb{F}(X,2)$ such that $\delta_0^1(f) =_{\mathcal{P}(X)} \delta_0^1(g)$ i.e., there are $e \in \mathbb{F}\big(\delta_0^1(f), \delta_0^1(g)\big)$ and $k \in \mathbb{F}\big(\delta_0^1(g), \delta_0^1(f)\big)$ such that $(e,k) \colon \delta_0^1(f) =_{\mathcal{P}(X)} \delta_0^1(g)$

$$\begin{tikzcd}
\delta_0^1(f) \arrow[rr, hook, bend left, "e"] \arrow[dr, hook, "\mathcal{E}_f^{1,X}"'] & & \delta_0^1(g) \arrow[ll, hook, bend left, "k"] \arrow[dl, hook, "\mathcal{E}_g^{1,X}"] \\
& X. &
\end{tikzcd}$$

Let $x \in X$. By the commutativity of the above diagram $x := \mathcal{E}_g^{1,X}(x) =_X \mathcal{E}_f^{1,X}\big(k(x)\big) := k(x)$. Hence, if $f(x) =_2 1$, then $f(k(x)) =_2 1$. Since $k(x) \in \delta_0^1(g)$, we get $g(k(x)) =_2 1$, and since $x =_X k(x)$, we get $g(x) =_2 1$. If $f(x) =_2 0$, we use proceed similarly. $\square$

Clearly, $\delta_0^1(\overline{1}) = X$, $\delta_0^1(f) \cap \delta_0^1(g) = \delta_0^1(f \cdot g)$, and $\delta_0^1(f) \cup \delta_0^1(g) = \delta_0^1(f + g - f \cdot g)$.

Proposition 4.6.10. *Let the family $\Delta^1(X) := \big(\delta_0^1, \mathcal{E}^{1,X}, \delta_1^1\big)$ of detachable subsets of X. If $\mathtt{compl} \colon \mathbb{F}(X,2) \to \mathbb{F}(X,2)$ is defined by $f \mapsto 1 - f$, for every $f \in \mathbb{F}(X,2)$, then the operation $\mathtt{Compl} \colon [\delta\mathbb{F}(X,2)](X) \rightsquigarrow [\delta\mathbb{F}(X,2)](X)$, defined by*

$$\mathtt{Compl}(\delta_0^1(f)) := \delta(\mathtt{compl}(f)) =: \delta_0^1(1 - f) = \delta_0^0(f); \qquad \delta_0^1(f) \in [\delta\mathbb{F}(X,2)](X),$$

is a function such that the following conditions hold:
(a) $\mathtt{Compl}(\mathtt{Compl}(\delta_0^1(f)) = \delta_0^1(f)$.
(b) $\mathtt{Compl}(\delta_0^1(f) \cap \delta_0^1(g)) = \mathtt{Compl}(\delta_0^1(f)) \cup \mathtt{Compl}_0^1(\delta(g))$.
(c) $\mathtt{Compl}(\delta_0^1(f) \cup \delta_0^1(g)) = \mathtt{Compl}(\delta_0^1(f)) \cap \mathtt{Compl}(\delta_0^1(g))$.

Proof. (i) By Proposition 4.6.4 the operation $\mathtt{Compl}$ is the unique function from $[\delta_0^1\mathbb{F}(X,2)](X)$ to $[\delta_0^1\mathbb{F}(X,2)](X)$ that makes the following diagram commutative

$$\begin{tikzcd}
\mathbb{F}(X,2) \arrow[r, "\mathtt{compl}"] \arrow[d, "\delta_0^1"'] & \mathbb{F}(X,2) \arrow[d, "\delta_0^1"] \\
{[\delta_0^1\mathbb{F}(X,2)](X)} \arrow[r, dashed, "\mathtt{Compl}"'] & {[\delta_0^1\mathbb{F}(X,2)](X).}
\end{tikzcd}$$

The proofs of conditions (a)-(c) are easy to show. $\square$

Proposition 4.6.11. *Let X, Y be sets, and let the sets of detachable subsets $\Delta^1(X) := \big(\delta_0^{1,X}, \mathcal{E}^{1,X}, \delta_1^X\big)$, $\Delta^1(Y) := \big(\delta_0^{1,Y}, \mathcal{E}^{1,Y}, \delta_1^{1,Y}\big)$ of X and Y, respectively. If $h : Y \to X$, then the operation $\tilde{h} : \mathbb{F}(X,2) \rightsquigarrow \mathbb{F}(Y,2)$, defined by $f \mapsto f \circ h$, for every $f \in \mathbb{F}(X,2)$, is a function, and there is a unique function $\delta_0^1\tilde{h} : [\delta_0^1\mathbb{F}(X,2)](X) \to [\delta_0^1 DF(Y,2)](Y)$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{F}(Y, 2) \arrow[r, "\delta_0^1 \tilde{h}"] \arrow[d, "\delta_{0,Y}^1"'] & \mathbb{F}(X, 2) \arrow[d, "\delta_{0,X}^1"] \\
{[\delta_0^1 \mathbb{F}(X, 2)](X)} \arrow[r, dashed, "\tilde{h}"'] & {[\delta_0^1 \mathbb{F}(Y, 2)](Y).}
\end{tikzcd}$$

Proof. It follows immediately from Proposition 4.6.8. $\square$

Proposition 4.6.12. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $M(Y) := (\mu_0, \mathcal{Z}^Y \mu_1) \in \mathtt{Set}(J, Y)$. If $f^* \colon \lambda_0 I(X) \to \mu_0 J(Y)$, there is a unique $f \colon I \to J$, such that the following diagram commutes*

$$\begin{tikzcd}
I \arrow[r, dashed, "f"] \arrow[d, "\lambda_0"'] & J \arrow[d, "\mu_0"] \\
\lambda_0 I \arrow[r, "f^*"'] & \mu_0 J.
\end{tikzcd}$$

Moreover, f^* is equal to the function from $\lambda_0 I(X)$ to $\mu_0 J(Y)$ generated by f.

Corollary 4.6.13. *Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Set}(I, X)$ and $M(Y) := (\mu_0, \mathcal{Z}^Y \mu_1) \in \mathtt{Fam}(J, Y)$. The operation $\Theta \colon \mathbb{F}(I, J) \rightsquigarrow \mathbb{F}(\lambda_0 I, \mu_0 J)$, defined by $f \mapsto f^*$, for every $f \in \mathbb{F}(I, J)$, is a function. If $M(Y) \in \mathtt{Set}(J, Y)$, then Θ is an embedding, and a surjection.*

Proof. By definition of the corresponding equalities we have that

$$\begin{aligned}
f =_{\mathbb{F}(I,J)} g &\Leftrightarrow \forall_{i \in I}\big(f(i) =_J g(i)\big) \\
&\Rightarrow \forall_{i \in I}\big(\mu_0(f(i)) =_{\mathcal{P}(Y)} \mu_0(g(i))\big) \\
&\Leftrightarrow \forall_{i \in I}\big(f^*(\lambda_0(i)) =_{\mathcal{P}(Y)} g^*(\lambda_0(i))\big) \\
&\Leftrightarrow f^* =_{\mathbb{F}(\lambda_0 I(X), \mu_0 J(Y))} g^*.
\end{aligned}$$

If $M(Y) \in \mathtt{Set}(J, Y)$, the above implication is also an equivalence, hence Θ is an embedding. By Proposition 4.6.5 we have that Θ is a surjection. $\square$

The notions of fiber and cofiber of a function were introduced in Definition 2.3.4.

Proposition 4.6.14. *Let the sets $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$, and let $f \colon X \to Y$.*
(i) *Let $\mathtt{fib}^f(X) := \big(\mathtt{fib}_0^f, \mathcal{E}^{\mathtt{fib},X}, \mathtt{fib}_1^f\big)$, where $\mathtt{fib}_0^f \colon Y \rightsquigarrow \mathbb{V}_0$ is defined by the rule $\mathtt{fib}_0^f(y) := \mathtt{fib}^f(y)$, for every $y \in Y$, and the dependent operations $\mathcal{E}^{\mathtt{fib},X} \colon \curlywedge_{y \in Y} \mathbb{F}(\mathtt{fib}_0^f(y), X)$ and $\mathtt{fib}_1^f \colon \curlywedge_{(y,y') \in D(Y)} \mathbb{F}(\mathtt{fib}_0^f(y), \mathtt{fib}_0^f(y'))$ are defined, respectively, by*

$$\mathcal{E}^{\mathtt{fib},X} \colon \mathtt{fib}_0^f(y) \hookrightarrow X \qquad x \mapsto x; \qquad x \in \mathtt{fib}_0^f(y),$$

$$\mathtt{fib}_1^1(y, y') := \mathtt{fib}_{yy'}^1 \colon \mathtt{fib}^f(y) \to \mathtt{fib}^f(y') \qquad x \mapsto x; \qquad x \in \mathtt{fib}_0^f(y).$$

Then $\mathtt{fib}^f(X) \in \mathtt{Fam}(Y, X)$ and if f is a surjection, then $\mathtt{fib}^f(X) \in \mathtt{Set}(Y, X)$.

(ii) *f is strongly extensional if and only if* $\mathtt{cofib}_0^f(y)]\![\![\mathtt{fib}_0^f(y)$*, for every* $y \in Y$.

(iii) *Let* $\mathtt{cofib}^f(X) := \big(\mathtt{cofib}_0^f, \mathcal{E}^{\mathtt{cofib},X}, \mathtt{cofib}_1^f\big)$*, where* $\mathtt{cofib}_0^f \colon Y \rightsquigarrow \mathbb{V}_0$ *is defined by the rule* $\mathtt{cofib}_0^f(y) := \mathtt{cofib}^f(y)$*, for every* $y \in Y$*, and* $\mathcal{E}^{\mathtt{cofib},X} \colon \bigcurlywedge_{y \in Y} \mathbb{F}(\mathtt{cofib}_0^f(y), X)$*,* $\mathtt{cofib}_1^f \colon \bigcurlywedge_{(y,y') \in D(Y)} \mathbb{F}(\mathtt{cofib}_0^f(y), \mathtt{cofib}_0^f(y'))$ *are defined, respectively, by*

$$\mathcal{E}_y^{\mathtt{cofib},X} \colon \mathtt{cofib}_0^f(y) \hookrightarrow X \qquad x \mapsto x; \qquad x \in \mathtt{cofib}_0^f(y),$$

$$\mathtt{cofib}_1^1(y, y') := \mathtt{cofib}_{yy'}^1 \colon \mathtt{cofib}^f(y) \to \mathtt{cofib}^f(y') \qquad x \mapsto x; \qquad x \in \mathtt{cofib}_0^f(y).$$

Then $\mathtt{cofib}^f(X) \in \mathtt{Fam}(Y, X)$*, and if f is a surjection, then* $\mathtt{cofib}^f(X) \in \mathtt{Set}(Y, X)$ *if and only if the inequality* $\neq_Y$ *is tight.*

Proof. (i) If $y =_Y y'$ and $x \in \mathtt{fib}^f(y)$, then $x \in \mathtt{fib}^f(y')$. Since the functions $\mathcal{E}_y^{\mathtt{fib},X}, \mathcal{E}_{y'}^{\mathtt{fib},X}$, and $\mathtt{fib}_{yy'}^1$ are defined through the identity map-rule, we get $\mathtt{fib}^f(X) \in \mathtt{Fam}(Y, X)$. Let $y, y' \in Y$ and functions $g \in \mathbb{F}\big(\mathtt{fib}^f(y), \mathtt{fib}^f(y')\big)$ and $h \in \mathbb{F}\big(\mathtt{fib}^f(y'), \mathtt{fib}^f(y)\big)$, such that $(g, h) \colon \mathtt{fib}^f(y) =_{\mathcal{P}(X)} \mathtt{fib}^f(y')$. Let $x \in X$ such that $f(x) =_Y y$ i.e., $x \in \mathtt{fib}^f(y)$. By the commutativity of one of the following left inner diagrams we have that $g(x) =_X x$, and, of course, $g(x) \in \mathtt{fib}^f(y')$ i.e., $f(g(x)) =_Y y'$. Hence, $y' =_Y f(g(x)) =_Y f(x) =_Y y$.

$$\begin{tikzcd}
\mathtt{fib}^f(y) \arrow[rr, bend left, "g"] \arrow[dr, "\mathcal{E}_y^{\mathtt{fib},X}"'] & & \mathtt{fib}^f(y') \arrow[ll, bend left, "h"] \arrow[dl, "\mathcal{E}_{y'}^{\mathtt{fib},X}"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathtt{cofib}^f(y) \arrow[rr, bend left, "g'"] \arrow[dr, "\mathcal{E}_y^{\mathtt{cofib},X}"'] & & \mathtt{cofib}^f(y') \arrow[ll, bend left, "h'"] \arrow[dl, "\mathcal{E}_{y'}^{\mathtt{cofib},X}"] \\
& X &
\end{tikzcd}$$

(ii) Suppose that f is strongly extensional and let $x \in \mathtt{cofib}^f(y)$ and $z \in \mathtt{fib}^f(y)$ i.e., $f(x) \neq_Y y$ and $f(x) =_Y y$. By the extensionality of $\neq_Y$ (Remark 2.2.6) we get $f(x) \neq_Y f(z)$, and as f is strongly extensional, we conclude that $x \neq_X z$. Suppose next that $\mathtt{cofib}_0^f(y)]\![\![\mathtt{fib}_0^f(y)$, for every $y \in Y$, and let $x, z \in X$ with $f(x) \neq_Y f(z)$. In this case, we get $x \in \mathtt{fib}^f(f(x))$ and $z \in \mathtt{cofib}^f(f(x))$. Since $\mathtt{cofib}_0^f(f(x))]\![\![\mathtt{fib}_0^f(f(x))$ and the corresponding embeddings into X are given by the identity map-rule, we get $x \neq_X z$.

(iii) If $y =_Y y'$ and $x \in \mathtt{cofib}^f(y)$, then $f(x) \neq_Y y$, and by the extensionality of $\neq_Y$, we get $f(x) \neq_Y y'$ i.e., $x \in \mathtt{fib}^f(y')$. Since the functions $\mathcal{E}_y^{\mathtt{cofib},X}, \mathcal{E}_{y'}^{\mathtt{cofib},X}$, and $\mathtt{cofib}_{yy'}^1$ are defined through the identity map-rule, we get $\mathtt{cofib}^f(X) \in \mathtt{Fam}(Y, X)$. Let f be a surjection. We suppose first that $\mathtt{cofib}^f(X) \in \mathtt{Set}(Y, X)$. If $\neg(y \neq_Y y')$, we show that $y =_Y y'$, by showing that $\mathtt{cofib}^f(y) =_{\mathcal{P}(X)} \mathtt{cofib}^f(y')$. If $x \in \mathtt{cofib}^f(y)$, then $f(x) \neq_Y y$. By condition (Ap_3) either $y' \neq_Y f(x)$ or $y' \neq_Y y$. Since the latter contradicts our hypothesis $\neg(y \neq_Y y')$, we conclude that $y' \neq_Y f(x)$ i.e., $x \in \mathtt{cofib}^f(y')$. Similarly we show that if $x \in \mathtt{cofib}^f(y')$, then $x \in \mathtt{cofib}^f(y)$. Hence, the functions between $\mathtt{cofib}^f(y)$ and $\mathtt{cofib}^f(y')$ that are given by the identity map-rule witness the equality $\mathtt{cofib}^f(y) =_{\mathcal{P}(X)} \mathtt{cofib}^f(y')$. Suppose next that the inequality $\neq_Y$ is tight. Let $y, y' \in Y$ and let functions $g' \in \mathbb{F}\big(\mathtt{cofib}^f(y), \mathtt{cofib}^f(y')\big)$ and $h' \in \mathbb{F}\big(\mathtt{cofib}^f(y'), \mathtt{cofib}^f(y)\big)$, such that $(g', h') \colon \mathtt{cofib}^f(y) =_{\mathcal{P}(X)} \mathtt{cofib}^f(y')$. We show that $y =_Y y'$ by showing $\neg(y \neq_Y y')$. For that suppose $y \neq_Y y'$, and let $x, x' \in X$ such that $f(x) =_Y y$ and $f(x') =_Y y'$. By the extensionality of $\neq_Y$ we get $f(x) \neq_Y y'$ i.e., $x \in \mathtt{cofib}^f(y')$.

Since $h'(x) \in \texttt{cofib}^f(y)$, and since by the commutativity of one of the above right inner diagrams $h'(x) =_X x$, we get $x \in \texttt{cofib}^f(y)$. Since $f(x) \neq_Y y$ and $y =_Y f(x)$, by the extensionality of $\neq_Y$ we get $f(x) \neq_Y f(x)$, which leads to the required contradiction. $\square$

If f is not a surjection, it is possible that $\texttt{fib}^f(y), \texttt{fib}^f(y')$ are not inhabited, and $y \neq_Y y'$. If f is not a surjection, like the function $f \colon X \to \{0, 1, 2\}$, defined by $f(x) := 0$, for every $x \in X$, then $\texttt{cofib}^f(1) = X = \texttt{cofib}^f(2)$ and $1 \neq 2$. Notice that it is not necessary that a family of subsets is a family of fibers or a family of cofibers, as the moduli of embeddings of the latter are given through the identity map-rule.

Definition 4.6.15. *An I-family of sets $\Lambda := (\lambda_0, \lambda_1)$ is a family of contractible sets, if $\lambda_0(i)$ is contractible, for every $i \in I$. A modulus of centres of contraction for Λ is a dependent operation $\texttt{centre}^\Lambda \colon \bigcurlywedge_{i \in I} \lambda_0(i)$, with $\texttt{centre}^\Lambda_i$ a centre of contraction for $\lambda_0(i)$, for every $i \in I$.*

In Proposition 2.4.1 we saw that if $(f, g) \colon X =_{\mathbb{V}_0} Y$, the set $\texttt{fib}^f(y)$ is contractible with $\texttt{centre}^f_y := g(y)$, for every $y \in Y$ i.e., the dependent operation $\texttt{centre}^f$ is a modulus of centres of contractions for the family $\texttt{fib}^f(X)$. Next follows a kind of inverse to Proposition 2.4.1.

Proposition 4.6.16. *Let the sets $(X, =_X, \neq_X)$, $(Y, =_Y, \neq_Y)$, and $f \colon X \to Y$. If $\texttt{fib}^f(X) := \big(\texttt{fib}^f_0, \mathcal{E}^{\texttt{fib},X}, \texttt{fib}^f_1\big)$ is a family of contractible subsets of X with $\texttt{centre}^{\texttt{fib}^f(X)} \colon \bigcurlywedge_{y \in Y} \texttt{fib}^f(y)$ a modulus of centres of contraction for $\texttt{fib}^f(X)$, there is $g \in \mathbb{F}(Y, X)$ with $(f, g) \colon X =_{\mathbb{V}_0} Y$.*

Proof. Let the operation $g \colon Y \rightsquigarrow X$, defined by $g(y) := \texttt{centre}^{\texttt{fib}^f(X)}(y)$, for every $y \in Y$. Since $g(y) \in \texttt{fib}^f(y)$, we have that $f(g(y)) =_Y y$. Since $g(y)$ is a centre of contraction for $\texttt{fib}^f(y)$, we have that $\forall_{x \in X}\big(f(x) =_Y y \Rightarrow x =_X g(y)\big)$. First we show that the operation g is a function. For that, let $y =_Y y'$, and we show that $g(y) =_X g(y')$. Since the map $\texttt{fib}^1_{yy'}$ in Proposition 4.6.14 is given by the identity map-rule, and since $g(y') \in \texttt{fib}^f(y')$, we get $g(y') \in \texttt{fib}^f(y)$. Since $g(y)$ is a centre of contraction for $\texttt{fib}^f(y)$, we get $g(y') =_X g(y)$. It remains to show that if $x \in X$, then $g(f(x)) =_X x$. By the definition of g we have that $g(f(x)) := \texttt{centre}^{\texttt{fib}^f(X)}(f(x))$. As $x \in \texttt{fib}^f(f(x))$, we get $x =_X g(f(x))$. $\square$

4.7 Families of equivalence classes

In this section we extend results on sets of subsets to families of equivalence classes. Although a family of equivalence classes is not, in general, a set of subsets, we can define functions on them, if we use appropriate functions on their index-set.

Definition 4.7.1. *If X is a set and $R_X(x, x')$ is an extensional property on $X \times X$ that satisfies the conditions of an equivalence relation, we call the pair (X, R_X) an equivalence structure. If (Y, S_Y) is an equivalence structure, a function $f \colon X \to Y$ is an equivalence preserving function, or an (R_X, S_Y)-function, if*

$$\forall_{x, x' \in X}\big(R(x, x') \Rightarrow S(f(x), f(x'))\big).$$

If, for every $x, x' \in X$, the converse implication holds, we say that f is an (R_X, S_Y)-embedding. Let $\mathbb{F}(R_X, S_Y)$ be the set of (R_X, S_Y)-functions[2].

[2]By the extensionality of S_Y the property of being an (R_X, S_Y)-function is extensional on $\mathbb{F}(X, Y)$.

Proposition 4.7.2. *If (X, R_X) is an equivalence structure, let $R(X) := \big(\rho_0, \mathcal{R}^X, \rho_1\big)$, where $\rho_0 \colon X \rightsquigarrow \mathbb{V}_0$ is defined by $\rho_0(x) := \{y \in X \mid R_X(y, x)\}$, for every $x \in X$, and the dependent operations $\mathcal{R}^X \colon \bigcurlywedge_{x \in X} \mathbb{F}\big(\rho_0(x), X\big)$, $\rho_1 \colon \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\big(\rho_0(x), \rho_0(x')\big)$ are defined by*

$$\mathcal{R}^X_x \colon \rho_0(x) \hookrightarrow X \qquad y \mapsto y; \qquad y \in \rho_0(x),$$

$$\rho_1(x, x') := \rho_{xx'} \colon \rho_0(x) \to \rho_0(x') \qquad y \mapsto y; \qquad y \in \rho_0(x).$$

Then $R(X) \in \mathtt{Fam}(X, X)$, such that $\forall_{xx' \in X}\big(\rho_0(x) =_{\mathcal{P}(X)} \rho_0(x') \Rightarrow R(x, x')\big)$.

Proof. By the extensionality of R_X the set $\rho_0(x)$ is a well-defined extensional subset of X. If $x =_X x'$ and $R_X(y, x)$, then by the extensionality of R_X we get $R_X(y, x')$, hence $\rho_{xx'}$ is well-defined. Let $(f, g) \colon \rho_0(x) =_{\mathcal{P}(X)} \rho_0(x')$

$$\begin{tikzcd}
\rho_0(x) \arrow[rr, bend left, "f"] \arrow[dr, hook, "\mathcal{R}^X_x"'] & & \rho_0(x') \arrow[ll, bend left, "g"] \arrow[dl, hook', "\mathcal{R}^X_{x'}"] \\
& X. &
\end{tikzcd}$$

If $y \in \rho_0(x) :\Leftrightarrow R_X(y, x)$, then $f(y) \in \rho_0(x') :\Leftrightarrow R_X(f(y), x')$, and by the commutativity of the corresponding above diagram we get $f(y) =_X y$. Hence by the extensionality of R_X we get $R_X(y, x')$. Since $R_X(y, x)$ implies $R_X(x, y)$, by transitivity we get $R_X(x, x')$. $\square$

Corollary 4.7.3. *Let $\mathtt{Eql}(X) := \big(\mathtt{eql}^X_0, \mathcal{E}^X, \mathtt{eql}^X_1\big)$ be the X-family of subsets of X induced by the equivalence relation $=_X$ i.e., $\mathtt{eql}^X_0(x) := \{y \in X \mid y =_X x\}$. Then $\mathtt{Eql}(X) \in \mathtt{Set}(X, X)$.*

Proof. It follows immediately from Proposition 4.7.2. $\square$

Proposition 4.7.4. *If (X, R_X) is an equivalence structure, and $f \colon X \to Y$ is an $(R_X, =_Y)$-function there is a unique $\rho_0 f : \rho_0 X(X) \to Y$ such that the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_0^*"'] & Y. \\
\rho_0 X(X) \arrow[ur, dashed, "\rho_0 f"'] &
\end{tikzcd}$$

Conversely, if $f \colon X \to Y$ and $f^* : \rho_0 X(X) \to Y$ such that the above diagram commutes, then f is an $(R_X, =_Y)$-function and f^* is equal to the function from $\rho_0 X(X)$ to Y generated by f.

Proposition 4.7.5. *If (X, R_X) is an equivalence structure, and $f^* : \rho_0 X(X) \to Y$, there is a unique $f \colon X \to Y$, which is an $(R_X, =_Y)$-function, such that the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, dashed, "f"] \arrow[d, "\rho_0^*"'] & Y. \\
\rho X(X) \arrow[ur, "f^*"'] &
\end{tikzcd}$$

Moreover, f^* is equal to the function from $\rho_0 X(X)$ to Y generated by f.

Proposition 4.7.6. *Let (X, R_X) and (Y, S_Y) be equivalence structures and $f\colon X \to Y$ an (R_X, S_Y)-function. If $R(X)$ and $S(Y)$ are the corresponding families of equivalence classes, there is a unique function $f^*\colon \rho_0 X(X) \to \sigma_0 Y(Y)$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_0^*"'] & Y \arrow[d, "\sigma_0^*"] \\ \rho_0 X(X) \arrow[r, dashed, "f^*"'] & \sigma_0 Y(Y). \end{tikzcd}$$

If $f\colon X \to Y$ and $f^* : \rho_0 X(X) \to \sigma_0 Y(Y)$ such that the above diagram commutes, then f is an (R_X, S_Y)-function and f^* is equal to the function from $\rho_0 X(X)$ to $\sigma_0 Y(Y)$ generated by f.

Proof. The assignment routinef^* from $\rho_0 X(X)$ to $\sigma_0 Y(Y)$ defined by $f^*(\rho_0(x)) := \sigma_0(f(x))$, for every $\rho_0(x) \in \rho_0 X(X)$ is extensional, since for every $x, x' \in X$ we have that $\rho_0(x) =_{\mathcal{P}(X)} \rho_0(x') \Rightarrow R_X(x, x') \Rightarrow S_Y(f(x), f(x'))$, hence $\sigma_0(f(x)) =_{\mathcal{P}(Y)} \sigma_0(f(x')) :\Leftrightarrow f^*(\rho_0(x)) =_{\mathcal{P}(Y)} f^*(\rho_0(x'))$. The uniqueness of f^* is immediate. For the converse, if $x, x' \in X$, then by the transitivity of $=_{\mathcal{P}(Y)}$ we have that $R_X(x, x') \Rightarrow \rho_0(x) =_{\mathcal{P}(X)} \rho_0(x') \Rightarrow f^*(\rho_0(x)) =_{\mathcal{P}(Y)} f^*(\rho_0(x')) \Rightarrow \sigma_0(f(x)) =_{\mathcal{P}(Y)} \sigma_0(f(x'))$, hence $S_Y(f(x), f(x'))$. The proof that f^* is equal to the function from $\rho_0 X(X)$ to $\sigma_0 Y(Y)$ generated by f is immediate. $\square$

The previous is the constructive analogue to a standard classical fact (see [45], p. 17). A function $f^*\colon \rho_0 X(X) \to \sigma_0 Y(Y)$ does not generate a function from X to Y.

Proposition 4.7.7. *Let (X, R_X) and (Y, S_Y) be equivalence structures and $R(X), S(Y)$ the families of their equivalence classes. If $f^* : \rho_0 X(X) \to \sigma_0 Y(Y)$, there is $f\colon X \rightsquigarrow Y$, which is (R_X, S_Y)-preserving and $(=_X, S_Y)$-preserving, such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho^*"'] & Y \arrow[d, "\sigma^*"] \\ \rho_0 X(X) \arrow[r, "f^*"'] & \sigma_0 Y(Y). \end{tikzcd}$$

Proof. If $x \in X$, then $f^*(\rho_0(x)) := \sigma_0(y)$, for some $y \in Y$. We define the routine $f(x) := y$ i.e., the output of f^* determines the output of f. Since $R(x, x') \Rightarrow \rho_0(x) =_{\mathcal{P}(X)} \rho_0(x') \Rightarrow f^*(\rho_0(x)) =_{\mathcal{P}(Y)} f^*(\rho_0(x'))$, hence $\sigma_0(y) =_{\mathcal{P}(Y)} \sigma_0(y')$ and $S_Y(y, y')$, we get $S_Y(f(x), f(x'))$, and the operation f is (R_X, S_Y)-preserving. Although we cannot show that f is a function, we can show that it is $(=_X, S)$-preserving, since $x =_X x' \Rightarrow R_X(x, x')$, and we work as above. $\square$

4.8 Families of partial functions

Definition 4.8.1. *Let X, Y and I be sets. A family of partial functions from X to Y indexed by I, or an I-family of partial functions from X to Y, is a triplet $\Lambda(X, Y) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^Y)$, where $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $\mathcal{P}^Y : \bigcurlywedge_{i \in I} \mathbb{F}(\lambda_0(i), Y)$ with $\mathcal{P}^Y(i) := \mathcal{P}^Y_i$, for every $i \in I$, such that, for every $(i, j) \in D(I)$, the following inner diagrams commute*

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, bend left, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] \arrow[ddr, bend right, "\mathcal{P}_i^Y"'] & & \lambda_0(j) \arrow[ll, bend left, "\lambda_{ji}"] \arrow[dl, hook', "\mathcal{E}_j^X"] \arrow[ddl, bend left, "\mathcal{P}_j^Y"] \\
& X & \\
& Y. &
\end{tikzcd}$$

We call $\mathcal{P}^Y$ a modulus of partial functions for λ_0, and $\Lambda(X)$ the I-family of domains of $\Lambda(X,Y)$. If $M(X,Y) := (\mu_0, \mathcal{Z}^X, \mu_1, \mathcal{Q}^Y)$ and $N(X,Y) := (\nu_0, \mathcal{H}^X, \nu_1, \mathcal{R}^Y)$ are I-families of partial functions from X to Y, a family of partial functions-map $\Psi\colon \Lambda(X,Y) \Rightarrow M(X,Y)$ from $\Lambda(X,Y)$ to $M(X,Y)$ is a dependent operation $\Psi\colon \bigcurlywedge_{i\in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$, where $\Psi(i) := \Psi_i$, for every $i \in I$, such that, for every $i \in I$, the following inner diagrams commute

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\Psi_i"] \arrow[dr, hook, "\mathcal{E}_i^X"'] \arrow[ddr, bend right, "\mathcal{P}_i^Y"'] & & \mu_0(i) \arrow[dl, hook', "\mathcal{Z}_i^X"] \arrow[ddl, bend left, "\mathcal{Q}_i^Y"] \\
& X & \\
& Y. &
\end{tikzcd}$$

The totality $\texttt{Map}_I(\Lambda(X,Y), M(X,Y))$ of the family of partial functions-maps from $\Lambda(X,Y)$ to $M(X,Y)$ is equipped with the pointwise equality. If $\Psi\colon \Lambda(X,Y) \Rightarrow M(X,Y)$ and if $\Xi\colon M(X,Y) \Rightarrow N(X,Y)$, the composition family of partial functions-map $\Xi \circ \Psi\colon \Lambda(X,Y) \Rightarrow N(X,Y)$ is defined by $(\Xi \circ \Psi)(i) := \Xi_i \circ \Psi_i$,

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\Psi_i"] \arrow[rr, bend left=50, "(\Xi \circ \Psi)_i"] \arrow[dr, hook, "\mathcal{E}_i"'] \arrow[ddr, bend right, "\mathcal{P}_i^Y"'] & \mu_0(i) \arrow[r, hook, "\Xi_i"] \arrow[d, hook, "\mathcal{Z}_i"] \arrow[dd, dashed, bend left, "\mathcal{Q}_i^Y"] & \nu_0(i) \arrow[dl, hook', "\mathcal{H}_i"] \arrow[ddl, bend left, "\mathcal{R}_i^Y"] \\
& X & \\
& Y &
\end{tikzcd}$$

for every $i \in I$. The identity family of partial functions-map $\mathrm{Id}_{\Lambda(X,Y)}\colon \Lambda(X,Y) \Rightarrow \Lambda(X,Y)$ and the equality on the totality $\texttt{Fam}(I,X,Y)$ of I-families of partial functions from X to Y are defined as in Definition 3.1.3.

Clearly, if $\Lambda(X,Y) \in \texttt{Fam}(I,X,Y)$ and $(i,j) \in D(I)$, then $(\lambda_{ij}, \lambda_{ji})\colon \mathcal{P}_i^Y =_{\mathfrak{F}(X,Y)} \mathcal{P}_j^Y$.

Proposition 4.8.2. *Let* $\Lambda(X,Y) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^Y) \in \mathtt{Fam}(I,X,Y)$ *and let* $M(Y,Z) := (\mu_0, \mathcal{Z}^Y, \mu_1, \mathcal{Q}^Z) \in \mathtt{Fam}(I,Y,Z)$*. Their composition* $(M \odot \Lambda)(X,Z)$ *is defined by*

$$(M \odot \Lambda)(X,Z) := \big(\mu_0 \odot \lambda_0, \mathcal{Z}^Y \odot \mathcal{E}^X, \mu_1 \odot \lambda_1, (\mathcal{Q} \odot \mathcal{P})^Z\big)$$

$$(\mu_0 \odot \lambda_0)(i) := \big(\mathcal{P}_i^Y\big)^{-1}(\mu_0(i)); \qquad i \in I,$$

$$\big(\mathcal{Z}^Y \odot \mathcal{E}^X\big)_i := \mathcal{E}_i^X \circ e^{\lambda_0(i)}_{(\mathcal{P}_i^Y)^{-1}(\mu_0(i))} \colon (\mu_0 \odot \lambda_0)(i) \hookrightarrow X,$$

$$(\mu_1 \odot \lambda_1)_{ij} \colon \big(\mathcal{P}_i^Y\big)^{-1}(\mu_0(i)) \to \big(\mathcal{P}_j^Y\big)^{-1}(\mu_0(j)),$$

$$(\mu_1 \odot \lambda_1)_{ij}(u,w) := \big(\lambda_{ij}(u), \mu_{ij}(w)\big); \qquad (u,w) \in \big(\mathcal{P}_i^Y\big)^{-1}(\mu_0(i)),$$

$$(\mathcal{Q} \odot \mathcal{P})_i^Z := \mathcal{Q}_i^Z \odot \mathcal{P}_i^Y, \qquad i \in I.$$

Then $M(Y,Z) \in \mathtt{Fam}(I,X,Z)$.

Proof. By Definition 2.6.9 we have that

$$\big(\mathcal{P}_i^Y\big)^{-1}(\mu_0(i)) := \big\{(u,w) \in \lambda_0(i) \times \mu_0(i) \mid \mathcal{P}_i^Y(u) =_Y \mathcal{Z}_i^Y(w)\big\},$$

$$\big(\mathcal{P}_j^Y\big)^{-1}(\mu_0(j)) := \big\{(u',w') \in \lambda_0(j) \times \mu_0(j) \mid \mathcal{P}_j^Y(u') =_Y \mathcal{Z}_j^Y(w``)\big\}.$$

If $(i,j) \in D(I)$, then $\mathcal{P}_j^Y\big(\lambda_{ij}(u)\big) =_Y \mathcal{P}^Y(u) =_Y \mathcal{Z}_i^Y(w) =_Y \mathcal{Z}_j^Y\big(\mu_{ij}(w)\big)$,

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, bend left, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] \arrow[ddr, hook, bend right, "\mathcal{P}_i^Y"'] & & \lambda_0(j) \arrow[ll, bend left, "\lambda_{ji}"] \arrow[dl, hook', "\mathcal{E}_j^X"] \arrow[ddl, hook', bend left, "\mathcal{P}_j^Y"] \\
& X & \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu_0(i) \arrow[rr, bend left, "\mu_{ij}"] \arrow[dr, hook, "\mathcal{Z}_i^Y"'] \arrow[ddr, hook, bend right, "\mathcal{Q}_i^Z"'] & & \mu_0(j) \arrow[ll, bend left, "\mu_{ji}"] \arrow[dl, hook', "\mathcal{Z}_j^Y"] \arrow[ddl, hook', bend left, "\mathcal{Q}_j^Z"] \\
& Y & \\
& Z. &
\end{tikzcd}$$

hence the operation $(\mu_1 \odot \lambda_1)_{ij}$ is well-defined, and it is immediate to show that it is a function. For the commutativity of the following inner diagrams we have that

$$\begin{tikzcd}
(\mu_0 \odot \lambda_0)(i) \arrow[rr, bend left, "(\mu_1 \odot \lambda_1)_{ij}"] \arrow[dr, hook, "(\mathcal{Z}^Y \odot \mathcal{E}^X)_i"'] \arrow[ddr, hook, bend right, "(\mathcal{Q} \odot \mathcal{P})_i^Z"'] & & (\mu_0 \odot \lambda_0)(j) \arrow[ll, bend left, "(\mu_1 \odot \lambda_1)_{ji}"] \arrow[dl, hook', "(\mathcal{Z}^Y \odot \mathcal{E}^X)_j"] \arrow[ddl, hook', bend left, "(\mathcal{Q} \odot \mathcal{P})_j^Z"] \\
& X & \\
& Z &
\end{tikzcd}$$

$$\begin{aligned}
\big(\mathcal{Z}^Y \odot \mathcal{E}^X\big)_j\big((\mu_1 \odot \lambda_1)_{ij}(u,w)\big) &:= \big(\mathcal{Z}^Y \odot \mathcal{E}^X\big)_j\big(\lambda_{ij}(u), \mu_{ij}(w)\big)\\
&:= \Big[\mathcal{E}_j^X \circ e^{\lambda_0(j)}_{(\mathcal{P}_j^Y)^{-1}(\mu_0(j))}\Big]\big(\lambda_{ij}(u), \mu_{ij}(w)\big)\\
&:= \mathcal{E}_j^X\big(\lambda_{ij}(u)\big)\\
&=_X \mathcal{E}_i^X(u)\\
&:= \Big[\mathcal{E}_i^X \circ e^{\lambda_0(i)}_{(\mathcal{P}_i^Y)^{-1}(\mu_0(i))}\Big](u,w)\\
&:= \big(\mathcal{Z}^Y \odot \mathcal{E}^X\big)_i(u,w),
\end{aligned}$$

$$\begin{aligned}
\big(\mathcal{Q} \odot \mathcal{P}\big)_j^Z\big(\lambda_{ij}(u), \mu_{ij}(w)\big) &:= \big(\mathcal{Q}_j^Z \odot \mathcal{P}_j^Y\big)\big(\lambda_{ij}(u), \mu_{ij}(w)\big)\\
&:= \mathcal{Q}_j^Z\big(\mu_{ij}(w)\big)\\
&=_Z \mathcal{Q}_i^Z(w)\\
&:= \big(\mathcal{Q}_i^Z \odot \mathcal{P}_i^Y\big)(u,w)\\
&:= \big(\mathcal{Q} \odot \mathcal{P}\big)_i^Z(u,w).
\end{aligned}$$

For the other two inner diagrams we proceed similarly. $\square$

The basic properties of the composition of partial functions extend to equalities for the corresponding families of partial functions. E.g., we get

$$N(Z,W) \odot \big[M(Y,Z) \odot \Lambda(X,Y)\big] =_{\texttt{Fam}(I,X,W)} \big[N(Z,W) \odot M(Y,Z)\big] \odot \Lambda(X,Y).$$

Suppose that $\Lambda(X,Y) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^Y) \in \texttt{Fam}(I,X,Y)$ and $M(X,Y) := (\mu_0, \mathcal{Z}^X, \mu_1, \mathcal{Q}^Y) \in \texttt{Fam}(I,X,Y)$. We can define in the expected way the following families of partial functions:

$$(\Lambda \cap_l M)(X,Y) := \big(\lambda_0 \cap_l \mu_0, \mathcal{E}^X \cap_l \mathcal{Z}^X, \lambda_1 \cap_l \mu_1, (\mathcal{P} \cap_l \mathcal{Q})^Y\big),$$

$$(\Lambda \cap_r M)(X,Y) := \big(\lambda_0 \cap_r \mu_0, \mathcal{E}^X \cap_r \mathcal{Z}^X, \lambda_1 \cap_r \mu_1, (\mathcal{P} \cap_r \mathcal{Q})^Y\big),$$

$$(\Lambda \cup M)(X,Y) := \big(\lambda_0 \cup \mu_0, \mathcal{E}^X \cup \mathcal{Z}^X, \lambda_1 \cup \mu_1, (\mathcal{P} \cup \mathcal{Q})^Y\big).$$

The basic properties of the intersections and union of partial functions extend to equalities for the corresponding families of partial functions. E.g., we get

$$(\Lambda \cup M)(X,Y) =_{\texttt{Fam}(I,X,Y)} (M \cup \Lambda)(X,Y).$$

Various notions and results on families of subsets extend to families of partial functions.

4.9 Families of complemented subsets

Definition 4.9.1. *Let the sets $(X, =_X, \neq_X)$ and $(I, =_I)$. A family of complemented subsets of X indexed by I, or an I-family of complemented subsets of X, is a structure $\boldsymbol{\Lambda}(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big)$, such that $\Lambda^1(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1\big) \in \texttt{Fam}(I,X)$ and $\Lambda^0(X) := \big(\lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big) \in \texttt{Fam}(I,X)$ i.e., for every $(i,j) \in D(I)$, the following inner diagrams commute*

$$\begin{tikzcd}
\lambda_0^1(i) \arrow[dr, hook, bend left, "\mathcal{E}_i^{1,X}"] \arrow[dd, bend right, "\lambda_{ij}^1"'] & & \lambda_0^0(i) \arrow[dl, hook', bend right, "\mathcal{E}_i^{0,X}"'] \arrow[dd, bend right, "\lambda_{ij}^0"'] \\
& X & \\
\lambda_0^1(j) \arrow[ur, hook', bend right, "\mathcal{E}_j^{1,X}"'] \arrow[uu, bend right, "\lambda_{ji}^1"'] & & \lambda_0^0(j) \arrow[ul, hook, bend left, "\mathcal{E}_j^{0,X}"] \arrow[uu, bend right, "\lambda_{ji}^0"']
\end{tikzcd}$$

such that

$$\forall_{i \in I}\big(\boldsymbol{\lambda}_0(i) := \big(\lambda_0^1(i), \lambda_0^0(i)\big) \in \mathcal{P}^{][}(X)\big)$$

If $\boldsymbol{M}(X) := (\mu_0^1, \mathcal{Z}^{1,X}, \mu_1^1, \mu_0^0, \mathcal{Z}^{0,X}, \mu_1^0)$, $\boldsymbol{N}(X) := (\nu_0^1, \mathcal{H}^{1,X}, \nu_1^1, \nu_0^0, \mathcal{H}^{0,X}, \nu_1^0)$ *are* I*-families of complemented subsets of* X*, a family of complemented subsets-map* $\boldsymbol{\Psi} \colon \boldsymbol{\Lambda}(X) \Rightarrow \boldsymbol{M}(X)$ *from* $\boldsymbol{\Lambda}(X)$ *to* $\boldsymbol{M}(X)$ *is a pair* $\boldsymbol{\Psi} := (\Psi^1, \Psi^0)$*, where* $\Psi^1 \colon \Lambda^1(X) \Rightarrow M^1(X)$ *and* $\Psi^0 \colon \Lambda^0(X) \Rightarrow M^0(X)$ *i.e., for every* $i \in I$*, the following inner diagrams commute*

$$\begin{tikzcd}
\lambda_0^1(i) \arrow[dr, hook, bend left, "\mathcal{E}_i^{1,X}"] \arrow[dd, bend right, "\Psi_i^1"'] & & \lambda_0^0(i) \arrow[dl, hook', bend right, "\mathcal{E}_i^{0,X}"'] \arrow[dd, bend left, "\Psi_i^0"] \\
& X & \\
\mu_0^1(i) \arrow[ur, hook', bend right, "\mathcal{Z}_i^{1,X}"'] & & \mu_0^0(i). \arrow[ul, hook, bend left, "\mathcal{Z}_i^{0,X}"]
\end{tikzcd}$$

The totality $\mathtt{Map}_I(\boldsymbol{\Lambda}(X), \boldsymbol{M}(X))$ *of the family of complemented subsets-maps from* $\boldsymbol{\Lambda}(X)$ *to* $\boldsymbol{M}(X)$ *is equipped with the pointwise equality. If* $\boldsymbol{\Psi} \colon \boldsymbol{\Lambda}(X) \Rightarrow \boldsymbol{M}(X)$ *and if* $\boldsymbol{\Xi} \colon \boldsymbol{M}(X) \Rightarrow \boldsymbol{N}(X)$*, the composition family of complemented subsets-map* $\boldsymbol{\Xi} \circ \boldsymbol{\Psi} \colon \boldsymbol{\Lambda}(X) \Rightarrow \boldsymbol{N}(X)$ *is defined by* $\boldsymbol{\Xi} := \big((\Xi \circ \Psi)^1, (\Xi \circ \Psi)^0\big)$*, where* $(\Xi \circ \Psi)^1 := \Xi^1 \circ \Psi^1$ *and* $(\Xi \circ \Psi)^0 := \Xi^0 \circ \Psi^0$*. Moreover,* $\mathrm{Id}_{\boldsymbol{\Lambda}(X)} := (\mathrm{Id}_{\Lambda^1(X)}, \mathrm{Id}_{\Lambda^0(X)})$*, and the totality* $\mathtt{Fam}(I, \boldsymbol{X})$ *of families of complemented subsets of* X *over* I *is equipped with the equality* $\boldsymbol{\Lambda}(X) =_{\mathtt{Fam}(I,\boldsymbol{X})} \boldsymbol{M}(X)$ *if and only if*

$$\exists_{\boldsymbol{\Psi} \in \mathtt{Map}_I(\boldsymbol{\Lambda}(X), \boldsymbol{M}(X))} \exists_{\boldsymbol{\Xi} \in \mathtt{Map}_I(\boldsymbol{M}(X), \boldsymbol{\Lambda}(X))}\big(\boldsymbol{\Psi} \circ \boldsymbol{\Xi} = \mathrm{Id}_{\boldsymbol{M}(X)} \ \& \ \boldsymbol{\Xi} \circ \boldsymbol{\Psi} = \mathrm{Id}_{\boldsymbol{\Lambda}(X)}\big).$$

As in the case of $\mathtt{Fam}(I, X)$, we see no reason not to consider $\mathtt{Fam}(I, \boldsymbol{X})$ a set. Clearly, the obviously defined set $\mathtt{PrfEql}_0(\boldsymbol{\Lambda}(X), \boldsymbol{M}(X))$ is a subsingleton. A family $\boldsymbol{\Lambda}(X) \in \mathtt{Fam}(I, \boldsymbol{X})$ is in $\mathtt{Set}(I, \boldsymbol{X})$, if $\boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{][}(X)} \boldsymbol{\Lambda}_0(j) \Rightarrow i =_I j$, for every $i, j \in I$. Trivially, if $\Lambda^1(X) \in \mathtt{Set}(I, X)$, or if $\Lambda^0(X) \in \mathtt{Set}(I, X)$, then $\boldsymbol{\Lambda}(X) \in \mathtt{Set}(I, \boldsymbol{X})$. Clearly, if $\boldsymbol{\Lambda}(X) \in \mathtt{Set}(I, \boldsymbol{X})$ and $\boldsymbol{M}(X) \in \mathtt{Fam}(I, \boldsymbol{X})$ such that $\boldsymbol{M}(X) =_{\mathtt{Fam}(I,\boldsymbol{X})} \boldsymbol{\Lambda}(X)$, then $\boldsymbol{M}(X) \in \mathtt{Set}(I, \boldsymbol{X})$. The operations between complemented subsets induce new families of complemented subsets and family-maps between them. If $\boldsymbol{\Lambda}(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big)$ and $\boldsymbol{M}(X) := \big(\mu_0^1, \mathcal{Z}^{1,X}, \mu_1^1, \mu_0^0, \mathcal{Z}^{0,X}, \mu_1^0\big) \in \mathtt{Fam}(I, \boldsymbol{X})$, let the following new elements of $\mathtt{Fam}(I, \boldsymbol{X})$:

$$(-\boldsymbol{\Lambda})(X) := \big(\lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0, \lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1\big),$$

$$(\boldsymbol{\Lambda} \cap \boldsymbol{M})(X) := \big(\lambda_0^1 \cap \mu_0^1, \mathcal{E}^{1,X} \cap \mathcal{Z}^{1,X}, \lambda_1^1 \cap \mu_1^1, \lambda_0^0 \cup \mu_0^0, \mathcal{E}^{0,X} \cup \mathcal{Z}^{0,X}, \lambda_1^0 \cup \mu_1^0\big),$$

$$(\boldsymbol{\Lambda} \cup \boldsymbol{M})(X) := \big(\lambda_0^1 \cup \mu_0^1, \mathcal{E}^{1,X} \cup \mathcal{Z}^{1,X}, \lambda_1^1 \cup \mu_1^1, \lambda_0^0 \cap \mu_0^0, \mathcal{E}^{0,X} \cap \mathcal{Z}^{0,X}, \lambda_1^0 \cap \mu_1^0\big),$$

$$(\boldsymbol{\Lambda} - \boldsymbol{M})(X) := [\boldsymbol{\Lambda} \cap (-\boldsymbol{M})](X).$$

If $\boldsymbol{N}(Y) := (\nu_0^1, \mathcal{H}^{1,Y}, \nu_1^1, \nu_0^0, \mathcal{H}^{0,Y}, \nu_1^0) \in \mathtt{Fam}(I, \boldsymbol{Y})$ and $f\colon X \to Y$, then using Proposition 4.1.9 we define

$$f^{-1}(\boldsymbol{N})(X) := \big(f^{-1}(\nu_0^1), f^{-1}\big(\mathcal{H}^{1,Y}\big)^X, f^{-1}(\nu_1^1), f^{-1}(\nu_0^0), f^{-1}\big(\mathcal{H}^{0,Y}\big)^X, f^{-1}(\nu_1^0)\big) \in \mathtt{Fam}(I, \boldsymbol{X}),$$

$$f(\boldsymbol{\Lambda})(Y) := \big(f^{-1}(\lambda_0^1), f\big(\mathcal{E}^{1,X}\big)^Y, f(\lambda_1^1), f(\lambda_0^0), f\big(\mathcal{E}^{0,X}\big)^Y, f(\lambda_1^0)\big) \in \mathtt{Fam}(I, \boldsymbol{Y}).$$

Properties between complemented subsets induce equalities between their families e.g.,

$$\big[f^{-1}\big[(\boldsymbol{N} \cup \boldsymbol{K})(Y)\big]\big](X) =_{\mathtt{Fam}(I,\boldsymbol{X})} \big[f^{-1}(\boldsymbol{N})(Y) \cup f^{-1}(\boldsymbol{K})(Y)\big](X).$$

Using definitions from section 4.1, if $\boldsymbol{\Phi}\colon \boldsymbol{\Lambda}(X) \Rightarrow \boldsymbol{M}(X)$, let $-\boldsymbol{\Phi}\colon (-\boldsymbol{\Lambda})(X) \Rightarrow (-\boldsymbol{M})(X)$

$$-\boldsymbol{\Phi} := (\Phi^0, \Phi^1); \qquad \boldsymbol{\Phi} := (\Phi^1, \Phi^0).$$

If $\boldsymbol{\Psi}\colon \boldsymbol{P}(X) \Rightarrow \boldsymbol{Q}(X)$, then $\boldsymbol{\Phi} \cap \boldsymbol{\Psi}\colon (\boldsymbol{\Lambda} \cap \boldsymbol{P})(X) \Rightarrow (\boldsymbol{M} \cap \boldsymbol{R})(X)$, where

$$\boldsymbol{\Phi} \cap \boldsymbol{\Psi} := (\Phi^1 \cap \Psi^1, \Phi^0 \cup^0),$$

and $\boldsymbol{\Phi} \cup \boldsymbol{\Psi}\colon (\boldsymbol{\Lambda} \cup \boldsymbol{P})(X) \Rightarrow (\boldsymbol{M} \cup \boldsymbol{R})(X)$, where

$$\boldsymbol{\Phi} \cup \boldsymbol{\Psi} := (\Phi^1 \cup \Psi^1, \Phi^0 \cap^0),$$

and $\boldsymbol{\Phi} - \boldsymbol{\Psi}\colon (\boldsymbol{\Lambda} - \boldsymbol{P})(X) \Rightarrow (\boldsymbol{M} - \boldsymbol{R})(X)$, where $\boldsymbol{\Phi} - \boldsymbol{\Psi} := \boldsymbol{\Phi} \cap (-\boldsymbol{\Psi})$. If $\boldsymbol{S}(Y) \in \mathtt{Fam}(J, Y)$,

$$(\boldsymbol{\Lambda} \times \boldsymbol{S})(X \times Y) := \big(\lambda_0^1 \times s_0^1, \mathcal{E}^{1,X} \times \mathcal{S}^{1,Y}, \lambda_1^1 \times s_1^1, \lambda_0^0 \times s_0^0, \mathcal{E}^{0,X} \times \mathcal{S}^{0,Y}, \lambda_0^0 \times s_0^0\big) \in \mathtt{Fam}(I \times J, \boldsymbol{X} \times \boldsymbol{Y}),$$

$$(\boldsymbol{\lambda}_0 \times \boldsymbol{s}_0)(i, j) := \boldsymbol{\lambda}_0(i) \times \boldsymbol{s}_0(i).$$

If $\boldsymbol{\Xi}\colon \boldsymbol{S}(Y) \Rightarrow \boldsymbol{T}(Y)$, then $\boldsymbol{\Phi} \times \boldsymbol{\Xi}\colon (\boldsymbol{\Lambda} \times \boldsymbol{S})(X \times Y) \Rightarrow (\boldsymbol{M} \times \boldsymbol{T})(X \times Y)$, where

$$\boldsymbol{\Phi} \times \boldsymbol{\Xi} := \big(\Phi^1 \times \Xi^1, \Phi^0 \times \Xi^0\big).$$

Due to the above families of complemented subsets the following proposition is well-formulated.

Proposition 4.9.2. *Let* $\boldsymbol{\Lambda}(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big) \in \mathtt{Fam}(I, \boldsymbol{X})$, $i_0 \in I$, *and let*

$$\bigcup_{i \in I} \boldsymbol{\lambda}_0(i) := \left(\bigcup_{i \in I} \lambda_0^1(i), \bigcap_{i \in I} \lambda_0^0(i)\right) \quad \& \quad \bigcap_{i \in I} \boldsymbol{\lambda}_0(i) := \left(\bigcap_{i \in I} \lambda_0^1(i), \bigcup_{i \in I} \lambda_0^0(i)\right).$$

(i) $\bigcup_{i \in I} \boldsymbol{\lambda}_0(i), \bigcap_{i \in I} \boldsymbol{\lambda}_0(i) \in \mathcal{P}^{\rrbracket\llbracket}(X)$.
(ii) $-\bigcup_{i \in I} \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \bigcap_{i \in I} \big(-\boldsymbol{\lambda}_0(i)\big)$.
(iii) $-\bigcap_{i \in I} \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \bigcup_{i \in I} \big(-\boldsymbol{\lambda}_0(i)\big)$.
(iv) *If* $i \in I$, *then* $\boldsymbol{\lambda}_0(i) \subseteq \bigcup_{i \in I} \boldsymbol{\lambda}_0(i)$.
(v) *If* $\boldsymbol{A} \subseteq \boldsymbol{\lambda}_0(i)$, *for some* $i \in I$, *then* $\boldsymbol{A} \subseteq \bigcup_{i \in I} \boldsymbol{\lambda}_0(i)$.
(vi) *If* $\boldsymbol{\lambda}_0(i) \subseteq \boldsymbol{A}$, *for every* $i \in I$, *then* $\bigcup_{i \in I} \boldsymbol{\lambda}_0(i) \subseteq \boldsymbol{A}$.
(vii) *If* $\boldsymbol{\lambda}_0(i) \supseteq \boldsymbol{A}$, *for every* $i \in I$, *then* $\bigcap_{i \in I} \boldsymbol{\lambda}_0(i) \supseteq \boldsymbol{A}$.
(viii) *If* $\boldsymbol{M}(X) := \big(\mu_0^1, \mathcal{Z}^{1,X}, \mu_1^1, \mu_0^0, \mathcal{Z}^{0,X}, \mu_1^0\big) \in \mathtt{Fam}(I, \boldsymbol{Y})$ *and* $f\colon X \to Y$, *then*

$$f^{-1}\left(\bigcup_{i \in I} \boldsymbol{\mu}_0(i)\right) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \bigcup_{i \in I} f^{-1}(\boldsymbol{\mu}_0(i)) \quad \& \quad f^{-1}\left(\bigcap_{i \in I} \boldsymbol{\mu}_0(i)\right) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \bigcap_{i \in I} f^{-1}(\boldsymbol{\mu}_0(i)).$$

Proof. (i) We show the first membership only. If $(i,x) \in \bigcup_{i\in I} \lambda_0^1(i)$ and $\Phi \in \bigcap_{i\in I} \lambda_0^0(i)$, then $e_{\bigcup}^{\Lambda(X)}(i,x) := \mathcal{E}_i^X(x)$ and $e_{\bigcap}^{\Lambda(X)}(\Phi) := \mathcal{E}_{i_0}^X(\Phi_{i_0})$. Since $\mathcal{E}_i^X(\Phi_i) =_X \mathcal{E}_{i_0}^X(\Phi_{i_0})$ and $\lambda_0^1(i)][\lambda_0^0(i)$, we have that $\mathcal{E}_i^X(x) \neq_X \mathcal{E}_i^X(\Phi_i)$, and by the extensionality of $\neq_X$ we get $\mathcal{E}_i^X(x) \neq_X \mathcal{E}_{i_0}^X(\Phi_{i_0})$.
(ii) and (iii) are straightforward to show. For (iv) we need to show that $\lambda_0^1(i) \subseteq \bigcup_{i\in I} \lambda_0^1(i)$ and $\bigcap_{i\in I} \lambda_0^0(i) \subseteq \lambda_0^0(i)$, which follow from Propositions 4.2.8(ii) and 4.3.6(ii), respectively. Case (v) follows from (iv) and the transitivity of $\boldsymbol{A} \subseteq \boldsymbol{B}$.
(vi) If $\lambda_0^1(i) \subseteq A^1$, for every $i \in I$, then $\bigcup_{i\in I} \lambda_0^1(i) \subseteq A^1$, and if $A^0 \subseteq \lambda_0^0(i)$, for every $i \in I$, then $A^0 \subseteq \bigcap_{i\in I} \lambda_0^0(i)$. Case (vii) is shown similarly.
(viii) We show the first equality only. By Propositions 4.2.5 and 4.3.5 we have that

$$\begin{aligned}
f^{-1}\Big(\bigcup_{i\in I} \boldsymbol{\mu}_0(i)\Big) &:= \Big(f^{-1}\Big(\bigcup_{i\in I} \mu_0^1(i)\Big), f^{-1}\Big(\bigcap_{i\in I} \mu_0^0(i)\Big)\Big) \\
&=_{\mathcal{P}^{][}(X)} \Big(\bigcup_{i\in I} f^{-1}\big(\mu_0^1(i)\big), \bigcap_{i\in I} f^{-1}\big(\mu_0^0(i)\big)\Big) \\
&:= \bigcup_{i\in I} \Big(f^{-1}\big(\mu_0^1(i)\big), f^{-1}\big(\mu_0^0(i)\big)\Big) \\
&:= \bigcup_{i\in I} f^{-1}\big(\boldsymbol{\mu}_0(i)\big).
\end{aligned}$$

$\square$

Let $\boldsymbol{\Lambda}(X), \boldsymbol{M}(X)$ and $\Psi \colon \boldsymbol{\Lambda}(X) \Rightarrow \boldsymbol{M}(X)$. Since $\Psi^1 \colon \Lambda^1(X) \Rightarrow M^1(X)$ and $\Psi^0 \colon \Lambda^0(X) \Rightarrow M^0(X)$, the following maps between complemented subsets (see Definition 2.8.2) are defined

$$\bigcup \boldsymbol{\Psi} := \Big(\bigcup \Psi^1, \bigcap \Psi^0\Big) \colon \bigcup_{i\in I} \boldsymbol{\lambda}_0(i) \to \bigcup_{i\in I} \boldsymbol{\mu}_0(i),$$

$$\bigcap \boldsymbol{\Psi} := \Big(\bigcap \Psi^0, \bigcup \Psi^1\Big) \colon \bigcap_{i\in I} \boldsymbol{\lambda}_0(i) \to \bigcap_{i\in I} \boldsymbol{\mu}_0(i), \qquad \text{where}$$

$$\bigcup \Psi^1 \colon \bigcup_{i\in I} \lambda_0^1(i) \to \bigcup_{i\in I} \mu_0^1(i) \quad \& \quad \bigcap \Psi^0 \colon \bigcap_{i\in I} \lambda_0^0(i) \to \bigcap_{i\in I} \mu_0^0(i)$$

are defined according to Proposition 4.2.8(ii) and 4.3.6(ii).

Proposition 4.9.3. *Let $\boldsymbol{A} \in \mathcal{P}^{][}(X), \boldsymbol{B} \in \mathcal{P}^{][}(Y)$, $\boldsymbol{\Lambda}(X) \in \mathtt{Fam}(I, \boldsymbol{X})$ and $\boldsymbol{M}(Y) \in \mathtt{Fam}(J, \boldsymbol{Y})$. The following properties hold:*

$$\boldsymbol{A} \times \bigcup_{j\in J} \boldsymbol{\mu}_0(j) =_{\mathcal{P}^{][}(X\times Y)} \bigcup_{j\in J} (\boldsymbol{A} \times \boldsymbol{\mu}_0(j)),$$

$$\boldsymbol{A} \times \bigcap_{j\in J} \boldsymbol{\mu}_0(j) =_{\mathcal{P}^{][}(X\times Y)} \bigcap_{j\in J} (\boldsymbol{A} \times \boldsymbol{\mu}_0(j)),$$

$$\Big(\bigcup_{i\in I} \boldsymbol{\lambda}_0(i)\Big) \times \boldsymbol{B} =_{\mathcal{P}^{][}(X\times Y)} \bigcup_{i\in I} (\boldsymbol{\lambda}_0(i) \times \boldsymbol{B}),$$

$$\Big(\bigcap_{i\in I} \boldsymbol{\lambda}_0(i)\Big) \times \boldsymbol{B} =_{\mathcal{P}^{][}(X\times Y)} \bigcap_{i\in I} (\boldsymbol{\lambda}_0(i) \times \boldsymbol{B}).$$

Proof. We show the first equality, and for the rest we proceed similarly. By the equalities shown after Propositions 4.4.2 we have that

$$
\begin{aligned}
\boldsymbol{A} \times \bigcup_{j \in J} \boldsymbol{\mu}_0(j) &:= \bigg(A^1 \times \bigcup_{j \in J} \mu_0^1(j), (A^0 \times Y) \cup \Big[X \times \Big(\bigcap_{j \in J} \mu_0^0(j)\Big)\Big]\bigg) \\
&=_{\mathcal{P}\mathbb{I}\mathbb{I}(X \times Y)} \bigg(\bigcup_{j \in J} (A^1 \times \mu_0^1(j)), (A^0 \times Y) \cup \Big(\bigcap_{j \in J} (X \times \mu_0^0(j))\Big)\bigg) \\
&=_{\mathcal{P}\mathbb{I}\mathbb{I}(X \times Y)} \bigg(\bigcup_{j \in J} (A^1 \times \mu_0^1(j)), \bigcap_{j \in J} \big[(A^0 \times Y) \cup (X \times \mu_0^0(j))\big]\bigg) \\
&:= \bigcup_{j \in J} (\boldsymbol{A} \times \boldsymbol{\mu}_0(j)).
\end{aligned}
$$

□

4.10 Direct families of subsets

Definition 4.10.1. *Let $(I, \preccurlyeq)$ be a directed set, and $X \in \mathbb{V}_0$. A (covariant) direct family of subsets of X indexed by I, or an $(I, \preccurlyeq)$-family of subsets of X, is a triplet $\Lambda^{\preccurlyeq}(X) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\mathcal{E}^X$ is a modulus of embeddings for λ_0 (see Definition 4.1.1)*

$$\lambda_1^{\preccurlyeq} : \bigcurlywedge_{(i,j) \in D^{\preccurlyeq}(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1^{\preccurlyeq}(i,j) := \lambda_{ij}^{\preccurlyeq}, \quad (i,j) \in D^{\preccurlyeq}(I),$$

a modulus of covariant transport maps for λ_0, such that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$, for every $i \in I$, and, for every $(i,j) \in D^{\preccurlyeq}(I)$, the following left diagram commutes

$$
\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\lambda_{ij}^{\preccurlyeq}"] \arrow[dr, hook, "\mathcal{E}_i"'] & & \lambda_0(j) \arrow[dl, hook', "\mathcal{E}_j"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(i) \arrow[dr, hook, "\mathcal{E}_i"'] & & \lambda_0(j) \arrow[ll, hook', "\lambda_{ji}^{\succcurlyeq}"'] \arrow[dl, hook', "\mathcal{E}_j"] \\
& X. &
\end{tikzcd}
$$

A contravariant $(I, \succcurlyeq)$-family of subsets of X is defined dually i.e.,

$$\lambda_1^{\succcurlyeq} : \bigcurlywedge_{(i,j) \in \preccurlyeq(I)} \mathbb{F}\big(\lambda_0(j), \lambda_0(i)\big), \quad \lambda_1^{\succcurlyeq}(i,j) := \lambda_{ji}^{\succcurlyeq}, \quad (i,j) \in D^{\preccurlyeq}(I),$$

is a modulus of contravariant transport maps for λ_0, such that for every $(i,j) \in D^{\preccurlyeq}(I)$, the above right diagram commutes.

Proposition 4.10.2. *Let $X \in \mathbb{V}_0$, $(I, \preccurlyeq_I)$ a directed set, $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\mathcal{E}^X$ a modulus of embeddings for λ_0, and λ_1 a modulus of transport maps for λ_0. The following are equivalent.*
(i) $\Lambda^{\succcurlyeq}(X) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq})$ *is an $(I, \preccurlyeq_I)$-family of subsets of X.*
(ii) $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I, \preccurlyeq_I)$ *and $\mathcal{E}^X : \Lambda^{\preccurlyeq} \Rightarrow C^{\preccurlyeq, X}$, where $C^{\preccurlyeq, X}$ is the constant $(I, \preccurlyeq_I)$-family X (see Definition 3.8.1).*

Proof. We proceed exactly as in the proof of Proposition 4.1.2. □

If $\Lambda^{\preccurlyeq} := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq})$ is an $(I, \preccurlyeq_I)$-family of subsets of X, and if $i \preccurlyeq_I j$, then $\lambda_{ij}^{\preccurlyeq} \colon \lambda_0(i) \subseteq \lambda_0(j)$ i.e., $\lambda_1^{\preccurlyeq}$ is a *modulus of subset-witnesses* for λ_0.

Definition 4.10.3. *If $\Lambda^{\preccurlyeq}(X) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq}) and M^{\preccurlyeq}(X) := (\mu_0, \mathcal{Z}^X, \mu_1^{\preccurlyeq})$ are $(I, \preccurlyeq_I)$-families of subsets of X, a direct family of subsets-map $\Psi \colon \Lambda^{\preccurlyeq}(X) \Rightarrow M^{\preccurlyeq}(X)$ from $\Lambda^{\preccurlyeq}(X)$ to $M^{\preccurlyeq}(X)$ is a family of subsets-map $\Phi \colon \Lambda(X) \Rightarrow M(X)$. Their set $\mathtt{Map}_{(I,\preccurlyeq_I)}(\Lambda^{\preccurlyeq}(X), M^{\preccurlyeq}(X))$ is the set $\mathtt{Map}_I(\Lambda(X), M(X))$. The composition of direct family of subsets-maps, and the totality $\mathtt{Fam}(I, \preccurlyeq_I, X)$ of $(I, \preccurlyeq_I)$-families of subsets of X are defined as the composition of family of subsets-maps, and as the totality $\mathtt{Fam}(I, X)$, respectively. The totality $\mathtt{Fam}(I, \succcurlyeq, X)$ of contravariant direct families of subsets of X over (I, lt_I) and the corresponding family-maps are defined similarly.*

Proposition 4.10.4. *Let $\Lambda^{\preccurlyeq}(X) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq}), M(X) := (\mu_0, \mathcal{Z}^X, \mu_1^{\preccurlyeq}) \in \mathtt{Fam}(I, \preccurlyeq_I, X)$.*

(i) *If $\Psi \colon \Lambda^{\preccurlyeq}(X) \Rightarrow M^{\preccurlyeq}(X)$, then $\Psi \colon \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$.*

(ii) *If $\Psi \colon \Lambda^{\preccurlyeq}(X) \Rightarrow M^{\preccurlyeq}(X)$ and $\Phi \colon \Lambda^{\preccurlyeq}(X) \Rightarrow M^{\preccurlyeq}(X)$, then $\Phi =_{\mathtt{Map}_{(I,\preccurlyeq_I)}(\Lambda^{\preccurlyeq}(X), M^{\preccurlyeq}(X))} \Psi$.*

Proof. We proceed exactly as in the proof of Proposition 4.1.6 □

The interior union and intersection of $\Lambda^{\preccurlyeq}(X)(\Lambda^{\succcurlyeq}(M))$, are defined as for an I-family of subsets $\Lambda(X)$. As in the case of $\sum_{i \in I} \lambda_0(i)$ and $\bigcup_{i \in I} \lambda_0(i)$, the equality of $\bigcup_{i \in I} \lambda_0(i)$ does not imply the externally defined equality of $\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)$, only the converse is true i.e.,

$$(i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (j, y) \Rightarrow (i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y),$$

as, if there is some $k \in I$ such that $i \preccurlyeq_I k, j \preccurlyeq_I k$, and $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)$, then by the equalities $\mathcal{E}_i = \mathcal{E}_k \circ \lambda_{ik}^{\preccurlyeq}$ and $\mathcal{E}_j = \mathcal{E}_k \circ \lambda_{jk}^{\preccurlyeq}$ we get $\mathcal{E}_i(x) = \mathcal{E}_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) = \mathcal{E}_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) = \mathcal{E}_j(y)$.

4.11 Notes

Note 4.11.1. The definition of a family of subsets given by Bishop in [9], p. 65, was the rough description we gave at the beginning of this chapter. Our definition 4.1.1 highlights the witnessing data of the rough description, and it is in complete analogy to Richman's definition of a set-indexed family of sets, included later by Bishop and Bridges in [19], p. 78. In [19], p. 80, and in [9], p. 65, an alternative definition of a family of subsets of X indexed by I is given, as a subset Λ of $X \times I$. The fact that $(x, i) \in \Lambda$ can be interpreted as $x \in \lambda_0(i)$ This definition though, which was never used by Bishop, does not reveal the witnessing data for the equality $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$, if $i =_I j$, and it is not possible to connect with the notion of a family of sets. The definition of a set of subsets is given by Bishop in [9], p. 65, and it is repeated in [19], p. 69. The example of the set of detachable subsets of a set is given in [9], p. 65, where the term *free* subsets is used instead, and it is repeated in [19], p. 70.

Note 4.11.2. There are many examples of families of subsets in the literature of Bishop-style constructive mathematics. In topology a neighborhood space (in [19], p. 75, the reference to the indices is omitted for simplicity) is a pair (X, N), where X is a set and N is a family ν of subsets of X indexed by some set I such that

$$\forall_{i,j \in I} \forall_{x \in X} \big(x \in \nu(i) \cap \nu(j) \Rightarrow \exists_{k \in I} \big(x \in \nu(k) \subseteq \nu(i) \cap \nu(j)\big)\big).$$

The covering property is not mentioned there. If (X, F) is a Bishop space (see [19], chapter 3, and [88]), the neighborhood structure N_F on X generated by the Bishop topology F on X is the family U of subsets of X indexed by F that assigns to every element $f \in F$ the set

$$U(f) := \{x \in X \mid f(x) > 0\}.$$

If $f = g$, then $U(f) = U(g)$, while the converse is not true (take e.g., $f = \mathrm{id}_{\mathbb{R}}$ and $g = 2\mathrm{id}_{\mathbb{R}}$, where $X = \mathbb{R}$ and $F = \mathrm{Bic}(\mathbb{R})$). In real analysis sequences of bounded intervals of $\mathbb{R}$ are considered in [19] Problem 1, p. 292. In the theory of normed linear spaces a sequence of bounded, located, open, convex sets is constructed in the proof of the separation theorem (see [19], pp. 336–340). A family N_t^*, for every $t > 0$, of subsets of the unit sphere of the dual space X^* of a separable normed space X occurs in the proof of Theorem (6.8) in [19], p. 354. In constructive algebra families of ideals and families of submodules of an R-module are studied (see [76], p. 44, and p. 53, respectively).

Note 4.11.3. In [19], p. 69, the interior union $\bigcup_{i\in I}\lambda_0(i)$ is defined as the totality

$$\bigcup_{i\in I}\lambda_0(i) := \big\{x \in X \mid \exists_{i\in I}\big(x \in \lambda_0(i)\big)\big\}.$$

Using our notation though, in [19], pp. 69–70 it is written that

> ... to construct an element u of $\bigcup_{i\in I}\lambda_0(i)$ we first construct an element i of I, and then construct an element x of $\lambda_0(i)$.

Clearly, what is meant by the totality $\bigcup_{i\in I}\lambda_0(i)$ is what is written in Definition 4.2.1. The intersection of an I-family λ of subsets of X is roughly defined in [9], p. 70, as

$$\bigcap_{i\in I}\lambda_0(i) := \big\{x \in X \mid \forall_{i\in I}\big(x \in \lambda_0(i)\big)\big\},$$

while the more precise definition that follows this simplified notation is different, and it is based on the undefined in [9] and [19] notion of a dependent operation over λ, hence it is not that precise. Moreover, the definition of $\prod_{i\in I}^{X}\lambda_0(i)$, given in [19], p. 70, as the set

$$\Big\{f : I \to \bigcup_{i\in I}\lambda_0(i) \mid \forall_{i\in I}\big(f(i) \in \lambda_0(i)\big)\Big\}$$

is not compatible with the precise definition of $\bigcup_{i\in I}\lambda_0(i)$, and it is not included in [9].

Note 4.11.4. One could have defined an I-family of disjoint subsets of X with respect to given inequalities $\neq_I$ and $\neq_X$ (Definition 4.2.1) by

$$\forall_{i,j\in I}\big(i \neq_I j \Rightarrow \neg\big(\lambda_0(i) \between \lambda_0(j)\big)\big), \qquad \text{or}$$

$$\forall_{i,j\in I}\big(\lambda_0(i) \between \lambda_0(j) \Rightarrow i =_I j\big).$$

The first definition is negativistic, while the second, which avoids $\neq_I$, is too strong.

Note 4.11.5. The classical proof of the extension theorem of coverings (Theorem 4.2.6) is based on the definition of the interior union as the set $\bigcup_{i\in I}\lambda_0(i) := \big\{x \in X \mid \exists_{i\in I}\big(x \in \lambda_0(i)\big)\big\}$. As a result, the required function $f\colon X \to Y$ is defined as follows: If $x \in \bigcup_{i\in I}\lambda_0(i)$, there is $i \in I$ such that $x \in \lambda_0(i)$. Then, one defines $f(x) = f_i(x)$, and shows that the value $f(x)$ does not depend on the choice of i (see [45], p. 13). The use of choice is avoided in our proof, because of the embedding $e\colon X \hookrightarrow \bigcup_{i\in I}\lambda_0(i)$. Theorem 4.2.6 is related to the notion of a *sheaf of sets*. The sheaf-property added to the notion of a presheaf is exactly the main condition of Theorem 4.2.6, where the covering of X is an open covering i.e., a covering of open subsets (see [53]).

Note 4.11.6. If $P(I)$ is a partition of I, such that $p_0(k) \neq \emptyset$, for every $k \in K$, and if

$$T := \prod_{k \in K} p_0(k),$$

then the converse inclusion to the semi-distributivity of $\bigcap$ over $\bigcup$ (Proposition 4.5.5) holds classically, and the distributivity of $\bigcap$ over $\bigcup$ holds classically. The converse inclusion to the semi-distributivity of $\bigcap$ over $\bigcup$ is equivalent to the axiom of choice (see [45], p. 25). It is expected that this converse inclusion is constructively provable only if non-trivial data are added to the hypotheses.

Note 4.11.7. In the hypothesis of Proposition 4.6.16 we need to suppose the existence of a modulus of centres of contraction to avoid choice in the definition of function g. Proposition 4.6.16 is our translation of Theorem 4.4.3 of book-HoTT into BST. In the formulation of Theorem 4.4.3 of [124] no modulus of centres of contraction is mentioned, as the type-theoretic axiom of choice is provable in MLTT.

Note 4.11.8. As an equivalence structure (X, R_X) is the analogue to the set $(X, =_X)$, one can equip (X, R_X) with an extensional relation I_X on $X \times Y$ satisfying the properties of an inequality. In this way the structure (X, R_X, I_X) becomes the equivalence relation-analogue to the set $(X, =_X, \neq_X)$.

Note 4.11.9. Examples of families of partial functions are found in the predicative reconstruction of the Bishop-Cheng measure theory in [129] and [102].

Note 4.11.10. There are many examples of families of complemented subsets in the literature of Bishop-style constructive mathematics. In the theory of normed linear spaces, sequences of complemented subsets occur in the formulation of the constructive version of Lebesgue's decomposition of measures (see [19], pp. 329–331), and in the formulation of the constructive Radon-Nikodym theorem (see [19], pp. 333–334). In the integration theory of [19], the sequences of integrable sets in an integrable space X (see [19], pp. 234–235) are families of subsets of X indexed by $\mathbb{N}$. Sequences of measurable sets are considered in [19], pp. 269–271. Moreover, a measure space (see [19], p. 282) is defined as a triplet (X, M, μ), where M is a set of complemented sets in an inhabited set X. In the definition of complete measure space in [19], pp. 288–289, the notion of a sequence of elements of M is also used.

Note 4.11.11. In the measure theory developed in [9] certain families $\mathfrak{F}$ (and subfamilies $\mathfrak{M}$ of $\mathfrak{F}$) of complemented subsets of some set X are considered in the definition of a measure space (see [9], p. 183). For the definition of a measure space found in [9], p. 183, Myhill writes in [80], p. 351, the following:

> The only one of the classical set-existence axioms (not counting choice) which is missing[3] is power set. Certainly there is no hint of this axiom in Bishop's book (except for $\mathfrak{F}$ on p. 183, *surely a slip*[4]), or for that matter anywhere in Brouwer's writings prior to 1974.

In our view, Myhill is wrong to believe first, that the use of family of $\mathfrak{F}$ requires the powerset axiom, and, second, that its use from Bishop is surely a slip. The notion of family of subsets

[3] He means from his system CST.

[4] Our emphasis.

does not imply the use of the powerset as a set, since a family of subsets is a certain assignment routine from I to $\mathbb{V}_0$ that behaves like a function, without being one. Moreover, it is not a slip, as it is repeatedly used by Bishop in the new measure theory, also found in [19], and by practicioners of Bishop-style constructive mathematics, like Bridges and Richman. It is not a coincidence that the notion of family of subsets is not a fundamental function-like object in Myhill's system CST.

Note 4.11.12. In [9], p. 68, the following properties of complemented subsets are mentioned

$$(\boldsymbol{A} \cup -\boldsymbol{A}) \cap \left(\boldsymbol{A} \cup \bigcap_{i \in I} \boldsymbol{\lambda}_0(i)\right) =_{\mathcal{P}^{\mathbb{II}}(X)} (\boldsymbol{A} \cup -\boldsymbol{A}) \cap \left[\bigcap_{i \in I} \big(\boldsymbol{A} \cup \boldsymbol{\lambda}_0(i)\big)\right],$$

$$(\boldsymbol{A} \cap -\boldsymbol{A}) \cup \left(\boldsymbol{A} \cap \bigcup_{i \in I} \boldsymbol{\lambda}_0(i)\right) =_{\mathcal{P}^{\mathbb{II}}(X)} (\boldsymbol{A} \cap -\boldsymbol{A}) \cup \left[\bigcup_{i \in I} \big(\boldsymbol{A} \cap \boldsymbol{\lambda}_0(i)\big)\right].$$

These equalities are the constructive analogue of the classical properties

$$\boldsymbol{A} \cup \bigcap_{i \in I} \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{\mathbb{II}}(X)} \bigcap_{i \in I} \big(\boldsymbol{A} \cup \boldsymbol{\lambda}_0(i)\big),$$

$$\boldsymbol{A} \cap \bigcup_{i \in I} \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{\mathbb{II}}(X)} \bigcup_{i \in I} \big(\boldsymbol{A} \cap \boldsymbol{\lambda}_0(i)\big).$$

Note 4.11.13. In [18], pp. 16–17, and in [19], p. 73, the join and meet of a countable family of complemented subsets are defined by

$$\bigvee_{n=1}^{\infty} \boldsymbol{\lambda}_0(n) := \left(\left[\bigcap_{n=1}^{\infty} \big(\lambda_0^1(n) \cup \lambda_0^0(n)\big)\right] \cap \left[\bigcup_{n=1}^{\infty} \lambda_0^1(n)\right], \ \bigcap_{n=1}^{\infty} \lambda_0^0(n)\right),$$

$$\bigwedge_{n=1}^{\infty} \boldsymbol{\lambda}_0(n) := \left(\bigcap_{n=1}^{\infty} \lambda_0^1(n), \ \left[\bigcap_{n=1}^{\infty} \big(\lambda_0^1(n) \cup \lambda_0^0(n)\big)\right] \cap \left[\bigcup_{n=1}^{\infty} \lambda_0^0(n)\right]\right).$$

These definitions can be generalised to arbitrary families of complemented subsets and properties similar to the ones shown for $\bigcup_{i \in I} \boldsymbol{\lambda}_0(i)$ and $\bigcap_{i \in I} \boldsymbol{\lambda}_0(i)$ hold.

Note 4.11.14. Set-relevant families of subsets over some set I, and set-relevant direct families of subsets over some directed set $(I, \preccurlyeq_I)$ can be studied in a way similar to set-relevant families of sets over I and set-relevant direct families of sets over $(I, \preccurlyeq_I)$ in section 3.9. As a consequence, a theory of generalised direct spectra of subspaces can be developed. Families of families of subsets of X can also be studied, in analogy to families of families of sets (see Section 3.10). As $\texttt{Fam}(I, X)$ is in $\mathbb{V}_0$, the families of families of subsets of X are defined in $\mathbb{V}_0$.

Chapter 5

Proof-relevance in BISH

A form of proof-relevance is added to BISH through BST, which is both separate from its standard mathematical part, and also expressible in it. The distinctive feature of MLTT is its proof-relevance, the fact that proof-objects are considered as "first-class citizens". The various kinds of moduli, like the moduli of uniform continuity, of convergence etc., which witness that a function is uniformly continuous, a sequence converges etc., form a trace of proof-relevance in BISH. We make explicit the algorithmic content of several constructive proofs by defining a BHK-interpretation of certain formulas of BISH within BST. We define the notion of a set with a proof-relevant equality and the notion of a Martin-Löf set, which translates the first level of the identity type of intensional MLTT. As a result, notions and facts from homotopy type theory are translated in BISH.

5.1 On the BHK-interpretation of BISH within BST

In the next naive definition of the BHK-interpretation of BISH the notion of proof is not understood in the proof-theoretic sense. Although we agree with Streicher in [122] that the term witness is better, we use the symbol $\texttt{Prf}(\phi)$ for traditional reasons. We could have used the symbol $\texttt{Evd}(\phi)$, or $\texttt{Wtn}(\phi)$ instead. We choose not to reduce the rule for $\phi \vee \psi$ to the rest ones, as for example is done in [5], p. 156. The rule for $\neg\phi$ is usually reduced to the rule for implication.

Definition 5.1.1 (Naive BHK-interpretation of BISH)**.** *Let ϕ, ψ be formulas in* BISH*, such that it is understood what it means "q is a proof* (*or witness, or evidence*) *of ϕ" and "r is a proof of ψ".*

($\wedge$) *A proof of $\phi \wedge \psi$ is a pair (p_0, p_1) such that p_0 is a proof of ϕ and p_1 is a proof of ψ.*

($\Rightarrow$) *A proof of $\phi \Rightarrow \psi$ is a rule r that associates to any proof p of ϕ a proof $r(p)$ of ψ.*

($\vee$) *A proof of $\phi \vee \psi$ is a pair (i, p_i), where if $i := 0$, then p_0 is a proof of ϕ, and if $i := 1$, then p_1 is a proof of ψ.*

($\bot$) *There is no proof of $\bot$.*

For the next two rules let $\phi(x)$ be a formula on a set X, such that it is understood what it means "q is a proof of $\phi(x)$", for every $x \in X$.

($\forall$) *A proof of $\forall_{x \in X}\phi(x)$ is a rule R that associates to any given $x \in X$ a proof R_x of $\phi(x)$.*

($\exists$) *A proof of $\exists_{x \in X}\phi(x)$ is a pair (x, q), where $x \in X$ and q is a proof of $\phi(x)$.*

The notions of rule in the rules ($\Rightarrow$) and ($\forall$) are unclear. The nature of a proof or a witness is also unclear. The interpretation of atomic formulas is also not included (see Note 5.7.1). A formal version of the above naive BHK-interpretation of BISH is a corresponding realisability interpretation (see Note 5.7.2).

Following Feferman [49], Beeson declared in [5], p. 158, that "the fundamental relation in constructive set theory is not membership but membership-with-evidence" (MwE). All examples given by Feferman are certain extensional subsets of some set X. In MLTT this kind of (MwE) is captured by the type $\sum_{x:A} P(x)$, where $P\colon A \to \mathcal{U}$ is a family of types over $A\colon \mathcal{U}$. Here we explain how we can talk about (MwE) for extensional subsets of some set X within BST, showing that BISH, as MLTT, is capable of expressing (MwE). As all known to us such examples are extensional subsets, we do not consider the notion of a completely presented set X^*, for *every* set X, as it is done in the formal systems T_0^* of Feferman in [49], and in Beeson's system found in [5]. In the system of [5] proof-relevance is even more stressed, as to any formula ϕ a formula $\mathtt{Prf}_\phi(p)$ is associated by a certain rule, expressing that "p proves formula ϕ". The resulting formal set theory though, is, in our opinion, not attractive. The problem of the totality of proofs being a definite preset, hence the problem of quantifying over it (see [5], p. 177) is solved by our "internal" treatment of MwE within BST. Consequently, questionable principles, like Beeson's "(MwE) is self-evident" (see [5], p. 159), are avoided.

Proposition 5.1.2 (Membership-with-Evidence I (MwE-I))**.** *Let X, Y be sets, and let $P(x)$ be a property on X of the form*

$$P(x) :\Leftrightarrow \exists_{p \in Y}\big(Q(x,p)\big),$$

where $Q(x,p)$ is an extensional property on $X \times Y$ i.e., $\big[x =_X x' \ \&\ p =_Y p' \ \&\ Q(x,p)\big] \Rightarrow Q(x',p')$, for every $x, x' \in X$ and every $p, p' \in Y$. Let $\mathtt{PrfMemb}_0^P : X \rightsquigarrow \mathbb{V}_0$, defined by

$$\mathtt{PrfMemb}_0^P(x) := \{p \in Y \mid Q(x,p)\},$$

for every $x \in X$, and let $\mathtt{PrfMemb}_1^P : \bigwedge_{(x,x') \in D(X)} \mathbb{F}\big(\mathtt{PrfMemb}_0^P(x), \mathtt{PrfMemb}_0^P(x')\big)$, where $\mathtt{PrfMemb}_{xx'}^P := \mathtt{PrfMemb}_1^P(x,x') : \mathtt{PrfMemb}_0^P(x) \to \mathtt{PrfMemb}_0^P(x')$ is defined by the identity map-rule $\mathtt{PrfMemb}_{xx'}^P(p) := p$, for every $p \in \mathtt{PrfMemb}_0^P(x)$ and every $(x,x') \in D(X)$.

(i) *The property $P(x)$ is extensional.*

(ii) *The pair $\mathtt{PrfMemb}^P := \big(\mathtt{PrfMemb}_0^P, \mathtt{PrfMemb}_1^P\big) \in \mathtt{Fam}(X)$.*

Proof. (i) Let $x =_X x'$ and $p \in Y$ such that $Q(x,p)$. Since $p =_Y p$, by the extensionality of Q we get $Q(x',p)$, and hence $P(x')$.
(ii) First we show that the dependent operation $\mathtt{PrfMemb}_1^P$ is well-defined. If $x =_X x'$ and $p \in \mathtt{PrfMemb}_0^P(x)$ i.e., $Q(x,p)$, by the extensionality of Q we get $Q(x',p)$. Clearly, the operation $\mathtt{PrfMemb}_{xx'}^P$ is a function. As $\mathtt{PrfMemb}_{xx'}^P$ is given by the identity map- rule, the properties of a family of sets for $\mathtt{PrfMemb}_1^P$ are trivially satisfied. $\square$

Actually, $\mathtt{PrfMemb}^P$ can be seen as a family of subsets of Y over X, but now we want to work externally, and not internally. For the previous result it suffices to suppose that Q is X-extensional i.e., $\big[x =_X x' \ \&\ Q(x,p)\big] \Rightarrow Q(x',p)$, for every $x, x' \in X$ and every $p \in Y$. Notice that the extensionality of P alone does not imply neither the X-extensionality of Q, nor the extensionality of Q, and it is not enough to define a function from $\mathtt{PrfMemb}_0^P(x)$ to $\mathtt{PrfMemb}_0^P(x')$. If X_P is the extensional subset of X generated by P, we write $p : x \in X_P :\Leftrightarrow Q(x,p)$. The following obvious generalisation of (MwE-I) is shown similarly.

Proposition 5.1.3 (Membership-with-Evidence II (MwE-II)). *Let X, Y, Z be sets, and let $R(x)$ be a property on X of the form*

$$R(x) :\Leftrightarrow \exists_{p \in Y} \exists_{q \in Z} \big(Q(x,p,q)\big),$$

where $Q(x,p,q)$ is an extensional property on $X \times Y \times Z$ i.e., $\big[x =_X x' \ \& \ p =_Y p' \ \& \ q =_Z q' \ \& \ Q(x,p,q)\big] \Rightarrow Q(x',p',q')$, for every $x, x' \in X$, $p, p' \in Y$, and every $q, q' \in Y$. Let $\mathtt{PrfMemb}_0^R : X \rightsquigarrow \mathbb{V}_0$, defined by the rule

$$\mathtt{PrfMemb}_0^R(x) := \{(p,q) \in Y \times Z \mid Q(x,p,q)\},$$

for every $x \in X$, and let $\mathtt{PrfMemb}_1^R : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\big(\mathtt{PrfMemb}_0^R(x), \mathtt{PrfMemb}_0^R(x')\big)$, where

$$\mathtt{PrfMemb}_{xx'}^R := \mathtt{PrfMemb}_1^R(x,x') : \mathtt{PrfMemb}_0^R(x) \to \mathtt{PrfMemb}_0^R(x',$$

$$\mathtt{PrfMemb}_{xx'}^R(p,q) := (p,q); \qquad (p,q) \in \mathtt{PrfMemb}_0^R(x), \ (x,x') \in D(X).$$

(i) *The property $R(x)$ is extensional.*

(ii) *The pair $\mathtt{PrfMemb}^R := \big(\mathtt{PrfMemb}_0^R, \mathtt{PrfMemb}_1^R\big) \in \mathtt{Fam}(X)$.*

Again, $\mathtt{PrfMemb}^R$ can be seen as a family of subsets of Y over X. If X_R is the extensional subset of X generated by R, we write

$$(p,q) : x \in X_R :\Leftrightarrow Q(x,p,q).$$

Clearly, the schema MwE-II can be generalised to a property $S(x)$ on X of the form

$$S(x) :\Leftrightarrow \exists_{p_1 \in X_1} \ldots \exists_{p_n \in X_n} \big(T(x, p_1, \ldots, p_n)\big),$$

for some extensional property $T(p_1, \ldots, p_n)$ on $X_1 \times \ldots \times X_n$. The following scheme of defining functions on extensional subsets of sets given by existential formulas is immediate to prove.

Proposition 5.1.4. *Let X, Y, X', Y' be sets, and let $P(x)$ and $P(x')$ properties on X and X', respectively, of the form*

$$P(x) :\Leftrightarrow \exists_{p \in Y}\big(Q(x,p)\big) \qquad \& \qquad P'(x') :\Leftrightarrow \exists_{p' \in Y'}\big(Q'(x',p')\big),$$

where $Q(x,p)$ and $Q'(x',p')$ are extensional properties on $X \times Y$, and on $X' \times Y'$, respectively.

(i) *Let $f \colon X \rightsquigarrow X'$ and $\Phi_f \colon \bigcurlywedge_{x \in X} \bigcurlywedge_{p \in \mathtt{PrfMemb}_0^P(x)} \mathtt{PrfMemb}_0^{P'}(f(x))$. Then the operation $f_{PP'} \colon X_P \rightsquigarrow X'_{P'}$, defined by the rule $X_P \ni x \mapsto f(x) \in X'_{P'}$, is well-defined. If f is a function, then $f_{PP'}$ is a function.*

(ii) *Let $g \colon X \rightsquigarrow X'$ and $\Phi_g \colon \bigcurlywedge_{x \in X} \mathtt{PrfMemb}_0^{P'}(g(x))$. Then the operation $g_{P'} \colon X \rightsquigarrow X'_{P'}$, defined by the rule $X \ni x \mapsto g(x) \in X'_{P'}$, is well-defined. If g is a function, then $g_{P'}$ is a function.*

The above results MwE-I and MwE-II are useful, when a mathematical concept is defined as a property on a given set, and not as an element of the set together with some extra data. E.g., in [19], p. 38, and in [9], p. 34, a function $f : [a,b] \to \mathbb{R}$ is called *continuous*, if there is a function $\omega_f \colon \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers, such that

$$\forall_{\varepsilon > 0} \forall_{x,y \in [a,b]} \big(|x - y| \leq \omega_f(\varepsilon) \Rightarrow |f(x) - f(y)| \leq \varepsilon\big) :\Leftrightarrow \omega_f \colon \mathtt{Cont}(f).$$

It is also mentioned that the function ω, the so-called *modulus of (uniform) continuity* of f is "an indispensable part of the definition of a continuous function". The same concept can be defined though, through a property on the set $\mathbb{F}([a,b])$, given by an existential formula i.e.,

$$\mathtt{Cont}(f) :\Leftrightarrow \exists_{\omega_f \in \mathbb{F}\left(\mathbb{R}^+,\mathbb{R}^+\right)}\big(\omega_f \colon \mathtt{Cont}(f)\big).$$

It is this kind of definition of a mathematical notion that facilitates the definition of a set of witnesses of some membership of an extensional subset of a set.

Example 5.1.5 (Convergent sequences at $x \in \mathbb{R}$)**.** Let $X := \mathbb{F}(\mathbb{N},\mathbb{R})$, $Y := \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$. If $x \in \mathbb{R}$, let, for every $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},\mathbb{R})$

$$\mathrm{Conv}_x\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \exists_{C\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}\big(C\colon x_n \xrightarrow{n} x\big),$$

$$C\colon x_n \xrightarrow{n} x :\Leftrightarrow \forall_{k\in\mathbb{N}^+}\forall_{n\geq C(k)}\left(|x_n - x| \leq \frac{1}{k}\right).$$

If $C\colon x_n \xrightarrow{n}$, we say that C is a modulus of convergence of $(x_n)_{n\in\mathbb{N}}$ at $x \in \mathbb{R}$.

By the compatibility of the operation $-$, the function $|.|$, and the relation $\leq$ with the equality of real numbers we get the extensionality of $Q_x\big((x_n)_{n\in\mathbb{N}}, C\big) :\Leftrightarrow C : x_n \xrightarrow{n} x$ on $\mathbb{F}(\mathbb{N},\mathbb{R}) \times \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$, as

$$\big[(x_n)_{n\in\mathbb{N}} =_{\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)} (y_n)_{n\in\mathbb{N}} \ \&\ C\colon x_n \xrightarrow{n} x\big] \Rightarrow C\colon y_n \xrightarrow{n} x.$$

By Proposition 5.1.2 $\mathtt{PrfMemb}^{\mathrm{Conv}_x} := \big(\mathtt{PrfMemb}_0^{\mathrm{Conv}_x}, \mathtt{PrfMemb}_1^{\mathrm{Conv}_x}\big) \in \mathtt{Fam}\big(\mathbb{F}(\mathbb{N},\mathbb{R})\big)$, where

$$\mathtt{PrfMemb}_0^{\mathrm{Conv}_x}\big((x_n)_{n\in\mathbb{N}}\big) := \big\{C \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid C\colon x_n \xrightarrow{n} x\big)\big\}.$$

Example 5.1.6 (Cauchy sequences)**.** Let $X := \mathbb{F}(\mathbb{N},\mathbb{R})$, $Y := \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$, and let

$$\mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \exists_{C\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}\big(C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big),$$

$$C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \forall_{k\in\mathbb{N}^+}\forall_{n,m\geq C(k)}\left(|x_n - x_m| \leq \frac{1}{k}\right),$$

for every $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},\mathbb{R})$. If $C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big)$, we say that C is a modulus of Cauchyness for $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},\mathbb{R})$.

The extensionality of $R\big((x_n)_{n\in\mathbb{N}}, C\big) :\Leftrightarrow \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big)$ on $\mathbb{F}(\mathbb{N},\mathbb{R}) \times \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$ follows as above. By Proposition 5.1.2 $\mathtt{PrfMemb}^{\mathrm{Cauchy}} := \big(\mathtt{PrfMemb}_0^{\mathrm{Cauchy}}, \mathtt{PrfMemb}_1^{\mathrm{Cauchy}}\big) \in \mathtt{Fam}\big(\mathbb{F}(\mathbb{N},\mathbb{R})\big)$, where

$$\mathtt{PrfMemb}_0^{\mathrm{Cauchy}}\big((x_n)_{n\in\mathbb{N}}\big) := \big\{C \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big)\big\}.$$

Example 5.1.7 (Convergent sequences)**.** Let $X := \mathbb{F}(\mathbb{N},\mathbb{R})$, $Y := \mathbb{R}$, $Z := \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$, and

$$\mathrm{Conv}\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \exists_{x\in\mathbb{R}}\exists_{C\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}\big((x,C)\colon \mathrm{Conv}\big((x_n)_{n\in\mathbb{N}}\big),$$

$$(x,C)\colon \mathrm{Conv}\big((x_n)_{n\in\mathbb{N}} :\Leftrightarrow \big(C\colon x_n \xrightarrow{n} x\big),$$

for every $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},\mathbb{R})$. If $(x,C)\colon \mathrm{Conv}\big((x_n)_{n\in\mathbb{N}}\big)$, we say that (x,C) is a modulus of convergence of $(x_n)_{n\in\mathbb{N}}$.

The extensionality of $S\big((x_n)_{n\in\mathbb{N}}, x, C\big) :\Leftrightarrow C\colon x_n \xrightarrow{n} x$ on $\mathbb{F}(\mathbb{N},\mathbb{R})\times\mathbb{R}\times\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$ follows from the compatibility of convergence with equality i.e.,

$$\big[(x_n)_{n\in\mathbb{N}} =_{\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)} (y_n)_{n\in\mathbb{N}} \ \&\ x =_{\mathbb{R}} y \ \&\ C\colon x_n \xrightarrow{n} x\big] \Rightarrow C\colon y_n \xrightarrow{n} y.$$

By Proposition 5.1.3 $\mathtt{PrfMemb}^{\mathrm{Conv}} := \big(\mathtt{PrfMemb}_0^{\mathrm{Conv}}, \mathtt{PrfMemb}_1^{\mathrm{Conv}}\big) \in \mathtt{Fam}(\mathbb{F}(\mathbb{N},\mathbb{R}))$, where

$$\mathtt{PrfMemb}_0^{\mathrm{Conv}}\big((x_n)_{n\in\mathbb{N}}\big) := \big\{(x,C)\in\mathbb{R}\times\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid (x,C)\colon \mathrm{Conv}\big((x_n)_{n\in\mathbb{N}}\big)\big\}.$$

Similar PrfMemb-sets can be defined for the set $C([a,b])$ of (uniformly) continuous real-valued functions on a compact interval $[a,b]$, and for the set $D([a,b])$ of (uniformly) differentiable functions on a compact interval $[a,b]$. In this framework the Riemann-integral is not a mapping $\int_a^b\colon C([a,b])\to\mathbb{R}$, given by the rule $f\mapsto\int_a^b f$. As the definition of $\int_a^b f$ depends on the modulus of continuity ω_f for f (see [19], pp. 51–52), the Riemman-integral is a dependent operation

$$\int_a^b\colon \bigcurlywedge_{f\in\mathbb{F}([a,b])} \mathbb{F}\Big(\mathtt{PrfMemb}_0^{\mathtt{Cont}(f)}, \mathbb{R}\Big).$$

The standard writing

$$\int_a^b f := \int_a^b (f,\omega_f)$$

expresses the independence of the integral from the choice of a modulus of continuity i.e.,

$$\int_a^b (f,\omega_f) =_{\mathbb{R}} \int_a^b (f,{\omega'}_f),$$

for every $\omega_f, {\omega'}_f \in \mathtt{PrfMemb}_0^{\mathtt{Cont}(f)}$, but it is not the accurate writing of a function from $C([a,b])$ to $\mathbb{R}$, only a notational convention, compatible with the classical one. The following obvious generalisation (MwE-III) of (MwE-II) to relations an a set given by an existential formula is shown similarly. A variation of (MwE-III) concerns relations on finitely many different sets.

Proposition 5.1.8 (Membership-with-Evidence III (MwE-III)). *Let X, Y, Z be sets, and let $S(x,y)$ be a relation on X of the form*

$$S(x,y) :\Leftrightarrow \exists_{p\in Y}\big(Q(x,y,p)\big),$$

where $Q(x,y,p)$ is an extensional property on $X\times X\times Y$. Let $\mathtt{PrfRel}_0^R : X\times X \rightsquigarrow \mathbb{V}_0$, where

$$\mathtt{PrfEql}_0^S(x,y) := \{p\in Y \mid Q(x,y,p)\},$$

for every $x\in X$, and let $\mathtt{PrfRel}_1^S\colon \bigcurlywedge_{((x,x'),(y,y'))\in D(X\times X)} \mathbb{F}\big(\mathtt{PrfRel}_1^S(x,x'), \mathtt{PrfRel}_1^S(x',y')\big)$, where $\mathtt{PrfRel}_1^S((x,x')(y,y'))\colon \mathtt{PrfRel}_1^S(x,x') \to \mathtt{PrfRel}_1^S(x',y')$ is defined by the identity map-rule $\big[\mathtt{PrfRel}_1^S(x,x')\big](p) := p$, for every $p\in\mathtt{PrfRel}_1^S(x,x')$.

(i) *The property $S(x,y)$ is extensional.*

(ii) *The pair $\big(\mathtt{PrfRel}_0^S, \mathtt{PrfRel}_1^S\big)\in\mathtt{Fam}(X\times X)$.*

The "extension" of the BHK-interpretation to what usually corresponds to atomic formulas like the equality formulas (see also the comment of Aczel and Rathjen in Note 5.7.1), is the first part of the following definition.

Definition 5.1.9 (BHK-interpretation of BISH in BST - Part I)**.** *Let a membership condition, like e.g., in Propositions 5.1.2 and 5.1.3. We define*

$$\texttt{Prf}(x \in X_P) := \texttt{PrfMemb}_0^P(x),$$

$$\texttt{Prf}(x \in X_R) := \texttt{PrfMemb}_0^R(x).$$

Let a relation $S(x, y)$ on a set X, as e.g., in Proposition 5.1.8. We define

$$\texttt{Prf}\big(S(x, y)\big) := \texttt{PrfRel}_0^S(x, y).$$

Let ϕ, ψ be formulas in BISH *such that $\texttt{Prf}(\phi)$ and $\texttt{Prf}(\psi)$ are already defined. We define*

$$\texttt{Prf}(\phi \ \& \ \psi) := \texttt{Prf}(\phi) \times \texttt{Prf}(\psi),$$

$$\texttt{Prf}(\phi \vee \psi) := \texttt{Prf}(\phi) + \texttt{Prf}(\psi),$$

$$\texttt{Prf}(\phi \Rightarrow \psi) := \mathbb{F}\big(\texttt{Prf}(\phi), \texttt{Prf}(\psi)\big).$$

Let $\phi(x)$ be a formula on a set X, and let $\texttt{Prf}^\phi := \big(\texttt{Prf}_0^\phi, \texttt{Prf}_1^\phi\big) \in \texttt{Fam}(X)$, where $\texttt{Prf}_0^\phi \colon X \rightsquigarrow \mathbb{V}_0$ is given by the rule $x \mapsto \texttt{Prf}_0^\phi(x) := \texttt{Prf}\big(\phi(x)\big)$, for every $x \in X$. The $\texttt{Prf}$-sets of the formulas $\forall_{x \in X}\phi(x)$ and $\exists_{x \in X}\phi(x)$ with respect to the given family $\texttt{Prf}^\phi$, where $\exists_{x \in X}\phi(x)$ is a formula that does not express a membership condition or a relation, are defined by

$$\texttt{Prf}\bigg(\forall_{x \in X}\phi(x)\bigg) := \prod_{x \in X} \texttt{Prf}_0^\phi(x) := \prod_{x \in X} \texttt{Prf}\big(\phi(x)\big),$$

$$\texttt{Prf}\bigg(\exists_{x \in X}\phi(x)\bigg) := \sum_{x \in X} \texttt{Prf}_0^\phi(x) := \sum_{x \in X} \texttt{Prf}\big(\phi(x)\big).$$

Due to the definition of the coproduct of two sets in Definition 3.2.1, and because of Remark 3.3.3(i), the definitions of the $\texttt{Prf}$-set for $\exists_{x \in X}\phi(x)$ and for $\forall_{x \in X}\phi(x)$ are generalisations of the definitions for $\phi \vee \psi$ and for $\phi \ \& \ \psi$, respectively.

Example 5.1.10. Let the following proposition: if $(x_n)_{n \in \mathbb{N}^+} \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$ and $x_0 \in \mathbb{R}$, then

$$x_n \xrightarrow{n} x_0 \Rightarrow (x_n)_{n \in \mathbb{N}^+} \text{ is Cauchy.}$$

If $\chi(x_n, x_0)$ is the above implication, then $\chi(x_n, x_0)$ of the form $\phi(x_n, x_0) \Rightarrow \psi(x_n)$. Its proof (see [19], p. 29) can be seen as a rule that sends a modulus of convergence $C \colon x_n \xrightarrow{n} x_0$ of $(x_n)_{n \in \mathbb{N}^+}$ at x_0 to a modulus of Cauchyness $D \colon \mathrm{Cauchy}\big((x_n)_{n \in \mathbb{N}^+}\big)$ for $(x_n)_{n \in \mathbb{N}^+}$, where $D(k) := C(2k)$, for every $k \in \mathbb{N}^+$. This operation from $\texttt{PrfMemb}_0^{\mathrm{Conv}_{x_0}}\big((x_n)_{n \in \mathbb{N}^+}\big)$ to $\texttt{PrfMemb}_0^{\mathrm{Cauchy}}\big((x_n)_{n \in \mathbb{N}^+}\big)$ is a function, and

$$\texttt{Prf}(\chi(x_n, x_0)) := \mathbb{F}\Bigg(\texttt{Prf}\big(\phi(x_n, x_0)\big), \texttt{Prf}\big(\psi(x_n)\big)\Bigg),$$

$$\texttt{Prf}\big(\phi(x_n, x_0)\big) := \texttt{PrfMemb}_0^{\mathrm{Conv}_{x_0}}\big((x_n)_{n \in \mathbb{N}^+}\big),$$

$$\texttt{Prf}\big(\psi(x_n)\big) := \texttt{PrfMemb}_0^{\mathrm{Cauchy}}\big((x_n)_{n \in \mathbb{N}^+}\big).$$

Example 5.1.11. Let the following proposition: if $x_0 \in \mathbb{R}$, then

$$\forall_{(x_n)_{n\in\mathbb{N}^+} \in \mathbb{F}(\mathbb{N}^+,\mathbb{R})}\big(x_n \xrightarrow{n} x_0 \Rightarrow (x_n)_{n\in\mathbb{N}^+} \text{ is Cauchy }\big).$$

The formula corresponding to this proposition is

$$\chi^*(x_0) :\Leftrightarrow \forall_{x_n \in \mathbb{F}(\mathbb{N}^+,\mathbb{R})}\chi(x_n, x_0),$$

where the $\mathtt{Prf}$-set of $\chi(x_n, x_0) :\Leftrightarrow \big(\phi(x_n, x_0) \Rightarrow \psi(x_n)\big)$ is determined in the previous example. To determine the $\mathtt{Prf}$-set of $\chi^*(x_0)$ we need to determine first a family of $\mathtt{Prf}$-sets over $\mathbb{F}(\mathbb{N}^+, \mathbb{R})$. Using Definition 3.1.6(ii), let

$$\mathtt{Prf}^{\chi^*(x_0)} := \mathbb{F}\big(\mathtt{Prf}^{\phi(x_n,x_0)}, \mathtt{Prf}^{\psi(x_n)}\big),$$

and by Definition 5.1.9, we get

$$\mathtt{Prf}\big(\chi^*(x_0)\big) := \prod_{x_n \in \mathbb{F}(\mathbb{N}^+,\mathbb{R})} \mathtt{Prf}\big(\chi(x_n, x_0)\big).$$

Example 5.1.12. Let the following proposition: if $(x_n)_{n\in\mathbb{N}^+} \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$, then

$$(x_n)_{n\in\mathbb{N}^+} \text{ is Cauchy} \Rightarrow \exists_{y\in\mathbb{R}}\big(x_n \xrightarrow{n} y\big).$$

The formula corresponding to this proposition is

$$\theta(x_n) :\Leftrightarrow \big[\psi(x_n) \Rightarrow \exists_{y\in Y}\big(\phi(x_n, y)\big)\big].$$

Its proof generates a rule that associates to every $C : \text{Cauchy}\big((x_n)_{n\in\mathbb{N}^+}\big)$ a pair (y, D), where $y \in \mathbb{R}$ and $D\colon x_n \xrightarrow{n} y$, and y is defined by the rule $y_k := \big[x_{D(k)}\big]_{2k}$, and $D(k) := 3k \vee C(2k)$, for every $k \in \mathbb{N}^+$. The use of the modulus of Cauchyness in the definition of a Cauchy sequence is responsible for the avoidance of choice in the proof. Clearly, the rule $C \mapsto (y, D)$ of the proof of $\theta(x_n)$ determines a function from $\mathtt{Prf}(\psi(x_n))$ to the $\mathtt{Prf}$-set of the formula $\exists_{y\in\mathbb{R}}\phi(x_n, y)$. Since $\mathtt{Prf}(\phi(x_n, y)$ is already determined above, and as a corresponding family over $\mathbb{F}(\mathbb{N}^+, \mathbb{R})$ is determined in Example 5.1.5, then, using Definition 3.5.2(iii), from Definition 5.1.9 we get

$$\mathtt{Prf}(\theta(x_n)) := \sum_{y\in\mathbb{R}} \mathtt{PrfMemb}^{\mathrm{Conv}_y}(x_n).$$

From the last two examples, we see how the schemes of defining new families of sets from given ones that were established in Chapter 3 can be used in order to define canonical families of $\mathtt{Prf}$-sets from given such families. These canonical families of $\mathtt{Prf}$-sets are determined in the second part of our definition of the BHK-interpretation of BISH within BST. As we have already seen in the previous two examples, the following extension of Definition 5.1.9 refers to Definitions 3.1.6 and 3.5.2.

Definition 5.1.13 (BHK-interpretation of BISH in BST - Part II)**.** *Let X, Y be sets. Let $\phi_1(x), \phi_2(x)$ be formulas in* BISH *such that* $\mathtt{Prf}^{\phi_1} := \big(\mathtt{Prf}_0^{\phi_1}, \mathtt{Prf}_1^{\phi_1}\big) \in \mathtt{Fam}(X)$ *and* $\mathtt{Prf}^{\phi_2} := \big(\mathtt{Prf}_0^{\phi_2}, \mathtt{Prf}_1^{\phi_2}\big) \in \mathtt{Fam}(X)$ *are given. To the formulas*

$$(\phi_1 \ \&\ \phi_2)(x) :\Leftrightarrow \phi_1(x) \ \&\ \phi_2(x),$$

$$(\phi_1 \Rightarrow \phi_2)(x) :\Leftrightarrow \phi_1(x) \Rightarrow \phi_2(x),$$

$$(\phi_1 \vee \phi_2)(x) :\Leftrightarrow \phi_1(x) \vee \phi_2(x),$$

on X we associate in a canonical way the following families of sets over X, respectively:

$$\mathtt{Prf}^{\phi_1 \& \phi_2} := \mathtt{Prf}^{\phi_1} \times \mathtt{Prf}^{\phi_2},$$

$$\mathtt{Prf}^{\phi_1 \Rightarrow \phi_2} := \mathbb{F}\big(\mathtt{Prf}^{\phi_1}, \mathtt{Prf}^{\phi_2}\big),$$

$$\mathtt{Prf}^{\phi_1 \vee \phi_2} := \mathtt{Prf}^{\phi_1} + \mathtt{Prf}^{\phi_2}.$$

Let $\theta(x,y)$ be a formula on $X \times Y$ and $\mathtt{Prf}^{\theta} := \big(\mathtt{Prf}^{\theta}_0, \mathtt{Prf}^{\theta}_1\big) \in \mathtt{Fam}(X \times Y)$ i.e., $\mathtt{Prf}^{\theta}_0 : X \times Y \rightsquigarrow \mathbb{V}_0$, with $(x,y) \mapsto \mathtt{Prf}^{\theta}_0(x,y) := \mathtt{Prf}\big(\theta(x,y)\big)$, for every $(x,y) \in X \times Y$. To the formulas

$$\big(\forall_y \theta\big)(x) :\Leftrightarrow \forall_{y \in Y} \theta(x,y),$$

$$\big(\exists_y \theta\big)(x) :\Leftrightarrow \exists_{y \in Y} \theta(x,y),$$

on X we associate in a canonical way the following families of sets over X, respectively:

$$\mathtt{Prf}^{\forall_y \theta} := \prod^{1} \mathtt{Prf}^{\theta},$$

$$\mathtt{Prf}^{\exists_y \theta} := \sum^{1} \mathtt{Prf}^{\theta}.$$

By Definitions 3.1.6 and 3.5.2 we get

$$\mathtt{Prf}^{\forall_y \theta} := \bigg(\prod^{1} \mathtt{Prf}^{\theta}_0, \prod^{1} \mathtt{Prf}^{\theta}_1\bigg),$$

$$\bigg(\prod^{1} \mathtt{Prf}^{\theta}_0\bigg)(x) := \prod_{y \in Y} \mathtt{Prf}^{\theta}_0(x,y) := \prod_{y \in Y} \mathtt{Prf}\big(\theta(x,y)\big),$$

$$\mathtt{Prf}^{\exists_y \theta} := \bigg(\sum^{1} \mathtt{Prf}^{\theta}_0, \sum^{1} \mathtt{Prf}^{\theta}_1\bigg),$$

$$\bigg(\sum^{1} \mathtt{Prf}^{\theta}_0\bigg)(x) := \sum_{y \in Y} \mathtt{Prf}^{\theta}_0(x,y) := \sum_{y \in Y} \mathtt{Prf}\big(\theta(x,y)\big).$$

5.2 Examples of totalities with a proof-relevant equality

So far we have seen many examples of totalities equipped with an equality defined through an existential formula. The universe $\mathbb{V}_0$, the powerset $\mathcal{P}(X)$ of a set X, the impredicative set $\mathtt{Fam}(I)$ of families of sets indexed by I, the set $\mathtt{Fam}(I,X)$ of families of subsets of X indexed by I, and all other sets of set-indexed families of subsets examined in Chapter 4. Next we describe some more motivating examples.

Example 5.2.1 (The Richman ordinals)**.** The equality on the totality of Richman ordinals, as this is defined in [76], pp. 24-28, behaves similarly to the equality on the powerset. Notice that the following definition of a well-founded relation is impredicative, as it requires quantification over the powerset of a set. If $<$ is a binary relation on a set W, a subset H of W is called hereditary, if

$$\forall_{w \in W}\bigg(\forall_{u \in W}\big(u < w \Rightarrow u \in H\big) \Rightarrow w \in H\bigg).$$

The relation $<$ is well-founded if

$$\forall_{H \in \mathcal{P}(X)}\big(H \text{ is hereditary } \Rightarrow H = W\big).$$

A *Richman ordinal* is a pair $(\alpha, \leq)$, where α is a discrete set, $\leq$ is a linear order (i.e., $x \leq y \vee y \leq x$, for every $x, y \in \alpha$), and $<$ is well founded, where $x < y :\Leftrightarrow x \leq y \ \&\ x \neq_\alpha y$. If α, β are ordinals, an *injection* $\rho : \alpha \leq \beta$ from α to β is a function $\rho : \alpha \to \beta$ such that

(i) $\forall_{x,y \in \alpha}\big(x < y \Rightarrow \rho(x) < \rho(y)\big)$.

(ii) $\forall_{z \in \beta}\forall_{y \in \alpha}\big(z < \rho(y) \Rightarrow \exists_{x \in \alpha}(\rho(x) =_\beta z)\big)$.

In this case we write $\alpha \leq \beta$. In [76], p. 28, it is shown that there is at most one injection from α to β. If Ord_R is the totality (class) of Richman ordinals, then in analogy to Proposition 2.6.2 we have the following.

Proposition 5.2.2. *If $\alpha, \beta \in \mathrm{Ord}_R$, $\rho : \alpha \leq \beta$, and $\sigma : \alpha \leq \beta$, then ρ is an embedding, and $\rho =_{\mathbb{F}(\alpha,\beta)} \sigma$.*

Proof. Let $x, y \in \alpha$ such that $\rho(x) =_\beta \rho(y)$. If $x \neq_\alpha y$, by the linearity of $\leq$ either $x \leq y$ or $y \leq x$. In the first case we get $x < y$, hence $\rho(x) < \rho(y)$, and in the second we get $y < x$, hence $\rho(y) < \rho(x)$ i.e., in both cases we get a contradiction. Hence, $x =_\alpha y$. For the rest, one shows that the set $H := \{x \in \alpha \mid \rho(x) =_\beta \sigma(x)\}$ is hereditary (see [76], p. 28). $\square$

As in the case of $\mathcal{P}(X)$, we define $\alpha =_{\mathrm{Ord}_R} \beta :\Leftrightarrow \alpha \leq \beta \ \&\ \beta \leq \alpha$, and

$$\texttt{PrfEql}_0(\alpha, \beta) := \{(\rho, \sigma) \in \mathbb{F}(\alpha, \beta) \times \mathbb{F}(\beta, \alpha) \mid \rho : \alpha \leq \beta \ \&\ \sigma : \beta \leq \alpha\}.$$

Since the composition of injections is an injection, let

$$\texttt{refl}(\alpha) := \big(\mathrm{id}_\alpha, \mathrm{id}_\alpha\big) \ \ \&\ \ (\rho, \sigma)^{-1} := (\sigma, \rho) \ \ \&\ \ (\rho, \sigma) * (\tau, v) := (\tau \circ \rho, \sigma \circ v),$$

and the groupoid properties for $\texttt{PrfEql}_0(\alpha, \beta)$ hold trivially by the equality of all its elements.

Example 5.2.3 (The direct sum of a direct family of sets)**.** If $\Lambda^{\preccurlyeq} := \big(\lambda_0, \lambda_1^{\preccurlyeq}\big) \in \texttt{Fam}(I, \preccurlyeq_I)$, and if $i, x), (j, y) \in \sum_{i \in I}^{\preccurlyeq} \lambda_0(i)$, and since by Definition 3.8.2

$$(i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (j, y) :\Leftrightarrow \exists_{k \in I}\big(i \preccurlyeq_I k \ \&\ j \preccurlyeq_I k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big),$$

let

$$\texttt{PrfEql}_0\big((i, x), (j, y)\big) := \big\{m \in I_{ij} \mid \lambda_{im}^{\preccurlyeq}(x) =_{\lambda_0(m)} \lambda_{jm}^{\preccurlyeq}(y)\big\},$$
$$I_{ij} := \{k \in I \mid i \preccurlyeq_I k \ \&\ j \preccurlyeq_I k\}.$$

To show the extensionality of $\texttt{PrfEql}_0\big((i, x), (j, y)\big)$, let $m' =_{I_{ij}} m :\Leftrightarrow m' =_I m$ and $\lambda_{im}^{\preccurlyeq}(x) =_{\lambda_0(m)} \lambda_{jm}^{\preccurlyeq}(y)$. As $\preccurlyeq_I$ is extensional and reflexive, $m \preccurlyeq_I m'$, and by Definition 3.8.1(b)

$$\lambda_{im'}^{\preccurlyeq}(x) = \lambda_{mm'}^{\preccurlyeq}\big(\lambda_{im}^{\preccurlyeq}(x)\big) = \lambda_{mm'}^{\preccurlyeq}\big(\lambda_{jm}^{\preccurlyeq}(y)\big) = \lambda_{jm'}^{\preccurlyeq}(y).$$

To define an operation of composition, we work with directed sets equipped with a modulus of directedness δ. In the case of a partial order like the standard relation $\leq$ on $\mathbb{R}$, the functions $\delta(x, y) := x \vee y := \max\{x, y\}$ is such a modulus (see section 9.2).

Proposition 5.2.4. *Let δ be a modulus of directedness on a poset $(I, \preccurlyeq_I)$, and let $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$ be a family of sets over $(I, \preccurlyeq_I)$.*
(i) *$\delta(i, i) =_I i$, for every $i \in I$.*
(ii) *$\delta(i, j) =_I \delta(j, i)$, for every $i, j \in I$.*
(iii) *If $(i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (j, y) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (k, z)$, then*

$$m \in \texttt{PrfEql}_0\big((i, x), (j, y)\big) \ \&\ l \in \texttt{PrfEql}_0\big((j, y), (k, z)\big) \ \Rightarrow\ \delta(m, l) \in \texttt{PrfEql}_0\big((i, x), (k, z)\big).$$

Proof. (i) Since $i \preccurlyeq_I i$, we use the definitional clause (δ_1) of a modulus of directedness.
(ii) By (δ_3) we have that $\delta\big(\delta(i, j), i\big) =_I \delta\big(i, \delta(j, i)\big)$. By (δ_1) and (δ_2) we get $\delta\big(\delta(i, j), i\big) =_I \delta(i, j)$ and $\delta\big(i, \delta(j, i)\big) =_I \delta(j, i)$.
(iii) If $m \in \texttt{PrfEql}_0\big((i, x), (j, y)\big) \Leftrightarrow m \in I_{ij} \ \&\ \lambda_{im}^{\preccurlyeq}(x) =_{\lambda_0(m)} \lambda_{jm}^{\preccurlyeq}(y)$, and

$$l \in \texttt{PrfEql}_0\big((j, y), (k, z)\big) \Leftrightarrow l \in I_{jk} \ \&\ \lambda_{jl}^{\preccurlyeq}(y) =_{\lambda_0(l)} \lambda_{kl}^{\preccurlyeq}(z),$$

we show that $\delta(m, l) \in I_{ik}$ and $\lambda_{i\delta(m,l)}^{\preccurlyeq}(x) =_{\lambda_0(\delta(m,l)} \lambda_{k\delta(m,l)}^{\preccurlyeq}(z)$. By our hypotheses, $i \preccurlyeq_I m \preccurlyeq_I \delta(m, l)$ and $k \preccurlyeq_I l \preccurlyeq_I \delta(m, l)$. Moreover,

$$\begin{aligned}
\lambda_{i\delta(m,l)}^{\preccurlyeq}(x) &\overset{i \preccurlyeq_I m \preccurlyeq_I \delta(m,l)}{=} \lambda_{m\delta(m,l)}^{\preccurlyeq}\big(\lambda_{im}^{\preccurlyeq}(x)\big) \\
&= \lambda_{m\delta(m,l)}^{\preccurlyeq}\big(\lambda_{jm}^{\preccurlyeq}(y)\big) \\
&\overset{j \preccurlyeq_I m \preccurlyeq_I \delta(m,l)}{=} \lambda_{j\delta(m,l)}^{\preccurlyeq}(y) \\
&\overset{j \preccurlyeq_I l \preccurlyeq_I \delta(m,l)}{=} \lambda_{l\delta(m,l)}^{\preccurlyeq}\big(\lambda_{jl}^{\preccurlyeq}(y)\big) \\
&= \lambda_{l\delta(m,l)}^{\preccurlyeq}\big(\lambda_{kl}^{\preccurlyeq}(z)\big) \\
&\overset{k \preccurlyeq_I l \preccurlyeq_I \delta(m,l)}{=} \lambda_{k\delta(m,l)}^{\preccurlyeq}(z).
\end{aligned}$$

□

If $m \in \texttt{PrfEql}_0\big((i, x), (j, y)\big)$ and $l \in \texttt{PrfEql}_0\big((j, y), (k, z)\big)$ it is natural to define

$$\texttt{refl}(i, x) := i \ \ \&\ \ m^{-1} := m \ \ \&\ \ m * l := \delta(m, l).$$

Then, $\texttt{refl}(i, x) * m := i * m := \delta(i, m) =_I m$, and similarly $m * \texttt{refl}(i, x) =_I m$, for every $m \in \texttt{PrfEql}_0\big((i, x), (j, y)\big)$. The associativity $(m * l) * n =_I m * (l * n)$ is just the condition (δ_3), and if $m, m' \in \texttt{PrfEql}_0\big((i, x), (j, y)\big)$ and $l \in \texttt{PrfEql}_0\big((j, y), (k, z)\big)$ such that $m =_I m'$ and $l =_I l'$, then $m * l =_I m' * l'$ is reduced to $\delta(m, l) = \delta(m', l')$, which follows from the fact that δ is a function. If $m \in \texttt{PrfEql}_0\big((i, x), (j, y)\big)$, to show $m * m^{-1} = \texttt{refl}(i, x) := i$, we need to use as equality on $\texttt{PrfEql}_0\big((i, x), (i, x)\big)$ not the equality inherited from I, but the equality

$$m =_{\texttt{PrfEql}_0\big((i,x),(i,x)\big)} m' :\Leftrightarrow i =_I i,$$

according to which all elements of $\texttt{PrfEql}_0\big((i, x), (i, x)\big)$ are equal to each other. Similarly we get $m^{-1} * m := \delta(m^{-1}, m) =_{\texttt{PrfEql}_0\big((i,x),(i,x)\big)} j := \texttt{refl}(j, y)$. Hence, the equality on

$\texttt{PrfEql}_0\big((i,x),(j,y)\big)$ is defined as above, if $i := j$ and $x := y$, and it is inherited from I otherwise. In order to make such a distinction though, we need to know that the previous equalities are possible, something which is not always the case without some further assumptions on the general equality $:=$. Of course, all aforementioned groupoid properties of $*$ and $^{-1}$ hold, if we define all elements of any set $\texttt{PrfEql}_0\big((i,x),(j,y)\big)$ to be equal.

Example 5.2.5 (The set of reals)**.** In [19], p. 18, the set of reals $\mathbb{R}$ is defined as an extensional subset of $\mathbb{F}(\mathbb{N}^+,\mathbb{Q})$. Specifically,

$$\mathbb{R} := \bigg\{x \in \mathbb{F}(\mathbb{N}^+,\mathbb{Q}) \mid \forall_{m,n\in\mathbb{N}^+}\bigg(|x_m - x_n| \leq \frac{1}{m} + \frac{1}{n}\bigg)\bigg\},$$

where $\mathbb{N}^+$ is the set of non-zero natural numbers. The equality on $\mathbb{R}$ is defined as follows:

$$x =_{\mathbb{R}} y :\Leftrightarrow \forall_{n\in\mathbb{N}^+}\bigg(|x_n - y_n| \leq \frac{2}{n}\bigg).$$

To prove though that $x =_{\mathbb{R}} y$ is transitive, one needs the following characterisation:

$$x =_{\mathbb{R}} y \Leftrightarrow \forall_{j\in\mathbb{N}^+}\exists_{N_j\in\mathbb{N}^+}\forall_{n\geq N_j}\bigg(|x_n - y_n| \leq \frac{1}{j}\bigg).$$

Using countable choice, we get the equivalence

$$x =_{\mathbb{R}} y \Leftrightarrow \exists_{\omega\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}\forall_{j\in\mathbb{N}^+}\forall_{n\geq\omega(j)}\bigg(|x_n - y_n| \leq \frac{1}{j}\bigg).$$

If $\omega : \mathbb{N}^+ \to \mathbb{N}^+$ witnesses the equality $x =_{\mathbb{R}} y$, then $\omega \vee \mathrm{id}_{\mathbb{N}^+}$, where $(\omega \vee \mathrm{id}_{\mathbb{N}^+})(j) := \omega(j) \vee \mathrm{id}_{\mathbb{N}^+}(j) := \max\{\omega(j), \mathrm{id}_{\mathbb{N}^+}(j)\}$, for every $j \in \mathbb{N}^+$, also witnesses the equality $x =_{\mathbb{R}} y$. Hence, without loss of generality we can assume that $\omega \geq \mathrm{id}_{\mathbb{N}^+}$. We define

$$\texttt{PrfEql}_0(x,y) := \big\{\omega \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid \omega : x =_{\mathbb{R}} y\big\},$$

$$\omega : x =_{\mathbb{R}} y :\Leftrightarrow \omega \geq \mathrm{id}_{\mathbb{N}^+} \ \& \ \forall_{j\in\mathbb{N}^+}\forall_{n\geq\omega(j)}\bigg(|x_n - y_n| \leq \frac{1}{j}\bigg).$$

If $\omega \in \texttt{PrfEql}_0(x,y)$ and $\delta \in \texttt{PrfEql}_0(y,z)$, we define

$$\texttt{refl}(x) := \mathrm{id}_{\mathbb{N}^+} \ \ \& \ \ \omega^{-1} := \omega \ \ \& \ \ (\omega * \delta)(j) := \omega(2j) \vee \delta(2j),$$

for every $j \in \mathbb{N}^+$. In this case $\omega * \delta \in \texttt{PrfEql}_0(x,z)$, since if $n \geq \omega(2j) \vee \delta(2j)$, then

$$|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \leq \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j}.$$

It is easy to see that $*$ is associative, and it also compatible with the canonical equality of the sets $\texttt{PrfEql}_0(x,y)$, the one inherited from $\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$. The rest of the groupoid properties of $*$ and $^{-1}$ do not hold if we keep the canonical equality of the sets $\texttt{PrfEql}_0(x,y)$. In other words, the set $\texttt{PrfEql}_0^{\mathbb{R}}(x,y)$, equipped with its canonical equality, is not a (-1)-set. It becomes, if we truncate it i.e., if we equip $\texttt{PrfEql}_0^{\mathbb{R}}(x,y)$ with the equality

$$\omega \ {}_{||=_{\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}||} \ \delta :\Leftrightarrow \omega =_{\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)} \omega \ \ \& \ \ \delta =_{\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)} \delta.$$

If (X, d) is a metric space, hence $x =_X y \Leftrightarrow d(x, y) = 0$, for every $x, y \in X$, we define

$$\texttt{PrfEql}_0(x, y) := \texttt{PrfEql}_0\big(d(x, y), 0\big).$$

If F is a set of real-valued functions on a set X, like a Bishop topology on X, that separates the points of X i.e., $x =_X y \Leftrightarrow \forall_{f \in F}\big(f(x) =_{\mathbb{R}} f(y)\big)$, we define

$$\texttt{PrfEql}_0(x, y) := \bigcurlywedge_{f \in F} \texttt{PrfEql}_0\big(f(x), f(y)\big).$$

If $\phi : \mathbb{R} \to \mathbb{R}$, let a dependent operation

$$\phi_1 : \bigcurlywedge_{x,y \in \mathbb{R}} \bigcurlywedge_{\omega \in \texttt{PrfEql}_0(x,y)} \texttt{PrfEql}_0(\phi(x), \phi(y).$$

For example, let $[\phi_1(x, y, \omega)](j) := 2j$, for every $j \in \mathbb{N}^+$. This element of $\texttt{PrfEql}_0\big(f(x), f(y)$ though, does not depend on ω and it is not compatible with $*$ and $^{-1}$.

Example 5.2.6 (Sets of integrable and measurable functions in BCMT)**.** In (BCMT) Bishop and Cheng define the set of integrable functions of an integration space $\mathcal{L} := (X, L, \int)$ (see [19], p. 222) as the totality

$$L_1 := \{f \in \mathfrak{F}(X) \mid f \text{ has a representation in } L\},$$

where $\mathfrak{F}(X)$ is the totality of real-valued partial functions on the set X, which are strongly extensional i.e., if $f(x) \neq_{\mathbb{R}} f(x')$, then $x \neq_X x'$, for every $x, x' \in X$. An element f of $\mathfrak{F}(X)$ has a representation in L, if there is a sequence $(f_n)_{n=1}^{\infty}$ of partial functions in L such that

$$\sum_{n \in \mathbb{N}^+} \int |f_n| < +\infty, \quad \text{and}$$

$$\forall_{x \in X}\bigg(\sum_{n \in \mathbb{N}^+} |f_n(x)| < +\infty \Rightarrow f(x) = \sum_{n \in \mathbb{N}^+} f_n(x)\bigg).$$

A subset F of X is full, if there is $g \in L_1$ such that the domain of (the partial function) g is included in F. The equality on L_1 is defined in [19], p. 224, by

$$f =_{L_1} g :\Leftrightarrow \exists_{F \in \mathcal{P}(X)}\big(F \text{ is full } \& \ f_{|F} = g_{|F}\big).$$

Unfortunately, this presentation of L_1 within BCMT is highly problematic from a predicative point of view. The totality L_1 is defined through separation on $\mathfrak{F}(X)$, which, because of the definition of a partial function from X to $\mathbb{R}$, is a class, like $\mathcal{P}(X)$, and not a set. Moreover, the above equality $f =_{L_1} g$ requires quantification over the class $\mathcal{P}(X)$. The impredicative character of BCMT hinders its computational content (see sections 7.3 and 7.5). Within this impredicative theory BCMT though, one can define

$$\texttt{PrfEql}_0(f, g) := \{F \in \mathcal{P}(X) \mid F \text{ is full } \& \ f_{|F} = g_{|F}\}.$$

If $f, g, h \in L_1$, $F \in \texttt{PrfEql}_0(f, g)$, and $G \in \texttt{PrfEql}_0(g, h)$, it is natural to define

$$\texttt{refl}(f) := \text{dom}(f) \ \ \& \ \ F^{-1} := F \ \ \& \ \ F * G := F \cap G,$$

since the intersection of full sets is a full set, and $f_{|F} = g_{|F}$ & $g_{|G} = h_{|G} \Rightarrow f_{|F\cap G} = h_{|F\cap G}$. It is not hard to see that if we equip the sets $\texttt{PrfEql}_0(f, g)$ with the equality inherited from $\mathcal{P}(X)$, we get the same groupoid properties of $*$ and $^{-1}$ as in the case of $\mathbb{R}$ in the previous example. If $\int$ is a completely extended (see [19], p. 223), and σ-finite integral on X (see [19], p. 269), and if $p \geq 1$, the set L_p is defined as follows (see [19], p. 315):

$$L_1 := \{f \in \mathfrak{F}(X) \mid f \text{ is measurable } \& \ |f|^p \in L_1\},$$

where a partial function $f : X \rightharpoonup \mathbb{R}$ is measurable, if its domain $\mathrm{dom}(f)$ is a full set, and it is appropriately approximated by elements of L_1 (for the exact definition see [19], p. 259). Similarly to L_1, $f =_{L_p} g :\Leftrightarrow \exists_{F\in\mathcal{P}(X)}\big(F \text{ is full } \& \ f_{|F} = g_{|F}\big)$. If $\int$ is a σ-finite integral on X, the set L_∞ is defined as follows (see [19], p. 346):

$$L_\infty := \big\{f \in \mathfrak{F}(X) \mid f \text{ is measurable and essentially bounded relative to } \textstyle\int\big\},$$

where a real-valued function defined on a full subset of X is essentially bounded relative to a σ-finite integral $\int$ on X, if there are $c > 0$ and a full set F, such that $|f|_{|F} \leq c$ (see [19], p. 346). The equality on L_∞ is defined as in L_p, for $p \geq 1$, and the corresponding sets $\texttt{PrfEql}_0(f, g)$ behave analogously. A complemented subset $\boldsymbol{A} := (A^1, A^0)$ of X is called integrable, if its characteristic function $\chi_{\boldsymbol{A}}$ is in L_1, and then the measure on $\boldsymbol{A}$ is defined by $\mu(\boldsymbol{A}) := \int \chi_{\boldsymbol{A}}$. If $\mathcal{A}$ is the totality of integrable sets with positive measure, $=_{\mathcal{A}}$ is defined in [19], p. 346, by $\boldsymbol{A} =_{\mathcal{A}} \boldsymbol{B} :\Leftrightarrow \chi_{\boldsymbol{A}} =_{L_1} \chi_{\boldsymbol{B}}$, and one can define $\texttt{PrfEql}_0(\boldsymbol{A}, \boldsymbol{B}) := \texttt{PrfEql}_0(\chi_{\boldsymbol{A}}, \chi_{\boldsymbol{B}})$. All these totalities though, are defined impredicatively.

5.3 Martin-Löf sets

We give an abstract description of the previous examples of sets with a proof-relevant equality.

Definition 5.3.1. *Let Y be a set, and $(X, =_X)$ a set with an equality condition of the form*

$$x =_X x' :\Leftrightarrow \exists_{p\in Y}\big(p : x =_X x'\big),$$

where $\theta^{xx'}(p) :\Leftrightarrow p : x =_Y x'$ is an extensional property on Y. Let also the non-dependent assignment routine $\texttt{PrfEql}_0^X : X \times X \rightsquigarrow \mathbb{V}_0$, defined by

$$\texttt{PrfEql}_0^X(x, x') := \{p \in Y \mid p : x =_X x'\}; \qquad (x, x') \in X \times X,$$

together with dependent operations

$$\texttt{refl}^X : \bigcurlywedge_{x\in X} \texttt{PrfEql}_0^X(x, x),$$

$$^{-1_X} : \bigcurlywedge_{x,x'\in X} \mathbb{F}\Big(\texttt{PrfEql}_0^X(x, x'), \texttt{PrfEql}_0^X(x', x)\Big),$$

$$*_X : \bigcurlywedge_{x,x',x''\in X} \mathbb{F}\Big(\texttt{PrfEql}_0^X(x, x') \times \texttt{PrfEql}_0^X(x', x''), \texttt{PrfEql}_0^X(x, x'')\Big).$$

We call the structure $\widehat{X} := (X, =_X, \texttt{PrfEql}_0^X, \texttt{refl}^X, ^{-1_X}, *_X)$ a set with a proof-relevant equality. If X is clear from the context, we may omit the subscript X from the above dependent *operations. We call* $\widehat{X}$ *a Martin-Löf set, if the following conditions hold:*

(ML$_1$) $\mathtt{refl}_x * p =_{\mathtt{PrfEql}_0^X(x,x')} p$ *and* $p * \mathtt{refl}_y =_{\mathtt{PrfEql}_0^X(x,x')} p$, *for every* $p \in \mathtt{PrfEql}_0^X(x,x')$.

(ML$_2$) $p*p^{-1} =_{\mathtt{PrfEql}_0^X(x,x)} \mathtt{refl}_x$ *and* $p^{-1}*p =_{\mathtt{PrfEql}_0^X(y,y)} \mathtt{refl}_y$, *for every* $p \in \mathtt{PrfEql}_0^X(x,x')$.

(ML$_3$) $(p * q) * r =_{\mathtt{PrfEql}_0^X(x,x''')} p * (q * r)$, *for every* $p \in \mathtt{PrfEql}_0^X(x,x'), q \in \mathtt{PrfEql}_0^X(x',x'')$ *and* $r \in \mathtt{PrfEql}_0^X(x'',x''')$.

(ML$_4$) *If* $p, q \in \mathtt{PrfEql}_0^X(x,x')$ *and* $r, s \in \mathtt{PrfEql}_0^X(x',x'')$ *such that* $p =_{\mathtt{PrfEql}_0^X(x,x')} q$ *and* $r =_{\mathtt{PrfEql}_0^X(x',x'')} s$, *then* $p * r =_{\mathtt{PrfEql}_0^X(x,x'')} q * s$.

If $\widehat{X}$ is a set with a proof-relevant equality, by Definition 5.1.9 we get

$$\mathtt{Prf}(x =_X x') := \mathtt{PrfEql}_0^X(x,x').$$

Conditions (ML$_1$)-(ML$_3$) express that the proof-relevant equality of X has a groupoid-structure, see [82], while condition (ML$_4$) expresses the extensionality of the composition $*_X$. Next proposition is straightforward to show.

Proposition 5.3.2. *Let* $\widehat{X}$ *be a Martin-Löf set,* $x, x' \in X$, *and* $p, q \in \mathtt{PrfEql}_0(x,x')$.

(i) $\mathtt{refl}_x^{-1} =_{\mathtt{PrfEql}_0(x,x)} \mathtt{refl}_x$.

(ii) $(p^{-1})^{-1} =_{\mathtt{PrfEql}_0(x,x')} p$.

(iii) *If* $p =_{\mathtt{PrfEql}_0(x,x')} q$, *then* $p^{-1} =_{\mathtt{PrfEql}_0(x',x)} q^{-1}$.

Definition 5.3.3. *Let* $\widehat{X}, \widehat{Y}$ *be sets with proof-relevant equalities. A map from* $\widehat{X}$ *to* $\widehat{Y}$ *is a pair* $\widehat{f} := (f, f_1)$, *where* $f : X \to Y$ *and*

$$f_1 : \bigcurlywedge_{x,x' \in X} \mathbb{F}\Big(\mathtt{PrfEql}_0^X(x,x'), \mathtt{PrfEql}_0^Y(f(x), f(x'))\Big).$$

We write $\widehat{f} : \widehat{X} \to \widehat{Y}$ *to denote a map from* $\widehat{X}$ *to* $\widehat{Y}$. *We call the dependent operation* f_1 *the first associate of* $\widehat{f}$. *If, for every* $x, x' \in X$ *and every* $p, p' \in \mathtt{PrfEql}_0^X(x,x')$, *we have that*

$$p =_{\mathtt{PrfEql}_0^X(x,x')} p' \Rightarrow f_1(x,x',p) =_{\mathtt{PrfEql}_0^Y(f(x),f(x'))} f_1(x,x',p'),$$

we say that f_1 *is a function-like first associate of* $\widehat{f}$. *If* $\widehat{X}$ *and* $\widehat{Y}$ *are Martin-Löf sets, a map* $\widehat{f} : \widehat{X} \to \widehat{Y}$ *is a Martin-Löf map, if the following conditions hold:*

(i) $f_1(x,x,\mathtt{refl}_x) =_{\mathtt{PrfEql}_0^Y(f(x),f(x))} \mathtt{refl}_{f(x)}$, *for every* $x \in X$.

(ii) *If* $x =_X x' =_X x''$, *then* $f_1(x,x'',p*q) =_{\mathtt{PrfEql}_0^Y(f(x),f(x''))} f_1(x,x',p) * f_1(x',x'',q)$, *for every* $p \in \mathtt{PrfEql}_0^X(x,x')$ *and* $q \in \mathtt{PrfEql}_0^X(x',x'')$.

Definition 5.3.4. *Let* $\widehat{I}$ *be a set with a proof-relevant equality. A family of sets over* $\widehat{I}$ *is a triplet* $\widehat{\Lambda} := (\lambda_0, \mathtt{PrfEql}_0^I, \lambda_2)$, *where* $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, *and*

$$\lambda_2 : \bigcurlywedge_{(i,j) \in D(I)} \bigcurlywedge_{p \in \mathtt{PrfEql}_0^I(i,j)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_2\big((i,j),p\big) := \lambda_{ij}^p, \ (i,j) \in D(I), \ p \in \mathtt{PrfEql}_0^I(i,j),$$

such that the following conditions hold:

(i) *For every* $i \in I$ *we have that* $\lambda_{ii}^{\mathtt{refl}_i} = \mathrm{id}_{\lambda_0(i)}$.
(ii) *If* $i =_I j =_I k$, *for every* $p \in \mathtt{PrfEql}_0^I(i, j)$ *and* $q \in \mathtt{PrfEql}_0^I(j, k)$ *the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\lambda_{ij}^p"'] \arrow[dr, "\lambda_{ik}^{p*q}"] & \\
\lambda_0(j) \arrow[r, "\lambda_{jk}^q"'] & \lambda_0(k).
\end{tikzcd}$$

(iii) If $i =_I j$, *then for for every* $p \in \mathtt{PrfEql}_0^I(i, j)$ *the following diagrams commute*

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\lambda_{ij}^p"'] \arrow[dr, "\mathrm{id}_{\lambda_0(i)}"] & \\
\lambda_0(j) \arrow[r, "\lambda_{ji}^{p^{-1}}"'] & \lambda_0(i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(j) \arrow[d, "\lambda_{ji}^{p^{-1}}"'] \arrow[dr, "\mathrm{id}_{\lambda_0(j)}"] & \\
\lambda_0(i) \arrow[r, "\lambda_{ij}^p"'] & \lambda_0(i).
\end{tikzcd}$$

A family-map $\Phi \colon \widehat{\Lambda} \Rightarrow \widehat{M}$ *is defined as in Definition 3.9.2. We denote by* $\mathtt{Fam}(\widehat{I})$ *the totality of families of sets over* $\widehat{I}$, *which is equipped with the obvious equality. We call* $\widehat{\Lambda}$ *proof-irrelevant, if for every* $(i, j) \in D(I)$ *and* $p, p' \in \mathtt{PrfEql}_0^I(i, j)$ *we have that* $\lambda_{ij}^p =_{\mathbb{F}(\lambda_0(i), \lambda_0(j))} \lambda_{ij}^{p'}$.

If $\widehat{\Lambda} \in \mathtt{Fam}(\widehat{I})$, then $\widehat{\Lambda} \in \mathtt{Fam}^*(I)$ (see Definition 3.9.1). If $\widehat{\Lambda}$ is function-like family over $\widehat{I}$, condition (iii) of the previous definition is provable, while if $\widehat{\Lambda}$ is proof-irrelevant, then $\widehat{\Lambda}$ is function-like. Following Definition 3.9.3, we denote the $\sum$-set of $\widehat{\Lambda}$ by $\widehat{\sum}_{i \in I} \lambda_0(i)$, where

$$(i, x) =_{\widehat{\sum}_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \exists_{p \in \mathtt{PrfEql}_0^I(i,j)} \big(\lambda_{ij}^p(x) =_{\lambda_0(j)} y\big),$$

and we denote the $\prod$-set of $\widehat{\Lambda}$, equipped with the pointwise equality, by $\widehat{\prod}_{i \in I} \lambda_0(i)$, where

$$\Theta \in \widehat{\prod}_{i \in I} \lambda_0(i) :\Leftrightarrow \Theta \in \mathbb{A}(I, \lambda_0) \ \& \ \forall_{p \in \mathtt{PrfEql}_0^I(i,j)} \big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}^p(\Theta_i)\big).$$

Proposition 5.3.5. *If* $\widehat{\Lambda} := (\lambda_0, \mathtt{PrfEql}_0^I, \lambda_2)$ *is a function-like family of sets over the Martin-Löf set* $\widehat{I}$, *then a structure of a Martin-Löf set is defined on* $\widehat{\sum}_{i \in I} \lambda_0(i)$.

Proof. Since $\widehat{\Lambda}$ is function-like, the property $Q_{ij}^{xy}(p) :\Leftrightarrow \lambda_{ij}^p(x) = y$ is extensional on the set $\mathtt{PrfEql}_0^I(i, j)$, and we can define by separation its subset

$$\mathtt{PrfEql}_0^{\widehat{\sum}}\big((i, x), (j, y)\big) := \big\{p \in \mathtt{PrfEql}_0^I(i, j) \mid \lambda_{ij}^p(x) = y\big\}.$$

Let $\mathtt{refl}(i, x) := \mathtt{refl}_i$, for every $(i, x) \in \widehat{\sum}_{i \in I} \lambda_0(i)$. If $p \in \mathtt{PrfEql}_0^{\widehat{\sum}}\big((i, x), (j, y)\big)$, then by the condition (iii) of Definition 5.3.4 we get $p^{-1} \in \mathtt{PrfEql}_0^{\widehat{\sum}}\big((j, y), (i, x)\big)$. If $r \in \mathtt{PrfEql}_0^{\widehat{\sum}}\big((j, y), (k, z)\big)$, then by condition (iii) of Definition 5.3.4 we have that $p * r \in \mathtt{PrfEql}_0^I \widehat{\sum}\big((i, x), (k, z)\big)$. The clauses of Definition 5.3.1 for $\mathtt{PrfEql}_0^{\widehat{\sum}}\big((i, x), (j, y)\big)$ follow from the corresponding clauses for $\mathtt{PrfEql}_0^I(i, j)$. $\square$

If $\widehat{I}$ and $\widehat{\sum}_{i\in I}\lambda_0(i)$ are Martin-Löf sets as above, it is straightforward to show that the pair $\widehat{\mathtt{pr}_1} := \big(\mathtt{pr}_1^{\widehat{\Lambda}}, \varpi_1\big)$ is a map from $\widehat{\sum}_{i\in I}\lambda_0(I)$ to $\widehat{I}$, where

$$\mathtt{pr}_1^{\widehat{\Lambda}} \colon \widehat{\sum}_{i\in I}\lambda_0(i) \to I, \quad (i,x)\mapsto i; \quad i\in I, \text{ and}$$

$$\varpi_1 : \bigcurlywedge_{(i,x),(j,y)\in\widehat{\sum}_{i\in I}\lambda_0(i)} \mathbb{F}\bigg(\mathtt{PrfEql}_0^{\widehat{\Sigma}}\big((i,x),(j,y)\big), \mathtt{PrfEql}_0^{I}(i,j)\bigg),$$

$$\big[\varpi_1\big((i,x),(j,y)\big)\big](p) := p; \quad p\in \mathtt{PrfEql}_0^{\widehat{\Sigma}}\big((i,x),(j,y)\big),$$

is a function-like first associate of $\widehat{\mathtt{pr}_1}$.

Lemma 5.3.6. *Let $\widehat{X}$ be a Martin-Löf set, $x_0\in X$ and let $\mathtt{PrfEql}_0^{x_0} : X \rightsquigarrow \mathbb{V}_0$ be defined by $x\mapsto \mathtt{PrfEql}_0^X(x,x_0)$, for every $x\in X$. Moreover, let*

$$\mathtt{PrfEql}_1^{x_0} \colon \bigcurlywedge_{(x,y)\in D(X)} \bigcurlywedge_{p\in \mathtt{PrfEql}_0^X(x,y)} \mathbb{F}\big(\mathtt{PrfEql}_0^X(x,x_0), \mathtt{PrfEql}_0^X(y,x_0)\big),$$

be defined, for every $(x,y)\in D(X)$, $p\in \mathtt{PrfEql}_0^X(x,y)$ and $r\in \mathtt{PrfEql}_0^X(x,x_0)$, by

$$\mathtt{PrfEql}_1^{x_0}\big((x,y),p\big) := \mathtt{PrfEql}_{xy}^{x_0} \colon \mathtt{PrfEql}_0^X(x,x_0) \to \mathtt{PrfEql}_0^X(y,x_0)$$

$$r\mapsto p^{-1} * r.$$

Then $\widehat{\mathtt{PrfEql}}^{x_0} := (\mathtt{PrfEql}_0^{x_0}, \mathtt{PrfEql}_1^{x_0})$ is a function-like family of sets over $\widehat{X}$.

Proof. If $x\in X$, then $\mathtt{PrfEql}_{xx}^{\mathtt{refl}_x}(r) := \mathtt{refl}_x^{-1} * r = \mathtt{refl}_x * r = r$, for every $r\in \mathtt{PrfEql}_0^X(x,x_0)$. If $x =_X y =_X z$, $p\in \mathtt{PrfEql}_0^X(x,y), q\in \mathtt{PrfEql}_0^X(y,z)$, then for every $r\in \mathtt{PrfEql}_0^X(x,x_0)$ we have that

$$\big(\mathtt{PrfEql}_{yz}^{q}\circ \mathtt{PrfEql}_{xy}^{p}\big)(r) := q^{-1} * (p^{-1} * r) = (q^{-1} * p^{-1}) * r = (p*q)^{-1} * r := \mathtt{PrfEql}_{xz}^{p*q}(r).$$

If $p =_{\mathtt{PrfEql}_0^X(x,y)} p'$, then by Proposition 5.3.2(iii) and condition (ML_4) we get $\mathtt{PrfEql}_{xy}^{p}(r) := p^{-1} * r = (p')^{-1} * r := \mathtt{PrfEql}_{xy}^{p'}(r)$, for every $r\in \mathtt{PrfEql}_0^X(x,x_0)$. $\square$

Theorem 5.3.7 (Contractibility of singletons)**.** *Let $\widehat{X}$ be a proof-relevant set, $x_0\in X$ and let $\widehat{\mathtt{PrfEql}}^{x_0} := (\mathtt{PrfEql}_0^{x_0}, \mathtt{PrfEql}_1^{x_0})$ be the function-like family of sets over $\widehat{X}$ from Lemma 5.3.6. Let $\widehat{\sum}_{x\in X}\mathtt{PrfEql}_0^X(x,x_0)$ be equipped with its canonical structure of a Martin-Löf set, according to Proposition 5.3.5. Then for every $(x,p)\in \widehat{\sum}_{x\in X}\mathtt{PrfEql}_0^X(x,x_0)$ we have that*

$$(x,p) =_{\widehat{\sum}_{x\in X}\mathtt{PrfEql}_0^X(x,x_0)} \big(x_0, \mathtt{refl}_{x_0}\big).$$

Proof. By the definition of equality on the $\sum$-set of some $\widehat{\Lambda}\in \mathtt{Fam}(\widehat{I})$ we have that

$$(x,p) =_{\widehat{\sum}_{x\in X}\mathtt{PrfEql}_0^X(x,x_0)} \big(x_0, \mathtt{refl}_{x_0})\big) :\Leftrightarrow x =_X x_0 \ \&\ \exists_{q\in \mathtt{PrfEql}_0^X(x,x_0)}\big(\mathtt{PrfEql}_{xx_0}^{q}(p) = \mathtt{refl}_{x_0}\big).$$

If $(x,p)\in \widehat{\sum}_{x\in X}\mathtt{PrfEql}_0^X(x,x_0)$, then $p\in \mathtt{PrfEql}_0^X(x,x_0)$, hence $x =_X x_0$. If we take $q := p$, then $\mathtt{PrfEql}_{xx_0}^{p}(p) := p^{-1} * p = \mathtt{refl}_{x_0}$. $\square$

A map between Martin-Löf sets can generate the family of its fibers over its codomain.

Theorem 5.3.8. *Let $\widehat{X}, \widehat{Y}$ be Martin-Löf sets, and $\widehat{f} := (f, f_1) \colon \widehat{X} \to \widehat{Y}$ a map from $\widehat{X}$ to $\widehat{Y}$ with a function-like first associate f_1.*
(i) *If $y \in Y$, the pair $\mathtt{PrfEql}f := \big(\mathtt{PrfEql}f_0^y, \mathtt{PrfEql}f_1^y\big)$, where $\mathtt{PrfEql}f_0^y : X \rightsquigarrow \mathbb{V}_0$ is defined by the rule $x \mapsto \mathtt{PrfEql}_0^Y(f(x), y)$,, for every $x \in X$, and*

$$\mathtt{PrfEql}f_1^y \colon \bigcurlywedge_{(x,x') \in D(X)} \bigcurlywedge_{p \in \mathtt{PrfEql}_0^X(x,x')} \mathbb{F}\big(\mathtt{PrfEql}_0^Y(f(x), y), \mathtt{PrfEql}_0^Y(f(x'), y)\big),$$

$$\mathtt{PrfEql}f_1^y\big((x,x'), p\big) := \mathtt{PrfEql}f_{xx'}^{y,p} \colon \mathtt{PrfEql}_0^Y(f(x), y) \to \mathtt{PrfEql}_0^Y(f(x'), y),$$

$$r \mapsto [f_1(x,x',p)]^{-1} * r; \qquad r \in \mathtt{PrfEql}_0^Y(f(x), y),$$

is a function-like family of sets over $\widehat{X}$.
(ii) *The pair $\mathtt{Prfib} := (\mathtt{Prfib}_0, \mathtt{Prfib}_1)$, where $\mathtt{Prfib}_0 : Y \rightsquigarrow \mathbb{V}_0$ is defined by the rule*

$$y \mapsto \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y); \qquad y \in Y, \quad \textit{and}$$

$$\mathtt{Prfib}_1 : \bigcurlywedge_{(y,y') \in D(Y)} \bigcurlywedge_{q \in \mathtt{PrfEql}_0^Y(y,y')} \mathbb{F}\bigg(\widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y), \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0({}^Y f(x), y')\bigg),$$

$$\mathtt{Prfib}_1^y\big((y,y'), q\big) := \mathtt{Prfib}_{yy'}^q : \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0(f(x), y) \to \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y'),$$

$$(x, p) \mapsto (x, p * q); \qquad (x,p) \in \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y),$$

is a function-like family of sets over $\widehat{Y}$.

Proof. (i) If $r \in \mathtt{PrfEql}_0^Y(f(x), y)$, then by Proposition 5.3.2(iii) we get

$$\mathtt{PrfEql}_0 f_{xx'}^{y,\mathtt{refl}_x}(r) := [f_1(x,x,\mathtt{refl}_x)]^{-1} * r = \big[\mathtt{refl}\big(f(x)\big)\big]^{-1} * r = \mathtt{refl}\big(f(x)\big) * r = r.$$

If $p \in \mathtt{PrfEql}_0^X(x,x')$ and $p' \in \mathtt{PrfEql}_0^X(x',x'')$, then for every $r \in \mathtt{PrfEql}_0^Y(f(x), y)$ we get

$$\begin{aligned}
\mathtt{PrfEql}_0 f_{x'x''}^{y,p'}\bigg(\mathtt{PrfEql}_0 f_{xx'}^{y,p}(r)\bigg) &= [f_1(x',x'',p')]^{-1} * \big([f_1(x,x',p)]^{-1} * r\big) \\
&= \big([f_1(x',x'',p')]^{-1} * [f_1(x,x',p)]^{-1}\big) * r \\
&= [f_1(x,x',p) * f_1(x',x'',p')]^{-1} * r \\
&= [f_1(x,x'',p * q)]^{-1} * r \\
&:= \mathtt{PrfEql}_0 f_{xx''}^{y,p*p'}(r).
\end{aligned}$$

If $p =_{\mathtt{PrfEql}_0^X(x,x')} s$, and if $r \in \mathtt{PrfEql}_0^Y(f(x), y)$, by the function-likeness[1] of f_1 we get

$$\mathtt{PrfEql}_0 f_{xx'}^{y,p}(r) := [f_1(x,x',p)]^{-1} * r = [f_1(x,x',s)]^{-1} * r := \mathtt{PrfEql}_0 f_{xx'}^{y,s}(r).$$

[1]The function-likeness of f_1 is also needed in the proof of condition (iii) of Definition 5.3.4.

(ii) First we show that for every $p, p' \in \mathtt{PrfEql}_0^Y(f(x), y)$ we have that

$$p =_{\mathtt{PrfEql}_0^Y(f(x),y)} p' \Rightarrow (x, p) =_{\widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x),y)} (x, p'), \tag{5.1}$$

since

$$\mathtt{PrfEql}_0 f_{xx}^{y,\mathtt{refl}_x}(p) := [f_1(x, x, \mathtt{refl}_x)]^{-1} * p = [\mathtt{refl}_{f(x)}]^{-1} * p = \mathtt{refl}_{f(x)} * p = p = q.$$

If $y \in Y$, then by (5.1), for every $(x, p) \in \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y)$, we get

$$\mathtt{Prfib}_{yy}^{\mathtt{refl}_y}(x, p) := \big(x, p * \mathtt{refl}_y\big) =_{\widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x),y)} (x, p).$$

If $q \in \mathtt{PrfEql}_0^Y(y, y'$ and $q' \in \mathtt{PrfEql}_0^Y(y', y'')$, then for every $(x, p) \in \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y)$

$$\mathtt{Prfib}_{y'y''}^{q'}\Big(\mathtt{Prfib}_{yy'}^{q}(x, p)\Big) := \mathtt{Prfib}_{y'y''}^{q'}(x, p * q) := (x, (p * q) * q')$$

$$\overset{(5.1)}{=} (x, p * (q * q') := \mathtt{Prfib}_{yy''}^{q*q'}(x, p).$$

If $q =_{\mathtt{PrfEql}_0^Y(y,y')} s$, then $\mathtt{Prfib}_{yy'}^q = \mathtt{Prfib}_{yy'}^s$, since for every $(x, p) \in \widehat{\sum}_{x \in X} \mathtt{PrfEql}_0^Y(f(x), y)$

$$\mathtt{Prfib}_{yy'}^q(x, p) := (x, p * q) \overset{(5.1)}{=} (x, p * s) := \mathtt{Prfib}_{yy'}^s(x, p). \qquad \square$$

5.4 On the Yoneda lemma for $\mathtt{Fam}(\widehat{I})$

Within MLTT Rijke viewed in [107] a type family $P : I \to \mathcal{U}$ over a type $I : \mathcal{U}$ as a presheave of a locally small category $\mathcal{C}$ i.e., as an element of $\mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$, and proved a type-theoretic version of the Yoneda lemma using the J-rule and the axiom of function extensionality. In BST the J-rule, in the form of the transport, is built in the definition of an I-family of sets, and the axiom of function extensionality is built in the definition of pointwise equality on $\mathtt{Map}(\Lambda, M)$. Here we present the Yoneda lemma when the function-like elements of $\mathtt{Fam}(\widehat{I})$ replace $\mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Theorem 5.4.1 (Yoneda lemma for $\mathtt{Fam}(\widehat{I})$)**.** *Let $\widehat{I} := \big(I, =_I, \mathtt{PrfEql}_0^I, \mathtt{refl}^I, ^{-1_I}, *_I\big)$ be a Martin-Löf set, $i_0 \in I$, and let $\widehat{\mathtt{PrfEql}^{i_0}} := (\mathtt{PrfEql}_0^{i_0}, \mathtt{PrfEql}_1^{i_0})$ be the function-like family of sets over $\widehat{I}$ defined in Lemma 5.3.6. If $\widehat{\Lambda} := \big(\lambda_0, \mathtt{PrfEql}_0^I, \lambda_2\big) \in \mathtt{Fam}(\widehat{I})$ is function-like, there are functions*

$$e_{i_0,\widehat{\Lambda}} : \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big) \to \lambda_0(i_0) \quad \& \quad e_{\widehat{\Lambda},i_0} : \lambda_0(i_0) \to \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big),$$

such that $e_{i_0,\widehat{\Lambda}} \circ e_{\widehat{\Lambda},i_0} = \mathrm{id}_{\lambda_0(i)}$. Moreover, if $\widehat{\Lambda}$ is proof-irrelevant, then $e_{\widehat{\Lambda},i_0} \circ e_{i_0,\widehat{\Lambda}} = \mathrm{id}_{\mathtt{Map}_{\widehat{I}}\left(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\right)}$, and hence $\big(e_{i_0,\widehat{\Lambda}}, e_{\widehat{\Lambda},i_0}\big) : \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big) =_{\mathbb{V}_0} \lambda_0(i_0)$.

Proof. Let the operation $e_{i_0,\widehat{\Lambda}} : \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big) \rightsquigarrow \lambda_0(i_0)$, defined by

$$e_{i_0,\widehat{\Lambda}}(\Phi) := \Phi_{i_0}(\mathtt{refl}_{i_o}); \quad \Phi \in \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big).$$

Since $\Phi\colon \widehat{\mathtt{PrfEql}^{i_0}} \Rightarrow \widehat{\Lambda}$, for its i_0-component $\Phi_{i_0}\colon \mathtt{PrfEql}_0^I(i_0,i_0) \to \lambda_0(i_0)$ we have that $\Phi(i_0)\big(\mathtt{refl}_{i_0}\big) \in \lambda_0(i_0)$, hence $e_{i_0,\widehat{\Lambda}}$ is well-defined. Clearly, $e_{i_0,\widehat{\Lambda}}$ is a function. Let the operation $e_{\widehat{\Lambda},i_0}\colon \lambda_0(i_0) \rightsquigarrow \mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big)$, defined by

$$e_{\widehat{\Lambda},i_0}(x) := \Phi^x; \qquad x \in \lambda_0(i_0),$$

$$\Phi^x\colon \bigcurlywedge_{i\in I} \mathbb{F}\big(\mathtt{PrfEql}_0^I(i,i_0), \lambda_0(i)\big), \qquad \Phi_i^x\colon \mathtt{PrfEql}_0^I(i,i_0) \to \lambda_0(i); \qquad i \in I,$$

$$\Phi_i^x(p) := \lambda_{i_0 i}^{p^{-1}}(x); \qquad p \in \mathtt{PrfEql}_0^I(i,i_0),$$

where, as $p\colon i =_I i_0$, we get $p^{-1}\colon i_0 =_I i$, and $\lambda_{i_0 i}^{p^{-1}}\colon \lambda_0(i_0) \to \lambda_0(i)$. To show that Φ^x is a function, we use the hypothesis that $\widehat{\Lambda}$ is function-like. Next we show that $e_{\widehat{\Lambda},i_0}$ is well-defined i.e., $\Phi^x\colon \widehat{\mathtt{PrfEql}^{i_0}} \Rightarrow \widehat{\Lambda}$. If $i,j \in I$ and $p \in \mathtt{PrfEql}_0^I(i,j)$, the following diagram commutes

$$\begin{tikzcd}
\mathtt{PrfEql}_0^I(i,i_0) \arrow[rr, "\mathtt{PrfEql}_1^{i_0}(i,j,p)"] \arrow[d, "\Phi_i^x"'] & & \mathtt{PrfEql}_0^I(j,i_0) \arrow[d, "\Phi_j^x"] \\
\lambda_0(i) \arrow[rr, "\lambda_{ij}^p"'] & & \lambda_0(j)
\end{tikzcd}$$

$$\begin{aligned}
\Phi_j^x\Big(\big[\mathtt{PrfEql}_1^{i_0}(i,j,p)\big](r)\Big) &:= \Phi_j^x(p^{-1} * r) \\
&:= \lambda_{i_0 j}^{(p^{-1}*r)^{-1}}(x) \\
&= \lambda_{i_0 j}^{r^{-1}*p}(x) \\
&= \lambda_{ij}^{p}\big(\lambda_{i_0 i}^{r^{-1}}(x)\big) \\
&:= \lambda_{ij}^{p}\big(\Phi_i^x(r)\big),
\end{aligned}$$

where the function-likeness of $\widehat{\Lambda}$ is repeatedly used in the previous equalities. It is straightforward to show that the operation $e_{\widehat{\Lambda},i_0}$ is a function. Moreover,

$$e_{i_0,\widehat{\Lambda}}\big(e_{\widehat{\Lambda},i_0}(x)\big) := e_{i_0,\widehat{\Lambda}}\big(\Phi^x\big) := \Phi_{i_0}^x\big(\mathtt{refl}_{i_0}\big) := \lambda_{i_0 i_0}^{\mathtt{refl}_{i_0}^{-1}}(x) = \lambda_{i_0 i_0}^{\mathtt{refl}_{i_0}}(x) = \mathrm{id}_{\lambda_0(i_0)}(x) := x.$$

For the converse composition we have that

$$e_{\widehat{\Lambda},i_0}\big(e_{i_0,\widehat{\Lambda}}(\Phi)\big) := e_{\widehat{\Lambda},i_0}\big(\Phi_{i_0}(\mathtt{refl}_{i_0})\big) := \Phi^{\Phi_{i_0}(\mathtt{refl}_{i_0})},$$

$$\Phi_i^{\Phi_{i_0}(\mathtt{refl}_{i_0})}(p) := \lambda_{i_0 i}^{p^{-1}}\big(\Phi_{i_0}(\mathtt{refl}_{i_0})\big); \qquad p \in \mathtt{PrfEql}_0^I(i,i_0).$$

We would like to show that $\Phi_i^{\Phi_{i_0}(\mathtt{refl}_{i_0})}(p) = \Phi_i(p)$, for every $p \in \mathtt{PrfEql}_0^I(i,i_0)$ and $i \in I$. From the hypothesis $\Phi\colon \widehat{\mathtt{PrfEql}^{i_0}} \Rightarrow \widehat{\Lambda}$, and since $\big[\mathtt{PrfEql}_1^{i_0}(i,i_0,p)\big](p) := p^{-1} * p = \mathtt{refl}_{i_0}$, there is $q \in \mathtt{PrfEql}_0^I(i,i_0)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathtt{PrfEql}_0^I(i, i_0) \arrow[r, "\mathtt{PrfEql}_1^{i_0}(i, i_0, p)"] \arrow[d, "\Phi_i"'] & \mathtt{PrfEql}_0^I(i_0, i_0) \arrow[d, "\Phi_{i_0}"] \\
\lambda_0(i) \arrow[r, "\lambda_{ii_0}^q"'] & \lambda_0(i_0).
\end{tikzcd}$$

Hence, $\Phi_i(p) = \lambda_{i_0 i}^{q^{-1}}\big(\Phi_{i_0}(\mathtt{refl}_{i_0})\big)$. To get $\Phi_i(p) = \Phi_i^{\Phi_{i_0}(\mathtt{refl}_{i_0})}(p) := \lambda_{i_0 i}^{p^{-1}}\big(\Phi_{i_0}(\mathtt{refl}_{i_0})\big)$, we need the equality $\lambda_{i_0 i}^{q^{-1}} = \lambda_{i_0 i}^{p^{-1}}$, which we get from the supposed proof-irrelevance of $\widehat{\Lambda}$. $\square$

In the previous proof we considered the totality $\mathtt{Map}_{\widehat{I}}\big(\widehat{\mathtt{PrfEql}^{i_0}}, \widehat{\Lambda}\big)$ of the corresponding covariant set-relevant family-maps. A Yoneda lemma of the same kind is shown similarly, if we consider the totality of the corresponding contravariant set-relevant family-maps.

5.5 Contractible sets and subsingletons in BST

The following results are translations of results from chapter 3 and 4 of book-HoTT, their proof of which in [124] often requires FunExt. As we have already seen in Definition 2.2.8(iv), the truncation $||X||$ of a set X, which is treated as a higher inductive type in HoTT, is the same totality X equipped with a new equality.

Proposition 5.5.1. *If $(f, g) : X =_{\mathbb{V}_0} Y$, then $(f^*, g^*) : \mathbb{F}(Z, X) =_{\mathbb{V}_0} \mathbb{F}(Z, Y)$, where the operations $f^* \colon \mathbb{F}(Z, X) \rightsquigarrow \mathbb{F}(Z, Y)$ and $g^* \colon \mathbb{F}(Z, Y) \rightsquigarrow \mathbb{F}(Z, X)$ are defined, respectively, by the commutativity of the following diagrams*

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[dr, "f^*(h)"'] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, "g"'] & Z \arrow[l, "k"'] \arrow[dl, "g^*(k)"] \\
X. &
\end{tikzcd}$$

Proof. Clearly, the operations f^* and g^* are functions. If $k \in \mathbb{F}(Z, Y)$ and $h \in \mathbb{F}(Z, X)$, then $f^*(g^*(k)) := f^*(g \circ k) := f \circ (g \circ k) := (f \circ g) \circ k := \mathrm{id}_Y \circ k := k$, and $g^*(f^*(h)) := g^*(f \circ h) := g \circ (f \circ h) := (g \circ f) \circ h := \mathrm{id}_X \circ h := h$. $\square$

Proposition 5.5.2. *If X is a set, the following are equivalent:*

(i) *X is contractible.*

(ii) *X is an inhabited subsingleton.*

(iii) *$X =_{\mathbb{V}_0} \mathbb{1}$.*

Proof. (i)$\Rightarrow$(ii) If x_0 is a centre of contraction for X, then x_0 inhabits X. If $x, y \in X$, then $x =_X x_0$ and $y =_X x_0$, hence $x =_X y$.
(ii)$\Rightarrow$(iii) Let $f \colon X \rightsquigarrow \mathbb{1}$, defined by $f(x) := 0$, for every $x \in X$, and $g \colon \mathbb{1} \colon X$, defined by $g(0) := x_0$, where x_0 inhabits X. Clearly, these operations are functions, and $(f, g) \colon X =_{\mathbb{V}_0} \mathbb{1}$.
(iii)$\Rightarrow$(i) Let $f \in \mathbb{F}(X, \mathbb{1})$ and $g \in \mathbb{F}(\mathbb{1}, X)$ such that $(f, g) \colon X =_{\mathbb{V}_0} \mathbb{1}$. If $x \in X$, then $x =_X g(f(x)) := g(0) \in X$. hence $g(0)$ is a centre of contraction for X. $\square$

Proposition 5.5.3. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$.*

(i) *If $\Theta \colon \bigcurlywedge_{i\in I} \lambda_0(i)$ is a modulus of centres of contraction for λ_0 i.e., Θ_i is a centre of contraction for $\lambda_0(i)$, then $\Theta \in \prod_{i\in I} \lambda_0(i)$ is a centre of contraction for $\prod_{i\in I} \lambda_0(i)$ and $\sum_{i\in I} \lambda_0(i) =_{\mathbb{V}_0} I$.*

(ii) *If $i_0 \in I$ is a centre of contraction for I, then $\sum_{i\in I} \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(i_0)$.*

Proof. (i) If $i =_I j$, then $\Theta_j =_{\lambda_0(j)} \lambda_{ij}(\Theta_i)$, as Θ_j is a centre of contraction for $\lambda_0(j)$. If $\Phi \in \prod_{i\in I} \lambda_0(i)$, then $\Phi_i =_{\lambda_0(i)} \Theta_i$, for every $i \in I$, hence $\Phi =_{\prod_{i\in I} \lambda_0(i)} \Theta$. Let $f \colon I \rightsquigarrow \sum_{i\in I} \lambda_0(i)$, defined by $f(i) := (i, \Theta_i)$, for every $i \in I$. It is immediate to show that f is a function, and $\big(\mathtt{pr}_1^{\Lambda}, f\big) \colon \sum_{i\in I} \lambda_0(i) =_{\mathbb{V}_0} I$.

(ii) Let $g \colon \lambda_0(i_0) \rightsquigarrow \sum_{i\in I} \lambda_0(i)$, defined by $g(x) := (i_0, x)$, for every $x \in \lambda_0(i_0)$, and $h \colon \sum_{i\in I} \lambda_0(i) \rightsquigarrow \lambda_0(i_0)$, defined by $h(i, x) := \lambda_{ii_0}(x)$, for every $(i, x) \in \sum_{i\in I} \lambda_0(i)$. It is straightforward to show that g, h are functions and $(g, h) \colon \sum_{i\in I} \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(i_0)$. $\square$

Proposition 5.5.4. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$, $\Theta \colon \bigcurlywedge_{i\in I} \lambda_0(i)$ a modulus of centres of contraction for λ_0, and X, Y sets.*

(i) *If $h \colon I \rightsquigarrow \sum_{i\in I} \lambda_0(i)$ is defined by $h(i) := \big(i, \Theta_i\big)$, for every $i \in I$, then h is a function and $(\mathtt{pr}_1^{\Lambda}, h) \colon \sum_{i\in I} \lambda_0(i) =_{\mathbb{V}_0} I$.*

(ii) *$\mathbb{F}\big(I, \sum_{i\in I} \lambda_0(i)\big) =_{\mathbb{V}_0} \mathbb{F}(I, I)$.*

(iii) *If X is contractible and Y is a retract of X, then Y is contractible.*

Proof. The proof of (i) is straightforward, and (ii) follows from (i) and Proposition 5.5.1. For the proof of the next theorem though, we write explicitly the witnesses of the required equality in $\mathbb{V}_0$, which are the witnesses provided by the proof of Proposition 5.5.1. Let $\phi \colon \mathbb{F}(I, \sum_{i\in I} \lambda_0(i)) \rightsquigarrow \mathbb{F}(I, I)$, defined by the rule $f \mapsto \phi(f)$, where $\phi(f) := \mathtt{pr}_1^{\Lambda} \circ f$

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[rr, bend right, "\phi(f)"'] & \sum_{i\in I} \lambda_0(i) \arrow[r, "\mathtt{pr}_1^{\Lambda}"] & I \arrow[r, "g"'] \arrow[rr, bend left, "\theta(g)"] & I \arrow[r, "h"'] & \sum_{i\in I} \lambda_0(i).
\end{tikzcd}$$

Clearly, ϕ is a function. Let $\theta \colon \mathbb{F}(I, I) \rightsquigarrow \mathbb{F}(I, \sum_{i\in I} \lambda_0(i))$, defined by the rule $g \mapsto \theta(g)$, where $\theta(g) := h \circ g$, where h is defined in case (i). Clearly, θ is a function. It is straightforward to show that $(\phi, \theta) \colon \mathbb{F}\big(I, \sum_{i\in I} \lambda_0(i)\big) =_{\mathbb{V}_0} \mathbb{F}(I, I)$.

(iii) Let $r \colon X \to Y$ and $s \colon Y \to X$ such that $r \circ s = \mathrm{id}_Y$. It is immediate to show that if $x_0 \in X$ is a centre of contraction for X, then $r(x_0)$ is a centre of contraction for Y. $\square$

Theorem 5.5.5. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$, and let $\Theta \colon \bigcurlywedge_{i\in I} \lambda_0(i)$ be a modulus of centres of contraction for λ_0. If $(\phi, \theta) \colon \mathbb{F}\big(I, \sum_{i\in I} \lambda_0(i)\big) =_{\mathbb{V}_0} \mathbb{F}(I, I)$, where ϕ and θ are defined in the proof of Proposition 5.5.4*(ii), *then $\prod_{i\in I} \lambda_0(i)$ is a retract of $\mathtt{fib}^{\phi}(\mathrm{id}_I)$.*

Proof. By Definition 2.3.4 we have that

$$\mathtt{fib}^{\phi}(\mathrm{id}_I) := \big\{ f \in \mathbb{F}\big(I, \sum_{i\in I} \lambda_0(i)\big) \mid \phi(f) =_{\mathbb{F}(I,I)} \mathrm{id}_I \big\}.$$

We need to find functions $r^{\phi} \colon \mathtt{fib}^{\phi}(\mathrm{id}_I) \to \prod_{i\in I} \lambda_0(i)$ and $s^{\phi} \colon \prod_{i\in I} \lambda_0(i) \to \mathtt{fib}^{\phi}(\mathrm{id}_I)$ such that the following diagram commutes

$$\begin{tikzcd} \prod_{i\in I}\lambda_0(i) \arrow[r, "s^{\phi}"] \arrow[rr, bend right, "\mathrm{id}_{\prod_{i\in I}\lambda_0(i)}"'] & \mathtt{fib}^{\phi}(\mathrm{id}_I) \arrow[r, "r^{\phi}"] & \prod_{i\in I}\lambda_0(i). \end{tikzcd}$$

Let the operation $r^{\phi}\colon \mathtt{fib}^{\phi}(\mathrm{id}_I) \rightsquigarrow \prod_{i\in I}\lambda_0(i)$, defined by the rule $f \mapsto r^{\phi}(f)$, where

$$r^{\phi}(f)\colon \bigcurlywedge_{i\in I}\lambda_0(i), \qquad \big[r^{\phi}(f)\big]_i := \lambda_{\mathtt{pr}_1^{\Lambda}(f(i))i}\Big(\mathtt{pr}_2^{\Lambda}\big(f(i)\big)\Big); \qquad i\in I.$$

As $\phi(f) := \mathtt{pr}_1^{\Lambda}\circ f = \mathrm{id}_I$, we get $[\phi(f)](i) := \mathtt{pr}_1^{\Lambda}(f(i)) =_I i$, hence $\big[r^{\phi}(f)\big]_i \in \lambda_0(i)$, for every $i\in I$. Next we show that $r^{\phi}(f)\in\prod_{i\in I}\lambda_0(i)$. If $i =_I j$, then $f(i) =_{\sum_{i\in I}\lambda_0(i)} f(j)$, and hence

$$\mathtt{pr}_1^{\Lambda}(f(i)) =_I \mathtt{pr}_1^{\Lambda}(f(j)) \quad \& \quad \lambda_{\mathtt{pr}_1^{\Lambda}(f(i))\mathtt{pr}_1^{\Lambda}(f(j))}\big(\mathtt{pr}_2^{\Lambda}(f(i)) =_{\lambda_0(\mathtt{pr}_1^{\Lambda}(f(j)))} \mathtt{pr}_2^{\Lambda}(f(j)).$$

Therefore,

$$\begin{aligned}
\lambda_{ij}\big(\big[r^{\phi}(f)\big]_i\big) &:= \lambda_{ij}\bigg(\lambda_{\mathtt{pr}_1^{\Lambda}(f(i))i}\Big(\mathtt{pr}_2^{\Lambda}\big(f(i)\big)\Big)\bigg)\\
&= \lambda_{\mathtt{pr}_1^{\Lambda}(f(i))j}\Big(\mathtt{pr}_2^{\Lambda}\big(f(i)\big)\Big)\\
&= \lambda_{\mathtt{pr}_1^{\Lambda}(f(j))j}\bigg(\lambda_{\mathtt{pr}_1^{\Lambda}(f(i))\mathtt{pr}_1^{\Lambda}(f(j))}\Big(\mathtt{pr}_2^{\Lambda}(f(i))\Big)\bigg)\\
&= \lambda_{\mathtt{pr}_1^{\Lambda}(f(j))j}\Big(\mathtt{pr}_2^{\Lambda}(f(j))\Big)\\
&:= \big[r^{\phi}(f)\big]_j.
\end{aligned}$$

Next we show that r^{ϕ} is a function. If $f = g$ and $i\in I$, then $f(i) =_{\sum_{i\in I}\lambda_0(i)} g(i)$ i.e.,

$$\mathtt{pr}_1^{\Lambda}(f(i)) =_I \mathtt{pr}_1^{\Lambda}(g(i)) \quad \& \quad \lambda_{\mathtt{pr}_1^{\Lambda}(f(i))\mathtt{pr}_1^{\Lambda}(g(i))}\big(\mathtt{pr}_2^{\Lambda}(f(i)) =_{\lambda_0(\mathtt{pr}_1^{\Lambda}(g(i)))} \mathtt{pr}_2^{\Lambda}(g(i)).$$

Therefore,

$$\begin{aligned}
\big[r^{\phi}(f)\big]_i &:= \lambda_{\mathtt{pr}_1^{\Lambda}(f(i))i}\Big(\mathtt{pr}_2^{\Lambda}\big(f(i)\big)\Big)\\
&= \lambda_{\mathtt{pr}_1^{\Lambda}(g(i))i}\bigg(\lambda_{\mathtt{pr}_1^{\Lambda}(f(i))\mathtt{pr}_1^{\Lambda}(g(i))}\Big(\mathtt{pr}_2^{\Lambda}(f(i))\Big)\bigg)\\
&= \lambda_{\mathtt{pr}_1^{\Lambda}(g(i))i}\Big(\mathtt{pr}_2^{\Lambda}(g(i))\Big)\\
&:= \big[r^{\phi}(g)\big]_i.
\end{aligned}$$

Let the operation $s^{\phi}\colon \prod_{i\in I}\lambda_0(i) \rightsquigarrow \mathtt{fib}^{\phi}(\mathrm{id}_I)$, defined by the rule $\Theta \mapsto s^{\phi}(\Theta)$, where

$$s^{\phi}(\Theta)\colon I \rightsquigarrow \sum_{i\in I}\lambda_0(i), \qquad \big[s^{\phi}(\Theta)\big](i) := (i, \Theta_i); \qquad i\in I.$$

First we show that $s^{\phi}(\Theta)$ is a function. If $i =_I j$, then $(i,\Theta_i) =_{\sum_{i\in I}\lambda_0(i)} (j,\Theta_j)$, as the equality $\Theta_j =_{\lambda_0(j)} \lambda_{ij}(\Theta)$ follows from the hypothesis $\Theta\in\prod_{i\in I}\lambda_0(i)$. To show $s^{\phi}(\Theta)\in\mathtt{fib}^{\phi}(\mathrm{id}_I)$, let $i \in I$, and then $\big(\phi\big(s^{\phi}(\Theta)\big)\big](i) := \mathtt{pr}_1^{\Lambda}(i, \Theta_i) := i$. To show that s^{ϕ} is a function, let $\Theta =_{\prod_{i \in I} \lambda_0(i)} \Theta'$. If $i \in I$, then $[s^{\phi}(\Theta)](i) := (i, \Theta_i) = (i, \Theta'{}_i) := [s^{\phi}(\Theta')](i)$. Finally, we show the commutativity of the initial diagram in the proof. If $i \in I$, then

$$\begin{aligned}\big[r^{\phi}\big(s^{\phi}(\Theta)\big)\big]_i &:= \lambda_{\mathtt{pr}_1^{\Lambda}\left([s^{\phi}(\Theta)](i)\right)i}\Big(\mathtt{pr}_2^{\Lambda}\big(\big[s^{\phi}(\Theta)\big](i)\big)\Big)\\ &:= \lambda_{\mathtt{pr}_1^{\Lambda}(i,\Theta_i)i}\big(\mathtt{pr}_2^{\Lambda}(i, \Theta_i)\big)\\ &:= \lambda_{ii}(\Theta_i)\\ &:= \Theta_i. \end{aligned}$$

$\square$

Corollary 5.5.6. *If $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$ and $\Theta\colon \bigcurlywedge_{i \in I} \lambda_0(i)$ is a modulus of centres of contraction for λ_0, then Θ is centre of contraction for $\prod_{i \in I} \lambda_0(i)$.*

Proof. Since $(\phi, \theta)\colon \mathbb{F}\big(I, \sum_{i \in I} \lambda_0(i)\big) =_{\mathbb{V}_0} \mathbb{F}(I, I)$, by Proposition 2.4.1 the set $\mathtt{fib}^{\phi}(\mathrm{id}_I)$ is contractible and $\theta(\mathrm{id}_I) := h \circ \mathrm{id}_I := h$ is a centre of contraction for $\mathtt{fib}^{\phi}(\mathrm{id}_I)$, where h is defined in Proposition 5.5.4(i). As $r^{\phi}\colon \mathtt{fib}^{\phi}(\mathrm{id}_I) \to \prod_{i \in I} \lambda_0(i)$ is a retraction, by the proof of Proposition 5.5.4(iv) we have that $\prod_{i \in I} \lambda_0(i)$ is contractible and $r^{\phi}(h)$ is a centre of contraction for $\prod_{i \in I} \lambda_0(i)$. If $i \in I$, then $\big[r^{\phi}(h)\big]_I := \lambda_{\mathtt{pr}_1^{\Lambda}(h(i))i}\big(\mathtt{pr}_2^{\Lambda}(h(i))\big) := \lambda_{ii}(\Theta_i) := \Theta_i$, hence $r^{\phi}(h) := \Theta$. $\square$

Proposition 5.5.7. *Let $||X||$ be the truncation of X, Y, Z subsingletons, and E a set.*

(i) *If $f \in \mathbb{F}(Y, Z)$ and $g \in \mathbb{F}(Z, Y)$, then $(f, g)\colon Y =_{\mathbb{V}_0} Z$.*

(ii) *If X is inhabited, then $||X||$ is inhabited.*

(iii) *If $f\colon X \to E$, there is $||f||\colon ||X|| \to ||E||$, such that $||f||(x) := f(x)$, for every $x \in X$.*

(iv) *$Y =_{\mathbb{V}_0} ||Y||$.*

Proof. (i) and (ii) follow immediately from cases (iv) and (i) nof Definition 2.2.8. For the proof of (iii), we define the operation $||f||\colon ||X|| \rightsquigarrow ||E||$ by the rule $||f||(x) := f(x)$, for every $x \in X$. As $||E||$ is a subsingleton, if $x \,{}_{||=_X||}\, x'$, then $||f||(x) := f(x) \,{}_{||=_E||}\, f(x') := ||f||(x')$, and $||f||$ is a function. For the proof of (iv) it is straightforward to show that the operations of type $Y \to ||Y||$ and $||Y|| \to Y$, defined by the identity map-rule, respectively, are functions that witness the equality $Y =_{\mathbb{V}_0} ||Y||$. $\square$

Corollary 5.5.8. *Let $\Lambda := (\lambda_0, \lambda_1) \in \mathtt{Fam}(I)$.*

(i) *$||\Lambda|| := \big(||\lambda_0||, ||\lambda_1||\big) \in \mathtt{Fam}(I)$, where $||\lambda_0||(i)\colon I \rightsquigarrow \mathbb{V}_0$ is defined by*

$$||\lambda_0||(i) := ||\lambda_0(i)||; \quad i \in I, \quad \textit{and}$$

$$||\lambda_1||(i, j) := ||\lambda||_{ij}\colon ||\lambda_0(i)|| \to ||\lambda_0(j)||, \quad ||\lambda||_{ij} := ||\lambda_{ij}||; \quad (i, j) \in D(I).$$

(ii) If $\lambda_0(i)$ is a subsingleton, for every $i \in I$, and $\Theta\colon \prod_{i \in I} ||\lambda_0(i)||$, then $\Theta\colon \prod_{i \in I} \lambda_0(i)$.

(iii) *If $\lambda_0(i)$ is a subsingleton, for every $i \in I$, then $\prod_{i \in I} \lambda_0(i)$ is a subsingleton.*

Proof. (i) To show that $||\lambda||_{ij}$ is well-defined, we use Proposition 5.5.7(iii). To show the properties of a family of sets over I for $||\Lambda||$ we use the corresponding properties for Λ.
(ii) By case (i), if $i =_I j$, then $\Theta_j \in ||\lambda_0(j)||$. As $||\lambda_0(j)||$ is the set $\lambda_0(j)$, we get $\Theta_j \in \lambda_0(j)$. Since $\lambda_0(j)$ is a subsingleton, we get $\Theta_i) =_{\lambda_0(j)} \lambda_{ij}(\Theta_i)$.
(iii) It follows immediately from the definition of the canonical equality on $\prod_{i \in I} \lambda_0(i)$. $\square$

5.6 On 0-sets in BST

Through the notion of set with a proof-relevant equality, Voevodsky's notion of 0-set can be formulated in BST. We need the notion of Martin-Löf set with an inhabited proof-relevant structure to translate some basic facts from Voevodsky's theory of 0-sets in BST.

Definition 5.6.1. *A Martin-Löf set* $\widehat{X} := \big(X, =_X, \texttt{PrfEql}_0^X, \texttt{refl}^X, ^{-1_X}, *_X\big)$ *has an inhabited proof-relevant structure, if there is a dependent operation*

$$\Theta \colon \bigcurlywedge_{(x,y)\in D(X)} \texttt{PrfEql}_0^X(x,y),$$

which we call a modulus of inhabitedness for $\widehat{X}$*, such that the following conditions hold:*
(i) $\Theta_{(x,x)} := \texttt{refl}_x$*, for every* $x \in X$*.*
(ii) $\Theta_{(x,y)}^{-1} = \Theta_{(y,x)}$*, for every* $(x,y) \in D(X)$*.*
(iii) $\Theta_{(x,y)} * \Theta_{(y,z)} = \Theta_{(x,z)}$*, for every* $(x,y),(y,z) \in D(X)$*.*

Proposition 5.6.2. *Let* $\widehat{X} := \big(X, =_X, \texttt{PrfEql}_0^X, \texttt{refl}^X, ^{-1_X}, *_X\big)$ *be a Martin-Löf set and* Θ *a modulus of inhabitedness for* $\widehat{X}$*. If* $(x,x') =_{X\times X} (y,y')$*, let*

$$\texttt{PrfEql}_1^X\big((x,x'),(y,y')\big) := \texttt{PrfEql}_{(x,x')(y,y')}^X \colon \texttt{PrfEql}_0^X(x,x') \to \texttt{PrfEql}_0^X(y,y'),$$

$$\texttt{PrfEql}_{(x,x')(y,y')}^X(r) := \Theta_{(x,y)}^{-1} * r * \Theta_{(x',y')}; \qquad r \in \texttt{PrfEql}_0^X(x,x').$$

(i) $\texttt{PrfEql}^X := \big(\texttt{PrfEql}_0^X, \texttt{PrfEql}_1^X\big) \in \texttt{Fam}(X \times X)$.
(ii) If $x_0 \in X$, the dependent operation $\Theta^{x_0} \colon \bigcurlywedge_{x\in X} \texttt{PrfEql}_0^X(x,x_0)$, defined by $\Theta_x^{x_0} := \Theta_{(x,x_0)}$, for every $x \in X$, is in $\prod_{x\in X} \texttt{PrfEql}_0^X(x,x_0)$.
(iii) *If* X *is a subsingleton, then*[2] $\Theta \colon \prod_{(x,y)\in X\times X} \texttt{PrfEql}_0^X(x,y)$.

Proof. (i) By Definition 5.6.1 we have that[3]

$$\texttt{PrfEql}_{(x,x')(x,x')}^X(r) := \Theta_{(x,x)}^{-1} * r * \Theta_{(x',x')} = \texttt{refl}_x^{-1} * r * \texttt{refl}_{x'} = \texttt{refl}_x * r = r,$$

$$\begin{aligned}
\texttt{PrfEql}_{(y,y')(z,z')}^X\big(\texttt{PrfEql}_{(x,x')(y,y')}^X(r)\big) &:= \texttt{PrfEql}_{(y,y')(z,z')}^X\big(\Theta_{(x,y)}^{-1} * r * \Theta_{(x',y')}\big)\\
&:= \Theta_{(y,z)}^{-1} * \big(\Theta_{(x,y)}^{-1} * r * \Theta_{(x',y')}\big) * \Theta_{(y',z')}\\
&:= \big(\Theta_{(y,z)}^{-1} * \Theta_{(x,y)}^{-1}\big) * r * \big(\Theta_{(x',y')} * \Theta_{(y',z')}\big)\\
&= \big(\Theta_{(x,y)} * \Theta_{(y,z)}\big)^{-1} * r * \big(\Theta_{(x',y')} * \Theta_{(y',z')}\big)\\
&= \Theta_{(x,z)}^{-1} * r * \Theta_{(x',z')},\\
&:= \texttt{PrfEql}_{(x,x')(z,z')}^X(r).
\end{aligned}$$

[2] In intensional MLTT the existence of an object of this type is the definition of X being a subsingleton.
[3] This proof is in complete analogy to the proof that $\mathbb{F}(\Lambda, M) \in \texttt{Fam}(I)$, if $\Lambda, M \in \texttt{Fam}(I)$.

(ii) By Definition 3.5.2(ii), if $x =_X y$, we have that

$$\begin{aligned}\texttt{PrfEql}^X_{(x,x_0)(y,x_0)}\big(\Theta^{x_0}_x\big) &:= \Theta^{-1}_{(x,y)} * \Theta^{x_0}_x * \Theta_{(x_0,x_0)}\\ &= \Theta_{(y,x)} * \Theta_{(x,x_0)} * \texttt{refl}_{x_0}\\ &= \Theta_{(y,x)} * \Theta_{(x,x_0)}\\ &= \Theta_{(y,x_0)}\\ &:= \Theta^{x_0}_y.\end{aligned}$$

(iii) X is a subsingleton, then $D(X) =_{\mathcal{P}(X\times X)} X\times X$ and hence $\Theta\colon \curlywedge_{(x,y)\in X\times X}\texttt{PrfEql}^X_0(x,y)$. If $(x,y),(x',y')\in X\times X$, then

$$\begin{aligned}\texttt{PrfEql}^X_{(x,y)(x'y')}\big(\Theta_{(x,y)}\big) &:= \Theta^{-1}_{(x,x')} * \Theta_{(x,y)} * \Theta_{(y,y')}\\ &= \Theta_{(x',x)} * \Theta_{(x,y)} * \Theta_{(y,y')}\\ &= \Theta_{(x',y)} * \Theta_{(y,y')}\\ &= \Theta_{(x',y')}.\end{aligned}$$ □

Definition 5.6.3. *A set with a proof-relevant equality $\widehat{X}$ is a* 0*-set, if*

$$\forall_{x,y\in X}\forall_{p,q\in \texttt{PrfEql}^X_0(x,y)}\big(p =_{\texttt{PrfEql}^X_0(x,y)} q\big).$$

Proposition 5.6.4. *Let $\widehat{X} := \big(X, =_X, \texttt{PrfEql}^X_0, \texttt{refl}^X, ^{-1_X}, *_X\big)$ be a Martin-Löf set and Θ a modulus of inhabitedness for $\widehat{X}$. If X is a subsingleton, the following are equivalent.*
(i) $\Theta^{x_0}\in\widehat{\prod}_{x\in X}\texttt{PrfEql}^{x_0}(x)$*, for some $x_0\in X$.*
(ii) $\widehat{X}$ *is a* 0*-set.*

Proof. (i)⇒(ii) If $x,y\in X$, the hypothesis (i) means that for every $p\in\texttt{PrfEql}^X_0(x,y)$ we get

$$\Theta_{(y,x_0)} := \Theta^{x_0}_y = \big[\texttt{PrfEql}^{x_0}_1(x,y,p)\big](\Theta^{x_0}_x) := p^{-1} * \Theta_{(x,x_0)},$$

hence $p^{-1} = \Theta_{(y,x_0)} * \Theta_{(x_0,x)} = \Theta_{(y,x)}$, and consequently $p = \Theta_{(x,y)}$. Since p is arbitrary, the required equality follows trivially.
(ii)⇒(i) It follows immediately from the equality $p = \Theta_{(x,y)}$, for every $p\in\texttt{PrfEql}^X_0(x,y)$. □

5.7 Notes

Note 5.7.1. In [1], p. 12, the following criticism to the naive BHK-interpretation is given:

> Many objections can be raised against the above definition. The explanations offered for implication and universal quantification are notoriously imprecise because the notion of function (or rule) is left unexplained. Another problem is that the notions of set and set membership are in need of clarification. But in practice these rules suffice to codify the arguments which mathematicians want to call constructive. Note also that the above interpretation (except for ⊥) does not address the case of atomic formulas.

Note 5.7.2. According to Feferman (see [49], p. 207), the *formal*, or *internal realisability interpretation* of the language $\mathfrak{L}(T)$ of a formal theory T in the language $\mathfrak{L}(T')$ of a formal theory T' is an assignment $\phi \mapsto f\ \boldsymbol{r}\ \phi$ of any formula ϕ of $\mathfrak{L}(T)$ to a formula $\phi_r :\Leftrightarrow f\ \boldsymbol{r}\ \phi$ in $\mathfrak{L}(T')$, where ϕ_r has at most one additional free variable f. This interpretation is *sound* if

$$T \vdash \phi \Rightarrow \exists_{\tau \in \mathtt{Term}(\mathfrak{L}(T'))}\big(T' \vdash \tau\ \boldsymbol{r}\ \phi\big),$$

for every formula ϕ of $\mathfrak{L}(T)$. The added axiom-scheme (A$-$r) "to assert is to realise"

$$\phi \Leftrightarrow \exists_f\big(f\ \boldsymbol{r}\ \phi\big),$$

which expresses the equivalence of the assertion of ϕ with its realisability, reflects the basic tenet of constructive reasoning that a statement is to be asserted only if it is proved. Note that in Feferman's refined theory with MwE the axiom-scheme (A$-$r) implies the principle of dependent choice DC and the presentation axiom! (see [49], pp. 214-215). It is also expected that one can show inductively that the scheme (A$-$r) is itself realisable in some theory S i.e.,

$$\forall_\phi \exists_\tau \bigg(S \vdash \tau\ \boldsymbol{r}\ \big[\phi \Leftrightarrow \exists_f\big(f\ \boldsymbol{r}\ \phi\big)\big]\bigg).$$

In the *informal*, or *external realisability interpretation* of $\mathfrak{L}(T)$ one defines a relation $R(f, \phi)$ between mathematical objects f of some sort and a formula ϕ. E.g., Kleene defined such a relation for $f \in \mathbb{N}$ and ϕ a formula of arithmetic. External realisability interpretations can often be regarded as the reading of a formal $f\ \boldsymbol{r}\ \phi$ in a specific model. Here we described an external realisability interpretation of some part of the language of the informal theory BISH in itself, where the corresponding realisability relation is

$$\mathtt{Prf}(p, \phi) :\Leftrightarrow p \in \mathtt{Prf}(\phi).$$

Why one would choose to work within an informal framework? Maybe because to realise some formula ϕ does not necessarily imply that ϕ is constructively acceptable. E.g., in [49], pp. 207-8, Feferman defined a formal realisability interpretation of $\mathfrak{L}(T_0)$ in itself such that the corresponding axiom scheme (A$-$r) implies the full axiom of choice. Moreover, even if one works with a realisability interpretation that avoids the realisability of the full AC, it is not certain that whatever this theory realises is constructively acceptable, or faithful to some motivating informal constructive theory like BISH. E.g., the realisability of the presentation axiom in T_0^*, which, as we have explained in Note 1.3.2, it holds also in the setoid-interpretation of Bishop sets in intensional MLTT, does not make it necessarily constructively acceptable. In the informal level of BISH there is no reason to accept it. If the main philosophical question regarding Bishop-style constructive mathematics (BCM), in general, is "what is constructive?", an answer provided from a formal treatment of BCM that cannot be "captured" by BISH itself, is not necessarily the "right" answer.

Note 5.7.3. In [49], p. 177, Feferman criticises Bishop for a "certain casualness about mentioning the witnessing information. . . . one is looser in practice in order to keep that from getting too heavy. Practice then looks very much like everyday analysis and it is hard to see what the difference is unless one takes the official definitions seriously". In our opinion, Feferman is right on spotting this casualness in Bishop's account, which is though on purpose, as his crucial comment in [12], p. 67, shows (see Note 7.6.7). One could also say that, if the difference between constructive analysis and everyday, *classical* analysis is difficult to see, then this is an indication of the success of Bishop's way of writing. What we find that is missing when *some* official definitions are not taken seriously is the proof-relevant character of Bishop's analysis and its proximity to *proof-relevant* mathematical analysis, like analysis within MLTT. An important consequence of revealing the witnessing information is the avoidance of choice (see next note).

Note 5.7.4. *The use of the axiom of choice in constructive mathematics is an indication of missing data.* As we have seen already in many cases, and also in Example 5.1.12, the inclusion of witnessing data, like a modulus of some sort, facilitates the avoidance of choice in the corresponding constructive proof. The standard view regarding the use of choice in BISH is that some weak form of choice, countable choice, or dependent choice, is necessary. This is certainly true when the witnessing data are ignored. Richman criticised the use of countable choice in BISH (see [106], and also [115]). The revealing of witnessing data or not in BISH "oscillates" between the two extremes, regarding proof-relevance, which are also the two extremes, regarding choice. The first extreme is classical mathematics based on ZFC, where the complete lack of proof-relevance is combined with the use of a powerful choice axiom, and the second extreme is type-theoretic mathematics based on intensional MLTT, where proof-relevance is "everywhere" and the axiom of choice, the distributivity of $\sum$ over $\prod$, is provable! When the witnessing data are ignored, then some form of weak choice is necessary for BISH, while when the witnessing data are highlighted, then choice is avoided. A similar phaenomemon occurs in univalent type theory. The univalent version of the axiom of choice, in the formulation of which truncation is involved, is not provable. And what truncation does, is to suppress the evidence.

Note 5.7.5. The proof-relevance of BISH is not a priori part of it, but it can be revealed a posteriori. In MLTT and its univalent extensions proof-relevance is a priori part of it. Moreover, many facts are generated or hold automatically by the presence of the J-rule, or the univalence axiom of Voevodsky. As it was pointed out to me by T. Coquand, this feature of MLTT and HoTT was criticised by Deligne in his talk at the memorial meeting of Voevodsky.

Note 5.7.6. The BISH-analogue to (A$-$r) is the following: if ϕ is a BISH-formula and a set $\texttt{Prf}(\phi)$ is predetermined, then

$$\phi \Leftrightarrow \texttt{Prf}(\phi) \text{ is inhabited.}$$

If $\texttt{Prf}(\phi)$ is a set with a proof-relevant equality, this equivalence can be realised in BISH. The BISH-analogue to the soundness of a formal realisability interpretation is

$$\text{BISH} \vdash \phi \Rightarrow \text{BISH} \vdash \texttt{Prf}(\phi) \text{ is inhabited,}$$

for every formula of BISH with a well-defined set $\texttt{Prf}(\phi)$. This should follow from the inductive definition of the BHK-interpretation, which is a definition in the extension BISH** of BISH with inductive definitions with rules of $|X|$-many premisses. Clearly, such an inductive proof requires, in general, a much stronger extension of BISH than BISH*.

Note 5.7.7. A BHK-interpretation of a negated formula $\neg\phi$ is missing from Definitions 5.1.9 and 5.1.13. If $\texttt{Prf}(\phi)$ is given, and we apply the rule of implication for $\neg\phi :\Leftrightarrow \phi \Rightarrow \bot$, then $\texttt{Prf}(\neg\phi) := \mathbb{F}\big(\texttt{Prf}(\phi), \texttt{Prf}(\bot)\big)$. If we accept the clause of the naive BHK-interpretation that $\bot$

has no witness, then we need to state $\texttt{Prf}(\bot) := \emptyset$, and then we get $\texttt{Prf}(\neg\phi) := \mathbb{F}\big(\texttt{Prf}(\phi), \emptyset\big)$. As we have already remarked in Note 2.9.17, the use of the empty set in BISH is problematic, and so is the status of the object $\mathbb{F}\big(\texttt{Prf}(\phi), \emptyset\big)$. As negated formulas are rare in BISH, we find safer at the moment to exclude them from our account of a BHK-interpretation of BISH.

Note 5.7.8. A proof-relevant membership-condition for $\mathbb{R}$ can be defined, if we treat a real number as a (general) Cauchy sequence of rationals, namely

$$x \in \mathbb{R} :\Leftrightarrow x \in \mathbb{F}(\mathbb{N}^+, \mathbb{Q}) \ \ \& \ \ \exists_{C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)}\big(C : x \in \mathbb{R}\big),$$

$$C : x \in \mathbb{R} :\Leftrightarrow \forall_{k \in \mathbb{N}^+} \forall_{m,n \geq C(k)} \left(|x_m - x_n| \leq \frac{1}{k} \right).$$

If $x \in \mathbb{F}(\mathbb{N}^+, \mathbb{Q})$, we define then

$$\texttt{PrfMemb}_0^{\mathbb{R}}(x) := \big\{ C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid C : x \in \mathbb{R} \big\}.$$

Note 5.7.9. In intensional MLTT the groupoid properties of $*$ and $^{-1}$ hold always by the J-rule. This is "good" and "bad". It is "good", because something very useful holds. On the other hand, as we have seen in the previous examples, it is not always the case in the practice of BISH that all these conditions hold simultaneously. Hence, it is "bad" that intensional MLTT is not as "flexible" as BISH (this is related to Deligne's critique mentioned in Note 5.7.5).

Note 5.7.10. The definition of equality on $\widehat{\sum}_{i \in I} \lambda_0(i)$ can be seen as a definitional translation of Theorem 2.7.2 of book-HoTT [124], where if $w, w' \in \sum_{i \colon I} P(i)$, then

$$w = w' \simeq \sum_{p \colon \texttt{pr}_1(w) = \texttt{pr}_1(w')} p_*(\texttt{pr}_2(w)) = \texttt{pr}_2(w').$$

The definition of $\widehat{\prod}_{i \in} \lambda_0(i)$ can be seen as a definitional translation of Lemma 2.3.4 of book-HoTT, where if $\Phi \colon \prod_{i \in I} P(i)$, there is a term

$$\texttt{apd}_\Phi \colon \prod_{p \colon i = j} \big(p_*(\Phi_i) = \Phi_j\big).$$

These definitions motivated Definition 3.9.3.

Note 5.7.11. Theorem 5.3.7 is a translation of the type-theoretic contractibility of the singleton type (see [42]) into BST. If M is the judgement (or the term) expressing this contractibility (see also [96]), Martin-Löf's J-rule trivially implies M, and it is equivalent to M and the transport (see [42]). In BISH we do not have the J-rule, but we have transport in a definitional way only. As Theorem 5.3.7 indicates, a definitional form of M is provable in BST, although there exists no direct translation of the J-rule in BST.

Note 5.7.12. In analogy to the category of setoids and setoid-maps, several categorical constructions for Martin-Löf sets and maps between them can be carried out. A family of sets over a Martin-Löf set $\widehat{I}$ corresponds to Palmgren's notion of a proof-relevant family of setoids. As it is noted by Palmgren in [82], p. 47, proof-relevant families of setoids are very common in MLTT, and as he explains in [82], pp. 37-38, such families are "difficult to use for certain purposes", like the construction of categories with equality on objects. In our language Palmgren's argument is reformulated as follows.

Definition 5.7.13. *Let $\widehat{\Lambda}$ be a family of sets over a Martin-Löf set $\widehat{I}$. Let the collection of objects be the set $\widehat{I}$, and for every $i, j \in I$ let* $\mathrm{Hom}(i,j)$ *be the set of triplets (i, f, j), where $f \in \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big)$. Moreover, let $(i, f, j) \sim (k, g, l)$, if there are $p \in \mathtt{PrfEql}_0^I(i,k)$ and $q \in \mathtt{PrfEql}_0^I(j,l)$ such that the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "f"] \arrow[d, "\lambda_{ik}^p"'] & \lambda_0(j) \arrow[d, "\lambda_{jk}^q"] \\
\lambda_0(k) \arrow[r, "g"'] & \lambda_0(l).
\end{tikzcd}$$

Two arrows (i, f, j) and (k, g, l) are composable, if there is $t \in \mathtt{PrfEql}_0^I(l,i)$

$$\lambda_0(k) \xrightarrow{g} \lambda_0(l) \xrightarrow{\lambda_{li}^t} \lambda_0(i) \xrightarrow{f} \lambda_0(j),$$

and then their composition is defined by $r \circ s := (k, f \circ \lambda_{li}^t \circ g, j)$

$$\begin{tikzcd}
\lambda_0(k) \arrow[r, "g"] \arrow[d, "\lambda_{kk'}^p"] & \lambda_0(l) \arrow[r, "\lambda_{li}^t"] \arrow[d, "\lambda_{ll'}^q"'] & \lambda_0(i) \arrow[r, "f"] \arrow[d, "\lambda_{ii'}^r"] & \lambda_0(j) \arrow[d, "\lambda_{jj'}^s"] \\
\lambda_0(k') \arrow[r, "g'"'] & \lambda_0(l') \arrow[r, "\lambda_{li}^{t'}"'] & \lambda_0(i') \arrow[r, "f'"'] & \lambda_0(j').
\end{tikzcd}$$

If $\widehat{\Lambda}$ is not proof-irrelevant, we cannot show, for arbitrary $t \in \mathtt{PrfEql}_0^I(l,i)$ and $t' \in \mathtt{PrfEql}_0^I(l',i')$, that the above outer diagram commutes. If some $t \in \mathtt{PrfEql}_0^I(l,i)$ is given though, there is $t' \in \mathtt{PrfEql}_0^I(l',i')$ such that the above outer diagram commutes.

Proposition 5.7.14. *Let $\widehat{\Lambda}$ be a family of sets over the Martin-Löf set $\widehat{I}$, $r := (i, f, j), s := (k, g, l)$, and let $r' := (i', f', j'), s' := (k', g', l')$ be arrows according to Definition 5.7.13. If $r \sim r'$, $s \sim s'$, and r and s are composable, then r' and s' are composable, and $s \circ r \sim s' \circ r'$.*

Proof. Let $p \in \mathtt{PrfEql}_0^I(k,k')$ and $q \in \mathtt{PrfEql}_0^I(l,l')$ such that the above left diagram commutes, and let $r \in \mathtt{PrfEql}_0^I(i,i')$ and $s \in \mathtt{PrfEql}_0^I(j,j')$ such that the above right diagram commutes. Since r and s are composable, there is $t \in \mathtt{PrfEql}_0^I(l,i)$. Since $t' := q^{-1} * t * r \in \mathtt{PrfEql}_0(l',i')$, we conclude that r' and s' are composable, and $r' \circ s' := \big(k', f' \circ \lambda_{li}^{t'} \circ g', j'\big)$. By Definition 5.3.4 we have that

$$\begin{aligned}
\lambda_{li}^{t'} \circ \lambda_{ll'}^q &= \big(\lambda_{ii'}^r \circ \lambda_{li}^t \circ \lambda_{l'l}^{q^{-1}}\big) \circ \lambda_{ll'}^q \\
&= \lambda_{ii'}^r \circ \lambda_{li}^t \circ \big(\lambda_{l'l}^{q^{-1}} \circ \lambda_{ll'}^q\big) \\
&= \lambda_{ii'}^r \circ \lambda_{li}^t \circ \mathrm{id}_{\lambda_0(l)} \\
&= \lambda_{ii'}^r \circ \lambda_{li}^t
\end{aligned}$$

i.e., the middle above diagram commutes. Since all the above inner diagrams commute, the above outer diagram commutes, hence $s \circ r \sim s' \circ r'$. □

As Palmgren commented on this issue in a personal communication, what we finally get is an *almost category*, and not a category.

Note 5.7.15. As we have already remarked in Note 2.9.13, Proposition 5.5.1 is an example of a result in BST the analogue of which in HoTT is shown with the axiom of univalence $\texttt{UA}$ in book-HoTT (the axiom FunExt can also be used instead). Theorem 5.5.5 is the translation of Theorem 4.9.4 in book-HoTT, where the universe in its hypothesis is supposed to be univalent. Corollary 5.5.6 is the translation in BST of the fact that $\texttt{UA}$ implies the weak function extensionality.

Note 5.7.16. Further results from book-HoTT can be translated in BISH through BST. E.g., Lemmata 4.8.1 and 4.8.2 in book-HoTT take the following form in BST. If $\widehat{\Lambda} := (\lambda_0, \lambda_1) \in \texttt{Fam}(\widehat{I})$, where $\widehat{I}$ is a Martin-Löf set, then, for every $i \in I$, we have that $\texttt{fib}^{\texttt{pr}_1^{\widehat{\Lambda}}}(i) =_{\mathbb{V}_0} \lambda_0(i)$, while if $\widehat{f} : \widehat{X} \to \widehat{Y}$, then $X =_{\mathbb{V}_0} \widehat{\sum_{y \in Y}} \texttt{fib}^f(y)$. Following the book-HoTT, we can use the translation of the "left universal property of identity types" in BST, namely the equality

$$\left(\sum_{j \in I} \sum_{p \in \texttt{PrfEql}_0^I(j,i)} \lambda_0(j) \right) =_{\mathbb{V}_0} \lambda_0(i).$$

Families of Martin-Löf sets over some Martin-Löf set $\widehat{I}$ can also be studied along this direction.

Chapter 6

Families of sets and spectra of Bishop spaces

We connect various notions and results from the theory of families of sets and subsets to the theory of Bishop spaces, a function-theoretic approach to constructive topology. Associating in an appropriate way to each set $\lambda_0(i)$ of an I-family of sets Λ a Bishop topology F_i a spectrum $S(\Lambda)$ of Bishop spaces is generated. The $\sum$-set and the $\prod$-set of a spectrum $S(\Lambda)$ are equipped with canonical Bishop topologies. A direct spectrum of Bishop spaces is a family of Bishop spaces associated to a direct family of sets. The direct and inverse limits of direct spectra of Bishop spaces are studied. Direct spectra of Bishop subspaces are also examined. For all notions and facts on Bishop spaces mentioned in this chapter we refer to section 9.1 of the Appendix. Many Bishop topologies are defined inductively within the extension BISH* of BISH with inductive definitions with rules of countably many premises. For all notions and facts on directed sets mentioned in this chapter we refer to section 9.2 of the Appendix.

6.1 Spectra of Bishop spaces

Roughly speaking, if S is a structure on some set, an S-spectrum is an I-family of sets Λ such that each set $\lambda_0(i)$ is equipped with a structure S_i, which is compatible with the transport maps λ_{ij} of Λ. Accordingly, a spectrum of Bishop spaces is an I-family of sets Λ such that each set $\lambda_0(i)$ is equipped with a Bishop topology, which is compatible with the transport maps of Λ. As expected, in the case of a spectrum of Bishop spaces this compatibility condition is that the transport maps λ_{ij} are Bishop morphisms i.e. $\lambda_{ij} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$. It is natural to associate to Λ an I-family of sets $\Phi := (\phi_0^{\Lambda}, \phi_1^{\Lambda})$ such that $\mathcal{F}_i := \big(\lambda_0(i), \phi_0^{\Lambda}(i)\big)$ is the Bishop space corresponding to $i \in I$. If $i =_I j$, and if we put no restriction to the definition of $\phi_{ij}^{\Lambda} : F_i \to F_j$, we need to add extra data in the definition of a map between spectra of Bishop spaces. Since the map $\lambda_{ji}^* : F_i \to F_j$, where λ_{ji}^* is the element of $\mathbb{F}(F_i, F_j)$ induced by the Bishop morphism $\Lambda_{ji} \in \mathrm{Mor}(\mathcal{F}_j, \mathcal{F}_i)$, is generated by the data of Λ, it is natural to define $\phi_{ij} := \lambda_{ji}^*$. In this way proofs of properties of maps between spectra of Bishop spaces become easier. If X is a set, we use the notation $\mathbb{F}(X) := \mathbb{F}(X, \mathbb{R})$, and every subset of $\mathbb{F}(X)$ considered in this chapter is an extensional subset of it.

Definition 6.1.1. *Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$. A family of Bishop topologies associated to Λ is a pair $\Phi^{\Lambda} := \big(\phi_0^{\Lambda}, \phi_1^{\Lambda}\big)$, where $\phi_0^{\Lambda} : I \rightsquigarrow \mathbb{V}_0$ and $\phi^{\Lambda} \colon \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\phi_0^{\lambda}(i), \phi_0^{\Lambda}(j)\big)$,*

such that the following conditions hold:

(i) $\phi_0^{\Lambda}(i) := F_i \subseteq \mathbb{F}(\lambda_0(i))$*, and* $\mathcal{F}_i := (\lambda_0(i), F_i)$ *is a Bishop space, for every* $i \in I$.

(ii) $\lambda_{ij} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$*, for every* $(i,j) \in D(I)$.

(iii) $\phi_1^{\Lambda}(i,j) := \lambda_{ji}^*$, for every $(i,j) \in D(I)$, where, if $f \in F_i$, the induced map $\lambda_{ji}^* \colon F_i \to F_j$ from λ_{ji} is defined by $\lambda_{ji}^*(f) := f \circ \lambda_{ji}$, for every $f \in F_i$.

The structure $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^{\Lambda}, \phi_1^{\Lambda})$ is called a *spectrum of Bishop spaces* over I, or an *I-spectrum* with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j)\in D(I)}$. If $S(M) := (\mu_0, \mu_1, \phi_0^{M}, \phi_1^{M})$ is an I-spectrum with Bishop spaces $(\mathcal{G}_i)_{i\in I}$ and Bishop isomorphisms $(\mu_{ij})_{(i,j)\in D(I)}$, a *spectrum-map* Ψ from $S(\Lambda)$ to $S(M)$, in symbols $\Psi \colon S(\Lambda) \Rightarrow S(M)$, is a family-map $\Psi \colon \Lambda \Rightarrow M$. The totality of spectrum-maps from $S(\Lambda)$ to $S(M)$ is denoted by $\mathtt{Map}_I(S(\Lambda), S(M))$ and it is equipped with the equality of $\mathtt{Map}_I(\Lambda, M)$. A spectrum-map $\Phi \colon S(\Lambda) \Rightarrow S(M)$ is called *continuous*, if $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)$, for every $i \in I$, and we denote by $\mathtt{Cont}_I(S(\Lambda), S(M))$ their totality, which is equipped with the equality of $\mathtt{Map}_I(\Lambda, M)$. The totality $\mathtt{Spec}(I)$ of I-spectra of Bishop spaces is equipped with the equality $S(\Lambda) =_{\mathtt{Spec}(I)} S(M)$ if and only if there exist continuous spectrum-maps $\Phi \colon S(\Lambda) \Rightarrow S(M)$ and $\Psi \colon S(M) \Rightarrow S(\Lambda)$ such that $\Phi \circ \Psi =_{\mathtt{Map}_I(M,M)} \mathrm{Id}_M$ and $\Psi \circ \Phi =_{\mathtt{Map}_I(\Lambda,\Lambda)} \mathrm{Id}_{\Lambda}$.

As the identity map $\mathrm{id}_X \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, where $\mathcal{F} := (X, F)$ is a Bishop space, the identity family-map $\mathrm{id}_{\Lambda} \colon \Lambda \Rightarrow \Lambda$ is a continuous spectrum-map from $S(\Lambda)$ to $S(\Lambda)$. As the composition of Bishop morphism is a Bishop morphism, if $\Phi \colon S(\Lambda) \Rightarrow S(M)$ and $\Xi \colon S(M) \Rightarrow S(N)$ are continuous spectrum-maps, then $\Xi \circ \Phi \colon S(\Lambda) \Rightarrow S(N)$ is a continuous spectrum-map.

Definition 6.1.2. *The structure* $S(2) := \big(\lambda_0^{2}, \lambda_1^{2}, \phi_0^{\Lambda^{2}}, \phi_1^{\Lambda^{2}}\big)$*, where* $\Lambda^{2} := (\lambda_0^{2}, \lambda_1^{2})$ *is the 2-family of* X *and* Y*, and* $\Phi^{\Lambda^{2}} := \big(\phi_0^{\Lambda^{2}}, \phi_1^{\Lambda^{2}}\big)$ *is the 2-family of the sets* F *and* G*,* $\phi_0^{\Lambda^{2}}(0) := F$ *is a topology on* X*, and* $\phi_0^{\Lambda^{2}}(1) := G$ *is a topology on* Y*, is the 2-spectrum of* $\mathcal{F}$ *and* $\mathcal{G}$.

Since $\mathrm{id}_X \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, $\mathrm{id}_Y \in \mathrm{Mor}(\mathcal{G}, \mathcal{G})$, $\phi_1^{\Lambda^{2}}(0,0) := \mathrm{id}_X^*$ with $\mathrm{id}_X^* := \mathrm{id}_F$, and similarly, $\phi_1^{\Lambda^{2}}(1,1) := \mathrm{id}_Y^*$ with $\mathrm{id}_Y^* := \mathrm{id}_G$, we conclude that $S(2)$ is a 2-spectrum with Bishop spaces $\mathcal{F}, \mathcal{G}$ and Bishop isomorphisms $\mathrm{id}_X, \mathrm{id}_Y$.

Remark 6.1.3. *Let* $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^{\Lambda}, \phi_1^{\Lambda})$ *be an* I*-spectrum with Bishop spaces* $\mathcal{F}_i$ *and Bishop isomorphisms* λ_{ij}*,* $S(M) := (\mu_0, \mu_1, \phi_0^{M}, \phi_1^{M})$ *an* I*-spectrum with Bishop spaces* $\mathcal{G}_i$ *and Bishop isomorphisms* μ_{ij}*, and* $\Psi \colon S(\Lambda) \Rightarrow S(M)$*. Then* $\Phi^{\Lambda} := \big(\phi_0^{\Lambda}, \phi_1^{\Lambda}\big) \in \mathtt{Fam}(I)$*, and if* Ψ *is continuous, then, for every* $(i,j) \in D(I)$*, the following diagram commutes*

$$\begin{tikzcd} G_i \arrow[r, "\mu_{ji}^*"] \arrow[d, "\Psi_i^*"'] & G_j \arrow[d, "\Psi_j^*"] \\ F_i \arrow[r, "\lambda_{ji}^*"'] & F_j. \end{tikzcd}$$

Proof. If $i \in I$, then $\phi_{ii}^{\Lambda}(f) := f \circ \lambda_{ii} := f \circ \mathrm{id}_{\lambda_0(i)} := f$. If $i =_I j =_I k$ and $f \in F_i$, then

$$\begin{tikzcd} F_i \arrow[d, "\lambda_{ji}^*"'] \arrow[dr, "\lambda_{ki}^*"] & \\ F_j \arrow[r, "\lambda_{kj}^*"'] & F_k \end{tikzcd}$$

$\lambda_{kj}^*\big(\lambda_{ji}^*(f)\big) := \lambda_{kj}^*(f \circ \lambda_{ji}) := (f \circ \lambda_{ji}) \circ \lambda_{kj} = f \circ (\lambda_{ji} \circ \lambda_{kj}) = f \circ \lambda_{ki} := \lambda_{ki}^*(f)$. By the definition of a continuous spectrum-map we have that if $(j,i) \in D(I)$, then

$$\Psi_j^*(\mu_{ji}^*(g)) := \Psi_j^*(g \circ \mu_{ji}) := (g \circ \mu_{ji}) \circ \Psi_j = g \circ (\mu_{ji} \circ \Psi_j) = g \circ (\Psi_i \circ \lambda_{ji})$$
$$= (g \circ \Psi_i) \circ \lambda_{ji} := \lambda_{ji}^*(g \circ \Psi_i) := \lambda_{ji}^*(\Psi_i^*(g)). \qquad \square$$

6.2 The topology on the $\sum$- and the $\prod$-set of a spectrum

Remark 6.2.1. *Let $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^\Lambda, \phi_1^\Lambda) \in \mathtt{Spec}(I)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j)\in D(I)}$. If $\Theta \in \prod_{i\in I} F_i$, the following operation is a function*

$$f_\Theta : \bigg(\sum_{i\in I} \lambda_0(i)\bigg) \rightsquigarrow \mathbb{R}, \qquad f_\Theta(i,x) := \Theta_i(x); \qquad (i,x) \in \sum_{i\in I} \lambda_0(i).$$

Proof. If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y$, by the definition of $\prod_{i\in I} F_i$ we have that $\Theta_i =_{F_i} \phi_{ji}^\Lambda(\Theta_j) := \lambda_{ij}^*(\Theta_j) := \Theta_j \circ \lambda_{ij}$, hence $f_\Theta(i,x) := \Theta_i(x) =_{\mathbb{R}} [\Theta_j \circ \lambda_{ij}](x) =_{\mathbb{R}} \Theta_j(y) := f_\Theta(j,y)$. $\square$

Definition 6.2.2. *Let $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^\Lambda, \phi_1^\Lambda) \in \mathtt{Spec}(I)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j)\in D(I)}$. The sum Bishop space of $S(\Lambda)$ is the pair*

$$\sum_{i\in I} \mathcal{F}_i := \bigg(\sum_{i\in I} \lambda_0(i), \int_{i\in I} F_i\bigg), \qquad \textit{where} \quad \int_{i\in I} F_i := \bigvee_{\Theta\in\prod_{i\in I} F_i} f_\Theta,$$

and the dependent product Bishop space of $S(\Lambda)$ is the pair

$$\prod_{i\in I} \mathcal{F}_i := \bigg(\prod_{i\in I} \lambda_0(i), \oint_{i\in I} F_i\bigg), \qquad \textit{where} \quad \oint_{i\in I} F_i := \bigvee_{i\in I}^{f\in F_i} (f \circ \pi_i^\Lambda),$$

and π_i^Λ is the projection function defined in Proposition 3.3.5(i).

Proposition 6.2.3. *Let $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^\Lambda, \phi_1^\Lambda) \in \mathtt{Spec}(I)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j)\in D(I)}$, $S(M) := (\mu_0, \mu_1, \phi_0^M, \phi_1^M) \in Spec(I)$ with Bishop spaces $(\mathcal{G}_i)_{i\in I}$ and Bishop isomorphisms $(\mu_{ij})_{(i,j)\in D(I)}$, and $\Psi\colon S(\Lambda) \Rightarrow S(M)$.*

(i) *If $i \in I$, then $e_i^\Lambda \in \mathrm{Mor}\big(\mathcal{F}_i, \sum_{i\in I} \mathcal{F}_i\big)$.*

(ii) *If Ψ is continuous, then $\Sigma\Psi \in \mathrm{Mor}\big(\sum_{i\in I} \mathcal{F}_i, \sum_{i\in I} \mathcal{G}_i\big)$.*

(iii) *If Ψ is continuous, then $\Pi\Psi \in \mathrm{Mor}\big(\prod_{i\in I} \mathcal{F}_i, \prod_{i\in I} \mathcal{G}_i\big)$.*

Proof. (i) By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{\Theta\in\prod_{i\in I} F_i}\big(f_\Theta \circ e_i^\Lambda \in F_i\big)$. If $x \in \lambda_0(i)$, then $\big(f_\Theta \circ e_i^\Lambda\big)(x) := f_\Theta(i,x) := \Theta_i(x)$, and $f_\Theta \circ e_i^\Lambda := \Theta_i \in F_i$.
(ii) By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{\Theta'\in\prod_{i\in I} G_i}\big(f_{\Theta'} \circ \Sigma\Psi \in \int_{i\in I} F_i\big)$. If $i \in I$ and $x \in \lambda_0(i)$, we have that $\big(f_{\Theta'} \circ \Sigma\Psi\big)(i,x) := f_{\Theta'}\big(i, \Psi_i(x)\big) := {\Theta_i}'\big(\Psi_i(x)\big) := f_\Theta(i,x)$, where $\Theta : \curlywedge_{i\in I} F_i$ is defined by $\Theta_i := {\Theta_i}' \circ \Psi_i$, for every $i \in I$. By the continuity of Ψ we get $\Theta_i \in F_i$. We show that $\Theta \in \prod_{i\in I} F_i$. If $i =_I j$, by the commutativity of the diagram of Remark 6.1.3 we get $\phi_{ij}^\Lambda(\Theta_i) := \lambda_{ji}^*(\Theta_i) := \Theta_i \circ \lambda_{ji} := ({\Theta_i}' \circ \Psi_i) \circ \lambda_{ji} := ({\Theta_i}' \circ \mu_{ji}) \circ \Psi_j = {\Theta_j}' \circ \Psi_j := \Theta_j$.
(iii) By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{i\in I}\forall_{g\in G_i}\big((g \circ \pi_i^M) \circ \prod \Psi \in \oint_{i\in I} F_i\big)$.

If $\Theta \in \prod_{i\in I}\lambda_0(i)$, then $\big[(g\circ\pi_i^M)\circ\Pi\Psi\big](\Theta) := g(\Psi_i(\Theta_i)) := \big[(g\circ\Psi_i)\circ\pi_i^\Lambda\big](\Theta)$, hence $(g\circ\pi_i^M)\circ\Pi\Psi = (g\circ\Psi_i)\circ\pi_i^\Lambda$. By the continuity of Ψ we have that $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i,\mathcal{G}_i)$, hence $g\circ\Psi_i\in F_i$, and $(g\circ\Psi_i)\circ\pi_i^\Lambda \in \fint_{i\in I}F_i$. □

If $S(\Lambda^{\mathbb{2}})$ is the spectrum of the Bishop spaces $\mathcal{F}$ and $\mathcal{G}$, its sum Bishop space

$$\mathcal{F}+\mathcal{G} := \left(\sum_{i\in\mathbb{2}}\lambda_0^{\mathbb{2}}(i), \int_{i\in\mathbb{2}}\phi_0^{\Lambda^{\mathbb{2}}}(i)\right) := (X+Y, F+G)$$

is called the *coproduct* of $\mathcal{F}$ and $\mathcal{G}$. By definition of the sum Bishop topology

$$F+G := \bigvee_{f\in F, g\in G} f\oplus g,$$

$$(f\oplus g)(w) := \begin{cases} f(x) & , \exists_{x\in X}\big(w := (0,x)\big) \\ g(y) & , \exists_{y\in Y}\big(w := (1,y)\big) \end{cases} ; \qquad f\in F,\ g\in G.$$

The coproduct Bishop space is the coproduct in the category of Bishop spaces.

Proposition 6.2.4. *Let $\mathcal{F} := (X,F)$ and $\mathcal{G} := (Y,G)$ be Bishop spaces.*
(i) *The function $i_X : X \to X+Y$, defined by $x \mapsto (0,x)$, for every $x\in X$, is in* $\mathrm{Mor}(\mathcal{F},\mathcal{F}+\mathcal{G})$.
(ii) *The function $i_Y : Y \to X+Y$, defined by $y \mapsto (1,y)$, for every $y\in Y$, is in* $\mathrm{Mor}(\mathcal{G},\mathcal{F}+\mathcal{G})$.
(iii) *If $\mathcal{H} := (Z,H)$ is a Bishop space, $\phi_X \in \mathrm{Mor}(\mathcal{F},\mathcal{H})$ and $\phi_Y\in\mathrm{Mor}(\mathcal{G},\mathcal{H})$, there is a unique $\phi\in\mathrm{Mor}(\mathcal{F}+\mathcal{G},\mathcal{H})$ such that the following inner diagrams commute*

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\phi_X"] \arrow[r, "i_X"'] & X+Y \arrow[u, dashed, "\phi"] & Y. \arrow[ul, "\phi_Y"'] \arrow[l, "i_Y"] \end{tikzcd}$$

Proof. (i) By definition $i_X\in\mathrm{Mor}(\mathcal{F},\mathcal{F}+\mathcal{G})$ if and only if $\forall_{f\in F}\forall_{g\in G}\big((f\oplus g)\circ i_X \in F\big)$. It is immediate to see that $(f\oplus g)\circ i_X := f\in F$. Case (ii) is shown similarly.
(iii) We define $\phi : X+Y\to\mathbb{R}$ by

$$\phi(w) := \begin{cases} \phi_X(x) & , \exists_{x\in X}\big(w := (0,x)\big) \\ \phi_Y(y) & , \exists_{y\in y}\big(w := (1,y)\big), \end{cases}$$

and since $\phi\circ i_X := \phi_X$ and $\phi\circ i_Y := \phi_Y$, the diagrams commute. If $h\in H$, then

$$(h\circ\phi)(w) := \begin{cases} h(\phi_X(x)) & , \exists_{x\in X}\big(w := (0,x)\big) \\ h(\phi_Y(y)) & , \exists_{y\in Y}\big(w := (1,y)\big), \end{cases}$$

and since $h\circ\phi_X\in F$ and $h\circ\phi_Y\in G$, we get $h\circ\phi := (h\circ\phi_X)\oplus(h\circ\phi_Y)\in F+G$. The uniqueness of ϕ is immediate to show. □

Proposition 6.2.5. *If F is a topology on X, G is a topology on Y, $F_0\subseteq\mathbb{F}(X,\mathbb{R})$, and $G_0\subseteq\mathbb{F}(Y,\mathbb{R})$ are inhabited, then*

$$\left(\bigvee F_0\right)+G = \bigvee_{f_0\in F_0, g\in G} f_0\oplus g := F_0+G,$$

$$F + \Big(\bigvee G_0\Big) = \bigvee_{f \in F, g_0 \in G_0} f \oplus g_0 := F + G_0.$$

Proof. We prove only the first equality, and the proof of the second is similar. Clearly, $F_0 + G \subseteq \big(\bigvee F_0\big) + G$. Since $\big(\bigvee F_0\big) + G := \bigvee_{f \in \bigvee F_0, g \in G} f \oplus g$, and since $F_0 + G$ is a topology, for the converse inclusion it suffices to show inductively that $\forall_{f \in \bigvee F_0} P(f)$, where $P(f) :\Leftrightarrow \big(\forall_{g \in G}\big(f \oplus g \in F_0 + G\big)\big)$. If $f_0 \in F_0$, then $P(f_0)$ follows immediately. If $a \in \mathbb{R}$ and $g \in G$, we show that $\overline{a}^X \oplus g \in F_0 + G$. Since $\frac{g}{3} - \overline{a}^Y \in G$, by the inductive hypothesis on $f_0 \in F_0$, we get $f_0 \oplus \big(\frac{g}{3} - \overline{a}^Y\big) \in F_0 + G$. Since

$$(*) \quad (f_1 \oplus g_1) + (f_2 \oplus g_2) = (f_1 + f_2) \oplus (g_1 + g_2),$$

and since $\overline{a}^X \oplus \overline{a}^Y = \overline{a}^{X+Y} \in F_0 + G$, by $(*)$ we get $\big(f_0 + \overline{a}^X\big) \oplus \frac{g}{3} = \big(f_0 \oplus \big(\frac{g}{3} - \overline{a}^Y\big)\big) + \big(\overline{a}^X \oplus \overline{a}^Y\big) \in F_0 + G$. Since by the inductive hypothesis $\big(f_0 \oplus -\frac{2g}{3}\big) \in F_0 + G$, and since $-(f \oplus g) = (-f) \oplus (-g)$, we also get $\big(-f_0 \oplus \frac{2g}{3}\big) \in F_0 + G$, hence by $(*)$

$$\overline{a} \oplus g = \big[(f_0 + \overline{a}^X) \oplus \frac{g}{3}\big] + \big[\big(-f_0 \oplus \frac{2g}{3}\big)\big] \in F_0 + G.$$

Let $f_1, f_2 \in \bigvee F_0$ such that $P(f_1)$ and $P(f_2)$. If $g \in G$, by these hypotheses we get $f_1 \oplus \frac{g}{2} \in F_0 + G$ and $f_2 \oplus \frac{g}{2} \in F_0 + G$. Hence by $(*)$

$$(f_1 + f_2) \oplus g = \big(f_1 \oplus \frac{g}{2}\big) + \big(f_2 \oplus \frac{g}{2}\big) \in F_0 + G.$$

If $\phi \in \mathrm{Bic}(\mathbb{R})$ and $f \in \bigvee F_0$ such that $P(f)$, we show $P(\phi \circ f)$. If $g \in G$, then

$$(**) \quad \phi \circ (f \oplus g) := (\phi \circ f) \oplus (\phi \circ g).$$

By $P(f)$ we get $f \oplus \overline{0}^Y \in F_0 + G$, and since $\phi \circ \overline{0}^Y = \overline{\phi(0)}^Y \in F_0 + G$, by $(**)$

$$(\phi \circ f) \oplus \overline{\phi(0)}^Y = \big(\phi \circ f\big) \oplus \big(\phi \circ \overline{0}^Y\big) = \phi \circ \big(f \oplus \overline{0}^Y\big) \in F_0 + G.$$

By the case of constant functions $\overline{0}^X \oplus \big(g - \overline{\phi(0)}^Y\big) \in F_0 + G$, hence by $(*)$

$$(\phi \circ f) \oplus g = \big[(\phi \circ f) \oplus \overline{\phi(0)}^Y\big] + \big[\overline{0}^X \oplus \big(g - \overline{\phi(0)}^Y\big)\big] \in F_0 + G.$$

If $f \in \bigvee F_0$ such that for every $n \geq 1$ there is some $f_n \in \bigvee F_0$ such that $P(f_n)$ and $U\big(X; f, f_n, \frac{1}{n}\big)$, then, for every $g \in G$, we get $U\big(X + Y; f \oplus g, f_n \oplus g, \frac{1}{n}\big)$, and since $F_0 + G$ is a Bishop topology, by BS_4 we get $f \oplus g \in F_0 + G$, hence $P(f)$. $\square$

6.3 Direct spectra of Bishop spaces

As in the case of a family of Bishop spaces associated to an I-family of sets, the family of Bishop spaces associated to an $(I, \preccurlyeq)$-family of sets is defined in a minimal way from the data of $\Lambda^{\preccurlyeq}$. According to these data, the corresponding functions $\phi_{ij}^{\preccurlyeq}$ behave necessarily in a contravariant manner i.e., $\phi_{ij}^{\Lambda^{\preccurlyeq}} : F_j \to F_i$. Moreover, the transport maps $\lambda_{ij}^{\preccurlyeq}$ are Bishop morphisms, and not necessarily Bishop isomorphisms.

Definition 6.3.1. *Let $(I, \preccurlyeq)$ be a directed set, and let $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}), M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq}) \in$ $\mathtt{Fam}(I, \preccurlyeq_I)$. A family of Bishop topologies associated to $\Lambda^{\preccurlyeq}$ is a pair $\Phi^{\Lambda^{\preccurlyeq}} := \big(\phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}\big)$, where $\phi_0^{\Lambda^{\preccurlyeq}} : I \rightsquigarrow \mathbb{V}_0$ and $\phi_1^{\Lambda^{\preccurlyeq}} : \curlywedge_{(i,j)\in \preccurlyeq(I)} \mathbb{F}\big(\phi_0^{\Lambda^{\preccurlyeq}}(j), \phi_0^{\Lambda^{\preccurlyeq}}(i)\big)$, such that the following conditions hold:*

(i) *$\phi_0^{\Lambda^{\preccurlyeq}}(i) := F_i \subseteq \mathbb{F}(\lambda_0(i))$, and $\mathcal{F}_i := (\lambda_0(i), F_i)$ is a Bishop space, for every $i \in I$.*

(ii) *$\lambda_{ij}^{\preccurlyeq} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i,j) \in D^{\preccurlyeq}(I)$.*

(iii) *$\phi_1^{\Lambda^{\preccurlyeq}}(i,j) := \big(\lambda_{ij}^{\preccurlyeq}\big)^*$, for every $(i,j) \in D^{\preccurlyeq}(I)$, where, if $f \in F_j$, $\big(\lambda_{ij}^{\preccurlyeq}\big)^*(f) := f \circ \lambda_{ij}^{\preccurlyeq}$.*

The structure $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ is called a direct spectrum over $(I, \preccurlyeq)$, or an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$. If $S(M^{\preccurlyeq}) :=$ $(\mu_0, \mu_1, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ is an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, a direct spectrum-map Ψ from $S(\Lambda^{\preccurlyeq})$ to $S(M^{\preccurlyeq})$, in symbols $\Psi\colon S(\Lambda^{\preccurlyeq}) \Rightarrow$ $S(M^{\preccurlyeq})$, is a direct family-map $\Psi : \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$. The totality of direct spectrum-maps from $S(\Lambda^{\preccurlyeq})$ to $S(M^{\preccurlyeq})$ is denoted by $\mathtt{Map}_{(I,\preccurlyeq_I)}(S(\Lambda^{\preccurlyeq}), S(M^{\preccurlyeq}))$ and it is equipped with the equality of $\mathtt{Map}_I(\Lambda^{\preccurlyeq}, M^{\preccurlyeq})$. A direct spectrum-map $\Psi : S(\Lambda^{\preccurlyeq}) \Rightarrow S(M^{\preccurlyeq})$ is called continuous, if $\forall_{i\in I}\big(\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)\big)$, and let $\mathtt{Cont}_{(I,\preccurlyeq_I)}(S(\Lambda^{\preccurlyeq}), S(M^{\preccurlyeq}))$ be their totality, equipped with the equality of $\mathtt{Map}_I(\Lambda^{\preccurlyeq}, M^{\preccurlyeq})$. The totality $\mathtt{Spec}(I, \preccurlyeq_I)$ of direct spectra over $(I, \preccurlyeq_I)$ is equipped with an equality defined similarly to the equality on $\mathtt{Spec}(I)$. A contravariant direct spectrum $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ over $(I, \preccurlyeq)$, a contravariant direct spectrum-map $\Psi : S(\Lambda^{\succcurlyeq}) \Rightarrow$ $S(M^{\succcurlyeq})$, and their totalities $\mathtt{Map}_{(I,\preccurlyeq_I)}(S(\Lambda^{\succcurlyeq}), S(M^{\succcurlyeq}))$, $\mathtt{Spec}(I, \succcurlyeq_I)$ are defined similarly.

Remark 6.3.2. *Let $(I, \preccurlyeq)$ be a directed set, $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, $S(M^{\preccurlyeq}) := (\mu_0, \mu_1, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in$ $\mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, and $\Psi\colon S(\Lambda^{\preccurlyeq}) \Rightarrow$ $S(M^{\preccurlyeq})$. Then $\Phi^{\Lambda^{\preccurlyeq}} := \big(\phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}\big)$ is an $(I, \succcurlyeq)$-family of sets, and if Ψ is continuous, then, for every $(i,j) \in D^{\preccurlyeq}(I)$, the following diagram commutes*

$$\begin{tikzcd} G_j \arrow[r, "(\mu_{ij}^{\preccurlyeq})^*"] \arrow[d, "(\Psi_j)^*"'] & G_i \arrow[d, "(\Psi_i)^*"] \\ F_j \arrow[r, "(\lambda_{ij}^{\preccurlyeq})^*"'] & F_i. \end{tikzcd}$$

Proof. Since $(\lambda_{ii}^{\preccurlyeq})^*(f) := f \circ \lambda_{ii}^{\preccurlyeq} := f \circ \mathrm{id}_{\lambda_0(i)} := f$, for every $f \in F_i$, we get $(\lambda_{ii}^{\preccurlyeq})^* := \mathrm{id}_{F_i}$. If $i \preccurlyeq j \preccurlyeq k$ and $f \in F_k$, the required commutativity of the following diagram is shown:

$$\begin{tikzcd} & F_i \\ F_j \arrow[ur, "(\lambda_{ij}^{\preccurlyeq})^*"] & & F_k \arrow[ul, "(\lambda_{ik}^{\preccurlyeq})^*"'] \arrow[ll, "(\lambda_{jk}^{\preccurlyeq})^*"] \end{tikzcd}$$

$$\big(\lambda_{ij}^{\preccurlyeq}\big)^*\big((\lambda_{jk}^{\preccurlyeq})^*(f)\big) := \big(\lambda_{ij}^{\preccurlyeq}\big)^*(f \circ \lambda_{jk}^{\preccurlyeq}) := (f \circ \lambda_{jk}^{\preccurlyeq}) \circ \lambda_{ij}^{\preccurlyeq} := f \circ (\lambda_{jk}^{\preccurlyeq} \circ \lambda_{ij}^{\preccurlyeq}) = f \circ \lambda_{ik}^{\preccurlyeq} := \big(\lambda_{ik}\big)^*(f).$$

To show the required commutativity, if $g \in G_j$, then

$$\big(\lambda_{ij}^{\preccurlyeq}\big)^*\big((\Psi_j^{\preccurlyeq})^*(g)\big) := \big(\Psi_j\big)^*(g) \circ \lambda_{ij}^{\preccurlyeq} := \big(g \circ \Psi_j\big) \circ \lambda_{ij}^{\preccurlyeq} =_{F_i} g \circ \big(\Psi_j \circ \lambda_{ij}^{\preccurlyeq}\big) =_{F_i} g \circ \big(\mu_{ij}^{\preccurlyeq} \circ \Psi_i\big)$$

$$=_{F_i} \big(g \circ \mu_{ij}^{\preccurlyeq}\big) \circ \Psi_i := \big(\Psi_i\big)^*\big(g \circ \mu_{ij}^{\preccurlyeq}\big) := \big(\Psi_i\big)^*\big(\big(\mu_{ij}^{\preccurlyeq}\big)^*(g)\big). \qquad \square$$

6.4 The topology on the $\sum$-set of a direct spectrum

Remark 6.4.1. *Let $(I, \preccurlyeq)$ be a directed set and $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, D^{\preccurlyeq}(I))$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in\ D^{\preccurlyeq}(I)}$. If $\Theta \in \prod_{i\in I}^{\succcurlyeq} F_i$, the following operation is a function*

$$f_\Theta : \left(\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)\right) \rightsquigarrow \mathbb{R}, \qquad f_\Theta(i,x) := \Theta_i(x), \qquad (i,x) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i).$$

Proof. Let $(i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y) :\Leftrightarrow \exists_{k \succcurlyeq i,j}\big(\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big)$. Since $\Theta_i = \phi_{ki}^{\succcurlyeq}(\Theta_k) := (\lambda_{ik}^{\preccurlyeq})^*(\Theta_k) := \Theta_k \circ \lambda_{ik}^{\preccurlyeq}$, and similarly $\Theta_j = \Theta_k \circ \lambda_{jk}^{\preccurlyeq}$, we have that

$$\Theta_i(x) = \big[\Theta_k \circ \lambda_{ik}^{\preccurlyeq}\big](x) := \Theta_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) = \Theta_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) := \big[\Theta_k \circ \lambda_{jk}^{\preccurlyeq}\big](y) = \Theta_j(y). \qquad \square$$

Definition 6.4.2. *Let $(I, \preccurlyeq)$ be a directed set and $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$. The Bishop space*

$$\sum_{i\in I}^{\preccurlyeq} \mathcal{F}_i := \left(\sum_{i\in I}^{\preccurlyeq} \lambda_0(i), \int_{i\in I}^{\preccurlyeq} F_i\right) \quad \textit{where} \quad \int_{i\in I}^{\preccurlyeq} F_i := \bigvee_{\Theta \in \prod_{i\in I}^{\succcurlyeq} F_i} f_\Theta,$$

is the sum Bishop space of $S(\Lambda^{\preccurlyeq})$. If $S^{\succcurlyeq}$ is a contravariant direct spectrum over $(I, \preccurlyeq)$, the sum Bishop space of $S(\Lambda^{\succcurlyeq})$ is defined dually.

Lemma 6.4.3. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}), S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I$), and let $\Psi : S(\Lambda^{\preccurlyeq}) \to S(M^{\preccurlyeq})$ be continuous. If $H \in \prod_{i\in I}^{\succcurlyeq} G_i$, the dependent operation $H^* : \curlywedge_{i\in I} F_i$, defined by $H_i^* := \Psi_i^*(H_i) := H_i \circ \Psi_i$, for every $i \in I$, is in $\prod_{i\in I}^{\succcurlyeq} F_i$.*

Proof. If $i \preccurlyeq j$, we need to show that $H_i^* = (\lambda_{ij}^{\preccurlyeq})^*(H_j^*) = H_j^* \circ \lambda_{ij}^{\preccurlyeq}$. Since $H \in \prod_{i\in I}^{\succcurlyeq} G_i$, we have that $H_i = H_j \circ \mu_{ij}^{\preccurlyeq}$, and by the continuity of Ψ and the commutativity of the diagram

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ij}^{\preccurlyeq}"] \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"] \\
\mu_0(i) \arrow[r, "\mu_{ij}^{\preccurlyeq}"'] & \mu_0(j),
\end{tikzcd}$$

$$H_j^* \circ \lambda_{ij}^{\preccurlyeq} := \Psi_j^*(H_j) \circ \lambda_{ij}^{\preccurlyeq} := \big(H_j \circ \Psi_j\big) \circ \lambda_{ij}^{\preccurlyeq} := H_j(0) \circ \big(\Psi_j \circ \lambda_{ij}^{\preccurlyeq}\big)$$

$$= H_j \circ \big(\mu_{ij}^{\preccurlyeq} \circ \Psi_i\big) = \big(H_j \circ \big(\mu_{ij}^{\preccurlyeq}\big) \circ \Psi_i = H_i \circ \Psi_i := \Psi_i^*(H_i) := H_i^*. \qquad \square$$

Proposition 6.4.4. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ and $S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ be spectra over $(I, \preccurlyeq_I)$, and let $\Psi : S(\Lambda^{\preccurlyeq}) \Rightarrow S(M^{\preccurlyeq})$.*

(i) *If $i \in I$, then $e_i^{\Lambda^{\preccurlyeq}} \in \mathrm{Mor}\big(\mathcal{F}_i, \sum_{i\in I}^{\preccurlyeq} \mathcal{F}_i\big)$.*

(ii) *If Ψ is continuous, then $\Sigma^{\preccurlyeq}\Psi \in \mathrm{Mor}\big(\sum_{i\in I}^{\preccurlyeq} \mathcal{F}_i, \sum_{i\in I}^{\preccurlyeq} \mathcal{G}_i\big)$.*

Proof. (i) By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{\Theta \in \prod_{i\in I}^{\succcurlyeq} F_i}\big(f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}} \in F_i\big)$. If $x \in \lambda_0(i)$, then $\big(f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}}\big)(x) := f_\Theta(i, x) := \Theta_i(x)$, hence $f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}} := \Theta_i \in F_i$.
(ii) By the $\bigvee$-lifting of morphisms it suffices to show that

$$\forall_{H \in \prod_{i\in I}^{\succcurlyeq} G_i}\bigg(g_H \circ \Sigma^{\preccurlyeq}\Psi \in \int_{i\in I}^{\preccurlyeq} F_i\bigg).$$

If $i \in I$ and $x \in \lambda_0(i)$, and if $H^* \in \prod_{i\in I}^{\succcurlyeq} F_i$, defined in Lemma 6.4.3, then $\big(g_H \circ \Sigma^{\preccurlyeq}\Psi\big)(i, x) := g_H(i, \Psi_i(x)) := H_i(\Psi_i(x)) := (H_i \circ \Psi_i)(x) := f_{H^*}(i, x)$, and $g_H \circ \Sigma^{\preccurlyeq}\Psi := f_{H^*} \in \int_{i\in I}^{\preccurlyeq} F_i$. $\square$

6.5 Direct limit of a covariant spectrum of Bishop spaces

If X is a set, by Corollary 4.7.3 the family $\mathtt{Eql}(X) := \big(\mathtt{eql}_0^X, \mathcal{E}^X, \mathtt{eql}_1^X\big) \in \mathtt{Set}(X, X)$, where $\mathtt{eql}_0^X(x) := \{y \in X \mid y =_X x\}$. Consequently, if $f : X \to Y$, there is unique $\mathtt{eql}_0 f : \mathtt{eql}_0 X(X) \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathtt{eql}_0^*"'] & Y, \\ \mathtt{eql}_0 X(X) \arrow[ur, dashed, "\mathtt{eql}_0 f"'] & \end{tikzcd}$$

where $\mathtt{eql}_0 X(X)$ is the totality X with the equality $x =_{\mathtt{eql}_0 X(X)} x' :\Leftrightarrow \mathtt{eql}_0^X(x) =_{\mathcal{P}(X)} \mathtt{eql}_0^X(x')$. As $\mathtt{Eql}(X) \in \mathtt{Set}(X, X)$, we get $\mathtt{eql}_0^X(x) =_{\mathcal{P}(X)} \mathtt{eql}_0^X(x') \Leftrightarrow x =_X x'$. The map $\mathtt{eql}_0^* : X \to \mathtt{eql}_0 X(X)$ is defined by the identity map-rule, written in the form $x \mapsto \mathtt{eql}_0^X(x)$, for every $x \in X$. We use the set $\mathtt{eql}_0 X(X)$ to define the direct limit of a direct spectrum of Bishop spaces. In what follows we avoid including the superscript X in our notation.

Definition 6.5.1. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ and $\mathtt{eql}_0 : \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \rightsquigarrow \mathbb{V}_0$, defined by*

$$\mathtt{eql}_0(i, x) := \bigg\{(j, y) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \mid (j, y) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (i, x)\bigg\}; \qquad (i, x) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i), .$$

The direct limit $\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ of $S(\Lambda^{\preccurlyeq})$ is the set

$$\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) := \mathtt{eql}_0 \sum_{i\in I}^{\preccurlyeq} \lambda_0(i)\bigg(\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)\bigg),$$

$$\mathtt{eql}_0(i, x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0(j, y) :\Leftrightarrow \mathtt{eql}_0(i, x) =_{\mathcal{P}\big(\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)\big)} \mathtt{eql}_0(j, y) \Leftrightarrow (i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j, y).$$

We write $\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}$ when we need to express the dependence of $\mathtt{eql}_0$ from $\Lambda^{\preccurlyeq}$.

Remark 6.5.2. *If $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ and $i \in I$, the operation $\mathtt{eql}_i \colon \lambda_0(i) \rightsquigarrow \underset{\to}{\mathrm{Lim}}\lambda_0(i)$, defined by $\mathtt{eql}_i(x) := \mathtt{eql}_0(i, x)$, for every $x \in \lambda_0(i)$, is a function.*

Proof. If $x, x' \in \lambda_0(i)$ such that $x =_{\lambda_0(i)} x'$, then

$$\begin{aligned}
\mathtt{eql}_i(x) =_{\underset{\to}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_i(x') &:\Leftrightarrow \mathtt{eql}_0(i, x) =_{\underset{\to}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0(i, x') \\
&\Leftrightarrow (i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (i, x') \\
&:\Leftrightarrow \exists_{k \in I}\big(i \preccurlyeq k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{ik}^{\preccurlyeq}(x')\big),
\end{aligned}$$

which holds, since $\lambda_{ik}^{\preccurlyeq}$ is a function, and hence if $x =_{\lambda_0(i)} x'$, then $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{ik}^{\preccurlyeq}(x')$, for every $k \in I$ such that $i \preccurlyeq k$. Such a $k \in I$ always exists e.g., one can take $k := i$. □

Definition 6.5.3. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(F_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$. The direct limit of $S(\Lambda^{\preccurlyeq})$ is the Bishop space*

$$\underset{\to}{\mathrm{Lim}}\mathcal{F}_i := \big(\underset{\to}{\mathrm{Lim}}\lambda_0(i), \underset{\to}{\mathrm{Lim}}F_i\big), \qquad \textit{where}$$

$$\underset{\to}{\mathrm{Lim}}F_i := \bigvee_{\Theta \in \prod_{i \in I}^{\succcurlyeq} F_i} \mathtt{eql}_0 f_{\Theta},$$

$$\mathtt{eql}_0 f_{\Theta}\big(\mathtt{eql}_0(i, x)\big) := f_{\Theta}(i, x) := \Theta_i(x); \qquad \mathtt{eql}_0(i, x) \in \underset{\to}{\mathrm{Lim}}\lambda_0(i)$$

$$\begin{tikzcd}
\sum_{i \in I}^{\preccurlyeq} \lambda_0(i) \arrow[r, "f_{\Theta}"] \arrow[d, "\mathtt{eql}_0^*"'] & \mathbb{R}. \\
\underset{\to}{\mathrm{Lim}}\lambda_0(i) \arrow[ur, dashed, "\mathtt{eql}_0 f_{\Theta}"']
\end{tikzcd}$$

Remark 6.5.4. *If $(I, \preccurlyeq)$ is a directed set, $\mathcal{G} := (Y, G)$ is a Bishop space, and $S(\Lambda^{\preccurlyeq, Y})$ is the constant direct spectrum over $(I, \preccurlyeq_I)$ with Bishop space $\mathcal{G}$ and Bishop morphism id_Y, the direct limit $\underset{\to}{\mathrm{Lim}}\mathcal{G}$ of $S(\Lambda^{\preccurlyeq, Y})$ is Bishop-isomorphic to $\mathcal{G}$. Moreover, every Bishop space is Bishop-isomorphic to the direct limit of a direct spectrum over any given directed set.*

Proof. The proof is straightforward. □

Proposition 6.5.5 (Universal property of the direct limit)**.** *If $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$, its direct limit $\underset{\to}{\mathrm{Lim}}\mathcal{F}_i$ satisfies the universal property of direct limits i.e.,*

(i) *For every $i \in I$, we have that $\mathtt{eql}_i \in \mathrm{Mor}(\mathcal{F}_i, \underset{\to}{\mathrm{Lim}}\mathcal{F}_i)$.*

(ii) *If $i \preccurlyeq_I j$, the following left diagram commutes*

$$\begin{tikzcd}
& \underset{\to}{\mathrm{Lim}}\lambda_0(i) & \\
\lambda_0(i) \arrow[ur, "\mathtt{eql}_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"'] & & \lambda_0(j) \arrow[ul, "\mathtt{eql}_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y & \\
\lambda_0(i) \arrow[ur, "\varepsilon_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"'] & & \lambda_0(j). \arrow[ul, "\varepsilon_j"']
\end{tikzcd}$$

(iii) *If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varepsilon_i : \lambda_0(i) \to Y \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G})$, for every $i \in I$, such that if $i \preccurlyeq j$, the above right diagram commutes, there is a unique function $h \colon \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \to Y \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{G})$ such that the following diagrams commute*

$$\begin{tikzcd} & Y & \\ \lambda_0(i) \arrow[ur, "\varepsilon_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"] \arrow[dr, "\mathtt{eql}_i"'] & & \lambda_0(j), \arrow[ul, "\varepsilon_j"'] \arrow[dl, "\mathtt{eql}_j"] \\ & \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i). \arrow[uu, dashed, "h"] & \end{tikzcd}$$

Proof. For the proof of (i), we use the $\bigvee$-lifting of morphisms. We have that

$$\mathtt{eql}_i \in \mathrm{Mor}(\mathcal{F}_i, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{\Theta \in \prod_{i\in I}^{\preccurlyeq} F_i}\big(\mathtt{eql}_0 f_\Theta \circ \mathtt{eql}_i \in F_i\big).$$

If $x \in \lambda_0(i)$, then $\big(\mathtt{eql}_0 f_\Theta \circ \mathtt{eql}_i\big)(x) := \mathtt{eql}_0 f_\Theta\big(\mathtt{eql}_0(i, x)\big) := f_\Theta(i, x) := \Theta_i(x)$ hence $\mathtt{eql}_0 f_\Theta \circ \mathtt{eql}_i := \Theta_i \in F_i$. For the proof of (ii), if $x \in \lambda_0(i)$, then

$$\begin{aligned} \mathtt{eql}_j(\lambda_{ij}^{\preccurlyeq}(x)) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_i(x) &:\Leftrightarrow \mathtt{eql}_0\big(j, \lambda_{ij}^{\preccurlyeq}(x)\big) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0(i, x) \\ &\Leftrightarrow \big(j, \lambda_{ij}^{\preccurlyeq}(x)\big) =_{\sum_{i\in I}^{\preccurlyeq}\lambda_0(i)} (i, x) \\ &:\Leftrightarrow \exists_{k\in I}\big(i \preccurlyeq k \ \&\ j \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(\lambda_{ij}^{\preccurlyeq}(x))\big), \end{aligned}$$

which holds, since if $k \in I$ with $j \preccurlyeq k$, the equality $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(\lambda_{ij}^{\preccurlyeq}(x))$ holds by the definition of a direct family of sets, and by the definition of a directed set such a k always exists. To prove (iii) let the operation $h \colon \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow Y$, defined by $h\big(\mathtt{eql}_0(i, x)\big) := \varepsilon_i(x)$, for every $\omega(i, x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$. First we show that h is a function. Let

$$\mathtt{eql}_0(i, x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0(j, y) \Leftrightarrow \exists_{k\in I}\big(i, j \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big).$$

By the supposed commutativity of the following diagrams

$$\begin{tikzcd} & Y & \\ \lambda_0(i) \arrow[ur, "\varepsilon_i"] \arrow[rr, "\lambda_{ik}^{\preccurlyeq}"'] & & \lambda_0(k) \arrow[ul, "\varepsilon_k"'] \end{tikzcd} \qquad \begin{tikzcd} & Y & \\ \lambda_0(j) \arrow[ur, "\varepsilon_j"] \arrow[rr, "\lambda_{jk}^{\preccurlyeq}"'] & & \lambda_0(k) \arrow[ul, "\varepsilon_k"'] \end{tikzcd}$$

we get $h\big(\omega(i, x)\big) := \varepsilon_i(x) = \varepsilon_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) = \varepsilon_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) = \varepsilon_j(y) := h\big(\omega(j, y)\big)$. Next we show that h is a Bishop morphism. By the $\bigvee$-lifting of morphisms we have that $h \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{G}) \Leftrightarrow \forall_{g\in G}\big(g \circ h \in \underset{\rightarrow}{\mathrm{Lim}}F_i\big)$. If $g \in G$, we show that the dependent operation $\Theta_g : \bigcurlywedge_{i\in I} F_i$, defined by $\Theta_g(i) := g \circ \varepsilon_i$, for every $i \in I$, is well-defined, since $\varepsilon_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G})$, and $\Theta_g \in \prod_{i\in I}^{\succcurlyeq} F_i$. To prove the latter, if $i \preccurlyeq k$, we show that $\Theta_g(i) = \Theta_g(k) \circ \lambda_{ik}^{\preccurlyeq}$. By the commutativity of the above left diagram we have that $\Theta_g(k) \circ \lambda_{ik}^{\preccurlyeq} := (g \circ \varepsilon_k) \circ \lambda_{ik}^{\preccurlyeq} := g \circ (\varepsilon_k \circ \lambda_{ik}^{\preccurlyeq}) = g \circ \varepsilon_i := \Theta_g(i)$, Hence $f_{\Theta_g} \in \underset{\rightarrow}{\mathrm{Lim}}F_i$. Since $(g \circ h)\big(\mathtt{eql}_0(i, x)\big) := g(\varepsilon_i(x)) := (g \circ \varepsilon_i)(x) := [\Theta_g(i)](x) :=$

$f_{\Theta_g}\big((\mathtt{eql}_0(i,x)\big)$, we get $g \circ h := f_{\Theta_g} \in \underset{\rightarrow}{\mathrm{Lim}} F_i$. The uniqueness of h, and the commutativity of the diagram in property (iii) follow immediately. $\square$

The uniqueness of $\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$, up to Bishop isomorphism, is shown easily from its universal property. Note that if $i, j \in I$, $x \in \lambda_0(i)$ and $y \in \lambda_0(j)$, we have that

$$\begin{aligned}
\mathtt{eql}_i(x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_j(y) &:\Leftrightarrow \mathtt{eql}_0(i,x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0(j,y) \\
&\Leftrightarrow (i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y) \\
&:\Leftrightarrow \exists_{k\in I}\big(i \preccurlyeq k \ \& \ j \preccurlyeq k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big).
\end{aligned}$$

Definition 6.5.6. *Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ be a direct spectrum over $(I, \preccurlyeq)$. If $i \in I$, an element x of $\lambda_0(i)$ is a representative of $\omega(z) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$, if $\omega_i(x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \omega(z)$.*

Although an element $\mathtt{eql}_0(z) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ may not have a representative in every $\lambda_0(i)$, it surely has one at some $\lambda_0(i)$. Actually, the following holds.

Proposition 6.5.7. *For every $n \geq 1$ and every $\mathtt{eql}_0(z_1), \ldots, \mathtt{eql}_0(z_n) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ there are $i \in I$ and $x_1, \ldots, x_n \in \lambda_0(i)$ such that x_l represents $\mathtt{eql}_0(z_l)$, for every $l \in \{1, \ldots, n\}$.*

Proof. The proof is by induction on $\mathbb{N}^+$. We present only the case $n := 2$. Let $z := (j,y), z' := (j', y') \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$, and $k \in I$ with $j \preccurlyeq k$ and $j' \preccurlyeq k$. By definition we have that $\lambda_{jk}^{\preccurlyeq}(y) \in \lambda_0(k)$ and $\lambda_{j'k}^{\preccurlyeq}(y') \in \lambda_0(k)$. We show that $\lambda_{jk}^{\preccurlyeq}(y)$ represents $\mathtt{eql}_0(z)$ and $\lambda_{j'k}^{\preccurlyeq}(y')$ represents $\mathtt{eql}_0(z')$. By our remark right before Definition 6.5.6 for the first representation we need to show that

$$\omega_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \omega_j(y) \Leftrightarrow \exists_{k'\in I}\big(k \preccurlyeq k' \ \& \ j \preccurlyeq k' \ \& \ \lambda_{kk'}^{\preccurlyeq}(\lambda_{jk}^{\preccurlyeq}(y)) =_{\lambda_0(k')} \lambda_{jk'}^{\preccurlyeq}(y)\big).$$

By the composition of the transport maps it suffices to take any $k' \in I$ with $k \preccurlyeq k'$ & $j \preccurlyeq k'$, and for the second representation it suffices to take any $k'' \in I$ with $k \preccurlyeq k''$ & $j' \preccurlyeq k''$. $\square$

Theorem 6.5.8. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(F_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, $S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(G_i)_{i\in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, and $\Psi \colon S(\Lambda^{\preccurlyeq}) \Rightarrow S(M^{\preccurlyeq})$.*

(i) *There is a unique function $\Psi_{\rightarrow} \colon \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \to \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)$ such that, for every $i \in I$, the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "\mathtt{eql}_i^{\Lambda^{\preccurlyeq}}"'] & \mu_0(i) \arrow[d, "\mathtt{eql}_i^{M^{\preccurlyeq}}"] \\
\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[r, dashed, "\Psi_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i).
\end{tikzcd}$$

(ii) *If Ψ is continuous, then $\Psi_{\rightarrow} \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{G}_i)$.*

(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Psi_{\rightarrow}$ is an embedding.*

Proof. (i) The following well-defined operation $\Psi_{\rightarrow} \colon \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)$, given by

$$\Psi_{\rightarrow}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := \texttt{eql}_0^{M^{\preccurlyeq}}(i, \Psi_i(x)); \qquad \texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$$

is a function, since, if $\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \texttt{eql}_0^{\Lambda^{\preccurlyeq}}(j,y) \Leftrightarrow (i,x) =_{\sum_{i\in I}^{\preccurlyeq}\lambda_0(i)} (j,y)$, which is equivalent to $\exists_{k\in I}\big(i \preccurlyeq k \ \&\ j \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big)$, we show that

$$\begin{aligned}
\Psi_{\rightarrow}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) =_{\underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)} \Psi_{\rightarrow}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(j,y)\big) &:\Leftrightarrow \texttt{eql}_0^{M^{\preccurlyeq}}\big(i, \Psi_i^{\preccurlyeq}(x)\big) =_{\underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)} \texttt{eql}_0^{M^{\preccurlyeq}}\big(j, \Psi_j(y)\big)\\
&\Leftrightarrow (i, \Psi_i(x)) =_{\sum_{i\in I}^{\preccurlyeq}\mu_0(i)} (j, \Psi_j(y))\\
&:\Leftrightarrow \exists_{i\in I}\big(i, j \preccurlyeq k \ \&\ \mu_{ik}^{\preccurlyeq}(\Psi_i(x)) =_{\mu_0(k)} \mu_{jk}^{\preccurlyeq}(\Psi_j(y))\big).
\end{aligned}$$

By the commutativity of the following diagrams, and since Ψ_k is a function,

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ik}^{\preccurlyeq}"] \arrow[d, "\Psi_i"'] & \lambda_0(k) \arrow[d, "\Psi_k"] & \lambda_0(j) \arrow[r, "\lambda_{jk}^{\preccurlyeq}"] \arrow[d, "\Psi_j"'] & \lambda_0(k) \arrow[d, "\Psi_k"]\\
\mu_0(i) \arrow[r, "\mu_{ik}^{\preccurlyeq}"'] & \mu_0(k) & \mu_0(j) \arrow[r, "\mu_{jk}^{\preccurlyeq}"'] & \mu_0(k),
\end{tikzcd}$$

we get $\mu_{ik}^{\preccurlyeq}\big(\Psi_i(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) =_{\mu_0(k)} \mu_{jk}^{\preccurlyeq}\big(\Psi_j(y)\big)$.
(ii) By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{H\in\prod_{i\in I}^{\succcurlyeq} G_i}\big((\texttt{eql}_0^{M^{\preccurlyeq}} g_H) \circ \Psi_{\rightarrow} \in \underset{\rightarrow}{\mathrm{Lim}}F_i\big)$. By Definition 6.5.3 we have that

$$\big((\texttt{eql}_0^{M^{\preccurlyeq}} g_H) \circ \Psi_{\rightarrow}\big)\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := \big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}} g_H\big)\big(\texttt{eql}_0^{M^{\preccurlyeq}}(i, \Psi_i(x))\big) := g_H(i, \Psi_i(x))$$

$$:= H_i(\Psi_i(x)) = (H_i \circ \Psi_i)(x) := H_i^*(x) := f_{H^*}(i,x) := \big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}} f_{H^*}\big)\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big),$$

where $H^* \in \prod_{i\in I}^{\succcurlyeq} F_i$ is defined in Lemma 6.4.3, and $(\texttt{eql}_0^{M^{\preccurlyeq}} g_{H^*}) \circ \Psi_{\rightarrow} := \texttt{eql}_0^{\Lambda^{\preccurlyeq}} f_{H^*} \in \underset{\rightarrow}{\mathrm{Lim}}F_i$.
(iii) If $\Psi_{\rightarrow}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) =_{\underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)} \Psi_{\rightarrow}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(j,y)$ i.e., $\mu_{ik}^{\preccurlyeq}(\Psi_i(x)) =_{\mu_0(k)} \mu_{jk}^{\preccurlyeq}(\Psi_j(y))\big)\big)$, for some $k \in I$ with $i, j \preccurlyeq k$, by the proof of case (ii) we get $\Psi_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) =_{\mu_0(k)} \Psi_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big)$, and since Ψ_k is an embedding, we conclude that $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)$ i.e., $(i,x) =_{\sum_{i\in I}^{\preccurlyeq}\lambda_0(i)} (j,y)$. $\square$

Proposition 6.5.9. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \texttt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(F_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, $S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \texttt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(G_i)_{i\in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, and $S(N^{\preccurlyeq}) := (\nu_0, \nu_1^{\preccurlyeq}, \phi_0^{N^{\preccurlyeq}}, \phi_1^{N^{\preccurlyeq}}) \in \texttt{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(H_i)_{i\in I}$ and Bishop morphisms $(\nu_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$. If $\Psi \colon S(\Lambda^{\preccurlyeq}) \Rightarrow S(M^{\preccurlyeq})$ and $\Xi \colon S(M^{\preccurlyeq}) \Rightarrow S(N^{\preccurlyeq})$, then $(\Xi \circ \Psi)_{\rightarrow} := \Xi_{\rightarrow} \circ \Psi_{\rightarrow}$*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "\mathtt{eql}_i^{\Lambda^{\preccurlyeq}}"'] \arrow[rr, bend left, "(\Xi \circ \Psi)_i"] & \mu_0(i) \arrow[r, "\Xi_i"] \arrow[d, "\mathtt{eql}_i^{M^{\preccurlyeq}}"'] & \nu_0(i) \arrow[d, "\mathtt{eql}_i^{N^{\preccurlyeq}}"] \\
\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[r, "\Psi_{\rightarrow}"'] \arrow[rr, bend right, "(\Xi \circ \Psi)_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i) \arrow[r, "\Xi_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}}\nu_0(i).
\end{tikzcd}$$

Proof. If $\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$, then

$$\begin{aligned}
(\Xi \circ \Psi)_{\rightarrow}[\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x)] &:= \mathtt{eql}_0^{N^{\preccurlyeq}}(i, (\Xi \circ \Psi)_i(x)) \\
&:= \mathtt{eql}_0^{N^{\preccurlyeq}}(i, (\Xi_i \circ \Psi_i)(x)) \\
&:= \mathtt{eql}_0^{N^{\preccurlyeq}}(i, (\Xi_i(\Psi_i(x)))) \\
&:= \Xi_{\rightarrow}\big(\mathtt{eql}_0^{M^{\preccurlyeq}}(i, \Psi_i(x))\big) \\
&:= \Xi_{\rightarrow}\big(\Psi_{\rightarrow}\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x)\big)\big) \\
&:= (\Xi_{\rightarrow} \circ \Psi_{\rightarrow})\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x)\big).
\end{aligned}$$

□

Definition 6.5.10. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$ and $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$, a cofinal subset of I with modulus of cofinality $e\colon J \hookrightarrow I$. The relative spectrum of $S(\Lambda^{\preccurlyeq})$ to J is the e-subfamily $S(\Lambda^{\preccurlyeq}) \circ e := \big(\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^{\preccurlyeq}} \circ e, \phi_1^{\Lambda^{\preccurlyeq}} \circ e\big)$ of $S(\Lambda^{\preccurlyeq})$, where $\Phi^{\Lambda^{\preccurlyeq}} \circ e := \big(\phi_0^{\Lambda^{\preccurlyeq}} \circ e, \phi_1^{\Lambda^{\preccurlyeq}} \circ e\big)$ is the e-subfamily of $\Phi^{\Lambda^{\preccurlyeq}}$.*

Lemma 6.5.11. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$, $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$, and $S(\Lambda^{\preccurlyeq}) \circ e := \big(\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^{\preccurlyeq}} \circ e, \phi_1^{\Lambda^{\preccurlyeq}} \circ e\big)$ the relative spectrum of $S(\Lambda^{\preccurlyeq})$ to J.*

(i) *If $\Theta \in \prod_{i\in I}^{\succcurlyeq} F_i$, then $\Theta^J \in \prod_{j\in J}^{\succcurlyeq} F_j$, where for every $j \in J$ we define $\Theta_j^J := \Theta_{e(j)}$.*

(ii) *If $H^J \in \prod_{j\in J}^{\succcurlyeq} F_j$, then $H \in \prod_{i\in I}^{\succcurlyeq} F_i$, where, for every $i \in I$, let $H_i := H^J_{\mathtt{cof}_J(i)} \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}$*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}"] \arrow[dr, "H_i"'] & \lambda_0(e(\mathtt{cof}_J(i))) \arrow[d, "H^J_{\mathtt{cof}_J(i)}"] \\
& \mathbb{R}.
\end{tikzcd}$$

Proof. (i) It suffices to show that if $j \preccurlyeq j' :\Leftrightarrow e(j) \preccurlyeq e(j')$, then $\Theta_j^J = \Theta_{j'}^J \circ \lambda_{jj'}^{\preccurlyeq}$. Since $\Theta \in \prod_{i\in I}^{\succcurlyeq} F_i$ we have that $\Theta_j^J := \Theta_{e(j)} = \Theta_{e(j')} \circ \lambda^{\preccurlyeq}_{e(j)e(j')} := \Theta_{j'}^J \circ \lambda_{jj'}^{\preccurlyeq}$.

(ii) By definition $H^J_{\mathtt{cof}_J(i)} \in F_{\mathtt{cof}_J(i)} := F_{e(\mathtt{cof}_J(i))}$, and since $i \preccurlyeq e(\mathtt{cof}_J(i))$, we get $H_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{R}) = \mathcal{F}_i$ i.e., $H : \curlywedge_{i\in I} F_i$. Next we show that if $i \preccurlyeq i'$, then $H_i = H_{i'} \circ \lambda_{ii'}^{\preccurlyeq}$. By ($\mathtt{Cof}_3$) and ($\mathtt{Cof}_2$) we have that

$$i \preccurlyeq i' \preccurlyeq e(\mathtt{cof}_J(i')), \tag{6.1}$$

and $i \preccurlyeq i' \Rightarrow \mathtt{cof}_J(i) \preccurlyeq \mathtt{cof}_J(i') :\Leftrightarrow e(\mathtt{cof}_J(i)) \preccurlyeq e(\mathtt{cof}_J(i'))$, hence we also get

$$i \preccurlyeq e(\mathtt{cof}_J(i)) \preccurlyeq e(\mathtt{cof}_J(i')). \tag{6.2}$$

Since $H^J \in \prod_{j\in J}^{\succcurlyeq} F_j$, we have that

$$\begin{aligned}
H_{i'} \circ \lambda_{ii'}^{\preccurlyeq} &:= \big[H^J_{\mathtt{cof}_J(i')} \circ \lambda^{\preccurlyeq}_{i'e(\mathtt{cof}_J(i'))}\big] \circ \lambda^{\preccurlyeq}_{ii'} \\
&:= H^J_{\mathtt{cof}_J(i')} \circ \big[\lambda^{\preccurlyeq}_{i'e(\mathtt{cof}_J(i'))} \circ \lambda^{\preccurlyeq}_{ii'}\big] \\
&\overset{(6.1)}{=} H^J_{\mathtt{cof}_J(i')} \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i'))} \\
&\overset{(6.2)}{=} H^J_{\mathtt{cof}_J(i')} \circ \big[\lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i))e(\mathtt{cof}_J(i'))} \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}\big] \\
&:= \big[H^J_{\mathtt{cof}_J(i')} \circ \lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i))e(\mathtt{cof}_J(i'))}\big] \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))} \\
&:= \big[H^J_{\mathtt{cof}_J(i')} \circ \lambda^{\preccurlyeq}_{\mathtt{cof}_J(i)\mathtt{cof}_J(i')}\big] \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))} \\
&:= H^J_{\mathtt{cof}_J(i)} \circ \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))} \\
&:= H_i.
\end{aligned}$$

□

Theorem 6.5.12. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq_I)$, $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$, and $S(\Lambda^{\preccurlyeq}) \circ e := \big(\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^{\preccurlyeq}} \circ e, \phi_1^{\Lambda^{\preccurlyeq}} \circ e\big)$ the relative spectrum of $S(\Lambda^{\preccurlyeq})$ to J. Then*

$$\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_j \simeq \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i.$$

Proof. We define the operation $\phi : \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j) \rightsquigarrow \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ by $\phi\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j, y)\big) := \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j), y)$

$$\begin{tikzcd}
& \lambda_0(j) \arrow[dl, "\mathtt{eql}_j^{\Lambda^{\preccurlyeq}\circ e}"'] \arrow[dr, "\mathtt{eql}_{e(j)}^{\Lambda^{\preccurlyeq}}"] & \\
\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j) \arrow[rr, "\phi"'] & & \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i),
\end{tikzcd}$$

for every $\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j, y) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j)$, where, if $j \in J$ and $y \in \lambda_0(j)$, we have that

$$\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j, y) := \bigg\{(j', y') \in \sum_{j\in J}^{\preccurlyeq} \lambda_0(j) \mid (j', y') =_{\sum_{j\in J}^{\preccurlyeq} \lambda_0(j)} (j, y)\bigg\},$$

$$\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j), y) := \bigg\{(i, x) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \mid (i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (e(j), y)\bigg\}.$$

First we show that ϕ is a function. By definition we have that

$$\begin{aligned}
\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j, y) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j', y') &\Leftrightarrow (j, y) =_{\sum_{j\in J}^{\preccurlyeq} \lambda_0(j)} (j', y') \\
&\Leftrightarrow \exists_{j''\in J}\big(j, j' \preccurlyeq j'' \ \&\ \lambda^{\preccurlyeq}_{jj''}(y) =_{\lambda_0(j'')} \lambda^{\preccurlyeq}_{j'j''}(y')\big) \quad (1)
\end{aligned}$$

$$\begin{aligned}
\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j), y) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j'), y') &\Leftrightarrow (e(j), y) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (e(j'), y') \\
&\Leftrightarrow \exists_{k\in I}\big(e(j), e(j') \preccurlyeq k \ \&\ \lambda^{\preccurlyeq}_{e(j)k}(y) =_{\lambda_0(k)} \lambda^{\preccurlyeq}_{e(j')k}(y')\big). \ (2)
\end{aligned}$$

If $k := e(j'')$, then (1) implies (2), and hence ϕ is a function. To show that ϕ is an embedding, we show that (2) implies (1). Since $e(j), e(j') \preccurlyeq k \preccurlyeq e(\mathtt{cof}_J(k))$, we get $j, j' \preccurlyeq \mathtt{cof}_J(k) := j''$. By the commutativity of the following diagrams

$$\begin{tikzcd}
\lambda_0(e(j)) \arrow[d, "\lambda^{\preccurlyeq}_{e(j)k}"'] \arrow[dr, "\lambda^{\preccurlyeq}_{e(j)e(\mathtt{cof}_J(k))}"] & \\
\lambda_0(k) \arrow[r, "\lambda^{\preccurlyeq}_{ke(\mathtt{cof}_J(k))}"'] & \lambda_0(e(\mathtt{cof}_J(k)))
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_0(e(j')) \arrow[d, "\lambda^{\preccurlyeq}_{e(j')k}"'] \arrow[dr, "\lambda^{\preccurlyeq}_{e(j')e(\mathtt{cof}_J(k))}"] & \\
\lambda_0(k) \arrow[r, "\lambda^{\preccurlyeq}_{ke(\mathtt{cof}_J(k))}"'] & \lambda_0(e(\mathtt{cof}_J(k)))
\end{tikzcd}$$

$$\begin{aligned}
\lambda^{\preccurlyeq}_{jj''}(y) &:= \lambda^{\preccurlyeq}_{e(j)e(\mathtt{cof}_J(k))}(y) \\
&= \big[\lambda_{ke(\mathtt{cof}_J(k))} \circ \lambda^{\preccurlyeq}_{e(j)k}\big](y) \\
&= \lambda_{ke(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{e(j)k}(y)\big) \\
&= \lambda_{ke(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{e(j')k}(y')\big) \\
&:= \big[\lambda_{ke(\mathtt{cof}_J(k))} \circ \lambda^{\preccurlyeq}_{e(j')k}\big](y') \\
&= \lambda^{\preccurlyeq}_{e(j')e(\mathtt{cof}_J(k))}(y') \\
&:= \lambda_{j'j''}(y').
\end{aligned}$$

By the $\bigvee$-lifting of morphisms we have that

$$\phi \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_j, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i) :\Leftrightarrow \forall_{\Theta \in \prod^{\succcurlyeq}_{i \in I} F_i}\big(\mathtt{eql}_0 f_\Theta \circ \phi \in \underset{\rightarrow}{\mathrm{Lim}} F_j\big).$$

If $\Theta \in \prod^{\succcurlyeq}_{i \in I} F_i$, we have that

$$(\mathtt{eql}_0 f_\Theta \circ \phi)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq} \circ e}(j, y)\big) := (\mathtt{eql}_0 f_\Theta)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j), y)\big)$$

$$:= \Theta_{e(j)}(y) := \Theta^J_j(y) := (\mathtt{eql}_0 f_{\Theta^J})\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq} \circ e}(j, y)\big),$$

where $\Theta^J \in \prod^{\preccurlyeq}_{j \in J} F_j$ is defined in Lemma 6.5.11(i). Hence, $\mathtt{eql}_0 f_\Theta \circ \phi = \mathtt{eql}_0 f_{\Theta^J} \in \underset{\rightarrow}{\mathrm{Lim}} F_j$. Next we show that ϕ is a surjection. If $\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$, we find $\mathtt{eql}_0^{\Lambda^{\preccurlyeq} \circ e}(j, y) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j)$ such that $\phi\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq} \circ e}(j, y)\big) := \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(e(j), y) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x)$ i.e., we find $k \in I$ such that $i, e(j) \preccurlyeq k$ and $\lambda^{\preccurlyeq}_{ik}(x) =_{\lambda_0(k)} \lambda^{\preccurlyeq}_{e(j)k}(y)$. If $j := \mathtt{cof}_J(i)$, by $(\mathtt{Cof}_3)$ we have that $i \preccurlyeq e(\mathtt{cof}_J(i))$, and by the reflexivity of $\preccurlyeq$ we have that $e(\mathtt{cof}_J(i)) \preccurlyeq e(\mathtt{cof}_J(i)) := k$. If $y := \lambda_{ie(\mathtt{cof}_J(i))}{}^{\preccurlyeq}(x) \in \lambda_0(e(\mathtt{cof}_J(i))) := (\lambda_0 \circ e)(\mathtt{cof}_J(i))$, then

$$\lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i))e(\mathtt{cof}_J(i))}\big(\lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}(x)\big) =_{\lambda_0(k)} \lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}(x).$$

We can use the $\bigvee$-lifting of openness to show that ϕ is an open morphism, and hence a Bishop isomorphism, but it is better to define directly its inverse Bishop morphism using the previous proof of the surjectivity of ϕ. Let the operation $\theta\colon \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j)$, defined by

$$\theta\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x)\big) := \mathtt{eql}_0^{\Lambda^{\preccurlyeq} \circ e}\big(\mathtt{cof}_J(i), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big); \qquad \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i, x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i).$$

First we show that θ is a function. We have that

$$\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i',x') \Leftrightarrow \exists_{k \in I}\big(i \preccurlyeq k \ \&\ i' \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{i'k}^{\preccurlyeq}(x')\big),$$

$$\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}\big(\mathtt{cof}_J(i), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}\big(\mathtt{cof}_J(i'), \lambda_{i'e(\mathtt{cof}_J(i'))}(x')\big) \Leftrightarrow$$

$$\exists_{j' \in J}\Bigg(\mathtt{cof}_J(i) \preccurlyeq j' \ \&\ \mathtt{cof}_J(i') \preccurlyeq j' \ \&$$

$$\lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i))e(j')}\big(\lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}(x)\big) =_{\lambda_0(e(j')} \lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i'))e(j')}\big(\lambda^{\preccurlyeq}_{i'e(\mathtt{cof}_J(i'))}(x')\big)\Bigg).$$

If $j' := \mathtt{cof}_J(k)$, then by $(\mathtt{Cof}_2)$ we get $\mathtt{cof}_J(i) \preccurlyeq j'$ and $\mathtt{cof}_J(i') \preccurlyeq j'$. Next we show that

$$\lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i))e(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(i))}(x)\big) =_{\lambda_0(e(\mathtt{cof}_J(k))} \lambda^{\preccurlyeq}_{e(\mathtt{cof}_J(i'))e(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{i'e(\mathtt{cof}_J(i'))}(x').$$

By the following order relations, the two terms of the required equality are written as

$$\begin{tikzcd}
i \arrow[d] \arrow[drr] & & & & i' \arrow[dll] \arrow[d] \\
e(\mathtt{cof}_J(i)) \arrow[drr] & & k \arrow[d] & & e(\mathtt{cof}_J(i')) \arrow[dll] \\
& & e(\mathtt{cof}_J(k)) & &
\end{tikzcd}$$

$\lambda^{\preccurlyeq}_{ie(\mathtt{cof}_J(k))}(x) = \lambda^{\preccurlyeq}_{ke(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{ik}(x)\big)$, and $\lambda^{\preccurlyeq}_{i'e(\mathtt{cof}_J(k))}(x') = \lambda^{\preccurlyeq}_{ke(\mathtt{cof}_J(k))}\big(\lambda^{\preccurlyeq}_{i'k}(x')\big)$. By the equality $\lambda^{\preccurlyeq}_{ik}(x) =_{\lambda_0(k)} \lambda^{\preccurlyeq}_{i'k}(x')$ we get the required equality. Next we show that

$$\theta \in \mathrm{Mor}\big(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_j\big) \Leftrightarrow \forall_{H^J \in \prod_{j\in J}^{\succcurlyeq} F_j}\Bigg(\mathtt{eql}_0 f_{H^J} \circ \theta \in \bigvee_{\Theta \in \prod_{i \in I}^{\succcurlyeq} F_i} \mathtt{eql}_0 f_{\Theta}\Bigg).$$

If we fix $H^J \in \prod_{j\in J}^{\succcurlyeq} F_j$, and if $H \in \prod_{i\in I}^{\succcurlyeq} F_i$, defined in Lemma 6.5.11(ii), then

$$\begin{aligned}
\big(\mathtt{eql}_0 f_{H^J} \circ \theta\big)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) &:= \mathtt{eql}_0 f_{H^J}\Bigg(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}\big(\mathtt{cof}_J(i), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big)\Bigg) \\
&:= f_{H^J}\big(\mathtt{cof}_J(i), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big) \\
&:= H^J_{\mathtt{cof}_J(i)}\big(\lambda_{ie(\mathtt{cof}_J(i))}(x)\big) \\
&:= \big[H^J_{\mathtt{cof}_J(i)} \circ \lambda_{ie(\mathtt{cof}_J(i))}\big](x) \\
&:= H_i(x) \\
&:= f_H(i,x) \\
&:= \mathtt{eql}_0 f_H\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big),
\end{aligned}$$

hence $\mathtt{eql}_0 f_{H^J} \circ \theta := \mathtt{eql}_0 f_H \in \underset{\rightarrow}{\mathrm{Lim}}F_i$. Next we show that ϕ and θ are inverse to each other.

$$\begin{aligned}
\phi\big(\theta\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big)\big) &:= \phi\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}\big(\mathtt{cof}_J(i), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big) \\
&:= \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}\big(e(\mathtt{cof}_J(i)), \lambda_{ie(\mathtt{cof}_J(i))}(x)\big),
\end{aligned}$$

which is equal to $\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)$ if and only if there is $k \in I$ with $i \preccurlyeq k$ and $e(\texttt{cof}_J(i)) \preccurlyeq k$ and

$$\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{e(\texttt{cof}_J(i))k}^{\preccurlyeq}\big(\lambda_{ie(\texttt{cof}_J(i))}(x)\big),$$

which holds for every such $k \in I$. As by $(\texttt{Cof}_3)$ we have that $i \preccurlyeq e(\texttt{cof}_J(i))$, the existence of such a $k \in I$ follows trivially. Similarly,

$$\begin{aligned}\theta\big(\phi\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j,y)\big)\big) &:= \theta\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}\big(e(j),y\big)\big)\\ &:= \texttt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}\big(\texttt{cof}_J(e(j)), \lambda_{e(j)e(\texttt{cof}_J(e(j)))}(y)\big),\end{aligned}$$

which is equal to $\texttt{eql}_0^{\Lambda^{\preccurlyeq}\circ e}(j,y)$ if and only if there is $j' \in J$ with $j \preccurlyeq j'$, $(\texttt{cof}_J(e(j))) \preccurlyeq j'$ and

$$\lambda_{e(j)e(j')}^{\preccurlyeq}(y) =_{\lambda_0(e(j'))} \lambda_{e(\texttt{cof}_J(e(j)))e(j')}^{\preccurlyeq}\big(\lambda_{e(j)e(\texttt{cof}_J(e(j)))}(y)\big),$$

which holds for every such $j' \in J$. As by $(\texttt{Cof}_1)$ we have that $j =_J \texttt{cof}_J(e(j))$, the existence of such a $j' \in J$ follows trivially. $\square$

For simplicity we use next the same symbol for different orderings.

Proposition 6.5.13. *If $(I,\preccurlyeq),(J,\preccurlyeq)$ are directed sets, $i \in I$ and $j \in J$, let*

$$(i,j) \preccurlyeq (i',j') :\Leftrightarrow i \preccurlyeq i' \ \&\ j \preccurlyeq j'.$$

If $(K, i_K, \texttt{cof}_K) \subseteq^{\texttt{cof}} I$ and $(L, i_L, \texttt{cof}_L) \subseteq^{\texttt{cof}} J$, let $i_{K\times L} : K \times L \hookrightarrow I \times J$ and $\texttt{cof}_{K\times L} : I \times J \to K \times L$, defined, for every $k \in K$ and $l \in L$, by

$$i_{K\times L}(k,l) := \big(i_K(k), i_L(l)\big) \quad \& \quad \texttt{cof}_{K\times L}(i,j) := \big(\texttt{cof}_K(i), \texttt{cof}_L(j)\big).$$

Let $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}) \in \texttt{Fam}(I,\preccurlyeq)$ and $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq}) \in \texttt{Fam}(J,\preccurlyeq)$ an $(J,\preccurlyeq)$. Let also $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \texttt{Spec}(I,\preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ii'})_{(i,i')\in D^{\preccurlyeq}(I\times J)}$, and $S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \texttt{Spec}(J,\preccurlyeq)$ with Bishop spaces $(\mathcal{G}_j)_{j\in J}$ and Bishop morphisms $(\mu_{jj'})_{(j,j')\in\preccurlyeq(J)}^{\preccurlyeq}$.

(i) *$(I \times J, \preccurlyeq)$ is a directed set, and $(K \times L, i_{K\times L}, \texttt{cof}_{K\times L}) \subseteq^{\texttt{cof}} I \times J$.*

(ii) *The pair $\Lambda^{\preccurlyeq} \times M^{\preccurlyeq} := (\lambda_0 \times \mu_0, (\lambda_1 \times \mu_1)^{\preccurlyeq}) \in \texttt{Fam}(I \times J, \preccurlyeq)$, where*

$$(\lambda_0 \times \mu_0)\big((i,j)\big) := \lambda_0(i) \times \mu_0(j),$$

$$(\lambda_1 \times \mu_1)^{\preccurlyeq}\big((i,j),(i',j')\big) := (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preccurlyeq},$$

$$(\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preccurlyeq}\big((x,y)\big) := \big(\lambda_{ii'}^{\preccurlyeq}(x), \mu_{jj'}^{\preccurlyeq}(y)\big).$$

(iii) *The structure $S(\Lambda^{\preccurlyeq} \times M^{\preccurlyeq}) := \big(\lambda_0 \times \mu_0, \lambda_1^{\preccurlyeq} \times \mu_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big) \in \texttt{Spec}(I \times J, \preccurlyeq)$ with Bishop spaces $(F_i \times G_j)_{(i,j)\in I\times J}$ and Bishop morphisms $(\lambda_1 \times \mu_1)_{((i,j)(i',j'))\in D^{\preccurlyeq}(I\times J)}$, where*

$$\phi_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}(i,j) := F_i \times G_j,$$

$$\phi_1^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i,j),(i',j')\big) := [(\lambda_1 \times \mu_1)_{(i,j)(i',j')}^{\preccurlyeq}]^* : F_{i'} \times G_{j'} \to F_i \times G_j.$$

Proof. (i) is immediate to show. For the proof of case (ii) we have that $(\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j),(i,j)}\big((x,y)\big) := \big(\lambda^{\preccurlyeq}_{ii}(x), \mu^{\preccurlyeq}_{jj}(y)\big) := (x,y)$, and if $(i,j) \preccurlyeq (i',j') \preccurlyeq (i'',j'')$, then the commutativity of the

$$\begin{tikzcd}
\lambda_0(i) \times \mu_0(j) \arrow[d, "(\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j),(i',j')}"'] \arrow[dr, "(\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j),(i'',j'')}"] & \\
\lambda_0(i') \times \mu_0(j') \arrow[r, "(\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i',j'),(i'',j'')}"'] & \lambda_0(i'') \times \mu_0(j'')
\end{tikzcd}$$

above diagram follows from the equalities $\lambda^{\preccurlyeq}_{ii''} = \lambda^{\preccurlyeq}_{i'i''} \circ \lambda^{\preccurlyeq}_{ii'}$ and $\mu^{\preccurlyeq}_{jj''} = \mu^{\preccurlyeq}_{j'j''} \circ \mu^{\preccurlyeq}_{jj'}$.
(iii) We show that $(\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')} \in \mathrm{Mor}(\mathcal{F}_i \times \mathcal{G}_j, \mathcal{F}_{i'} \times \mathcal{G}_{j'})$. By the $\bigvee$-lifting of morphisms it suffices to show that $\forall_{f \in F_{i'}}\big((f \circ \pi_1) \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')} \in F_i \times G_j\big)$ and $\forall_{g \in G_{j'}}\big((g \circ \pi_2) \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')} \in F_i \times G_j\big)$. If $f \in F_{i'}$, then $(f \circ \pi_1) \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')} := (f \circ \lambda^{\preccurlyeq}_{ii'}) \circ \pi_1 \in F_i \times G_j$, as $f \circ \lambda^{\preccurlyeq}_{ii'} \in F_i$ and $[(f \circ \pi_1) \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')}](x,y) := (f \circ \pi_1)\big(\lambda^{\preccurlyeq}_{ii'}(x), \mu^{\preccurlyeq}_{jj'}(y)\big) := f\big(\lambda^{\preccurlyeq}_{ii'}(x)\big) := [(f \circ \lambda^{\preccurlyeq}_{ii'}) \circ \pi_1](x,y)$. If $g \in G_{j'}$, we get $(g \circ \pi_2) \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j)(i',j')} := (g \circ \lambda^{\preccurlyeq}_{jj'}) \circ \pi_2 \in F_i \times G_j$. $\square$

Lemma 6.5.14. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda^{\preccurlyeq}_1, \phi^{\Lambda^{\preccurlyeq}}_0, \phi^{\Lambda^{\preccurlyeq}}_1) \in \mathtt{Spec}(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ii'})^{\preccurlyeq}_{(i,i') \in D^{\preccurlyeq}(I)}$, $S(M^{\preccurlyeq}) := (\mu_0, \mu^{\preccurlyeq}_1, \phi^{M^{\preccurlyeq}}_0, \phi^{M^{\preccurlyeq}}_1) \in \mathtt{Spec}(J, \preccurlyeq)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and Bishop morphisms $(\mu_{jj'})_{(j,j') \in D^{\preccurlyeq}(J)}$, $\Theta \in \prod^{\succcurlyeq}_{i \in I} F_i$ and $\Phi \in \prod^{\succcurlyeq}_{j \in J} G_j$,. Then*

$$\Theta_1 \in \prod^{\succcurlyeq}_{(i,j) \in I \times J} F_i \times G_j \quad \& \quad \Phi_2 \in \prod^{\succcurlyeq}_{(i,j) \in I \times J} F_i \times G_j,$$

$$\Theta_1(i,j) := \Theta_i \circ \pi_1 \in F_i \times G_j \quad \& \quad \Phi_2(i,j) := \Phi_j \circ \pi_2 \in F_i \times G_j; \qquad (i,j) \in I \times J.$$

Proof. We prove that $\Theta_1 \in \prod^{\succcurlyeq}_{(i,j) \in I \times J} F_i \times G_j$, and for Φ_2 we proceed similarly. If $(i,j) \preccurlyeq (i',j')$, we need to show that $\Theta_1(i,j) = \Theta_1(i',j') \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j),(i',j')}$. Since $\Theta \in \prod^{\succcurlyeq}_{i \in I} F_i$, we have that $\Theta_i = \Theta_{i'} \circ \lambda^{\preccurlyeq}_{ii'}$. If $x \in \lambda_0(i)$ and $y \in \mu_0(j)$, we have that

$$\begin{aligned}
\big[\Theta_1(i',j') \circ (\lambda_1 \times \mu_1)^{\preccurlyeq}_{(i,j),(i',j')}\big](x,y) &:= \big[\Theta_{i'} \circ \pi_1\big]\big(\lambda^{\preccurlyeq}_{ii'}(x), \mu^{\preccurlyeq}_{jj'}(y)\big) \\
&:= \Theta_{i'}\big(\lambda^{\preccurlyeq}_{ii'}(x)\big) \\
&:= \big[\big(\Theta_{i'} \circ \lambda^{\preccurlyeq}_{ii'}\big) \circ \pi_1\big](x,y) \\
&:= \big(\Theta_i \circ \pi_1\big)(x,y) \\
&:= \big[\Theta_1(i,j)\big](x,y).
\end{aligned}$$

$\square$

Proposition 6.5.15. *If $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda^{\preccurlyeq}_1, \phi^{\Lambda^{\preccurlyeq}}_0, \phi^{\Lambda^{\preccurlyeq}}_1) \in \mathtt{Spec}(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ii'})^{\preccurlyeq}_{(i,i') \in D^{\preccurlyeq}(I)}$, and $S(M^{\preccurlyeq}) := (\mu_0, \mu^{\preccurlyeq}_1, \phi^{M^{\preccurlyeq}}_0, \phi^{M^{\preccurlyeq}}_1) \in \mathtt{Spec}(J, \preccurlyeq)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and Bishop morphisms $(\mu_{jj'})_{(j,j') \in D^{\preccurlyeq}(J)}$, there is a bijection*

$$\theta : \underset{\to}{\mathrm{Lim}}\big(\lambda_0(i) \times \mu_0(j)\big) \to \underset{\to}{\mathrm{Lim}}\lambda_0(i) \times \underset{\to}{\mathrm{Lim}}\mu_0(j) \in \mathrm{Mor}\big(\underset{\to}{\mathrm{Lim}}(\mathcal{F}_i \times \mathcal{G}_j), \underset{\to}{\mathrm{Lim}}\mathcal{F}_i \times \underset{\to}{\mathrm{Lim}}\mathcal{G}_j\big).$$

Proof. Let the operation $\theta : \underset{\to}{\mathrm{Lim}}\big(\lambda_0(i) \times \mu_0(j)\big) \rightsquigarrow \underset{\to}{\mathrm{Lim}}\lambda_0(i) \times \underset{\to}{\mathrm{Lim}}\mu_0(j)$, defined by

$$\theta\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq} \times M^{\preccurlyeq}}\big((i,j),(x,y)\big)\big) := \big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), \mathtt{eql}_0^{M^{\preccurlyeq}}(j,y)\big).$$

First we show that θ is an embedding as follows:

$$\begin{aligned}
&\texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i,j),(x,y)\big) = \texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i',j'),(x',y')\big) :\Leftrightarrow\\
&:\Leftrightarrow \exists_{(k,l)\in I\times J}\big((i,j),(i',j') \preccurlyeq (k,l) \ \&\ (\lambda_1\times\mu_1)^{\preccurlyeq}_{(i,j)(k,l)}(x,y) = (\lambda_1\times\mu_1)^{\preccurlyeq}_{(i',j')(k,l)}(x',y')\big)\\
&:\Leftrightarrow \exists_{(k,l)\in I\times J}\big((i,j),(i',j') \preccurlyeq (k,l) \ \&\ \big(\lambda^{\preccurlyeq}_{ik}(x),\mu^{\preccurlyeq}_{jl}(y)\big) = \big(\lambda^{\preccurlyeq}_{i'k}(x'),\mu^{\preccurlyeq}_{j'k}(y')\big)\big)\\
&\Leftrightarrow \exists_{k\in I}\big(i,i' \preccurlyeq k \ \&\ \lambda^{\preccurlyeq}_{ik}(x) = \lambda^{\preccurlyeq}_{i'k}(x')\big) \ \&\ \exists_{l\in J}\big(j,j' \preccurlyeq l \ \&\ \lambda^{\preccurlyeq}_{jk}(y) = \lambda^{\preccurlyeq}_{j'k}(y')\big)\\
&:\Leftrightarrow \texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) = \texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i',x') \ \ \&\ \ \texttt{eql}_0^{M^{\preccurlyeq}}(j,y) = \texttt{eql}_0^{M^{\preccurlyeq}}(j',y')\\
&:\Leftrightarrow \big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x),\texttt{eql}_0^{M^{\preccurlyeq}}(j,y)\big) = \big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i',x'),\texttt{eql}_0^{M^{\preccurlyeq}}(j',y')\big)\\
&:\Leftrightarrow \theta\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i,j),(x,y)\big)\big) = \theta\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i',j'),(x',y')\big)\big).
\end{aligned}$$

The fact that θ is a surjection is immediate to show. By definition of the direct limit and by the $\vee$-lifting of the product Bishop topology we have that

$$\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F}_i\times\mathcal{G}_j) := \left(\underset{\rightarrow}{\mathrm{Lim}}\big(\lambda_0(i)\times\mu_0(j)\big), \bigvee_{\Xi\in\prod^{\succcurlyeq}_{(i,j)\in I\times J}F_i\times G_j}\texttt{eql}_0 f_{\Xi}\right),$$

$$\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i\times\underset{\rightarrow}{\mathrm{Lim}}\mathcal{G}_j := \left(\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)\times\underset{\rightarrow}{\mathrm{Lim}}\mu_0(j), \bigvee_{\Theta\in\prod^{\succcurlyeq}_{i\in I}F_i}^{H\in\prod^{\succcurlyeq}_{j\in J}G_j}\texttt{eql}_0 f_{\Theta}\circ\pi_1, \texttt{eql}_0 f_H\circ\pi_2\right).$$

To show that $\theta\in\mathrm{Mor}\big(\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F}_i\times\mathcal{G}_j),\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i\times\underset{\rightarrow}{\mathrm{Lim}}\mathcal{G}_j\big)$ it suffices to show that

$$\forall_{\Theta\in\prod^{\succcurlyeq}_{i\in I}F_i}\forall_{H\in\prod^{\succcurlyeq}_{j\in J}G_j}\big((\texttt{eql}_0 f_{\Theta}\circ\pi_1)\circ\theta\in\underset{\rightarrow}{\mathrm{Lim}}(F_i\times G_j) \ \&\ (\texttt{eql}_0 f_H\circ\pi_2)\circ\theta\in\underset{\rightarrow}{\mathrm{Lim}}(F_i\times G_j)\big).$$

If $\Theta\in\prod^{\succcurlyeq}_{i\in I}F_i$, we show that $(\texttt{eql}_0 f_{\Theta}\circ\pi_1)\circ\theta\in\underset{\rightarrow}{\mathrm{Lim}}(F_i\times G_j)$ From the equalities

$$\begin{aligned}
\big[(\texttt{eql}_0 f_{\Theta^{\succcurlyeq}}\circ\pi_1)\circ\theta\big]\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i,j),(x,y)\big)\big) &:= (\texttt{eql}_0 f_{\Theta^{\succcurlyeq}}\circ\pi_1)\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x),\texttt{eql}_0^{M^{\preccurlyeq}}(j,y)\big)\\
&:= \texttt{eql}_0 f_{\Theta^{\succcurlyeq}}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big)\\
&:= \Theta_i(x)\\
&:= \big(\Theta_i\circ\pi_1\big)(x,y)\\
&:= \big[\Theta_1(i,j)\big](x,y)\\
&:= \texttt{eql}_0 f_{\Theta_1}\big(\texttt{eql}_0^{\Lambda^{\preccurlyeq}\times M^{\preccurlyeq}}\big((i,j),(x,y)\big),
\end{aligned}$$

where $\Theta_1\in\prod^{\succcurlyeq}_{i\in I}F_i\times G_j$ is defined in Lemma 6.5.14, we conclude that $(\texttt{eql}_0 f_{\Theta^{\succcurlyeq}}\circ\pi_1)\circ\theta := \texttt{eql}_0 f_{\Theta_1}\in\underset{\rightarrow}{\mathrm{Lim}}(F_i\times G_j)$. For the second case we work similarly. $\square$

6.6 Inverse limit of a contravariant spectrum of Bishop spaces

Definition 6.6.1. *If $S(\Lambda^{\succcurlyeq}) := (\lambda_0,\lambda_1^{\succcurlyeq},\phi_0^{\Lambda^{\succcurlyeq}},\phi_1^{\Lambda^{\succcurlyeq}})$ is a contravariant $(I,\preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda^{\succcurlyeq}_{ji})_{(i,j)\in D^{\preccurlyeq}(I)}$, the inverse limit of $S\Lambda(^{\succcurlyeq})$ is the Bishop space*

$$\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i := \big(\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i),\ \underset{\leftarrow}{\mathrm{Lim}}F_i\big),$$

$$\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) := \prod_{i\in I}^{\succcurlyeq} \lambda_0(i) \quad \& \quad \underset{\leftarrow}{\mathrm{Lim}}F_i := \bigvee_{i \in I}^{f\in F_i} f \circ \pi_i^{\Lambda^{\succcurlyeq}}.$$

For simplicity we write π_i instead of $\pi_i^{\Lambda^{\succcurlyeq}}$ for the function $\pi_i^{\Lambda^{\succcurlyeq}} : \prod_{i\in I}^{\succcurlyeq} \lambda_0(i) \to \lambda_0(i)$, which is defined, as its dual $\pi_i^{\Lambda^{\preccurlyeq}}$ in the Proposition 3.8.4(iv), by the rule $\Phi \mapsto \Phi_i$, for every $i \in I$..

Proposition 6.6.2 (Universal property of the inverse limit). *If $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ is a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j)\in\preccurlyeq(I)}$, its inverse limit $\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i$ satisfies the universal property of inverse limits i.e.,*

(i) *For every $i \in I$, we have that $\pi_i \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F}_i)$.*

(ii) *If $i \preccurlyeq j$, the following left diagram commutes*

$$\begin{tikzcd}
 & \prod_{i\in I}^{\succcurlyeq} \lambda_0(i) \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
\lambda_0(i) & & \lambda_0(j) \arrow[ll, "\lambda_{ji}^{\succcurlyeq}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] & \\
\lambda_0(i) & & \lambda_0(j). \arrow[ll, "\lambda_{ji}^{\preccurlyeq}"]
\end{tikzcd}$$

(iii) *If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varpi_i : Y \to \lambda_0(i) \in \mathrm{Mor}(\mathcal{G}, \mathcal{F}_i)$, for every $i \in I$, such that if $i \preccurlyeq j$, the above right diagram commutes, there is a unique function $h : Y \to \prod_{i\in I} \lambda_0(i) \in \mathrm{Mor}(\mathcal{G}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i)$ such that the following diagrams commute*

$$\begin{tikzcd}
 & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] \arrow[dd, dashed, "h"] & \\
\lambda_0(i) & & \lambda_0(j), \arrow[ll, hook', "\lambda_{ji}^{\succcurlyeq}"] \\
 & \prod_{i\in I} \lambda_0(i) \arrow[ul, "\pi_i"] \arrow[ur, "\pi_j"'] &
\end{tikzcd}$$

Proof. The condition $\pi_i \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F}_i) :\Leftrightarrow \forall_{f\in F_i}\Big(f \circ \pi_i \in \bigvee_{i\in I}^{f\in F_i} f \circ \pi_i\Big)$ is trivially satisfied, and (i) follows. For (ii), the required equality $\lambda_{ji}^{\succcurlyeq}\big(\pi_j(\Phi)\big) =_{\lambda_0(i)} \pi_i(\Phi) :\Leftrightarrow \lambda_{ji}^{\succcurlyeq}(\Phi_j) =_{\lambda_0(i)} \Phi_i$ holds by the definition of $\prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$. To show (iii), let the operation $h : Y \rightsquigarrow \prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$, defined by $h(y) := \Phi_y$, where $\Phi_y(i) := \varpi_i(y)$, for every $y \in Y$ and $i \in I$. First we show that h is well-defined i.e., $h(y) \in \prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$. If $i \preccurlyeq j$, by the supposed commutativity of the above right diagram we have that $\lambda_{ji}^{\succcurlyeq}\big(\Phi_y(j)\big) := \lambda_{ji}^{\succcurlyeq}\big(\varpi_j(y)\big) = \varpi_i(y) := \Phi_y(i)$. Next we show that h is a function. If $y =_Y y'$, the last formula in the following equivalences

$$\Phi_y =_{\prod_{i\in I}^{\succcurlyeq} \lambda_0(i)} \Phi_{y'} :\Leftrightarrow \forall_{i\in I}\big(\Phi_y(i) =_{\lambda_0(i)} \Phi_{y'}(i)\big) :\Leftrightarrow \forall_{i\in I}\big(\varpi_i(y) =_{\lambda_0(i)} \varpi_i(y')\big)$$

holds by the fact that ϖ_i is a function, for every $i \in I$. By the $\bigvee$-lifting of morphisms we have that $h \in \mathrm{Mor}(\mathcal{G}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{i\in I}\forall_{f\in F_i}\big((f \circ \pi_i) \circ h \in G\big)$. If $i \in I$, $f \in F_i$, and $y \in Y$, then

$$\big[(f \circ \pi_i) \circ h\big](y) := (f \circ \pi_i)(\Phi_y) := f\big(\Phi_y(i)\big) := f\big(\varpi_i(y)\big) := (f \circ \varpi_i)(y),$$

hence $(f \circ \pi_i) \circ h := f \circ \varpi_i \in G$, since $\varpi_i \in \mathrm{Mor}(\mathcal{G}, \mathcal{F}_i)$. The required commutativity of the last diagram above, and the uniqueness of h follow immediately. $\square$

The uniqueness of $\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i)$, up to Bishop isomorphism, follows easily from its universal property. Next follows the inverse analogue to the Theorem 6.5.8.

Theorem 6.6.3. *Let $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ be a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, $S(M^{\succcurlyeq}) := (\mu_0, \mu_1, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mu_{ji}^{\succcurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$, and $\Psi \colon S(\Lambda^{\succcurlyeq}) \Rightarrow S(M^{\succcurlyeq})$.*
(i) *There is a unique function $\Psi_{\leftarrow} \colon \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) \to \underset{\leftarrow}{\mathrm{Lim}}\mu_0(i)$ such that, for every $i \in I$, the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[u, "\pi_i^{\Lambda^{\succcurlyeq}}"] \arrow[r, dashed, "\Psi_{\leftarrow}"'] & \underset{\leftarrow}{\mathrm{Lim}}\mu_0(i). \arrow[u, "\pi_i^{M^{\succcurlyeq}}"']
\end{tikzcd}$$

(ii) *If Ψ is continuous, then $\Psi_{\leftarrow} \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{G}_i)$.*
(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Psi_{\leftarrow}$ is an embedding.*

Proof. (i) Let the assignment routine $\Psi_{\leftarrow} \colon \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow \underset{\leftarrow}{\mathrm{Lim}}\mu_0(i)$, defined by

$$\Theta \mapsto \Psi_{\leftarrow}(\Theta), \qquad \big[\Psi_{\leftarrow}(\Theta)\big]_i := \Psi_i(\Theta_i); \qquad \Theta \in \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i),\ i \in I.$$

First we show that $\Psi_{\leftarrow}$ is well-defined i.e., $\Psi_{\leftarrow}(\Theta) \in \prod_{i\in I}^{\succcurlyeq} \mu_0(i)$. If $i \preccurlyeq j$, since $\Theta \in \prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$, we have that $\Theta_i = \lambda_{ji}^{\succcurlyeq}(\Theta_j)$, and since $\Psi \colon S(\Lambda^{\succcurlyeq}) \Rightarrow S(M^{\succcurlyeq})$

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[l, "\lambda_{ji}^{\succcurlyeq}"'] \arrow[d, "\Psi_j"] \\
\mu_0(i) & \mu_0(j), \arrow[l, "\mu_{ji}^{\succcurlyeq}"]
\end{tikzcd}$$

$\big[\Psi_{\leftarrow}(\Theta)\big]_i := \Psi_i(\Theta_i) = \Psi_i\big(\lambda_{ji}^{\succcurlyeq}(\Theta_j)\big) := \big(\Psi_i \circ \lambda_{ji}^{\succcurlyeq}\big)(\Theta_j) = \big(\mu_{ji}^{\succcurlyeq} \circ \Psi_j\big)(\Theta_j) := \mu_{ji}^{\succcurlyeq}\big(\Psi_j(\Theta_j)\big) := \mu_{ji}^{\succcurlyeq}\big(\big[\Psi_{\leftarrow}(\Theta)\big]_j\big)$. Next we show that $\Psi_{\leftarrow}$ is a function: $\Theta =_{\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i)} \Phi :\Leftrightarrow \forall_{i\in I}\big(\Theta_i =_{\lambda_0(i)} \Phi_i\big) \Rightarrow \forall_{i\in I}\big(\Psi_i(\Theta_i) =_{\mu_0(i)} \Psi_i(\Phi_i)\big) :\Leftrightarrow \forall_{i\in I}\big(\big[\Psi_{\leftarrow}(\Theta)\big]_i =_{\mu_0(i)} \big[\Psi_{\leftarrow}(\Phi)\big]_i\big) :\Leftrightarrow \Psi_{\leftarrow}(\Theta) =_{\underset{\leftarrow}{\mathrm{Lim}}\mu_0(i)} \Psi_{\leftarrow}(\Phi)$. The commutativity of the diagram and the uniqueness of $\Psi_{\leftarrow}$ are immediate to show.
(ii) By the $\bigvee$-lifting of morphisms we have that $\Psi_{\leftarrow} \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{G}_i) \Leftrightarrow \forall_{i\in I}\forall_{g\in G_i}\big((g \circ \pi_i^{M^{\succcurlyeq}}) \circ \Psi_{\leftarrow} \in \underset{\leftarrow}{\mathrm{Lim}}F_i\big)$. If $i \in I$ and $g \in G_i$, then $[(g \circ \pi_i^{M^{\succcurlyeq}}) \circ \Psi_{\leftarrow}](\Theta) := g\big([\Psi_{\leftarrow}(\Theta)]_i\big) := g\big(\Psi_i(\Theta_i)\big) := \big(g \circ \Psi_i\big)(\Theta_i) := \big[\big(g \circ \Psi_i\big) \circ \pi_i^{\Lambda^{\succcurlyeq}}\big](\Theta)$, and $g \circ \Psi_i \in F_i$, by the continuity of Ψ, hence $(g \circ \pi_i^{M^{\succcurlyeq}}) \circ \Psi_{\leftarrow} := \big(g \circ \Psi_i\big) \circ \pi_i^{\Lambda^{\succcurlyeq}} \in \underset{\leftarrow}{\mathrm{Lim}}F_i$.
(iii) By definition we have that $\Psi_{\leftarrow}(\Theta) =_{\underset{\leftarrow}{\mathrm{Lim}}\mu_0(i)} \Psi_{\leftarrow}(\Phi) :\Leftrightarrow \forall_{i\in I}\big(\Psi_i(\Theta_i) =_{\mu_0(i)} \Psi_i(\Phi_i)\big) \Rightarrow \forall_{i\in I}\big(\Theta_i =_{\lambda_0(i)} \Phi_i\big) :\Leftrightarrow \Theta =_{\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i)} \Phi$. $\square$

Proposition 6.6.4. *If $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$, $S(M^{\succcurlyeq}) := (\mu_0, \mu_1^{\succcurlyeq}, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ and $S(N^{\succcurlyeq}) := (\nu_0, \nu_1^{\succcurlyeq}, \phi_0^{N^{\succcurlyeq}}, \phi_1^{N^{\succcurlyeq}})$ are contravariant direct spectra over $(I, \preccurlyeq)$, and if $\Psi \colon S(\Lambda^{\succcurlyeq}) \Rightarrow S(M^{\succcurlyeq})$ and $\Xi \colon S(M^{\succcurlyeq}) \Rightarrow S(N^{\succcurlyeq})$, then $(\Xi \circ \Psi)_{\leftarrow} := \Xi_{\leftarrow} \circ \Psi_{\leftarrow}$*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[rr, bend left, "(\Xi \circ \Psi)_i"] & \mu_0(i) \arrow[r, "\Xi_i"] & \nu_0(i) \\
\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[u, "\pi_i^{\Lambda^{\preccurlyeq}}"] \arrow[r, "\Psi_{\leftarrow}"'] \arrow[rr, bend right, "(\Xi \circ \Psi)_{\leftarrow}"'] & \underset{\leftarrow}{\mathrm{Lim}}\mu_0(i) \arrow[u, "\pi_i^{M^{\preccurlyeq}}"] \arrow[r, "\Xi_{\leftarrow}"'] & \underset{\leftarrow}{\mathrm{Lim}}\nu_0(i). \arrow[u, "\pi_i^{N^{\preccurlyeq}}"']
\end{tikzcd}$$

Proof. The required equality is reduced to $\forall_{i \in I}\big(\big[(\Xi \circ \Psi)_{\leftarrow}(\Theta)\big]_i =_{\nu_0(i)} \big[\Xi_{\leftarrow}(\Psi_{\leftarrow}(\Theta))\big]_i\big)$. If $i \in I$, then $\big[(\Xi \circ \Psi)_{\leftarrow}(\Theta)\big]_i := (\Xi \circ \Psi)_i(\Theta_i) := \Xi_i(\Psi_i(\Theta_i)) := \Xi_i\big(\big[\Psi_{\leftarrow}(\Theta)\big]_i\big) := \big[\Xi_{\leftarrow}(\Psi_{\leftarrow}(\Theta))\big]_i$. $\square$

Theorem 6.6.5. *Let $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ be a contravariant direct spectrum over $(I, \preccurlyeq)$, $(J, e, \mathtt{cof}_J)$ a cofinal subset of I, and $S(\Lambda^{\succcurlyeq} \circ e) := \big((\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^{\succcurlyeq} \circ e}, \phi_1^{\Lambda^{\succcurlyeq} \circ e}\big)$ the relative spectrum of $S(\Lambda^{\succcurlyeq})$ to J. Then*

$$\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_j \simeq \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i.$$

Proof. If $\Theta \in \prod_{j \in J}^{\succcurlyeq} \lambda_0(j)$, then, if $j \preccurlyeq j'$, we have that $\Theta_j = \lambda_{j'j}^{\succcurlyeq}\big(\Theta_{j'}\big) := \lambda_{e(j')e(j)}^{\succcurlyeq}\big(\Theta_{j'}\big)$. If $i \in I$, then $\mathtt{cof}_J(i) \in J$ and $\Theta_{\mathtt{cof}_J(i)} \in \lambda_0(e(\mathtt{cof}_J(i)))$. Since $i \preccurlyeq e(\mathtt{cof}_J(i))$, we define the operation $\phi : \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(j) \rightsquigarrow \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i)$, by the rule $\Theta \mapsto \phi(\Theta)$, for every $\Theta \in \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(j)$, where

$$[\phi(\Theta)]_i := \lambda_{e(\mathtt{cof}_J(i))i}^{\succcurlyeq}\big(\Theta_{\mathtt{cof}_J(i)}\big) \in \lambda_0(i); \quad i \in I.$$

First we show that ϕ is well-defined i.e., $\phi(\Theta) \in \prod_{i \in I}^{\succcurlyeq} \lambda_0(i)$ i.e., for every $i, i' \in I$, $i \preccurlyeq i' \Rightarrow [\phi(\Theta)]_i = \lambda_{i'i}^{\succcurlyeq}\big([\phi(\Theta)]_{i'}\big)$. Working as in the proof of Lemma 6.5.11(ii), we get

$$\begin{aligned}
\lambda_{i'i}^{\succcurlyeq}\big([\phi(\Theta)]_{i'}\big) &:= \lambda_{i'i}^{\succcurlyeq}\big(\lambda_{e(\mathtt{cof}_J(i'))i'}^{\succcurlyeq}\big(\Theta_{\mathtt{cof}_J(i')}\big)\big) \\
&:= \big[\lambda_{i'i}^{\succcurlyeq} \circ \lambda_{e(\mathtt{cof}_J(i'))i'}^{\succcurlyeq}\big]\big(\Theta_{\mathtt{cof}_J(i')}\big) \\
&\overset{(6.1)}{=} \lambda_{e(\mathtt{cof}_J(i'))i}^{\succcurlyeq}\big(\Theta_{\mathtt{cof}_J(i')}\big) \\
&\overset{(6.2)}{=} \big[\lambda_{e(\mathtt{cof}_J(i))i}^{\succcurlyeq} \circ \lambda_{e(\mathtt{cof}_J(i'))e(\mathtt{cof}_J(i))}^{\succcurlyeq}\big]\big(\Theta_{\mathtt{cof}_J(i')}\big) \\
&:= \lambda_{e(\mathtt{cof}_J(i))i}^{\succcurlyeq}\big(\lambda_{e(\mathtt{cof}_J(i'))e(\mathtt{cof}_J(i))}^{\succcurlyeq}\big(\Theta_{\mathtt{cof}_J(i')}\big)\big) \\
&= \lambda_{e(\mathtt{cof}_J(i))i}^{\succcurlyeq}\big(\Theta_{\mathtt{cof}_J(i)}\big) \\
&:= [\phi(\Theta)]_i.
\end{aligned}$$

To show that ϕ is a function we consider the following equivalences:

$$\begin{aligned}\phi(\Theta) =_{\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i)} \phi(H) &:\Leftrightarrow \forall_{i\in I}\big(\big[\phi(\Theta)\big]_i =_{\lambda_0(i)} \big[\phi(H)\big]_i\big)\\ &:\Leftrightarrow \forall_{i\in I}\big(\lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(\Theta_{\mathtt{cof}_J(i)}\big) =_{\lambda_0(i)} \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(H_{\mathtt{cof}_J(i)}\big)\big), \quad (1)\end{aligned}$$

$$\Theta =_{\underset{\leftarrow}{\mathrm{Lim}}\lambda_0(j)} H :\Leftrightarrow \forall_{j\in J}\big(\Theta_j =_{\lambda_0(e(j))} H_j\big) \qquad (2).$$

To show that (1) $\Rightarrow$ (2) we use the fact that $e(\mathtt{cof}_J(j)) = j$, and since $j \preccurlyeq j$, by the extensionality of $\preccurlyeq$ we get $j \preccurlyeq e(\mathtt{cof}_J(j))$. Since $\Theta_j = \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(\Theta_{\mathtt{cof}_J(i)}\big)$, and $H_j = \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(H_{\mathtt{cof}_J(i)}\big)$, we get (2). By the $\bigvee$-lifting of morphisms $\phi \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_j, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{i\in I}\forall_{f\in F_i}\big((f \circ \pi_i^{S^{\succcurlyeq}}) \circ \phi \in \underset{\leftarrow}{\mathrm{Lim}}F_j\big)$. If $\Theta \in \prod^{\succcurlyeq}_{j\in J}\lambda_0(j)$, we have that

$$\begin{aligned}[(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}) \circ \phi](\Theta) &:= f\big(\pi_i^{S(\Lambda^{\succcurlyeq})}(\phi(\Theta))\big)\\ &:= f\big(\big[\phi(\Theta)\big]_i\big)\\ &:= f\big(\lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(\Theta_{\mathtt{cof}_J(i)}\big)\big)\\ &:= \big(f \circ \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big)\big(\Theta_{\mathtt{cof}_J(i)}\big)\\ &:= \big[\big(f \circ \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big) \circ \pi^{S(\Lambda^{\succcurlyeq})\circ e}_{\mathtt{cof}_J(i)}\big](\Theta),\end{aligned}$$

hence $\big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ \phi := \big(f \circ \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big) \circ \pi^{S(\Lambda^{\succcurlyeq})\circ e}_{\mathtt{cof}_J(i)} \in \underset{\leftarrow}{\mathrm{Lim}}F_j$, as by definition $\lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i} \in \mathrm{Mor}(\mathcal{F}_{e(\mathtt{cof}_J(i))}, \mathcal{F}_i)$, and hence

$$\begin{tikzcd} \lambda_0(e(\mathtt{cof}_J(i))) \arrow[r, "\lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}"] \arrow[dr, "f \circ \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}"'] & \lambda_0(i) \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$f \circ \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i} \in F_{e(\mathtt{cof}_J(i))} := F_{\mathtt{cof}_J(i)}$. Let the operation $\theta\colon \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow \underset{\leftarrow}{\mathrm{Lim}}\lambda_0(j)$, defined by the rule $H \mapsto \theta(H) := H^J$, for every $H \in \prod^{\succcurlyeq}_{i\in I}\lambda_0(i)$, where $H^J_j := H_{e(j)} \in \lambda_0(e(j))$, for every $j \in J$. We show that $H^J \in \prod_{j\in J}\lambda_0(j)$. If $j \preccurlyeq j'$, then

$$H^J_j = \lambda^{\succcurlyeq}_{e(j')e(j)}\big(H^J_{j'}\big) :\Leftrightarrow H_{e(j)} = \lambda^{\succcurlyeq}_{e(j')e(j)}\big(H_{e(j')}\big),$$

which holds by the hypothesis $H \in \prod^{\succcurlyeq}_{i\in I}\lambda_0(i)$. Moreover, we have that $\phi(H^J) = H :\Leftrightarrow \forall_{i\in I}\big(\big[\phi(H^J)\big]_i =_{\lambda_0(i)} H_i\big)$. If $i \in I$, and since $i \preccurlyeq e(\mathtt{cof}_J(i))$, we have that

$$\big[\phi(H^J)\big]_i := \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(H^J_{\mathtt{cof}_J(i)}\big) := \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(i))i}\big(H_{e(\mathtt{cof}_J(i))}\big) = H_i.$$

It is immediate to show that θ is a function. Moreover, $\theta(\phi(\Theta) = \Theta$, as if $j \in J$, then

$$\phi(\Theta)^J_j := \phi(\Theta)_{e(j)} := \lambda^{\succcurlyeq}_{e(\mathtt{cof}_J(e(j)))e(j)}\big(\Theta_{\mathtt{cof}_J(e(j))}\big) = \Theta_j,$$

as by hypothesis $\Theta_j = \lambda^{\succcurlyeq}_{e(j')e(j)}(\Theta_{j'})$, with $j \preccurlyeq j'$, and by $(\texttt{Cof}_1)$ we have that $j =_J (\texttt{cof}_J(e(j))$, hence by the extensionality of $\preccurlyeq$ we get $j \preccurlyeq (\texttt{cof}_J(e(j))$. Finally, $\theta \in \mathrm{Mor}\big(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_j\big) \Leftrightarrow$ $\forall_{j\in J}\forall_{f\in F_j}\big(\big(f\circ \pi_j^{S(\Lambda^{\succcurlyeq})}\big)\circ\theta \in \underset{\leftarrow}{\mathrm{Lim}}F_i\big)$, which follows from the equalities

$$\begin{aligned}\big[\big(f\circ \pi_j^{S(\Lambda^{\succcurlyeq})}\big)\circ\theta\big](H) &:= \big(f\circ \pi_j^{S(\Lambda^{\succcurlyeq})}\big)(H^J)\\ &:= f\big(H^J_j\big)\\ &:= f\big(H_{e(j)}\big)\\ &:= \big(f\circ \pi_{e(j)}^{S(\Lambda^{\succcurlyeq})}\big)(H).\end{aligned}$$ □

Proposition 6.6.6. *If $(I,\preccurlyeq),(J,\preccurlyeq)$ are directed sets, $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ is a contravariant direct spectrum over $(I,\preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda^{\succcurlyeq}_{i'i})_{(i,i')\in D^{\preccurlyeq}(I)}$, and $S(M^{\succcurlyeq}) := (\mu_0, \mu_1^{\succcurlyeq}, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ is a contravariant direct spectrum over $(J,\preccurlyeq)$ with Bishop spaces $(\mathcal{G}_j)_{j\in J}$ and Bishop morphisms $(\mu^{\succcurlyeq}_{j'j})_{(j,j')\in D^{\preccurlyeq}(J)}$, there is a function*

$$\times : \prod_{i\in I}^{\succcurlyeq}\lambda_0(i) \times \prod_{j\in J}^{\succcurlyeq}\mu_0(j) \to \prod_{(i,j)\in I\times J}^{\succcurlyeq}\lambda_0(i)\times\mu_0(j) \in \mathrm{Mor}\big(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i \times \underset{\leftarrow}{\mathrm{Lim}}\mathcal{G}_j, \underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i\times\mathcal{G}_j)\big).$$

Proof. We proceed as in the proof of Proposition 6.5.15. □

6.7 Duality between direct and inverse limits of spectra

Proposition 6.7.1. *Let $\mathcal{F} := (X,F), \mathcal{G} := (Y,G)$ and $\mathcal{H} := (Z,H)$ be Bishop spaces, and let $\lambda \in \mathrm{Mor}(\mathcal{G},\mathcal{H}), \mu \in \mathrm{Mor}(\mathcal{H},\mathcal{G})$. We define the mappings*

$$\lambda^+ \colon \mathrm{Mor}(\mathcal{H},\mathcal{F}) \to \mathrm{Mor}(\mathcal{G},\mathcal{F}), \quad \lambda^+(\phi) := \phi\circ\lambda; \quad \phi\in\mathrm{Mor}(\mathcal{H},\mathcal{F}),$$

$$\mu^- \colon \mathrm{Mor}(\mathcal{F},\mathcal{H}) \to \mathrm{Mor}(\mathcal{F},\mathcal{G}), \quad \mu^-(\theta) := \mu\circ\theta; \quad \theta\in\mathrm{Mor}(\mathcal{F},\mathcal{H}),$$

$$\begin{tikzcd} Z \arrow[r, "\phi"] & X \arrow[r, "\theta"] \arrow[dr, "\mu\circ\theta"'] & Z \arrow[d, "\mu"] \\ Y \arrow[u, "\lambda"] \arrow[ur, "\phi\circ\lambda"'] & & Y. \end{tikzcd}$$

$${}^+ \colon \mathrm{Mor}(\mathcal{G},\mathcal{H}) \to \mathrm{Mor}(\mathcal{H}\to\mathcal{F}, \mathcal{G}\to\mathcal{F}), \quad \lambda\mapsto\lambda^+; \quad \lambda\in\mathrm{Mor}(\mathcal{G},\mathcal{H}),$$

$${}^- \colon \mathrm{Mor}(\mathcal{H},\mathcal{G}) \to \mathrm{Mor}(\mathcal{F}\to\mathcal{H}, \mathcal{F}\to\mathcal{G}), \quad \mu\mapsto\mu^-; \quad \mu\in\mathrm{Mor}(\mathcal{H},\mathcal{G}).$$

Then ${}^+ \in \mathrm{Mor}\big(\mathcal{G}\to\mathcal{H}, (\mathcal{H}\to\mathcal{F})\to(\mathcal{G}\to\mathcal{F})\big)$ and ${}^- \in \mathrm{Mor}\big(\mathcal{H}\to\mathcal{G}, (\mathcal{F}\to\mathcal{H})\to(\mathcal{F}\to\mathcal{G})\big)$.

Proof. By definition and the $\bigvee$-lifting of the exponential topology we have that

$$\mathcal{G}\to\mathcal{H} := \Big(\mathrm{Mor}(\mathcal{G},\mathcal{H}), \bigvee_{y\in Y}^{h\in H}\phi_{y,h}\Big), \quad \mathcal{H}\to\mathcal{F} := \Big(\mathrm{Mor}(\mathcal{H},\mathcal{F}), \bigvee_{z\in Z}^{f\in F}\phi_{z,f}\Big),$$

$$\mathcal{G} \to \mathcal{F} := \bigg(\mathrm{Mor}(\mathcal{G}, \mathcal{F}), \bigvee_{y \in Y}^{f \in F} \phi_{y,f}\bigg),$$

$$(\mathcal{H} \to \mathcal{F}) \to (\mathcal{G} \to \mathcal{F}) := \bigg(\mathrm{Mor}((\mathcal{H}, \mathcal{F}) \to \mathcal{G} \to \mathcal{F})), \bigvee_{\varphi \in \mathrm{Mor}(\mathcal{H},\mathcal{F})}^{e \in G \to F} \phi_{\varphi,e}\bigg),$$

$$\bigvee_{\varphi \in \mathrm{Mor}(\mathcal{H},\mathcal{F})}^{e \in G \to F} \phi_{\varphi,e} = \bigvee_{\varphi \in \mathrm{Mor}(\mathcal{H},\mathcal{F})}^{y \in Y, f \in F} \phi_{\varphi,\phi_{y,f}}.$$

By the $\bigvee$-lifting of morphisms we have that

$$^{+} \in \mathrm{Mor}\big(\mathcal{G} \to \mathcal{H}, (\mathcal{H} \to \mathcal{F}) \to (\mathcal{G} \to \mathcal{F})\big) \Leftrightarrow \forall_{\varphi \in \mathrm{Mor}(\mathcal{H},\mathcal{F})} \forall_{y \in Y} \forall_{f \in F} \big(\phi_{\varphi,\phi_{y,f}} \circ {}^{+} \in G \to H\big).$$

If $\lambda \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$, we have that $\phi_{\varphi,\phi_{y,f}} \circ {}^{+}](\lambda) := \phi_{\varphi,\phi_{y,f}}(\lambda^{+}) := (\phi_{y,f} \circ \lambda^{+})(\varphi) := (\phi_{y,f}(\varphi \circ \lambda) := [f \circ (\varphi \circ \lambda)](y) := [(f \circ \varphi) \circ \lambda](y) := \phi_{y,f\circ\varphi}(\lambda)$ i.e., $\phi_{\varphi,\phi_{y,f}} \circ {}^{+} := \phi_{y,f\circ\varphi} \in G \to H$, since $\varphi \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ and hence $f \circ \varphi \in H$. For the mapping $^{-}$ we work similarly. $\square$

Next we see how with the use of the exponential Bishop topology we can get a contravatiant spectrum from a covariant one, and vice versa.

Proposition 6.7.2. (A) *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq)$ and $\mathcal{F} := (X, F)$ a Bishop space.*

(i) *If $S(\Lambda^{\preccurlyeq}) \to \mathcal{F} := (\mu_0, \mu_1^{\succcurlyeq}, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$, where $M^{\succcurlyeq} := (\mu_0, \mu_1^{\succcurlyeq})$ is a contravariant direct family of sets over $(I, \preccurlyeq)$ with $\mu_0(i) := \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ and*

$$\mu_1^{\succcurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}_j, \mathcal{F}), \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}), (\lambda_{ij}^{\preccurlyeq})^{+}\big),$$

and if $\phi_0^{M^{\succcurlyeq}}(i) := F_i \to F$ and $\phi_1^{M^{\succcurlyeq}}(i,j) := \big(F_i \to F, F_j \to F, [(\lambda_{ij}^{\preccurlyeq})^{+}]^{*}\big)$, then $S^{\preccurlyeq} \to \mathcal{F}$ is a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}))_{i \in I}$ and Bishop morphisms $\big((\lambda_{ij}^{\preccurlyeq})^{+}\big)_{(i,j) \in D^{\preccurlyeq}(I)}$.

(ii) *If $\mathcal{F} \to S^{\preccurlyeq} := (\nu_0, \nu_1^{\preccurlyeq}, \phi_0^{N^{\preccurlyeq}}, \phi_1^{N^{\preccurlyeq}})$, where $N^{\preccurlyeq} := (\nu_0, \nu_1^{\preccurlyeq})$ is a direct family of sets over $(I, \preccurlyeq)$ with $\nu_0(i) := \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$ and*

$$\nu_1^{\preccurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), \mathrm{Mor}(\mathcal{F}, \mathcal{F}_j), (\lambda_{ij}^{\preccurlyeq})^{-}\big),$$

and if $\phi_0^{N^{\preccurlyeq}}(i) := F \to F_i$ and $\phi_1^{N^{\preccurlyeq}}(i,j) := \big(F \to F_j, F \to F_i, [(\lambda_{ij}^{\preccurlyeq})^{-}]^{*}\big)$, then $\mathcal{F} \to S^{\preccurlyeq}$ is a covariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i))_{i \in I}$ and Bishop morphisms $\big((\lambda_{ij}^{\preccurlyeq})^{-}\big)_{(i,j) \in D^{\preccurlyeq}(I)}$.

(B) *Let $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ be a contravariant $(I, \preccurlyeq)$-spectrum, and $\mathcal{F} := (X, F)$ a Bishop space.*

(i) *If $S(\Lambda^{\succcurlyeq}) \to \mathcal{F} := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$, where $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq})$ is a direct family of sets over $(I, \preccurlyeq)$ with $\mu_0(i) := \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ and*

$$\mu_1^{\preccurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}), \mathrm{Mor}(\mathcal{F}_j, \mathcal{F}), (\lambda_{ji}^{\succcurlyeq})^{+}\big),$$

and if $\phi_0^{M^{\preccurlyeq}}(i) := F_i \to F$ and $\phi_1^{M^{\preccurlyeq}}(i,j) := \big(F_j \to F, F_i \to F, [(\lambda_{ji}^{\succcurlyeq})^{+}]^{*}\big)$, then $S^{\succcurlyeq} \to \mathcal{F}$ is an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}))_{i \in I}$ and Bishop morphisms $\big((\lambda_{ji}^{\succcurlyeq})^{+}\big)_{(i,j) \in D^{\preccurlyeq}(I)}$.

(ii) *If* $\mathcal{F} \to S(N^{\succcurlyeq}) := (\nu_0, \nu_1^{\succcurlyeq}, \phi_0^{N^{\succcurlyeq}}, \phi_1^{N^{\succcurlyeq}})$*, where* $N^{\succcurlyeq} := (\nu_0, \nu_1^{\succcurlyeq})$ *is a contravariant direct family of sets over* $(I, \preccurlyeq)$ *with* $\nu_0(i) := \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$ *and*

$$\nu_1^{\succcurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_j), \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), (\lambda_{ji}^{\succcurlyeq})^-\big),$$

and if $\phi_0^{N^{\succcurlyeq}}(i) := F \to F_i$ *and* $\phi_1^{N^{\succcurlyeq}}(i,j) := \big(F \to F_i, F \to F_j, [(\lambda_{ji}^{\succcurlyeq})^-]^*\big)$*, then* $\mathcal{F} \to S^{\preccurlyeq}$ *is a contravariant* $(I, \preccurlyeq)$*-spectrum with Bishop spaces* $(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)_{i \in I}$ *and Bishop morphisms* $\big((\lambda_{ij}^{\preccurlyeq})^-\big)_{(i,j) \in D^{\preccurlyeq}(I)}$.

Proof. We prove only the case (A)(i) and for the other cases we work similarly. It suffices to show that if $i \preccurlyeq j \preccurlyeq k$, then the following diagram commutes

$$\begin{tikzcd} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}) & \\ \mathrm{Mor}(F_j, \mathcal{F}) \arrow[u, "(\lambda_{ij}^{\preccurlyeq})^+"] & \mathrm{Mor}(\mathcal{F}_k, \mathcal{F}). \arrow[l, "(\lambda_{jk}^{\preccurlyeq})^+"] \arrow[ul, "(\lambda_{ik}^{\preccurlyeq})^+"'] \end{tikzcd}$$

If $\phi \in \mathrm{Mor}(\mathcal{F}_k, \mathcal{F})$, then $\big(\lambda_{ij}^{\preccurlyeq}\big)^+\big[\big(\lambda_{jk}^{\preccurlyeq}\big)^+(\phi)\big] := \big(\lambda_{ij}^{\preccurlyeq}\big)^+[\phi \circ \lambda_{jk}^{\preccurlyeq}] := (\phi \circ \lambda_{jk}^{\preccurlyeq}) \circ \lambda_{ij}^{\preccurlyeq} := \phi \circ (\lambda_{jk}^{\preccurlyeq} \circ \lambda_{ij}^{\preccurlyeq}) = \phi \circ \lambda_{ik}^{\preccurlyeq} := \big(\lambda_{ik}^{\preccurlyeq}\big)^+(\phi)$. $\square$

Similarly to the $\bigvee$-lifting of the product topology, if $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $\big(F_i = \bigvee F_{0i}\big)_{i \in I}$, then

$$\prod_{i \in I}^{\succcurlyeq} F_i = \bigvee_{i \in I}^{f \in F_{0i}} \big(f \circ \pi_i^{\Lambda^{\succcurlyeq}}\big).$$

Theorem 6.7.3 (Duality principle)**.** *Let* $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I, \preccurlyeq)$ *with Bishop spaces* $(\mathcal{F}_i)_{i \in I}$ *and Bishop morphisms* $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$*. If* $\mathcal{F} := (X, F)$ *is a Bishop space and* $S(\Lambda^{\preccurlyeq}) \to \mathcal{F} := (\mu_0, \mu_1^{\succcurlyeq}, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ *is the contravariant direct spectrum over* $(I, \preccurlyeq)$ *defined in Proposition 6.7.2* (A)(i)*, then*

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}) \simeq [(\underset{\rightarrow}{\mathrm{Lim}} \mathcal{F}_i) \to \mathcal{F}].$$

Proof. First we determine the topologies involved in the required Bishop isomorphism. By definition and by the above remark on the $\bigvee$-lifting of the $\prod^{\succcurlyeq}$-topology we have that

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}) := \bigg(\prod_{i \in I}^{\succcurlyeq} \mu_0(i), \bigvee_{i \in I}^{g \in F_i \to F} g \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to \mathcal{F}}\bigg),$$

$$F_i \to F := \bigvee_{x \in \lambda_0(i)}^{f \in F} \phi_{x,f},$$

$$\bigvee_{i \in I}^{g \in F_i \to F} g \circ \pi_i^{S^{\preccurlyeq} \to \mathcal{F}} = \bigvee_{i \in I}^{x \in \lambda_0(i), f \in F} \phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to \mathcal{F}},$$

$$\underset{\to}{\mathrm{Lim}}\mathcal{F}_i := \bigg(\underset{\to}{\mathrm{Lim}}\lambda_0(i), \bigvee_{\Theta\in\prod_{i\in I}^{\preccurlyeq} F_i} \mathtt{eql}_0 f_\Theta\bigg),$$

$$\big(\underset{\to}{\mathrm{Lim}}\mathcal{F}_i\big) \to \mathcal{F} := \Bigg(\mathrm{Mor}(\underset{\to}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F}), \bigvee_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\in\underset{\to}{\mathrm{Lim}}\lambda_0(i)}^{f\in F} \phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x),f}\Bigg),$$

$$\phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x),f}(h) := (f\circ h)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big)$$

$$\begin{tikzcd}
\underset{\to}{\mathrm{Lim}}\lambda_0(i) \arrow[r, "h\in\mathrm{Mor}(\underset{\to}{\mathrm{Lim}}\mathcal{F}_i,\mathcal{F})"] \arrow[dr, "f\circ h"'] & X \arrow[d, "f\in F"] \\
& \mathbb{R}.
\end{tikzcd}$$

If $H \in \prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$, let the operation $\theta(H) : \underset{\to}{\mathrm{Lim}}\lambda_0(i) \rightsquigarrow X$, defined by

$$\theta(H)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := H_i(x); \quad \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) \in \underset{\to}{\mathrm{Lim}}\lambda_0(i).$$

We show that $\theta(H)$ is a function. If

$$\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) =_{\underset{\to}{\mathrm{Lim}}\lambda_0(i)} \mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(j,y) \Leftrightarrow \exists_{k\in I}\big(i,j \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big),$$

we show that $\theta(H)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := H_i(x) =_X H_j(y) =: \theta(H)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(j,y)\big)$. By the equalities $H_i = \big(\lambda_{ik}^{\preccurlyeq}\big)^+(H_k) = H_k \circ \lambda_{ik}^{\preccurlyeq}$ and $H_j = \big(\lambda_{jk}^{\preccurlyeq}\big)^+(H_k) = H_k \circ \lambda_{jk}^{\preccurlyeq}$ we get

$$H_i(x) = \big(H_k\circ\lambda_{ik}^{\preccurlyeq}\big)(x) := H_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) =_X H_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big) := \big(H_k\circ\lambda_{jk}^{\preccurlyeq}\big)(y) := H_j(y).$$

Next we show that $\theta(H) \in \mathrm{Mor}(\underset{\to}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F}) :\Leftrightarrow \forall_{f\in F}\big(f\circ\theta(H) \in \underset{\to}{\mathrm{Lim}}F_i\big)$. If $f \in F$, then the dependent assignment routine $\Theta : \bigcurlywedge_{i\in I} F_i$, defined by $\Theta_i := f \circ H_i$, for every $i \in I$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "H_i\in\mathrm{Mor}(\mathcal{F}_i,\mathcal{F})"] \arrow[dr, "f\circ H_i"'] & X \arrow[d, "f\in F"] \\
& \mathbb{R}
\end{tikzcd}$$

is in $\prod_{i\in I}^{\preccurlyeq} F_i$ i.e., if $i \preccurlyeq j$, then $\Theta_i = \big(\lambda_{ij}^{\preccurlyeq}\big)^*(\Theta_j) = \Theta_j \circ \lambda_{ij}^{\preccurlyeq}$, since $\Theta_i := f\circ H_i = f\circ\big(H_j\circ\lambda_{ij}^{\preccurlyeq}\big) = (f\circ H_j)\circ\lambda_{ij}^{\preccurlyeq} := \Theta_j\circ\lambda_{ij}^{\preccurlyeq}$. Hence $f\circ\theta(H) := \mathtt{eql}_0 f_\Theta \in \underset{\to}{\mathrm{Lim}}F_i$, since

$$[f\circ\theta(H)]\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := f\big(H_i(x)\big) := (f\circ H_i)(x) := f_\Theta(i,x) := \mathtt{eql}_0 f_\Theta\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big).$$

Consequently, the operation θ: $\prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}) \rightsquigarrow \mathrm{Mor}(\underset{\to}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F})$, defined by the rule $H \mapsto \theta(H)$, is well-defined. Next we show that θ is an embedding.

$$\begin{aligned}
\theta(H) = \theta(K) &:\Leftrightarrow \forall_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\in\underset{\to}{\mathrm{Lim}}\lambda_0(i)}\big(\theta(H)(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)) = \theta(K)(\omega_{S^{\preccurlyeq}}(i,x))\big)\\
&:\Leftrightarrow \forall_{i\in I}\big(H_i(x) =_X K_i(x)\big)\\
&:\Leftrightarrow H = K.
\end{aligned}$$

Next we show that $\theta \in \mathrm{Mor}\big(\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}), (\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i) \to \mathcal{F}\big)$ i.e.,

$$\forall_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \forall_{f \in F} \Bigg(\phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f} \circ \theta \in \bigvee_{i \in I, x \in \lambda_0(i)}^{f \in F} \phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to \mathcal{F}} \Bigg).$$

By the equalities

$$[\phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f} \circ \theta](H) := \phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f}(\theta(H)) := [f \circ \theta(H)]\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := f\big(H_i(x)\big),$$

$$[\phi_{x,f} \circ \pi_i^{S^{\preccurlyeq} \to \mathcal{F}}](H) := \phi_{x,f}\big(H_i)\big) := f\big(H_i(x)\big),$$

we get $\phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f} \circ \theta = \phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to \mathcal{F}}$. Let $\phi \colon \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F}) \rightsquigarrow \prod_{i \in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ be defined by $h \mapsto \phi(h) := H^h$, where $H^h : \curlywedge_{i \in I} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ is defined by $H_i^h := h \circ \mathtt{eql}_i$, for every $i \in I$

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\ \mathtt{eql}_i\ } & \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \\ & \searrow^{H_i^h} & \Big\downarrow h \\ & & X. \end{array}$$

By Proposition 6.5.5(i) $H_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$, as a composition of Bishop morphisms. To show that $H^h \in \prod_{i \in I} \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$, let $i \preccurlyeq j$, and by Proposition 6.5.5(ii) we get $H_i^h := h \circ \mathtt{eql}_i = h \circ \big(\mathtt{eql}_j \circ \lambda_{ij}^{\preccurlyeq}\big) := (h \circ \mathtt{eql}_j) \circ \lambda_{ij}^{\preccurlyeq} := H_j \circ \lambda_{ij}^{\preccurlyeq}$. Moreover, $\theta(H_h) := h$, since $\theta(H_h)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := H_i(x) := (h \circ \mathtt{eql}_i(x) := h\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big)$. Clearly, ϕ is a function. Moreover $H^{\theta(H)} := H$, as, for every $i \in I$ we have that $\big(H_i^{\theta(H)}\big)(x) := (\theta(H) \circ \mathtt{eql}_i)(x) := \theta(H)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := H_i(x)$. Finally we show that $\phi \in \mathrm{Mor}\big((\underset{]to}{\mathrm{Lim}}\mathcal{F}_i) \to \mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F})\big)$ if and only if

$$\forall_{i \in I} \forall_{x \in \lambda_0(i)} \forall_{f \in F} \big(\phi_{x,f} \circ \phi \in \bigvee_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x) \in \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)}^{f \in F} \phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f} \big).$$

If $h \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F})$, then

$$\big[\phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to F}) \circ \phi\big](h := (\phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \to F})(H^h) := \phi_{x,f}\big(H_i^h\big)$$

$$:= \phi_{x,f}(h \circ \mathtt{eql}_i) := f\big[(h \circ \mathtt{eql}_i)(x)\big] := (f \circ h)\big(\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x)\big) := \phi_{\mathtt{eql}_0^{\Lambda^{\preccurlyeq}}(i,x), f}(h). \qquad \square$$

With respect to the possible dual to the previous theorem i.e., the isomorphism $\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}) \simeq [(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \to \mathcal{F}]$, what we can show is the following proposition.

Proposition 6.7.4. *Let $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ be a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $S(\Lambda^{\succcurlyeq}) \to \mathcal{F} := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ is the $(I, \preccurlyeq)$-directed spectrum defined in Proposition 6.7.2* (B)(i), *there is a function $\widehat{\ } \colon \underset{\rightarrow}{\mathrm{Lim}}[\mathrm{Mor}(\mathcal{F}_i, \mathcal{F})] \to \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F})$ such that* *the following hold:*

(i) $\widehat{\ } \in \mathrm{Mor}\big(\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}), (\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \to \mathcal{F}\big)$.

(ii) *If for every* $j \in J$ *and every* $y \in \lambda_0(j)$ *there is* $\Theta_y \in \prod^{\succcurlyeq}_{i\in I} \lambda_0(i)$ *such that* $\Theta_y(j) =_{\lambda_0(j)} y$, *then* $\widehat{\ }$ *is an embedding of* $\underset{\rightarrow}{\mathrm{Lim}}[\mathrm{Mor}(\mathcal{F}_i, \mathcal{F})]$ *into* $\mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{F})$.

Proof. We proceed similarly to the proof of Theorem 6.7.3. $\square$

Theorem 6.7.5. *Let* $S(\Lambda^{\succcurlyeq}) := (\lambda_0, \lambda_1^{\succcurlyeq}, \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ *be a contravariant direct spectrum over* $(I, \preccurlyeq)$ *with Bishop spaces* $(\mathcal{F}_i)_{i\in I}$ *and Bishop morphisms* $(\lambda_{ji}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$. *If* $\mathcal{F} := (X, F)$ *is a Bishop space and* $\mathcal{F} \to S(\Lambda^{\succcurlyeq}) := (\nu_0, \nu_1^{\succcurlyeq}, \phi_0^{\succcurlyeq}, \phi_1^{N^{\succcurlyeq}})$ *is the contravariant direct spectrum over* $(I, \preccurlyeq)$, *defined in Proposition 6.7.2* (B)(ii), *then*

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i) \simeq [\mathcal{F} \to \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i].$$

Proof. First we determine the topologies involved in the required Bishop isomorphism:

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i) := \Bigg(\prod^{\succcurlyeq}_{i\in I} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), \bigvee_{i\in I}^{g\in F\to F_i} g \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})}\Bigg),$$

$$\bigvee_{i\in I}^{g\in F\to F_i} g \circ \pi_i^{\mathcal{F}\to S^{\succcurlyeq}} = \bigvee_{i\in I, x\in\lambda_0(i)}^{f\in F_i} \phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})},$$

$$\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i := \Bigg(\prod^{\succcurlyeq}_{i\in I} \lambda_0(i), \bigvee_{i\in I}^{f\in F_i} f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\Bigg),$$

$$\mathcal{F} \to \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i := \Bigg(\mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i), \bigvee_{x\in X}^{g\in \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i} \phi_{x,g}\Bigg),$$

$$\bigvee_{x\in X}^{g\in \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i} \phi_{x,g} = \bigvee_{x\in X, i\in I}^{f\in F_i} \phi_{x, f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}}.$$

If $H \in \prod^{\succcurlyeq}_{i\in I} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$, and if $i \preccurlyeq j$, then $H_i = \nu_{ji}^{\succcurlyeq}(H_j) = \big(\lambda_{ji}^{\succcurlyeq}\big)^{-}(H_j) = \lambda_{ji}^{\succcurlyeq} \circ H_j$

$$\begin{tikzcd} X \arrow[r, "H_j"] \arrow[dr, "H_i"'] & \lambda_0(j) \arrow[d, "\lambda_{ji}^{\succcurlyeq}"] \\ & \lambda_0(i). \end{tikzcd}$$

Let the operation $e(H) : X \rightsquigarrow \prod^{\succcurlyeq}_{i\in I} \lambda_0(i)$, defined by $x \mapsto [e(H)](x)$, where $\big[[e(H)](x)\big]_i := H_i(x)$, for every $i \in I$. First we show that $[e(H)](x) \in \prod^{\succcurlyeq}_{i\in I} \lambda_0(i)$. If $i \preccurlyeq j$, then $\big[[e(H)](x)\big]_i := H_i(x) = \big(\lambda_{ji}^{\succcurlyeq} \circ H_j\big)(x) := \lambda_{ji}^{\succcurlyeq}\big(H_j(x)\big) := \lambda_{ji}^{\succcurlyeq}\big(\big[[e(H)](x)\big]_j\big)$. Next we show that $e(H)$ is a function. If $x =_X x'$, then $\forall_{i\in I}\big(H_i(x) =_{\lambda_0(i)} H_i(x')\big) :\Leftrightarrow \forall_{i\in I}\big(\big[[e(H)](x)\big]_i =_{\lambda_0(i)} \big[[e(H)](x')\big]_i\big) :\Leftrightarrow [e(H)](x) =_{\prod^{\succcurlyeq}_{i\in I}\lambda_0(i)} [e(H)](x')$. By the $\bigvee$-lifting of morphisms $e(H) \in$

$\mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{i\in I}\forall_{f\in F_i}\big(\big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ e(H) \in F\big)$. Since $[(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}) \circ e(H)](x) := \big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big)\big([e(H)](x)\big) := f\big(H_i(x)\big) := (f \circ H_i)(x)$, we get $\big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ e(H) := f \circ H_i \in F$, since $f \in F_i$ and $H_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$. Hence, the operation $e : \prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i) \rightsquigarrow \mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i)$, defined by the rule $H \mapsto e(H)$, is well-defined. Next we show that e is an embedding. If $H, K \in \prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$, then

$$\begin{aligned}
e(H) = e(K) &:\Leftrightarrow \forall_{x\in X}\big([e(H)](x) =_{\prod_{i\in I}^{\succcurlyeq} \lambda_0(i)} [e(K)](x)\big)\\
&:\Leftrightarrow \forall_{x\in X}\forall_{i\in I}\big(H_i(x) =_{\lambda_0(i)} K_i(x)\big)\\
&:\Leftrightarrow \forall_{i\in I}\forall_{x\in X}\big(H_i(x) =_{\lambda_0(i)} K_i(x)\big)\\
&:\Leftrightarrow \forall_{i\in I}\big(H_i =_{\mathrm{Mor}(\mathcal{F},\mathcal{F}_i)} K_i\big)\\
&:\Leftrightarrow H =_{\prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F},\mathcal{F}_i)} K.
\end{aligned}$$

By the $\bigvee$-lifting of morphisms we show that

$$e \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i), \mathcal{F} \to \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{i\in I}\forall_{f\in F_i}\big(\phi_{x, f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}} \circ e \in \underset{\leftarrow}{\mathrm{Lim}}(F \to F_i)\big)$$

$$\begin{aligned}
\big(\phi_{x, f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}} \circ e\big)(H) &:= \phi_{x, f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}}\big(e(H)\big)\\
&:= \big[(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}) \circ e(H)\big](x)\\
&:= (f \circ \pi_i^{S(\Lambda^{\succcurlyeq})})\big([e(H)](x)\big)\\
&:= f\big([e(H)(x)]_i\big)\\
&:= f\big(H_i(x)\big)\\
&:= (f \circ H_i)(x)\\
&:= \phi_{x,f}\big(H_i\big)\\
&:= \big[\phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})}\big](H)
\end{aligned}$$

we get $\phi_{x, f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}} \circ e := \phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})} \in \underset{\leftarrow}{\mathrm{Lim}}(F \to F_i)$. Let $\phi\colon \mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i) \rightsquigarrow \prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$, defined by the rule $\mu \mapsto H^{\mu}$, where for every $\mu : X \to \prod_{i\in I}^{\succcurlyeq} \lambda_0(i) \in \mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i)$ i.e., $\forall_{i\in I}\forall_{f\in F_i}\big(\big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ \mu \in F\big)$, let

$$H^{\mu} : \bigcurlywedge_{i\in I} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), \quad [H_{\mu}]_i \colon X \to \lambda_0(i), \quad H_i^{\mu}(x) := [\mu(x)]_i; \quad x \in X,\ i \in I.$$

First we show that $H_i^{\mu} \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i) :\Leftrightarrow \forall_{f\in F_i}\big(f \circ H_i^{\mu} \in F\big)$. If $f \in F_i$, and $x \in X$, then $[f \circ H_i^{\mu}(x) := f\big(H_i^{\mu}\big) := f\big([\mu(x)]_i\big) := \big[\big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ \mu\big](x)$ i.e., $f \circ H_i^{\mu} := \big(f \circ \pi_i^{S(\Lambda^{\succcurlyeq})}\big) \circ \mu \in F$, as $\mu \in \mathrm{Mor}(\mathcal{F}, \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i)$. Since $\mu(x) \in \prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$, $[\mu(x)]_i = \lambda_{ji}^{\succcurlyeq}\big([\mu(x)]_j\big)$, for every $i, j \in I$ such that $i \preccurlyeq j$. To show that $H_{\mu} \in \prod_{i\in I}^{\succcurlyeq} \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$, let $i \preccurlyeq j$. Then

$$\begin{aligned}
H_i^{\mu} = \lambda_{ji}^{\succcurlyeq} \circ H_j^{\mu} &\Leftrightarrow \forall_{x\in X}\big(H_i^{\mu}(x) =_{\lambda_0(i)} \big[\lambda_{ji}^{\succcurlyeq} \circ H_j^{\mu}\big](x)\big)\\
&:\Leftrightarrow \forall_{x\in X}\big([\mu(x)]_i =_{\lambda_0(i)} \big[\lambda_{ji}^{\succcurlyeq}\big([\mu(x)]_j\big)\big),
\end{aligned}$$

which holds by the previous remark on $\mu(x)$. It is immediate to show that ϕ is a function. To show that $\phi \in \mathrm{Mor}\big([\mathcal{F} \to \underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i], \underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i)\big)$, we show that

$$\forall_{i\in I}\forall_{f\in F_i}\forall_{x\in\lambda_0(i)}\bigg(\big[\phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})}\big] \circ \phi \in \bigvee_{x\in X, i\in I}^{f\in F_i} \phi_{x,f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}}\bigg),$$

$$\begin{aligned}\big[\big[\phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})}\big] \circ \phi\big](\mu) &:= \big[\phi_{x,f} \circ \pi_i^{\mathcal{F}\to S(\Lambda^{\succcurlyeq})}\big](H^{\mu})\\ &:= \phi_{x,f}\big(H_i^{\mu}\big)\\ &:= (f \circ H_i^{\mu})(x)\\ &:= f\big(\mu(x)_i\big)\\ &:= (f \circ \pi_i^{S(\Lambda^{\succcurlyeq})} \circ \mu)(x)\\ &:= \big[\phi_{x,f\circ\pi_i^{S(\Lambda^{\succcurlyeq})}}\big](\mu).\end{aligned}$$

Moreover, $\phi(e(H)) := H$, as $H_i^{e(H)}(x) := [e(H)(x)]_i := H_i(x)$, and $e(\phi(\mu)) = \mu$, as

$$\begin{aligned}e(H^{\mu}) = \mu &:\Leftrightarrow \forall_{x\in X}\big([e(H^{\mu})](x) =_{\prod_{i\in I}^{\succcurlyeq}\lambda_0(i)} \mu(x)\big)\\ &:\Leftrightarrow \forall_{x\in X}\forall_{i\in I}\big(H_i^{\mu}(x) =_{\lambda_0(i)} [\mu(x)]_i\big)\\ &:\Leftrightarrow \forall_{x\in X}\forall_{i\in I}\big([\mu(x)]_i =_{\lambda_0(i)} [\mu(x)]_i\big).\end{aligned}$$

□

With respect to the possible dual to the previous theorem i.e., the isomorphism $\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i) \simeq (\mathcal{F} \to \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i)$, what we can show is the following proposition.

Proposition 6.7.6. *Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathtt{Spec}(I)$ with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j)\in D^{\preccurlyeq}(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $\mathcal{F} \to S(\Lambda^{\preccurlyeq}) := (\nu_0, \nu_1^{\preccurlyeq}, \phi_0^{N^{\preccurlyeq}}, \phi_1^{N^{\preccurlyeq}})$ is the $(I, \preccurlyeq)$-direct spectrum defined in Proposition 6.7.2* (A)(ii), *there is a map $\widehat{\ } : \underset{\rightarrow}{\mathrm{Lim}}[\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)] \to \mathrm{Mor}(\mathcal{F}, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i)$ with $\widehat{\ } \in \mathrm{Mor}\big((\underset{\rightarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i), \mathcal{F} \to \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i\big)$.*

Proof. We proceed similarly to the proof of Theorem 6.7.5. □

6.8 Spectra of Bishop subspaces

Definition 6.8.1. *If $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$, a family of Bishop subspaces of the Bishop space $\mathcal{F} := (X, F)$ associated to $\Lambda(X)$ is a pair $\Phi^{\Lambda(X)} := \big(\phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)}\big)$, where $\phi_0^{\Lambda(X)} \colon I \rightsquigarrow \mathbb{V}_0$ and $\phi_1^{\Lambda(X)} : \bigcurlywedge_{(i,j)\in D(I)} \mathbb{F}\big(\phi_0^{\Lambda(X)}(i), \phi_0^{\Lambda(X)}(j)\big)$ such that the following conditions hold:*

(i) $\phi_0^{\Lambda(X)}(i) := F_i := F_{|\lambda_0(i)} := \bigvee_{f\in F} f \circ \mathcal{E}_i^X$, *for every $i \in I$.*

(ii) $\phi_1^{\Lambda(X)}(i, j) := \lambda_{ji}^{*}$, *for every $(i, j) \in D(I)$.*

We call the structure $S_F(\Lambda(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1, F, \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)})$ a spectrum of subspaces of $\mathcal{F}$ over I, or an I-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i\in I}$. If $S_F(M(X)) := (\mu_0, \mathcal{Z}^X, \mu_1, F, \phi_0^{M(X)}, \phi_1^{M(X)})$ is an I-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{Z}_i^X)_{i\in I}$, a subspaces spectrum-map

Ψ from $S_F(\Lambda(X))$ to $S_F(M(X))$, in symbols $\Psi\colon S_F(\Lambda(X)) \Rightarrow S_F(M(X))$, is a family of subsets-map $\Psi\colon \Lambda(X) \Rightarrow M(X)$. If F is clear from the context, we may omit the symbol F as a subscript in the above notations.

The topology F_i on $\lambda_0(i)$ is the relative Bishop topology of F to $\lambda_0(i)$, and it is the least topology that makes the embedding $\mathcal{E}_i^X$ a Bishop morphism from $\mathcal{F}_i$ to $\mathcal{F}$. In contrast to the external framework of a spectrum of Bishop spaces, we can prove that the transport maps λ_{ij} of a spectrum of subspaces $S_F(\Lambda(X))$ are always Bishop morphisms. The extensionality of a Bishop topology F on a set X as a subset of $\mathbb{F}(X)$ is crucial to the next proof.

Remark 6.8.2. *Let $S_F(\Lambda(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1, F, \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)})$ be an I-spectrum of subspaces of $\mathcal{F} := (X, F)$ with Bishop subspaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i\in I}$, and $S_F(M(X)) := (\mu_0, \mathcal{Z}^X, \mu_1, F, \phi_0^{M(X)}, \phi_1^{M(X)})$ an I-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{Z}_i^X)_{i\in I}$.*

(i) *$S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)})$ is an I-spectrum with Bishop spaces $(\mathcal{F}_i)_{i\in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j)\in D(I)}$.*

(ii) *If $\Psi\colon S(\Lambda(X)) \Rightarrow S(M(X))$, then Ψ is continuous i.e., $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)$, for every $i \in I$.*

Proof. (i) It suffices to show that $\lambda_{ij} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i,j) \in D(I)$. By the $\bigvee$-lifting of morphisms we have that $\lambda_{ij} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j) \Leftrightarrow \forall_{f\in F}\big((f\circ\mathcal{E}_j^X)\circ\lambda_{ij} \in F_i\big) :\Leftrightarrow \forall_{f\in F}\big(f\circ(\mathcal{E}_j^X\circ\lambda_{ij}) \in F_i\big)$. If we fix some $f \in F$, and as $\mathcal{E}_j^X \circ \lambda_{ij} =_{\mathbb{F}(\lambda_0(i),X)} \mathcal{E}_i^X$, we get $f\circ(\mathcal{E}_j^X \circ \lambda_{ij}) =_{\mathbb{F}(\lambda_0(i))} f\circ\mathcal{E}_i^X$. Since $f \circ \mathcal{E}_i^X \in F_i$ by the extensionality of F_i we get $f \circ (\mathcal{E}_j^X \circ \lambda_{ij}) \in F_i$.

(ii) By the $\bigvee$-lifting of morphisms we have that $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i) \Leftrightarrow \forall_{f\in F}\big((f \circ \mathcal{Z}_i^X) \circ \Psi_i \in F_i\big) :\Leftrightarrow \forall_{f\in F}\big(f \circ (\mathcal{Z}_i^X \circ \Psi_i) \in F_i\big)$. Since $\Psi\colon \Lambda(X) \Rightarrow M(X)$, we get $\mathcal{Z}_i^X \circ \Psi_i =_{\mathbb{F}(\lambda_0(i),X)} \mathcal{E}_i^X$, and hence $f \circ (\mathcal{Z}_i^X \circ \Psi_i) =_{\mathbb{F}(\lambda_0(i))} f \circ \mathcal{E}_i^X$, for every $i \in I$ and $f \in F$. By the definition of F_i we have that $f \circ \mathcal{E}_i^X \in F_i$, and hence by the extensionality of F_i we conclude that $f \circ (\mathcal{Z}_i^X \circ \Psi_i) \in F_i$. ☐

Definition 6.8.3. *Let $\mathtt{Spec}_F(I, X)$ be the totality of spectra of subspaces of the Bishop space $\mathcal{F} := (X, F)$ over I, equipped with the equality of $\mathtt{Spec}(I, X)$.*

Definition 6.8.4. *Let $S_F(\Lambda(X)) := \big(\lambda_0, \mathcal{E}^X, \lambda_1, F, \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)}\big) \in \mathtt{Spec}_F(I, X)$ with Bishop subspaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i\in I}$. The canonical Bishop topology on the interior union $\bigcup_{i\in I}\lambda_0(i)$ is the relative topology of F to it i.e.,*

$$\bigcup_{i\in I}\mathcal{F}_i := \Big(\bigcup_{i\in I}\lambda_0(i), \bigcup_{i\in I}F_i\Big),$$

$$\bigcup_{i\in I}F_i := \bigvee_{f\in F} f \circ e_{\bigcup}^{\Lambda(X)},$$

$$\big(f \circ e_{\bigcup}^{\Lambda(X)}\big)(i,x) := f(\mathcal{E}_i^X(x)); \qquad (i,x) \in \bigcup_{i\in I}\lambda_0(i).$$

The canonical Bishop topology on $\bigcap_{i\in I}\lambda_0(i)$ is the relative topology of $\mathcal{F}$ to it i.e.,

$$\bigcap_{i\in I}\mathcal{F}_i := \Big(\bigcap_{i\in I}\lambda_0(i), \bigcap_{i\in I}F_i\Big),$$

$$\bigcap_{i\in I} F_i := \bigvee_{f\in F} f \circ e_{\cap}^{\Lambda(X)},$$

$$\big(f \circ e_{\cap}^{\Lambda(X)}\big)(\Phi) := f(\mathcal{E}_{i_0}^X(\Phi_{i_0})); \qquad \Phi \in \bigcap_{i\in I} \lambda_0(i).$$

Next follows the continuous-analogue to Proposition 4.3.6, using repeatedly the $\bigvee$-lifting of morphisms and the extensionality of a Bishop topology.

Proposition 6.8.5. *Let $S(\Lambda(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1, F, \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)}) \in \mathtt{Spec}_F(I, X)$ with Bishop subspaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i\in I}$, $S(M(X)) := (\mu_0, \mathcal{Z}^X, \mu_1, F, \phi_0^{M(X)}, \phi_1^{M(X)}) \in \mathtt{Spec}_F(I, X)$ with Bishop subspaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{Z}_i^X)_{i\in I}$, and $\Psi\colon S(\Lambda(X)) \Rightarrow S(M(X))$.*

(i) $e_i^{\Lambda(X)} \in \mathrm{Mor}(\mathcal{F}_i, \bigcup_{i\in I} \mathcal{F}_i)$*, for every* $i \in I$.

(ii) $\bigcup \Psi \in \mathrm{Mor}(\bigcup_{i\in I} \mathcal{F}_i, \bigcup_{i\in I} \mathcal{G}_i)$.

(iii) $\pi_i^{\Lambda(X)} \in \mathrm{Mor}(\bigcap_{i\in I} \mathcal{F}_i, \mathcal{F}_i)$*, for every* $i \in I$.

(iv) $\bigcap \Psi \in \mathrm{Mor}(\bigcap_{i\in I} \mathcal{F}_i, \bigcap_{i\in I} \mathcal{G}_i)$.

Proof. (i) $e_i^{\Lambda(X)} \in \mathrm{Mor}(\mathcal{F}_i, \bigcup_{i\in I} \mathcal{F}_i) \Leftrightarrow \forall_{f\in F}\big(\big(f \circ e_{\cup}^{\Lambda(X)}\big) \circ e_i^{\Lambda(X)} \in F_i\big)$. If $f \in F$, then $\big(f \circ e_{\cup}^{\Lambda(X)}\big) \circ e_i^{\Lambda(X)} := f \circ \mathcal{E}_i^X \in F_i$, since, for every $x \in \lambda_0(i)$, we have that $\big[\big(f \circ e_{\cup}^{\Lambda(X)}\big) \circ e_i^{\Lambda(X)}\big](x) := \big(f \circ e_{\cup}^{\Lambda(X)}\big)(i, x) := \big(f \circ \mathcal{E}_i^X\big)(x)$.
(ii) $\bigcup \Psi \in \mathrm{Mor}(\bigcup_{i\in I} \mathcal{F}_i, \bigcup_{i\in I} \mathcal{G}_i) \Leftrightarrow \forall_{f\in F}\big(\big(f \circ e_{\cup}^{M(X)}\big) \circ \bigcup \Psi \in \bigcup_{i\in I} F_i\big)$, and $\big(f \circ e_{\cup}^{M(X)}\big) \circ \bigcup \Psi = f \circ e_{\cup}^{\Lambda(X)} \in \bigcup_{i\in I} F_i$, as

$$\big[\big(f \circ e_{\cup}^{M(X)}\big) \circ \bigcup \Psi\big](i, x) := \big(f \circ e_{\cup}^{M(X)}\big)\big((i, \Psi_i(x))\big) := f\big(\mathcal{Z}_i^X(\Psi_i(x))\big)$$

$$= \big((f \circ \mathcal{E}_i^X) \circ \Psi_i\big)(x) = (f \circ \mathcal{E}_i^X)(x) := \big(f \circ e_{\cup}^{\Lambda(X)}\big)(i, x).$$

(iii) $\pi_i^{\Lambda(X)} \in \mathrm{Mor}(\bigcap_{i\in I} \mathcal{F}_i, \mathcal{F}_i) \Leftrightarrow \forall_{f\in F}\big((f \circ \mathcal{E}_i^X) \circ \pi_i^{\Lambda(X)} \in \bigcap_{i\in I} F_i\big)$, and $(f \circ \mathcal{E}_i^X) \circ \pi_i^{\Lambda(X)} = f \circ e_{\cap}^{\Lambda(X)} \in \bigcap_{i\in I} F_i$, as $\big[(f \circ \mathcal{E}_i^X) \circ \pi_i^{\Lambda(X)}\big](\Phi) := f(\mathcal{E}_i^X(\Phi_i)) = f(\mathcal{E}_{i_0}^X(\Phi_{i_0})) := \big(f \circ e_{\cap}^{\Lambda(X)}\big)(\Phi)$.
(iv) $\bigcap \Psi \in \mathrm{Mor}(\bigcap_{i\in I} \mathcal{F}_i, \bigcap_{i\in I} \mathcal{G}_i) \Leftrightarrow \forall_{f\in F}\big(\big(f \circ e_{\cap}^{M(X)}\big) \circ \bigcap \Psi \in \bigcap_{i\in I} F_i\big)$, and $\big(f \circ e_{\cap}^{M(X)}\big) \circ \bigcap \Psi = f \circ e_{\cup}^{\Lambda(X)} \in \bigcup_{i\in I} F_i$, as $\big[\big(f \circ e_{\cap}^{M(X)}\big) \circ \bigcap \Psi\big](\Phi) := \big(f \circ e_{\cap}^{M(X)}\big)\big(\mathcal{Z}_{i_0}^X(\Psi_{i_0}(\Phi_{i_0}))\big) = \big(f \circ e_{\cap}^{M(X)}\big)\big(\mathcal{E}_{i_0}^X(\Phi_{i_0})\big) := f\big(\mathcal{E}_{i_0}^X(\Phi_{i_0})\big) := \big(\big(f \circ e_{\cap}^{M(X)}\big)(\Phi)$. ☐

The notions mentioned in the next proposition were defined in Proposition 4.3.8.

Proposition 6.8.6. *Let $\mathcal{F} := (X, F)$, $(\mathcal{G} := (Y, G)$ be Bishop spaces, $h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, $S(\Lambda(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1, F; \phi_0^{\Lambda(X)}, \phi_1^{\Lambda(X)}) \in \mathtt{Spec}_F(I, X)$ with Bishop subspaces $(\mathcal{F}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i\in I}$, $S(M(Y)) := (\mu_0, \mathcal{Z}^Y, \mu_1, G, \phi_0^{M(Y)}, \phi_1^{M(Y)}) \in \mathtt{Spec}_G(I, Y)$ with Bishop subspaces $(\mathcal{G}_i)_{i\in I}$ and Bishop morphisms $(\mathcal{Z}_i^Y)_{i\in I}$, and $\Psi : \Lambda(X) \overset{h}{\Rightarrow} M(Y)$.*

(i) Ψ *is continuous i.e.,* $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)$*, for every* $i \in I$.

(ii) $\bigcup_h \Psi \in \mathrm{Mor}(\bigcup_{i\in I} \mathcal{F}_i, \bigcup_{i\in I} \mathcal{G}_i)$.

(iii) $\bigcap_h \Psi \in \mathrm{Mor}(\bigcap_{i\in I} \mathcal{F}_i, \bigcap_{i\in I} \mathcal{G}_i)$.

Proof. (i) $\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i) \Leftrightarrow \forall_{g \in G}\big((g \circ E_i) \circ \Psi_i \in F_i$, and if $g \in G$, then $(g \circ \mathcal{Z}_i^Y) \circ \Psi_i := g \circ (\mathcal{Z}_i^Y \circ \Psi_i) = g \circ (h \circ \mathcal{E}_i^X) := (g \circ h) \circ \mathcal{E}_i^X \in F_i$, as $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, and hence $g \circ h \in F$.
(ii) and (iii) Working as in the proof of the Proposition 6.8.5(ii) and (iv), we get $\big(g \circ e_{\bigcup}^{M(X)}\big) \circ \bigcup_h \Psi = (g \circ h) \circ e_{\bigcup}^{\Lambda(X)}$ and $\big(f \circ e_{\bigcap}^{M(X)}\big) \circ \bigcap_h \Psi = (g \circ h) \circ e_{\bigcap}^{\Lambda(X)}$, for every $g \in G$. □

6.9 Direct spectra of Bishop subspaces

Definition 6.9.1. *If $\Lambda^{\preccurlyeq}(X) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq}) \in \mathtt{Fam}(I, \preccurlyeq, X)$, a family of Bishop subspaces of the Bishop space $\mathcal{F} := (X, F)$ associated to $\Lambda^{\preccurlyeq}(X)$ is a pair $\Phi^{\Lambda^{\preccurlyeq}(X)} := \big(\phi_0^{\Lambda^{\preccurlyeq}(X)}, \phi_1^{\Lambda^{\preccurlyeq}(X)}\big)$, where $\phi_0^{\Lambda^{\preccurlyeq}(X)} \colon I \rightsquigarrow \mathbb{V}_0$ and $\phi_1^{\Lambda^{\preccurlyeq}(X)} : \curlywedge_{(i,j) \in D^{\preccurlyeq}(I)} \mathbb{F}\big(\phi_0^{\Lambda^{\preccurlyeq}(X)}(j), \phi_0^{\Lambda^{\preccurlyeq}(X)}(i)\big)$ such that the following conditions hold:*

(i) *$\phi_0^{\Lambda^{\preccurlyeq}(X)}(i) := F_i := F_{|\lambda_0(i)} := \bigvee_{f \in F} f \circ \mathcal{E}_i^X$, for every $i \in I$.*

(ii) *$\phi_1^{\Lambda^{\preccurlyeq}(X)}(i,j) := \big(\lambda_{ij}^{\preccurlyeq}\big)^*$, for every $(i,j) \in D^{\preccurlyeq}(I)$.*

We call the structure $S_F(\Lambda^{\preccurlyeq}(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq}, F, \phi_0^{\Lambda^{\preccurlyeq}(X)}, \phi_1^{\Lambda^{\preccurlyeq}(X)})$ a (covariant)direct spectrum of subspaces of $\mathcal{F}$ over I, or an $(I, \preccurlyeq_I)$-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i \in I}$. If $S_F(M^{\preccurlyeq}(X)) := (\mu_0, \mathcal{Z}^X, \mu_1^{\preccurlyeq}, F, \phi_0^{M^{\preccurlyeq}(X)}, \phi_1^{M^{\preccurlyeq}(X)})$ is an $(I, \preccurlyeq_I)$-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{Z}_i^X)_{i \in I}$, a subspaces direct spectrum-map Ψ from $S_F(\Lambda^{\preccurlyeq}(X))$ to $S_F(M^{\preccurlyeq}(X))$, in symbols $\Psi \colon S_F(\Lambda^{\preccurlyeq}(X)) \Rightarrow S_F(M^{\preccurlyeq}(X))$, is a direct family of subsets-map $\Psi \colon \Lambda^{\preccurlyeq}(X) \Rightarrow M^{\preccurlyeq}(X)$ (see Definition 4.10.3). If F is clear from the context, we may omit he symbol F as a subscript in the above notations. A contravariant direct spectrum $S_F(\Lambda^{\succcurlyeq}(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\succcurlyeq}, F, \phi_0^{\Lambda^{\succcurlyeq}(X)}, \phi_1^{\Lambda^{\succcurlyeq}(X)})$ of subspaces of $\mathcal{F}$ over $(I, \preccurlyeq_I)$ and a subspaces contravariant direct spectrum-map are defined similarly.

Remark 6.9.2. *Let $S_F(\Lambda^{\preccurlyeq}(X)) := (\lambda_0, \mathcal{E}^X, \lambda_1^{\preccurlyeq}, F, \phi_0^{\Lambda^{\preccurlyeq}(X)}, \phi_1^{\Lambda^{\preccurlyeq}(X)})$ be an $(I, \preccurlyeq_I)$-spectrum of subspaces of $\mathcal{F} := (X, F)$ with Bishop subspaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{E}_i^X)_{i \in I}$, and $S_F(M^{\preccurlyeq}(X)) := (\mu_0, \mathcal{Z}^X, \mu_1^{\preccurlyeq}, F, \phi_0^{M^{\preccurlyeq}(X)}, \phi_1^{M^{\preccurlyeq}(X)})$ an $(I, \preccurlyeq_I)$-spectrum of subspaces of $\mathcal{F}$ with Bishop subspaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{Z}_i^X)_{i \in I}$.*

(i) *$S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}(X)}, \phi_1^{\Lambda^{\preccurlyeq}(X)})$ is an $(I, \preccurlyeq_I)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$.*

(ii) *If $\Psi \colon S(\Lambda^{\preccurlyeq}(X)) \Rightarrow S(M^{\preccurlyeq}(X))$, then Ψ is continuous.*

Proof. We proceed as in the proof of Remark 6.8.2. □

Definition 6.9.3. *Let $\mathtt{Spec}_F(I, \preccurlyeq, X)$ be the totality of covariant direct spectra of subspaces of the Bishop space $\mathcal{F} := (X, F)$ and let $\mathtt{Spec}_F(I, \succcurlyeq, X)$ be the totality of contravariant direct spectra of subspaces of $\mathcal{F}$ over $(I, \preccurlyeq_I$, equipped with the equality of $\mathtt{Fam}(I, \preccurlyeq, X)$ and $\mathtt{Fam}(I, \succcurlyeq, X)$, respectively.*

Definition 6.9.4. *If $S(\Lambda^{\succcurlyeq}(X)) := (\lambda_0, \mathcal{E}, \lambda_1^{\succcurlyeq}, F, \phi_0^{\Lambda^{\succcurlyeq}(X)}, \phi_1^{\Lambda^{\succcurlyeq}(X)})$ is contravariant direct spectrum of subspaces of the Bishop space $\mathcal{F} := (X, F)$ over $(I, \preccurlyeq)$ with Bishop subspaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{E}_i)_{i \in I}$, its inverse limit is the following Bishop space*

$$\underset{\leftarrow}{\mathrm{Lim}} S(\Lambda^{\succcurlyeq}(X)) := \underset{\leftarrow X}{\mathrm{Lim}} \mathcal{F}_i := \bigg(\bigcap_{i \in I} \lambda_0(i), \bigvee_{f \in F} f \circ e_{\bigcap}^{\Lambda^{\succcurlyeq}(X)}\bigg).$$

Next we show the universal property of the inverse limit for $\underset{\leftarrow X}{\mathrm{Lim}}\mathcal{F}_i$

Proposition 6.9.5. *If* $S(\Lambda^{\succcurlyeq}(X)) := \big(\lambda_0, \mathcal{E}, \lambda_1^{\succcurlyeq}, F, \phi_0^{\Lambda^{\succcurlyeq}(X)}, \phi_1^{\Lambda^{\succcurlyeq}(X)}\big) \in \mathtt{Spec}(I, \succcurlyeq, X)$ *with Bishop subspaces* $(\mathcal{F}_i)_{i\in I}$ *and Bishop morphisms* $(\mathcal{E}_i^X)_{i\in I}$*, its inverse limit* $\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i$ *satisfies the universal property of inverse limits i.e., if* $i \preccurlyeq_j$*, the following left diagram commutes*

$$\begin{tikzcd} & \bigcap_{i\in I}\lambda_0(i) \arrow[dl, "\pi_i^{\Lambda^{\succcurlyeq}(X)}"'] \arrow[dr, "\pi_j^{\Lambda^{\succcurlyeq}(X)}"] & \\ \lambda_0(i) & & \lambda_0(j) \arrow[ll, hook', "\lambda_{ji}^{\succcurlyeq}"] \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] & \\ \lambda_0(i) & & \lambda_0(j), \arrow[ll, hook', "\lambda_{ji}^{\succcurlyeq}"] \end{tikzcd}$$

and for every Bishop space $\mathcal{G} := (Y, G)$ *and a family* $(\varpi_i)_{i\in I}$*, where* $\varpi_i \in \mathrm{Mor}(\mathcal{G}, \mathcal{F}_i)$*, for every* $i \in I$*, such that the above right diagram commutes, there is a unique Bishop morphism* $h : Y \to \bigcap_{i\in I}\lambda_0(i)$ *such that the following diagrams commute*

$$\begin{tikzcd} & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] \arrow[dd, dashed, "h"] & \\ \lambda_0(i) & & \lambda_0(j), \arrow[ll, hook', "\lambda_{ji}^{\succcurlyeq}"] \\ & \bigcap_{i\in I}\lambda_0(i) \arrow[ul, "\pi_i^{\Lambda^{\succcurlyeq}(X)}"] \arrow[ur, "\pi_j^{\Lambda^{\succcurlyeq}(X)}"'] & \end{tikzcd}$$

Proof. For the commutativity of the first diagram, we have that if $\Phi \in \bigcap_{i\in I}\lambda_0(i)$, then $\pi_i^{\Lambda^{\succcurlyeq}(X)}(\Phi) := \Phi_i$, and $\lambda_{ji}^{\succcurlyeq}\big(\pi_j^{\Lambda^{\succcurlyeq}}(\Phi)\big) := \lambda_{ji}^{\succcurlyeq}(\Phi_j)$, and since $\mathcal{E}_j^X = \mathcal{E}_i^X \circ \lambda_{ji}^{\succcurlyeq}$, we have that $\mathcal{E}_j^X(\Phi_j) = \mathcal{E}_i^X\big(\lambda_{ji}^{\succcurlyeq}(\Phi_j)\big)$, hence by the definition of $\bigcap_{i\in I}\lambda_0(i)$ we get $\mathcal{E}_i^X(\Phi_i) = \mathcal{E}_j^X(\Phi_j) = \mathcal{E}_i^X\big(\lambda_{ji}^{\succcurlyeq}(\Phi_j)\big)$, and since $\mathcal{E}_i^X$ is an embedding we get $\Phi_i = \lambda_{ji}^{\succcurlyeq}(\Phi_j) =: \lambda_{ji}^{\succcurlyeq}\big(\pi_j^{\Lambda^{\succcurlyeq}(X)}(\Phi)\big)$. Let a Bishop space $\mathcal{G} := (Y, G)$ and a family of Bishop morphisms $(\varpi_i)_{i\in I}$, where $\varpi_i : Y \to \lambda_0(i)$, for every $i \in I$, such that the above right diagram commutes. Let also the operation $h : Y \rightsquigarrow \bigcap_{i\in I}\lambda_0(i)$, defined by the rule $y \mapsto h(y)$, where

$$h(y) : \bigcurlywedge_{i\in I}\lambda_0(i), \quad h(y)_i := \varpi_i(y); \quad i \in I.$$

To show that $h(y) \in \bigcap_{i\in I}\lambda_0(i)$ we need to show that

$$\mathcal{E}_i^X\big(h(y)_i\big) =_X \mathcal{E}_{i'}^X\big(h(y)_{i'}\big) \Leftrightarrow \mathcal{E}_i^X(\varpi_i(y)) =_X \mathcal{E}_{i'}^X(\varpi_{i'}(y)),$$

for every $i, i' \in I$. Since $(I, \preccurlyeq_I)$ is directed, there is $k \in I$ such that $i \preccurlyeq_I k$ and $i' \preccurlyeq_I k$, hence

$$\mathcal{E}_i^X(\varpi_i(y)) =_X \mathcal{E}_i^X\big(\lambda_{ki}^{\succcurlyeq}(\varpi_k(y))\big) =_X \mathcal{E}_k^X\big(\varpi_k(y)\big) =_X \mathcal{E}_{i'}^X\big(\lambda_{ki'}^{\succcurlyeq}(\varpi_k(y))\big) =_X \mathcal{E}_{i'}^X(\varpi_{i'}(y)).$$

It is immediate to show that h is a function. Finally, we show that $h \in \mathrm{Mor}(\mathcal{G}, \underset{\leftarrow X}{\mathrm{Lim}}\mathcal{F}_i) \Leftrightarrow \forall_{f\in F}\big(\big(f \circ e_{\bigcap}^{\Lambda^{\succcurlyeq}(X)}\big) \circ h \in G\big)$. If $y \in Y$, then

$$\big[\big(f \circ e_{\bigcap}^{\Lambda^{\succcurlyeq}(X)}\big) \circ h\big](y) := f\big(\mathcal{E}_{i_0}^X(\varpi_{i_0}(y))\big) := [(f \circ \mathcal{E}_{i_0}^X) \circ \varpi_{i_0}](y),$$

hence $\big(f \circ e_{\bigcap}^{\Lambda^{\succcurlyeq}(X)}\big) \circ h := (f \circ \mathcal{E}_{i_0}^X) \circ \varpi_{i_0} \in G$, as by our hypothesis $\varpi_{i_0} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F}_{i_0})$. $\square$

6.10 Notes

Note 6.10.1. The theory of Bishop spaces, that was only sketched by Bishop in [9], and revived by Bridges in [26], and Ishihara in [63], was developed by the author in [88]-[96] and [98]-[100]. Since inductive definitions with rules of countably many premises are used, for the study of Bishop spaces we work within BST^*, which is BST extended with such inductive definitions. A formal system for BISH extended with such definitions is Myhill's formal system CST^* with dependent choice, where CST^* is Myhill's extension of his formal system of constructive set theory CST with inductive definitions (see [80]). A variation of CST^* is Aczel's system CZF together with a very weak version of Aczel's regular extension axiom (REA), to accommodate these inductive definitions (see [1]).

Note 6.10.2. In contrast to topological spaces, in the theory of Bishop spaces continuity of functions is an a priori notion, while the concept of an open set comes a posteriori, through the neighbourhood space induced by a Bishop topology. The theory of Bishop spaces can be seen as an abstract and constructive approach to the theory of the ring $C(X)$ of continuous functions of a topological space $(X, \mathcal{T})$ (see [52] for a classical treatment of this subject).

Note 6.10.3. The results on the direct and inverse limits of direct spectra of Bishop spaces are the constructive analogue to the classical theory of direct and inverse limits of (spectra of) topological spaces, as this is developed e.g., in the Appendix of [45]. As in the case of the classic textbook of Dugundji, we avoid here possible, purely categorical arguments in our proofs. One of the advantages of working with a proof-relevant definition of a cofinal subset is that the proof of the cofinality theorem 6.5.12 is choice-free.

Note 6.10.4. The notion of a spectrum of Bishop spaces can be generalised by considering a family of Bishop spaces associated to a set-relevant family of sets over some set I. In this case, all transport maps λ_{ij}^{m} are taken to be Bishop morphisms. The direct versions of set-relevant spectra of Bishop spaces can be defined, and their theory can be developed in complete analogy to the theory of direct spectra of Bishop spaces, as in the case of generalised direct spectra of topological spaces (see [45], p. 426).

Note 6.10.5. The formulation of the universal properties of the various limits of spectra of Bishop spaces included here is impredicative, as it requires quantification over the class of Bishop spaces. A predicative formulation of a universal property can be given, if one is restricted to a given set-indexed family of Bishop spaces.

Note 6.10.6. The study of the direct limit of a spectrum of Bishop subspaces is postponed for future work. The natural candidate $\bigcup_{i \in I} \lambda_0(i)$, equipped with the relative topology, "almost" satisfies the universal property of the direct limit.

Chapter 7

Families of subsets in measure theory

We study the Borel and Baire sets within Bishop spaces as a constructive counterpart to the study of Borel and Baire algebras within topological spaces. As we use the inductively defined least Bishop topology, and as the Borel and Baire sets over a family of F-complemented subsets are defined inductively, we work within the extension BISH* of BISH with inductive definitions with rules of countably many premises. In contrast to the classical theory, we show that the Borel and the Baire sets of a Bishop space coincide. Our reformulation within BST of the Bishop-Cheng definition of a measure space and of an integration space, based on the notions of families of complemented subsets and of families of partial functions, facilitates a predicative reconstruction of the originally impredicative Bishop-Cheng measure theory.

7.1 The Borel sets of a Bishop space

The Borel sets of a topological space $(X, \mathcal{T})$ is the least set of subsets of X that includes the open (or, equivalently the closed) sets in X and it is closed under countable unions, countable intersections and relative complements. The Borel sets of a Bishop space (X, F) is the least set of complemented subsets of X that includes the *basic* F-complemented subsets of X that are generated by F, and it is closed under countable unions and countable intersections. As the Borel sets of (X, F) are complemented subsets, it is not a coincidence that their closure under complements is provable. *In the next two sections* F denotes a Bishop topology on a set X and G a Bishop topology on a set Y. For simplicity, we denote the constant function on X with value $a \in \mathbb{R}$ also by a, and we may write equalities between elements of $\mathcal{P}^{]\![_F}(X)$ and equalities between elements of F without denoting the corresponding subscripts.

Definition 7.1.1. *If $a, b \in \mathbb{R}$, let $a \neq_{\mathbb{R}} b :\Leftrightarrow |a - b| > 0 \Leftrightarrow a > b \vee b < a$. For simplicity we may write $a \neq b$, instead of $a \neq_{\mathbb{R}} b$. The inequality $x \neq_X^F y$ on X generated by F is defined by*

$$x \neq_X^F y :\Leftrightarrow \exists_{f \in F}\big(f : x \neq_X^F y\big), \qquad \textit{where} \qquad f : x \neq_X^F y :\Leftrightarrow f(x) \neq_{\mathbb{R}} f(y).$$

A complemented subset $\boldsymbol{A}$ of X with respect to $\neq_X^F$ is called an F-complemented subset of X, and their totality is denoted by $\mathcal{P}^{]\![_F}(X)$. An F-complemented subset $\boldsymbol{A}$ of X is uniformly F-complemented, if

$$\exists_{f \in F}\big(f : A^1]\!][\![_F A^0\big), \qquad \textit{where} \qquad f : A^1]\!][\![_F A^0 :\Leftrightarrow \forall_{x \in A^1} \forall_{y \in A^0}\big(f : x \neq_X^F y\big),$$

and $\boldsymbol{A}$ is strongly F-complemented, if there is $f \in F$ such that $f\colon A^1]\![_F A^0$, $f(x) = 1$, for every $x \in A^1$, and $f(y) = 0$, for every $y \in A^0$.

Remark 7.1.2. *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\boldsymbol{A} \in \mathcal{P}^{]\![_G}(Y)$, then $h^{-1}(\boldsymbol{A}) \in \mathcal{P}^{]\![_F}(X)$.*

Proof. Let $x \in h^{-1}(A^1)$ and $y \in h^{-1}(A^0)$ i.e., $h(x) \in A^1$ and $h(y) \in A^0$. Let $g \in G$ such that $g(h(x)) \neq g(h(y))$. Hence, $g \circ h \in F$ and $(g \circ h)(x) \neq (g \circ h)(y)$. $\square$

Definition 7.1.3. *We denote by $\mathtt{Fam}(I, F, \boldsymbol{X})$ and $\mathtt{Set}(I, F, \boldsymbol{X})$ the sets of families and sets of F-complemented subsets of X, respectively. Let $\boldsymbol{O}_F(X) := \big(o_0^{1,F}, \mathcal{O}^{1,X}, o_1^{1,F}, o_0^{0,F}, \mathcal{O}^{0,X}, o_1^{0,F}\big) \in \mathtt{Fam}(F, F, \boldsymbol{X})$ be the family of basic open F-complemented subsets of X, where*

$$\boldsymbol{o}_F(f) := \big(o_0^{1,F}(f), o_0^{0,F}(f)\big) := \big([f > 0], [f \leq 0]\big),$$

$$[f > 0] := \{x \in X \mid f(x) > 0\}, \qquad [f \leq 0] := \{x \in X \mid f(x) \leq 0\},$$

and, as $[f > 0$, $[f \leq 0]$ are extensional subsets of X, the dependent operations $\mathcal{O}^{1,X}, \mathcal{O}^{0,X}, o_1^{1,F}$, and $o_1^{0,F}$ are defined by the identity map-rule. If F is clear from the context, we may write $\boldsymbol{O}_F(X) := \big(o_0^1, \mathcal{O}^{1,X}, o_1^1, o_0^0, \mathcal{O}^{0,X}, o_1^0\big)$.

Clearly, $f\colon o_0^1(f)]\![_F o_0^0(f)$, for every $f \in F$. Recall that a sequence of F-complemented subsets of X is a structure $\boldsymbol{B}(X) := \big(\beta_0^1, \mathcal{B}^{1,X}, \beta_1^1, \beta_0^0, \mathcal{B}^{0,X}, \beta_1^0\big) \in \mathtt{Fam}(\mathbb{N}^+, F, \boldsymbol{X})$, where $\boldsymbol{\beta}(n) := \big(\beta_0^1(n), \beta_0^0(n)\big) \in \mathcal{P}^{]\![_F}(X)$, and $\beta_{nn}^1\colon \beta_0^1(n) \to \beta_0^1(n)$ and $\beta_{nn}^0\colon \beta_0^0(n) \to \beta_0^0(n)$ are given by $\beta_{nn}^1 := \mathrm{id}_{\beta_0^1(n)}$ and $\beta_{nn}^0 := \mathrm{id}_{\beta_0^0(n)}$, respectively, for every $n \in \mathbb{N}^+$. We also write $\bigcup_{n=1}^\infty \boldsymbol{\beta}_0(n)$ and $\bigcap_{n=1}^\infty \boldsymbol{\beta}_0(n)$, instead of $\bigcup_{n\in\mathbb{N}^+} \boldsymbol{\beta}_0(n)$ and $\bigcap_{n\in\mathbb{N}^+} \boldsymbol{\beta}_0(n)$, respectively. A family $\boldsymbol{A}(X) := \big(\alpha_0^1, \mathcal{A}^{1,X}, \alpha_1^1, \alpha_0^0, \mathcal{A}^{0,X}, \alpha_1^0\big) \in \mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$ is defined similarly.

Definition 7.1.4. *If $\boldsymbol{\Lambda}(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big) \in \mathtt{Fam}(I, F, \boldsymbol{X})$, the set $\mathtt{Borel}(\boldsymbol{\Lambda}(X))$ of Borel sets generated by $\boldsymbol{\Lambda}(X)$ is defined inductively by the following rules:*

$$(\mathtt{Borel}_1) \qquad \frac{i \in I}{\boldsymbol{\lambda}_0(i) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X))},$$

$$(\mathtt{Borel}_2) \qquad \frac{\boldsymbol{\beta}_0(1) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X)),\ \boldsymbol{\beta}_0(2) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X)), \ldots}{\bigcup_{n=1}^\infty \boldsymbol{\beta}_0(n) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X))\ \ \&\ \ \bigcap_{n=1}^\infty \boldsymbol{\beta}_0(n) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X))}\,\boldsymbol{B}(X)\in\mathtt{Fam}(\mathbb{N}^+,F,\boldsymbol{X}),$$

$$(\mathtt{Borel}_3) \qquad \frac{\boldsymbol{B} \in \mathtt{Borel}(\boldsymbol{\Lambda}(X)),\ \boldsymbol{\alpha}_0(0) =_{\mathcal{P}^{]\![_F}(X)} \boldsymbol{B}}{\boldsymbol{\alpha}_0(0) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X))}\,\boldsymbol{A}(X)\in\mathtt{Fam}(\mathbb{1},F,\boldsymbol{X}).$$

The corresponding induction principle $\mathrm{Ind}_{\mathtt{Borel}(\boldsymbol{\Lambda}(X))}$ is the formula

$$\forall_{i\in I}\big(P(\boldsymbol{\lambda}_0(i))\big)\ \&\ \forall_{\boldsymbol{B}(X)\in\mathtt{Fam}(\mathbb{N}^+,F,\boldsymbol{X})}\Bigg[\forall_{n\in\mathbb{N}^+}\big(\boldsymbol{\beta}_0(n) \in \mathtt{Borel}(\boldsymbol{\Lambda}(X))\ \&\ P(\boldsymbol{\beta}_0(n))\big) \Rightarrow$$

$$P\bigg(\bigcup_{n=1}^\infty \boldsymbol{\beta}_0(n)\bigg)\ \&\ P\bigg(\bigcap_{n=1}^\infty \boldsymbol{\beta}_0(n)\bigg)\Bigg]\ \&$$

$$\forall_{\boldsymbol{A}(X)\in\mathtt{Fam}(\mathbb{1},F,\boldsymbol{X})}\forall_{\boldsymbol{B}\in\mathtt{Borel}(\boldsymbol{\Lambda}(X))}\Big([P(\boldsymbol{B})\ \&\ \boldsymbol{\alpha}_0(0) =_{\mathcal{P}^{]\![_F}(X)} \boldsymbol{B}] \Rightarrow P(\boldsymbol{\alpha}_0(0))\Big)$$

$$\Rightarrow \forall_{\boldsymbol{B}\in\mathtt{Borel}(\boldsymbol{\Lambda}(X))}\big(P(\boldsymbol{B})\big),$$

where P is any bounded formula. Let

$$\mathtt{Borel}(\mathcal{F}) := \mathtt{Borel}(\boldsymbol{O}_F(X)),$$

and we call its elements the Borel sets of $\mathcal{F}$.

In $\mathrm{Ind}_{\mathtt{Borel}(\boldsymbol{\Lambda}(X))}$ we quantify over the sets $\mathtt{Fam}(\mathbb{N}^+, F, \boldsymbol{X})$ and $\mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$, avoiding quantification over $\mathcal{P}^{\rrbracket\llbracket_F}(X)$ in condition ($\mathtt{Borel}_3$) and treating $\mathcal{P}^{\rrbracket\llbracket_F}(X)$ as a set in ($\mathtt{Baire}_2$).

Proposition 7.1.5. (i) *$\boldsymbol{O}_F(X)$ is not in $\mathtt{Set}(F, F, \boldsymbol{X})$.*
(ii) *If $f, g \in F$, then $\boldsymbol{o}(f) \cup \boldsymbol{o}(g) = \boldsymbol{o}(f \vee g)$.*
(iii) *If $\boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$, then $-\boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$.*
(iv) *There are Bishop space (X, F) and $f \in F$ such that $\neg[-\boldsymbol{o}(f) = \boldsymbol{o}(g)]$, for every $g \in F$.*
(v) *$\boldsymbol{o}(f) = \boldsymbol{o}([f \vee 0] \wedge 1)$.*

Proof. (i) If $f \in F$, then $\boldsymbol{o}(f) = \boldsymbol{o}(2f)$, but $\neg(f = 2f)$.
(iii) This equality is implied from the following properties for reals $a \vee b > 0 \Leftrightarrow a > 0 \vee b > 0$ and $a \vee b \leq 0 \Leftrightarrow a \leq 0 \wedge b \leq 0$.
(iv) If $a \in \mathbb{R}$, then $a \leq 0 \Leftrightarrow \forall_{n \geq 1}\big(a < \frac{1}{n}\big)$ and $a > 0 \Leftrightarrow \exists_{n \geq 1}\big(a \geq \frac{1}{n}\big)$, hence

$$\begin{aligned}
-\boldsymbol{o}(f) &:= \big([f \leq 0], [f > 0]\big) \\
&= \bigg(\bigcap_{n=1}^{\infty} \Big[\Big(\frac{1}{n} - f\Big) > 0\Big], \bigcup_{n=1}^{\infty} \Big[\Big(\frac{1}{n} - f\Big) \leq 0\Big]\bigg) \\
&:= \bigcap_{n=1}^{\infty} \boldsymbol{o}\Big(\frac{1}{n} - f\Big) \in \mathtt{Borel}(F).
\end{aligned}$$

If $P(\boldsymbol{B}) :\Leftrightarrow -\boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$, the above equality proves the first step of the corresponding induction on $\mathtt{Borel}(\mathcal{F})$. The rest of the inductive proof is straightforward.
(v) Let the Bishop space $(\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$. If we take $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) := \big([x > 0], [x \leq 0]\big)$, and if we suppose that $-\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) := \big([x \leq 0], [x > 0]\big) = \big([\phi > 0], [\phi \leq 0]\big) =: \boldsymbol{o}(\phi)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi(0) > 0$ and ϕ is not continuous at 0, which contradicts the fact that ϕ is uniformly continuous, hence pointwise continuous, on $[-1, 1]$.
(vi) The proof is based on basic properties of $\mathbb{R}$, like $a \wedge 1 = 0 \Rightarrow a = 0$. $\square$

Since $\mathtt{Borel}(\mathcal{F})$ is closed under intersections and complements, if $\boldsymbol{A}, \boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$, then $\boldsymbol{A} - \boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$. Constructively, we cannot show, in general, that $\boldsymbol{o}(f) \cap \boldsymbol{o}(g) = \boldsymbol{o}(f \wedge g)$. If $f := \mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$ and $g := -\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, then $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) \cap \boldsymbol{o}(-\mathrm{id}_{\mathbb{R}}) = \big([x > 0] \cap [x < 0], [x \leq 0] \cup [-x \leq 0]\big) = \big(\emptyset, [x \leq 0] \cup [x \geq 0]\big)$ Since $x \wedge (-x) = -|x|$, we get $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}} \wedge (-\mathrm{id}_{\mathbb{R}})) = \boldsymbol{o}(-|x|) = \big(\emptyset, [|x| \geq 0]\big)$. The supposed equality implies that $|x| \geq 0 \Leftrightarrow x \leq 0 \vee x \geq 0$. Since $|x| \geq 0$ is always the case, we get $\forall_{x \in \mathbb{R}}\big(x \leq 0 \vee x \geq 0\big)$, which implies LLPO (see [24], p. 20). If one add the condition $|f| + |g| > 0$, then $\boldsymbol{o}(f) \cap \boldsymbol{o}(g) = \boldsymbol{o}(f \wedge g)$ follows constructively. The condition (BS_4) in the definition of a Bishop space is crucial to the next proof.

Proposition 7.1.6. *If $(f_n)_{n=1}^{\infty} \subseteq F$, then $f := \sum_{n=1}^{\infty}(f_n \vee 0) \wedge 2^{-n} \in F$ and*

$$\boldsymbol{o}(f) = \bigcup_{n=1}^{\infty} \boldsymbol{o}(f_n) = \bigg(\bigcup_{n=1}^{\infty} [f_n > 0], \bigcup_{n=1}^{\infty} [f_n \leq 0]\bigg).$$

Proof. The function f is well-defined by the comparison test (see [19], p. 32). If $g_n := (f_n \vee 0) \wedge 2^{-n}$, for every $n \geq 1$, then

$$\bigg|\sum_{n=1}^{\infty} g_n - \sum_{n=1}^{N} g_n\bigg| = \bigg|\sum_{n=N+1}^{\infty} g_n\bigg| \leq \sum_{n=N+1}^{\infty} |g_n| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \xrightarrow{N} 0,$$

the sequence of the partial sums $\sum_{n=1}^{N} g_n \in F$ converges uniformly to f, hence by BS_4 we get $f \in F$. Next we show that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[f_n > 0]$. If $x \in X$ such that $f(x) > 0$, there is $N \geq 1$ such that $\sum_{n=1}^{N} g_n(x) > 0$. By Proposition (2.16) in [19], p. 26, there is $n \geq 1$ and $n \leq N$ with $g_n(x) > 0$, hence $(f_n(x) \vee 0) \geq g_n(x) > 0$, which implies $f_n(x) > 0$. For the converse inclusion, if $f_n(x) > 0$, for some $n \geq 1$, then $g_n(x) > 0$, hence $f(x) > 0$. To show $[f \leq 0] \subseteq \bigcup_{n=1}^{\infty}[f_n \leq 0]$, let $x \in X$ such that $f(x) \leq 0$, and suppose that $f_n(x) > 0$, for some $n \geq 1$. By the previous argument we get $f(x) > 0$, which contradicts our hypothesis $f(x) \leq 0$. For the converse inclusion, let $f_n(x) \leq 0$, for every $n \geq 1$, hence $f_n(x) \vee 0 = 0$ and $g_n(x) = 0$, for every $n \geq 1$. Consequently, $f(x) = 0$. $\square$

Proposition 7.1.7. *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\boldsymbol{B} \in \mathtt{Borel}(\mathcal{G})$, then $h^{-1}(\boldsymbol{B}) \in \mathtt{Borel}(\mathcal{F})$.*

Proof. By the definition of $h^{-1}(\boldsymbol{B})$, if $g \in G$, then

$$\begin{aligned} h^{-1}(\boldsymbol{o}_G(g)) &:= h^{-1}\big([g > 0], [g \leq 0]\big) \\ &:= \big(h^{-1}[g > 0], h^{-1}[g \leq 0]\big) \\ &= \big([(g \circ h) > 0], [(g \circ h) \leq 0]\big) \\ &:= \boldsymbol{o}_F(g \circ h) \in \mathtt{Borel}(\mathcal{F}). \end{aligned}$$

If $P(\boldsymbol{B}) := h^{-1}(\boldsymbol{B}) \in \mathtt{Borel}(\mathcal{F})$, the above equality is the first step of the corresponding inductive proof on $\mathtt{Borel}(\mathcal{G})$. The rest of the proof follows from the properties $h^{-1}\big(\bigcup_{n=1}^{\infty} \boldsymbol{B}_n\big) = \bigcup_{n=1}^{\infty} h^{-1}(\boldsymbol{B}_n)$ and $h^{-1}\big(\bigcap_{n=1}^{\infty} \boldsymbol{B}_n\big) = \bigcap_{n=1}^{\infty} h^{-1}(\boldsymbol{B}_n)$ of complemented subsets. $\square$

Definition 7.1.8. *If Φ is an extensional subset of F and if $\mathrm{id}_{\Phi}^{F} \colon \Phi \hookrightarrow F$ is defined by the identity map-rule, let $\boldsymbol{O}_{\Phi}(X) := \boldsymbol{O}_F(X) \circ \mathrm{id}_{\Phi}^{F}$ be the id_{Φ}^{F}-subfamily of $\boldsymbol{O}_F(X)$. We write $\boldsymbol{o}_{\Phi}(f) := \boldsymbol{o}_F(f)$, for every $f \in \Phi$, and let*

$$\mathtt{Borel}(\Phi) := \mathtt{Borel}(\boldsymbol{O}_{\Phi}(X)).$$

If F_0 is a subbase of F, then, $\mathtt{Borel}(F_0) \subseteq \mathtt{Borel}(F)$. More can be said on the relation between $\mathtt{Borel}(\Phi)$ and $\mathtt{Borel}(\mathcal{F})$, when Φ is a base of F.

Proposition 7.1.9. *Let Φ be a base of F.*

(i) *If for every $f \in F$, $\boldsymbol{o}_F(f) \in \mathtt{Borel}(\Phi)$, then $\mathtt{Borel}(\mathcal{F}) = \mathtt{Borel}(\Phi)$.*

(ii) *If for every $g \in \Phi$ and $f \in F$, $f \wedge g \in \Phi$, then $\mathtt{Borel}(\mathcal{F}) = \mathtt{Borel}(\Phi)$.*

(iii) *If for every $g \in B$ and every $n \geq 1$, $g - \frac{1}{n} \in \Phi$, then $\mathtt{Borel}(\mathcal{F}) = \mathtt{Borel}(\Phi)$.*

Proof. (i) It follows by a straightforward induction on $\mathtt{Borel}(F)$.
(ii) and (iii) Let $f \in F$ and $(g_n)_{n=1}^{\infty} \subseteq \Phi$ such that $\forall_{n \geq 1}\big(U(f, g_n, \frac{1}{n})\big)$. Then we have tha

$$\boldsymbol{o}_F(f) \subseteq \bigcup_{n=1}^{\infty} \boldsymbol{o}_{\Phi}(g_n) := \left(\bigcup_{n=1}^{\infty}[g_n > 0], \bigcap_{n=1}^{\infty}[g_n \leq 0]\right)$$

i.e., $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[g_n > 0]$ and $\bigcap_{n=1}^{\infty}[g_n \leq 0] \subseteq [f \leq 0]$; if $x \in X$ with $f(x) > 0$ there is $n \geq 1$ with $g_n(x) > 0$, and if $\forall_{n \geq 1}\big(g_n(x) \leq 0\big)$, then for the same reason $\neg[f(x) > 0$, hence $f(x) \leq 0$. Because of (i), for (ii), it suffices to show that $\boldsymbol{o}_F(f) \in \mathtt{Borel}(\Phi)$. We show that

$$\boldsymbol{o}_F(f) = \bigcup_{n=1}^{\infty} \boldsymbol{o}_{\Phi}(f \wedge g_n) := \left(\bigcup_{n=1}^{\infty}[(f \wedge g_n) > 0], \bigcap_{n=1}^{\infty}[(f \wedge g_n) \leq 0]\right) \in \mathtt{Borel}(\Phi).$$

If $f(x) > 0$, then we can find $n \geq 1$ such that $g_n(x) > 0$, hence $f(x) \wedge g_n(x) > 0$. Hence we showed that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[(f \wedge g_n) > 0]$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $(f \wedge g_n)(x) > 0$. Then $f(x) > 0$ and $x \in [f > 0]$. If $f(x) \leq 0$, then $\forall_{n \geq 1}\big(f(x) \wedge g_n(x) \leq 0\big)$. Suppose next that $\forall_{n \geq 1}\big(f(x) \wedge g_n(x) \leq 0\big)$. If $f(x) > 0$, there is $n \geq 1$ with $g_n(x) > 0$, hence $f(x) \wedge g_n(x) > 0$, which contradict the hypothesis $f(x) \wedge g_n(x) \leq 0$. Hence $f(x) \leq 0$.
Because of (i), for (iii), it suffices to show that $\boldsymbol{o}_F(f) \in \texttt{Borel}(\Phi)$. We show that

$$\boldsymbol{o}_F(f) = \bigcup_{n=1}^{\infty} \boldsymbol{o}_\Phi\big(g_n - \frac{1}{n}\big) := \bigg(\bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big], \bigcap_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) \leq 0\big]\bigg) \in \texttt{Borel}(\Phi).$$

First we show that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big]$. If $f(x) > 0$, there is $n \geq 1$ with $f(x) > \frac{1}{n}$, hence, since $-\frac{1}{2n} \leq g_{2n}(x) - f(x) \leq \frac{1}{2n}$, we get

$$g_{2n}(x) - \frac{1}{2n} \geq \bigg(f(x) - \frac{1}{2n}\bigg) - \frac{1}{2n} = f(x) - \frac{1}{n} > 0$$

i.e., $x \in \big[\big(g_{2n} - \frac{1}{2n}\big) > 0\big]$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $g_n(x) - \frac{1}{n} > 0$. Since $0 < g_n(x) - \frac{1}{n} \leq f(x)$, we get $x \in [f > 0]$. Next we show that $[f \leq 0] \subseteq \bigcap_{n=1}^{\infty}[\big(g_n - \frac{1}{n}\big) \leq 0]$. Let $x \in X$ with $f(x) \leq 0$, and suppose that $n \geq 1$ with $g_n(x) - \frac{1}{n} > 0$. Then $0 \geq f(x) > 0$. By this contradiction we get $g_n(x) - \frac{1}{n} \leq 0$. For the converse inclusion let $x \in X$ such that $g_n(x) - \frac{1}{n} \leq 0$, for every $n \geq 1$, and suppose that $f(x) > 0$. Since we have already shown that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big]$, there is some $n \geq 1$ with $g_n(x) - \frac{1}{n} > 0$, which contradicts our hypothesis, hence $f(x) \leq 0$. $\square$

7.2 The Baire sets of a Bishop space

One of the definitions[1] of the set of Baire sets in a topological space $(X, \mathcal{T})$, which was given by Hewitt in [60], is that it is the least σ-algebra of subsets of X that includes the zero sets of X i.e., the sets of the form $f^{-1}(\{0\})$, where $f \in C(X)$. Clearly, a Baire set in $(X, \mathcal{T})$ is a Borel set in $(X, \mathcal{T})$, and for many topological spaces, like the metrisable ones, the two classes coincide. In this section we adopt Hewitt's notion in Bishop spaces and the framework of F-complemented subsets.

Definition 7.2.1. *Let* $\boldsymbol{Z}_F(X) := \big(\zeta_0^{1,F}, \mathcal{Z}^{1,X}, \zeta_1^{1,F}, \zeta_0^{0,F}, \mathcal{Z}^{0,X}, \zeta_1^{0,F}\big) \in \texttt{Fam}(F, F, \boldsymbol{X})$ *be the family of zero* F*-complemented subsets of* X*, where*

$$\boldsymbol{\zeta}_F(f) := \big(\zeta_0^{1,F}(f), \zeta_0^{0,F}(f)\big) := \big([f = 0], [f \neq 0]\big),$$

$$[f = 0] := \{x \in X \mid f(x) = 0\}, \qquad [f \leq 0] := \{x \in X \mid f(x) \neq 0\},$$

and, as $[f = 0$, $[f \neq 0]$ *are extensional subsets of* X*, the dependent operations* $\mathcal{Z}^{1,X}, \mathcal{Z}^{0,X}, \zeta_1^{1,F}$*, and* $\zeta_1^{0,F}$ *are defined by the identity map-rule. If* F *is clear from the context, we may write* $\boldsymbol{Z}_F(X) := \big(\zeta_0^1, \mathcal{Z}^{1,X}, \zeta_1^1, \zeta_0^0, \mathcal{Z}^{0,X}, \zeta_1^0\big)$*. Let*

$$\texttt{Baire}(\mathcal{F}) := \texttt{Borel}(\boldsymbol{Z}_F(X)),$$

and we call its elements the Baire sets of $\mathcal{F}$*.*

[1] A different definition is given in [57]. See [109] for the relations between these two definitions.

Since $a \neq 0 :\Leftrightarrow |a| > 0 \Leftrightarrow a < 0 \vee a > 0$, for every $a \in \mathbb{R}$, we get

$$\boldsymbol{\zeta}(f) = \big([f = 0], [|f| > 0]\big) = \big([f = 0], [f > 0] \cup [f < 0]\big).$$

Proposition 7.2.2. (i) $\texttt{Baire}(\mathcal{F})$ *is not in* $\texttt{Set}(F, F, \boldsymbol{X})$.
(ii) *If* $f, g \in F$, *then* $\boldsymbol{\zeta}(f) \cap \boldsymbol{\zeta}(g) = \boldsymbol{\zeta}(|f| \vee |g|)$.
(iii) *If* $\boldsymbol{B} \in \texttt{Baire}(\mathcal{F})$, *then* $-\boldsymbol{B} \in \texttt{Baire}(\mathcal{F})$.
(iv) *There are Bishop space* (X, F) *and* $f \in F$ *such that* $\neg[-\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(g)$, *for every* $g \in F$.
(vi) $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(|f| \wedge 1)$.

Proof. (i) If $f \in F$, then $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(2f)$, but $\neg(f = 2f)$.
(ii) This equality is implied from the following property for reals $|a| \vee |b| = 0 \Leftrightarrow |a| = 0 \wedge |b| = 0$ and $|a| \vee |b| \neq 0 \Leftrightarrow |a| > 0 \vee |b| > 0$.
(iii) If $f \in F$, then $-\boldsymbol{\zeta}_0(f) := \big([f \neq 0], [f = 0]\big)$. If, for every $n \geq 1$,

$$g_n := \left(|f| \wedge \frac{1}{n}\right) - \frac{1}{n} \in F,$$

$$\bigcup_{n=1}^{\infty} \boldsymbol{\zeta}(g_n) := \left(\bigcup_{n=1}^{\infty} [g_n = 0], \bigcap_{n=1}^{\infty} [g_n \neq 0]\right) = -\boldsymbol{\zeta}(f) \in \texttt{Baire}(\mathcal{F}).$$

First we show that $[f \neq 0] = \bigcup_{n=1}^{\infty} [g_n = 0]$. If $|f(x)| > 0$, there is $n \geq 1$ such that $|f(x)| > \frac{1}{n}$, hence $|f(x)| \wedge \frac{1}{n} = \frac{1}{n}$, and $g_n(x) = 0$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $g_n(x) = 0 \Leftrightarrow |f(x)| \wedge \frac{1}{n} = \frac{1}{n}$, hence $|f(x)| \geq \frac{1}{n} > 0$. Next we show that $[f = 0] = \bigcap_{n=1}^{\infty} [g_n \neq 0]$. If $x \in X$ such that $f(x) = 0$, and $n \geq 1$, then $g_n(x) = -\frac{1}{n} < 0$. For the converse inclusion, let $x \in X$ such that for all $n \geq 1$ we have that $g_n(x) \neq 0$. If $|f(x)| > 0$, there is $n \geq 1$ such that $|f(x)| > \frac{1}{n}$, hence $g_n(x) = 0$, which contradicts our hypothesis. Hence, $|f(x)| \leq 0$, which implies that $|f(x)| = 0 \Leftrightarrow f(x) = 0$. If $P(\boldsymbol{B}) := -\boldsymbol{B} \in \texttt{Baire}(\mathcal{F})$, the above equality proves the first step of the corresponding induction on $\texttt{Baire}(F)$. The rest of the inductive proof is straightforward[2].
(v) Let the Bishop space $(\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$. If we take $\boldsymbol{\zeta}(\mathrm{id}_{\mathbb{R}}) := \big([x = 0], [x \neq 0]\big)$, and if we suppose that $-\boldsymbol{\zeta}(\mathrm{id}_{\mathbb{R}}) := \big([x \neq 0], [x = 0]\big) = \big([\phi = 0], [\phi \neq 0]\big) =: \boldsymbol{\zeta}(\phi)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi(0) > 0 \vee \phi(0) < 0$ and $\phi(x) = 0$, if $x < 0$ or $x > 0$. Hence ϕ is not continuous at 0, which contradicts the fact that ϕ is uniformly continuous on $[-1, 1]$.
(v) Using basic properties of $\mathbb{R}$, this proof is straightforward. $\square$

As in the case of $\texttt{Borel}(\mathcal{F})$, we cannot show constructively that $\boldsymbol{\zeta}(f) \cup \boldsymbol{\zeta}(g) = \boldsymbol{\zeta}(|f| \wedge |g|)$. If we add the condition $|f| + |g| > 0$ though, this equality is constructively provable.

Proposition 7.2.3. *If* $(f_n)_{n=1}^{\infty} \subseteq F$, *then* $f := \sum_{n=1}^{\infty} |f_n| \wedge 2^{-n} \in F$ *and*

$$\boldsymbol{\zeta}(f) = \bigcap_{n=1}^{\infty} \boldsymbol{\zeta}(f_n) = \left(\bigcap_{n=1}^{\infty} [f_n = 0], \bigcup_{n=1}^{\infty} [f_n \neq 0]\right).$$

[2] Hence, if we define the set of Baire sets over an arbitrary family Θ of functions from X to $\mathbb{R}$, a sufficient condition so that $\texttt{Baire}(\Theta)$ is closed under complements is that Θ is closed under $|.|$, under wedge with $\frac{1}{n}$ and under subtraction with $\frac{1}{n}$, for every $n \geq 1$. If $\Theta := \mathbb{F}(X, 2)$, then $-\boldsymbol{o}_{\mathbb{F}(X,2)}(f) = \boldsymbol{o}_{\mathbb{F}(X,2)}(1 - f) = \boldsymbol{\zeta}_{\mathbb{F}(X,2)}(f)$, hence by Proposition 7.1.7(ii) we get $\texttt{Borel}(\mathbb{F}(X, 2)) = \texttt{Baire}(\mathbb{F}(X, 2))$.

Proof. Proceeding as in the proof of Proposition 7.1.6, f is well-defined, and if $g_n := |f_n| \wedge 2^{-n}$, for every $n \geq 1$, the sequence of the partial sums $\sum_{n=1}^{N} g_n \in F$ converges uniformly to f, and by (BS$_4$) we get $f \in F$. Since $f(x) = 0 \Leftrightarrow \forall_{n\geq 1}(g_n(x) = 0) \Leftrightarrow \forall_{n\geq 1}(f_n(x) = 0)$, we get $[f = 0] = \bigcap_{n=1}^{\infty}[f_n = 0]$. To show $[f \neq 0] \subseteq \bigcup_{n=1}^{\infty}[f_n \neq 0]$, if $|f(x)| > 0$, there is $N \geq 1$ such that $\sum_{n=1}^{N} g_n(x) > 0$. By Proposition (2.16) in [19], p. 26, there is some $n \geq 1$ and $n \geq N$ such that $g_n(x) > 0$, hence $|f_n(x)| \geq g_n(x) > 0$. The converse inclusion follows trivially. $\square$

Let $\mathcal{F}^* := (X, F^*)$ be the Bishop space generated by the bounded functions F^* in F.

Theorem 7.2.4. (i) *If* $\boldsymbol{B} \in \mathtt{Baire}(\mathcal{F})$, *then* $\boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$.
(ii) *If* $\boldsymbol{o}(f) \in \mathtt{Baire}(\mathcal{F})$, *for every* $f \in F$, *then* $\mathtt{Baire}(\mathcal{F}) = \mathtt{Borel}(\mathcal{F})$.
(iii) *If* $f \in F$, *then* $\boldsymbol{o}(f) = -\boldsymbol{\zeta}\big((-f) \wedge 0\big)$.
(iv) $\mathtt{Baire}(\mathcal{F}^*) = \mathtt{Baire}(\mathcal{F}) = \mathtt{Borel}(\mathcal{F}) = \mathtt{Borel}(\mathcal{F}^*)$.

Proof. (i) By Proposition 7.1.5(iv) $-\boldsymbol{o}(f) = \big([f \leq 0], [f > 0]\big) \in \mathtt{Borel}(\mathcal{F})$, for every $f \in F$, hence $-\boldsymbol{o}(-f) = \big([f \geq 0], [f < 0] \in \mathtt{Borel}(\mathcal{F})$ too. Consequently

$$-\boldsymbol{o}(f) \cap -\boldsymbol{o}(-f) = \big([f \leq 0] \cap [f \geq 0], [f > 0] \cup [f < 0]\big) = \boldsymbol{\zeta}(f) \in \mathtt{Borel}(\mathcal{F}).$$

If $P(\boldsymbol{B}) := \boldsymbol{B} \in \mathtt{Borel}(\mathcal{F})$, the above equality is the first step of the corresponding inductive proof on $\mathtt{Baire}(\mathcal{F})$. The rest of the inductive proof is straightforward.
(ii) The hypothesis is the first step of the obvious inductive proof on $\mathtt{Borel}(\mathcal{F})$, which shows that $\mathtt{Borel}(\mathcal{F}) \subseteq \mathtt{Baire}(\mathcal{F})$. By (i) we get $\mathtt{Baire}(\mathcal{F}) \subseteq \mathtt{Borel}(\mathcal{F})$.
(iii) We show that

$$\big([f > 0], [f \leq 0]\big) = \big([(-f) \wedge 0 \neq 0], [(-f) \wedge 0 = 0]\big).$$

First we show that $[f > 0] \subseteq [(-f) \wedge 0 \neq 0]$; if $f(x) > 0$, then $-f(x) \wedge 0 = -f(x) < 0$. For the converse inclusion, let $-f(x) \wedge 0 \neq 0 \Leftrightarrow -f(x) \wedge 0 > 0$ or $-f(x) \wedge 0 < 0$. Since $0 \geq -f(x) \wedge 0$, the first option is impossible. If $-f(x) \wedge 0 < 0$, then $-f(x) < 0$ or $0 < 0$, hence $f(x) > 0$. Next we show that $[f \leq 0] = [(-f) \wedge 0 = 0]$; since $f(x) \leq 0 \Leftrightarrow -f(x) \geq 0 \Leftrightarrow -f(x) \wedge 0 = 0$ (see [24], p. 52), the equality follows.
(iv) Clearly, $\mathtt{Baire}(\mathcal{F}^*) \subseteq \mathtt{Baire}(\mathcal{F})$. By Proposition 7.2.2(vi) $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(|f| \wedge 1)$, where $|f| \wedge 1 \in F^*$. Continuing with the obvious induction we get $\mathtt{Baire}(\mathcal{F}) \subseteq \mathtt{Baire}(\mathcal{F}^*)$. By case (iii) and Proposition 7.2.2(iv) we get $\boldsymbol{o}(f) \in \mathtt{Baire}(\mathcal{F})$, hence by case (ii) we conclude that $\mathtt{Baire}(\mathcal{F}) = \mathtt{Borel}(\mathcal{F})$. Clearly, $\mathtt{Borel}(\mathcal{F}^*) \subseteq \mathtt{Borel}(\mathcal{F})$. By Proposition 7.1.5(vi) $\boldsymbol{o}(f) = \boldsymbol{o}((f \vee 0) \wedge 1)$, where $(f \vee 0) \wedge 1 \in F^*$. Continuing with the obvious induction we get $\mathtt{Borel}(\mathcal{F}) \subseteq \mathtt{Borel}(\mathcal{F}^*)$. $\square$

Either by definition, as in the proof of Proposition 7.1.7, or by Theorem 7.2.4(iii) and Proposition 7.1.7, if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\boldsymbol{B} \in \mathtt{Baire}(\mathcal{G})$, then $h^{-1}(\boldsymbol{B}) \in \mathtt{Baire}(\mathcal{F})$. Suppose next that $\boldsymbol{A}$ is strongly F-complemented i.e., there is $f \in F$ such that $f \colon A^1]\!][\![_F A^0$ and $f(x) = 1$, for every $x \in A^1$, and $f(y) = 0$, for every $y \in A^0$. If $g := (f \vee 0) \wedge 1 \in F$, then $0 \leq g \leq 1$ and $\forall_{x \in A^1} \forall_{y \in A^0} \big(g(x) = 1 \ \& \ g(y) = 0\big)$. In [18], p. 55, the following relation between complemented subsets is defined:

$$\boldsymbol{A} \leq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \ \& \ \ A^0 \subseteq B^0.$$

If $\boldsymbol{A}$ is strongly F-complemented, then $\boldsymbol{A} \leq \boldsymbol{o}(f)$. According to the classical Urysohn lemma for $C(X)$-zero sets, the disjoint zero sets of a topological space X are separated by some $f \in C(X)$

(see [52], p. 17). Next we show a constructive version of this result, where disjointness is replaced by a stronger, but positively defined form of it.

Theorem 7.2.5 (Urysohn lemma for zero complemented subsets). *If* $\boldsymbol{A} := (A^1, A^0) \in \mathcal{P}^{][_F}(X)$, *then* $\boldsymbol{A}$ *is strongly* F*-complemented if and only if*

$$\exists_{f,g \in F} \exists_{c>0} \big(\boldsymbol{A} \leq \boldsymbol{\zeta}(f) \ \& \ -\boldsymbol{A} \leq \boldsymbol{\zeta}(g) \ \& \ |f| + |g| \geq c\big).$$

Proof. ($\Rightarrow$) Let $h \in F$ such that $0 \leq h \leq 1$, $A^1 \subseteq [h = 1]$ and $A^0 \subseteq [h = 0]$. We take $f := 1 - h \in F, g := h$ and $c := 1$. First we show that $\boldsymbol{A} \leq \boldsymbol{\zeta}(f)$. If $x \in A^1$, then $h(x) = 1$, and $f(x) = 0$. If $y \in A^0$, then $h(y) = 0$, hence $f(y) = 1$ and $y \in [f \neq 0]$. Next we show that $-\boldsymbol{A} \leq \boldsymbol{\zeta}(g)$. If $y \in A^0$, then $h(y) = 0 = g(y)$. If $x \in A^1$, then $h(x) = 1 = g(y)$ i.e., $x \in [g \neq 0]$. If $x \in X$, then $1 = |1 - h(x) + h(x)| \leq |1 - h(x)| + |h(x)|$.
($\Leftarrow$) Let $h := 1 - \big(\frac{1}{c}|f| \wedge 1\big) \in F$. If $x \in A^1$, then $f(x) = 0$, and hence $h(x) = 1$. If $y \in A^0$, then $g(y) = 0$, hence $|f(y)| \geq c$, and consequently $h(y) = 0$. $\square$

The condition (BS$_3$) of a Bishop space is crucial to the next proof.

Corollary 7.2.6. *ILet*$\boldsymbol{A} := (A^1, A^0) \in \mathcal{P}^{][_F}(X)$ *and* $f \in F$. *If* $f(\boldsymbol{A}) := \big(f(A^1), f(A^0)\big)$ *is strongly* $\mathrm{Bic}(\mathbb{R})$*-complemented, then* $\boldsymbol{A}$ *is strongly* F*-complemented.*

Proof. By the Urysohn lemma for zero complemented subsets there are $\phi, \theta \in \mathrm{Bic}(\mathbb{R})$ and $c > 0$ with $f(\boldsymbol{A}) \leq \boldsymbol{\zeta}(\phi), -f(\boldsymbol{A}) \leq \boldsymbol{\zeta}(\theta)$ and $|\phi| + |\theta| \geq c$. Consequently, $\boldsymbol{A} \leq \boldsymbol{\zeta}(\phi \circ f), -\boldsymbol{A} \leq \boldsymbol{\zeta}(\theta \circ f)$ and $|\phi \circ f| + |\theta \circ f| \geq c$. Since by (BS$_3$) we have that $\phi \circ f \in F$ and $\theta \circ f \in F$, by the other implication of the Urysohn lemma for zero complemented subsets we conclude that $\boldsymbol{A}$ is strongly F-complemented. $\square$

7.3 Measure and pre-measure spaces

There are two, quite different, notions of measure space in traditional Bishop-style constructive mathematics. The first, which was introduced in [9] as part of *Bishop's measure theory* (BMT) (see Note 7.6.6), is an abstraction of the measure function $A \mapsto \mu(A)$, where A is a member of a family of complemented subsets of a locally compact metric space X. The use of complemented subsets in order to overcome the difficulties generated in measure theory by the use of negation and negatively defined concepts is one of Bishop's great conceptual achievements, while the use of the concept of a family of complemented subsets is crucial to the predicative character of this notion of measure space[3]. The indexing required behind this first notion of measure space is evident in [9], and sufficiently stressed in [12] (see Note 7.6.7). The second notion of measure space, introduced in [18] and repeated in [19] as part of the far more general *Bishop-Cheng measure theory* (BCMT), is highly impredicative, as the necessary indexing for its predicative reformulation is missing. A lack of predicative concern is evident also in the integration theory of BCMT. Next we define a predicative variation of the Bishop-Cheng notion of measure space using the predicative conceptual ingredients of the initial Bishop notion of measure space. We also keep the operations of complemented subsets introduced in [9], and not the operations used in [18] and [19]. Following Bishop's views in [12], we introduce the notion of pre-measure space, which is understood though, in a way different from the classical term.

[3]Myhill's impredicative interpretation in [80] of Bishop's first definition is discussed in Note 7.6.8.

As in Definition 4.6.3, if $\boldsymbol{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0) \in \mathtt{Fam}(I, \boldsymbol{X})$, the set $\lambda_0 I(\boldsymbol{X})$ of complemented subsets of X is the totality I, equipped with the equality $i =_I^{\Lambda(X)} j :\Leftrightarrow \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \boldsymbol{\lambda}_0(j)$, for every $i, j \in I$. For simplicity we write $\boldsymbol{\lambda}_0(i) := (\lambda_0^1(i), \lambda_0^0(i))$ instead of i for an element of $\lambda_0 I(\boldsymbol{X})$.

Definition 7.3.1 (Measure space within BST)**.** *Let $(X, =_X, \neq_X)$ be an inhabited set, $\boldsymbol{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1; \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0) \in \mathtt{Fam}(I, \boldsymbol{X})$, and let $\mu \colon \lambda_0 I(\boldsymbol{X}) \to [0, +\infty)$ such that the following conditions hold:*

$$\text{(MS}_1\text{)} \qquad \forall_{i,j \in I} \exists_{k,l \in I} \bigg(\boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(k) \ \& \ \boldsymbol{\lambda}_0(i) \cap \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(l) \ \& \ \mu\big(\boldsymbol{\lambda}_0(i)\big) + \mu\big(\boldsymbol{\lambda}_0(j)\big) = \mu\big(\boldsymbol{\lambda}_0(k)\big) + \mu\big(\boldsymbol{\lambda}_0(l)\big) \bigg).$$

$$\text{(MS}_2\text{)} \qquad \forall_{i \in I} \forall_{\boldsymbol{A}(X) \in \mathtt{Fam}(\mathbb{1}, \boldsymbol{X})} \bigg[\exists_{k \in I} \bigg(\boldsymbol{\lambda}_0(i) \cap \boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(k) \bigg) \Rightarrow \big(\exists_{l \in I} \big(\boldsymbol{\lambda}_0(i) - \boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(l) \big) \ \& \ \mu\big(\boldsymbol{\lambda}_0(i)\big) = \mu\big(\boldsymbol{\lambda}_0(k)\big) + \mu\big(\boldsymbol{\lambda}_0(l)\big) \bigg].$$

$$\text{(MS}_3\text{)} \qquad \exists_{i \in I} \big(\mu\big(\boldsymbol{\lambda}_0(i)\big) > 0.$$

$$\text{(MS}_4\text{)} \qquad \forall_{\alpha \in \mathbb{F}(\mathbb{N}, I)} \bigg\{ \forall_{\beta \in \mathbb{F}(\mathbb{N}, I)} \bigg[\forall_{m \in \mathbb{N}} \bigg(\bigcap_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0(\beta(m)) \bigg) \ \& \ \exists \lim_{m \to +\infty} \mu\big(\boldsymbol{\lambda}_0(\beta(m))\big) \ \& \ \lim_{m \to +\infty} \mu\big(\boldsymbol{\lambda}_0(\beta(m))\big) > 0 \Rightarrow \exists_{x \in X} \bigg(x \in \bigcap_{n \in \mathbb{N}} \lambda_0^1(\alpha(n)) \bigg) \bigg] \bigg\}.$$

The triplet $\mathcal{M} := (X, \lambda_0 I(\boldsymbol{X}), \mu)$ is called a measure space with $\lambda_0 I(\boldsymbol{X})$ its set of integrable, or measurable sets, and μ its measure.

With respect to condition (MS$_1$), we do not say that the set $\lambda_0 I(\boldsymbol{X})$ is closed under the union or intersection of complemented subsets (as Bishop-Cheng do in their definition). This amounts to the rather strong condition $\forall_{i,j \in I} \exists_{k,l \in I} \big(\boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j) := \boldsymbol{\lambda}_0(k) \ \& \ \boldsymbol{\lambda}_0(i) \cap \boldsymbol{\lambda}_0(j) := \boldsymbol{\lambda}_0(l) \big)$. The weaker condition (MS$_1$) states that the complemented subsets $\boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j)$ and $\boldsymbol{\lambda}_0(i) \cap \boldsymbol{\lambda}_0(j)$ "pseudo-belong" to $\lambda_0 I(\boldsymbol{X})$ i.e., there are elements of it, which are equal to them in $\mathcal{P}^{\rrbracket\llbracket}(X)$. In contrast to the formulation of condition (MS$_2$) by Bishop and Cheng, we avoid quantification over the class $\mathcal{P}^{\rrbracket\llbracket}(X)$, by quantifying over the set $\mathtt{Fam}(\mathbb{1}, \boldsymbol{X})$. In our formulation of (MS$_4$) we quantify over $\mathbb{F}(\mathbb{N}, I)$, in order to avoid the use of some choice principle. If we had written

$$\forall_{m \in \mathbb{N}} \exists_{k \in \mathbb{N}} \bigg(\bigcap_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) =_{\mathcal{P}^{\rrbracket\llbracket_{\neq}}} \boldsymbol{\lambda}_0(k) \bigg)$$

instead, we would need countable choice to express the limit to infinity of the terms $\mu\big(\boldsymbol{\lambda}_0(k)\big)$. Next we define the notion of a pre-measure space, giving an explicit formulation of Bishop's idea, expressed in [12], p. 67, and quoted in Note 7.6.7, to formalise his first definition of measure space, applied though, to Definition 7.3.1. The main idea is to define operations on I that correspond to the operations on complemented subsets, and reformulate accordingly the clauses for the measure μ. The fact that μ is defined on the index-set is already expressed in the definition of the set $\lambda_0 I(\boldsymbol{X})$. The notion of a pre-measure space provides us a method to generate measure spaces.

Definition 7.3.2 (Pre-measure space within BST)**.** *Let $(X, =_X, \neq_X)$ be an inhabited set, and let $(I, =_I)$ be equipped with operations $\vee \colon I \times I \rightsquigarrow I, \wedge \colon I \times I \rightsquigarrow I$, and $\sim \colon I \rightsquigarrow I$. If $i, j \in I$ and $i_1, \ldots, i_m \in I$, where $m \geq 1$, let*[4]

$$i \sim j := i \wedge (\sim j) \;\;\&\;\; i \leq j :\Leftrightarrow i \wedge j = i,$$

$$\bigvee_{n=1}^{m} i_n := i_1 \vee \ldots \vee i_m \;\;\&\;\; \bigwedge_{n=1}^{m} i_n := i_1 \wedge \ldots \wedge i_m.$$

Let $\boldsymbol{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1; \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0) \in \mathtt{Set}(I, \boldsymbol{X})$, and $\mu \colon I \to [0, +\infty)$ such that the following conditions hold:

(PMS$_1$) $$\forall_{i,j \in I}\Big(\boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(i \vee j) \;\&\; \boldsymbol{\lambda}_0(i) \cap \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(i \wedge j) \;\&\; -\boldsymbol{\lambda}_0(i) = \boldsymbol{\lambda}_0(\sim i) \;\&$$

$$\mu(i) + \mu(j) = \mu(i \vee j) + \mu(i \wedge j)\Big).$$

(PMS$_2$) $$\forall_{i \in I} \forall_{\boldsymbol{A}(X) \in \mathtt{Fam}(\mathbb{1}, \boldsymbol{X})}\Big[\exists_{k \in I}\Big(\boldsymbol{\lambda}_0(i) \cap \boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(k)\Big) \Rightarrow$$

$$\boldsymbol{\lambda}_0(i) - \boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(i \sim k) \;\&\; \mu(i) = \mu(k) + \mu(i \sim k)\Big].$$

(PMS$_3$) $$\exists_{i \in I}\big(\mu(i)\big) > 0.$$

(PMS$_4$) $$\forall_{\alpha \in \mathbb{F}(\mathbb{N}, I)}\Big[\exists \lim_{m \to +\infty} \mu\Big(\bigwedge_{n=1}^{m} \alpha(n)\Big) \;\&\; \lim_{m \to +\infty} \mu\Big(\bigwedge_{n=1}^{m} \alpha(n)\Big) > 0 \Rightarrow$$

$$\Rightarrow \exists_{x \in X}\Big(x \in \bigcap_{n \in \mathbb{N}} \lambda_0^1(\alpha(n))\Big)\Big].$$

The triplet $\mathcal{M}(\boldsymbol{\Lambda}(X)) := (X, I, \mu)$ is called a pre-measure space, the function μ a pre-measure, and the index-set I a set of integrable, or measurable indices.

Corollary 7.3.3. *Let $\mathcal{M}(\boldsymbol{\Lambda}) := (X, I, \mu)$ be a pre-measure space and $i, j \in I$.*

(i) *The operations $\vee$, $\wedge$ and $\sim$ are functions.*

(ii) *The triplet $(I, \vee, \wedge)$ is a distributive lattice.*

(iii) $\sim(\sim i) =_I i$.

(iv) $\sim(i \wedge j) =_I (\sim i) \vee (\sim j)$.

(v) $i \leq j \Leftrightarrow \sim j \leq \sim i$.

(vi) $i \leq j \Leftrightarrow \boldsymbol{\lambda}_0(i) \subseteq \boldsymbol{\lambda}_0(j)$.

(vii) $\boldsymbol{\lambda}_0(i) - \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(i \sim j)$.

Proof. We show that $\vee$ is a function, and for $\wedge$ and $\sim$ we proceed similarly.

$$\begin{aligned} i = i' \;\&\; j = j' &\Rightarrow \boldsymbol{\lambda}_0(i) = \boldsymbol{\lambda}_0(i') \;\&\; \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(j') \\ &\Rightarrow \boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(i') \cup \boldsymbol{\lambda}_0(j') \\ &\Rightarrow \boldsymbol{\lambda}_0(i \vee j) = \boldsymbol{\lambda}_0(i' \vee j') \\ &\Rightarrow i \vee j = i' \vee j'. \end{aligned}$$

[4] The operations $\bigvee_{n=1}^{m} i_n$ and $\bigvee_{n=1}^{m} i_n$ are actually recursively defined.

(ii) The defining clauses of a distributive lattice follow from the corresponding properties of complemented subsets for $\boldsymbol{A} \cap \boldsymbol{B}$ and $\boldsymbol{A} \cup \boldsymbol{B}$, from ($\mathrm{PMS}_1$), and from the fact that $\boldsymbol{\Lambda}(X) \in \mathtt{Set}(I, \boldsymbol{X})$. E.g., to show $i \vee j = j \vee i$, we use the equalities $\boldsymbol{\lambda}_0(i \vee j) = \boldsymbol{\lambda}_0(i) \cup \boldsymbol{\lambda}_0(j) = \boldsymbol{\lambda}_0(j) \cup \boldsymbol{\lambda}_0(i) = \boldsymbol{\lambda}_0(j \vee i)$. For the rest of the proof we proceed similarly. $\square$

In the next example of a pre-measure space the index-set I is a Boolean algebra.

Proposition 7.3.4. *Let* $\big(X, =_X, \neq_X^{\mathbb{F}(X,2)}\big)$ *be a set, and* $\boldsymbol{\Delta}(X) := \big(\delta_0^1, \mathcal{E}^{1,X}, \delta_1^1, \delta_0^0, \mathcal{E}^{0,X}, \delta_1^0\big) \in \mathtt{Set}(\mathbb{F}(X, 2), \boldsymbol{X})$ *the family of complemented detachable subsets of* X*, where by Remark 4.6.9*

$$\boldsymbol{\delta}_0(f) := \big(\delta_0^1(f), \delta_0^0(f)\big) := \big([f = 0], [f = 1]\big).$$

If $x_0 \in X$ *and* $\mu_{x_0} \colon \mathbb{F}(X, 2) \rightsquigarrow [0, +\infty)$ *is defined by the rule*

$$\mu_{x_0}(f) := f(x_0); \qquad f \in \mathbb{F}(X, 2),$$

then the triplet $\mathcal{M}(\boldsymbol{\Delta}(X)) := (X, \mathbb{F}(X, 2), \mu_{x_0})$ *is a pre-measure space.*

Proof. We define the maps $\vee, \wedge : \mathbb{F}(X, 2) \times \mathbb{F}(X, 2) \to \mathbb{F}(X, 2)$ and $\sim : \mathbb{F}(X, 2) \to \mathbb{F}(X, 2)$ by

$$f \vee g := f + g - fg, \quad f \wedge g := fg, \quad \sim f := 1 - f; \qquad f, g \in \mathbb{F}(X, 2),$$

where 1 also denotes the constant function on X with value 1. By definition of the union and intersection of complemented subsets we have that

$$\begin{aligned}
\boldsymbol{\delta}_0(f) \cup \boldsymbol{\delta}_0(g) &:= \big(\delta_0^1(f) \cup \delta_0^1(g), \delta_0^0(f) \cap \delta_0^0(g)\big) \\
&= \big(\delta_0^1(f) \cup \delta_0^1(g), \delta_0^1(1 - f) \cap \delta_0^1(1 - g)\big) \\
&= \big(\delta_0^1(f + g - fg), \delta_0^1((1 - f)(1 - g))\big) \\
&= \big(\delta_0^1(f + g - fg), \delta_0^1(1 - (f + g - fg))\big) \\
&= \big(\delta_0^1(f + g - fg), \delta_0^0(f + g - fg)\big) \\
&:= \boldsymbol{\delta}_0(f \vee g).
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}_0(f) \cap \boldsymbol{\delta}_0(g) &:= \big(\delta_0^1(f) \cap \delta_0^1(g), \delta_0^0(f) \cup \delta_0^0(g)\big) \\
&= \big(\delta_0^1(f) \cap \delta_0^1(g), \delta_0^1(1 - f) \cup \delta_0^1(1 - g)\big) \\
&= \big(\delta_0^1(fg), \delta_0^1((1 - f) + (1 - g) - (1 - f)(1 - g))\big) \\
&= \big(\delta_0^1(fg), \delta_0^1(1 - fg)\big) \\
&= \big(\delta_0^1(fg), \delta_0^0(fg)\big) \\
&:= \boldsymbol{\delta}_0(f \wedge g).
\end{aligned}$$

Clearly, $\boldsymbol{\delta}_0(\sim f) := \boldsymbol{\delta}_0(1 - f) = -\boldsymbol{\delta}_0(f)$. Clearly, the operation μ_{x_0} is a function. As

$$\mu_{x_0}(f) + \mu_{x_0}(g) = \mu_{x_0}(f + g - fg) + \mu_{x_0}(fg) \Leftrightarrow$$

$$f(x_0) + g(x_0) = f(x_0) + g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0),$$

which is trivially the case, (PMS_1) follows. Let $f \in \mathbb{F}(X, 2)$ and $\boldsymbol{B} := (B^1, B^0)$ a given complemented subset of X with $\boldsymbol{\alpha}_0(0) := \boldsymbol{B}$. If $g \in \mathbb{F}(X, 2)$ such that

$$\boldsymbol{\delta}_0(f) \cap \boldsymbol{B} := \big(\delta_0^1(f) \cap B^1, \delta_0^1(1 - f) \cup B^0\big) = \big(\delta_0^1(g), \delta_0^1(1 - g)\big) \Leftrightarrow$$

$$\delta_0^1(f) \cap B^1 = \delta_0^1(g) \ \ \& \ \ \delta_0^1(1-f) \cup B^0 = \delta_0^1(1-g),$$

$$\begin{aligned}
\boldsymbol{\delta}_0(f(1-g)) &:= \big(\delta_0^1(f(1-g)),\ \delta_0^0(f(1-g))\big)\\
&= \big(\delta_0^1(f) \cap \delta_0^1(1-g),\ \delta_0^0(f) \cup \delta_0^1(1-g)\big)\\
&= \big(\delta_0^1(f) \cap [\delta_0^1(1-f) \cup B^0],\ \delta_0^0(f) \cup [\delta_0^1(f) \cap B^1]\big)\\
&= \big([\delta_0^1(f) \cap \delta_0^1(1-f)] \cup [\delta_0^1(f) \cap B^0],\ [\delta_0^0(f) \cup \delta_0^1(f) \cap [\delta_0^0(f) \cup B^1]\big)\\
&= \big(\emptyset \cup [\delta_0^1(f) \cap B^0],\ X \cap [\delta_0^0(f) \cup B^1]\big)\\
&= \big(\delta_0^1(f) \cap B^0,\ \delta_0^0(f) \cup B^1\big)\\
&:= \boldsymbol{\delta}_0(f) - \boldsymbol{B}.
\end{aligned}$$

To complete the proof of (PMS_2), we need to show

$$\mu_{x_0}(f) = \mu_{x_0}(g) + \mu_{x_0}(f(1-g)) \Leftrightarrow f(x_0) = g(x_0) + f(x_0)(1-g(x_0)).$$

If $g(x_0) = 0$, the equality holds trivially. If $g(x_0) = 1$, and since $\delta_0^1(g) = \delta_0^1(f) \cap B^1$, we also have that $f(x_0) = 1$, and the required equality holds. As $\mu_{x_0}(1) = 1 > 0$, (PMS_3) follows. For the proof of (PMS_4) we fix $\alpha : \mathbb{N} \to \mathbb{F}(X, 2)$, and we suppose that

$$\begin{aligned}
&\exists \lim_{m\to+\infty} \mu_{x_0}\Big(\bigwedge_{n=0}^{m} \alpha_n\Big) \ \& \ \lim_{m\to+\infty} \mu_{x_0}\Big(\bigwedge_{n=0}^{m} \alpha_n\Big) > 0 \ \Leftrightarrow\\
&\exists \lim_{m\to+\infty} \Big(\bigwedge_{n=0}^{m} \alpha_n\Big)(x_0) \ \& \ \lim_{m\to+\infty} \Big(\bigwedge_{n=0}^{m} \alpha_n\Big)(x_0) > 0 \ \Leftrightarrow\\
&\exists \lim_{m\to+\infty} \prod_{n=0}^{m} \alpha_n(x_0) \ \& \ \lim_{m\to+\infty} \prod_{n=0}^{m} \alpha_n(x_0) > 0.
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
\lim_{m\to+\infty} \prod_{n=0}^{m} \alpha_n(x_0) > 0 &\Rightarrow \lim_{m\to+\infty} \prod_{n=0}^{m} \alpha_n(x_0) = 1\\
&\Leftrightarrow \exists_{m_0\in\mathbb{N}} \forall_{m\geq m_0} \Big(\prod_{n=0}^{m} \alpha_n(x_0) = 1\Big)\\
&\Rightarrow \forall_{n\in\mathbb{N}} \big(\alpha_n(x_0) = 1\big)\\
&\Leftrightarrow x_0 \in \bigcap_{n\in\mathbb{N}} \delta_0^1(\alpha_n).
\end{aligned}$$

□

Proposition 7.3.5. *Let $\mathcal{M}(\boldsymbol{\Lambda}(X)) := (X, I, \mu)$ be a pre-measure space. If $\mu^* \colon \boldsymbol{\lambda}_0 I \rightsquigarrow [0, +\infty)$, where $\mu^*(\boldsymbol{\lambda}_0(i)) := \mu(i)$, for every $\boldsymbol{\lambda}_0(i) \in \boldsymbol{\lambda}_0 I$, then $\mathcal{M} := (X, \lambda_0 I(\boldsymbol{X}), \mu^*)$ is a measure space.*

Proof. By Proposition 4.6.4 μ^* is a function. For the proof of (MS_1) we fix $i, j \in I$ and we take $k := i \vee j$ and $l := i \wedge j$. From (PMS_1) and (PMS_4) we get

$$\begin{aligned}
\mu^*(\boldsymbol{\lambda}_0(i)) + \mu^*(\boldsymbol{\lambda}_0(j)) &:= \mu(i) + \mu(j)\\
&= \mu(i \vee j) + \mu(i \wedge j)\\
&:= \mu^*(\boldsymbol{\lambda}_0(i \vee j)) + \mu^*(\boldsymbol{\lambda}_0(i \wedge j)).
\end{aligned}$$

For the proof of (MS_2) we fix $i \in I$ and $\boldsymbol{A} \in \mathtt{Fam}(\mathbb{1}, \boldsymbol{X})$ with $\boldsymbol{\alpha}_0(0) := \boldsymbol{B}$. If $\boldsymbol{\lambda}_0(i) \cap \boldsymbol{B} = \boldsymbol{\lambda}_0(k)$, for some $k \in I$, we take $l := i \sim k \in I$ and by (PMS_2) $\mu^*(\boldsymbol{\lambda}_0(i)) := \mu_0(i) = \mu(k) + \mu(i \sim k) := \mu^*(\boldsymbol{\lambda}_0(k)) + \mu^*(\boldsymbol{\lambda}_0(i \sim k))$. Condition (MS_3) follows immediately from (PMS_3). For the proof of (MS_4), we fix $\alpha, \beta \in \mathbb{F}(\mathbb{N}, I)$, and we suppose that

$$\forall_{m\in\mathbb{N}}\bigg(\bigcap_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0(\beta(m))\bigg) \ \&\ \exists \lim_{m\to+\infty} \mu^*\big(\boldsymbol{\lambda}_0(\beta(m))\big) \ \&\ \lim_{m\to+\infty} \mu^*\big(\boldsymbol{\lambda}_0(\beta(m))\big) > 0 \Leftrightarrow$$

$$\forall_{m\in\mathbb{N}}\bigg(\bigcap_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0(\beta(m))\bigg) \ \&\ \exists \lim_{m\to+\infty} \mu\big(\beta(m)\big) \ \&\ \lim_{m\to+\infty} \mu\big(\beta(m)\big) > 0.$$

If $m \geq 1$, by (PMS_1) we have that

$$\boldsymbol{\lambda}_0(\beta(m)) = \bigcap_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0\bigg(\bigwedge_{n=1}^{m} \alpha(n)\bigg),$$

hence, since $\boldsymbol{\lambda}_0$ is a set of complemented subsets, $\beta(m) = \bigwedge_{n=1}^{m} \alpha(n)$, and consequently $\mu\big(\beta(m)\big) = \mu\big(\bigwedge_{n=1}^{m} \alpha(n)\big)$. Hence

$$\exists \lim_{m\to+\infty} \mu\bigg(\bigwedge_{n=1}^{m} \alpha(n)\bigg) \ \&\ \lim_{m\to+\infty} \mu\bigg(\bigwedge_{n=1}^{m} \alpha(n)\bigg) > 0.$$

By (PMS_4) we conclude that there is some $x \in X$ such that $x \in \bigcap_{n\in\mathbb{N}} \lambda_0^1(\alpha(n))$. $\square$

Corollary 7.3.6. *Let $\mathcal{M}(\boldsymbol{\Delta}(X)) := (X, \mathbb{F}(X, 2), \mu_{x_0})$ be the pre-measure space of complemented detachable subsets of X. If $\mu^*_{x_0} : \delta_0\mathbb{F}(X, 2)(\boldsymbol{X}) \rightsquigarrow [0, +\infty)$ is defined by $\mu^*_{x_0}\big(\boldsymbol{\delta}_0(f)\big) := \mu_{x_0}(f) := f(x_0)$, for every $\delta_0(f) \in \delta_0\mathbb{F}(X, 2)(\boldsymbol{X})$, then $\mathcal{M}(X) := (X, \delta_0\mathbb{F}(X, 2)(\boldsymbol{X}), \mu^*_{x_0})$ is a measure space.*

Next we formulate in our framework the definition of a complete measure space given by Bishop and Cheng[5] (see Note 7.6.5).

Definition 7.3.7. *A measure space $\mathcal{M} := (X, \lambda_0 I(\boldsymbol{X}), \mu)$ is called complete, if the following conditions hold:*

(CM_1) $$\forall_{i\in I}\forall_{\boldsymbol{A}(X)\in\mathtt{Fam}(\mathbb{1},\boldsymbol{X})}\Big(\lambda_0^1(i) \subseteq \alpha_0^1(0) \ \&\ \lambda_0^0(i) \subseteq \alpha_0^0(0) \Rightarrow \exists_{k\in I}\big(\boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(k)\big)\Big).$$

(CM_2) $$\forall_{\alpha\in\mathbb{F}(\mathbb{N},I)}\bigg\{\forall_{\beta\in\mathbb{F}(\mathbb{N},I)}\forall_{l\in[0,+\infty]}\bigg[\forall_{m\in\mathbb{N}}\bigg(\bigcup_{n=1}^{m} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0(\beta(m))\bigg) \ \&$$
$$\exists \lim_{m\to+\infty} \mu\big(\boldsymbol{\lambda}_0(\beta(m))\big) \ \&\ \lim_{m\to+\infty} \mu\big(\boldsymbol{\lambda}_0(\beta(m))\big) = l$$
$$\Rightarrow \exists_{k\in I}\bigg(\bigcup_{n\in\mathbb{N}} \boldsymbol{\lambda}_0(\alpha(n)) = \boldsymbol{\lambda}_0(k) \ \&\ \mu\big(\boldsymbol{\lambda}_0(k)\big) = l\bigg)\bigg]\bigg\}.$$

(CM_3) $$\forall_{i,j\in I}\forall_{\boldsymbol{A}(X)\in\mathtt{Fam}(\mathbb{1},\boldsymbol{X})}\Big(\boldsymbol{\lambda}_0(i) \subseteq \boldsymbol{\alpha}_0(0) \subseteq \boldsymbol{\lambda}_0(j) \ \&\ \mu\big(\boldsymbol{\lambda}_0(i)\big) = \mu\big(\boldsymbol{\lambda}_0(j)\big)$$
$$\Rightarrow \exists_{k\in I}\big(\boldsymbol{\alpha}_0(0) = \boldsymbol{\lambda}_0(k)\big)\Big).$$

[5] In the definition of Bishop and Cheng the symbol of definitional equality $l := \lim_{m\to+\infty} \mu\big(\boldsymbol{\lambda}_0(\beta(m))\big)$ is used, but as this a convergence condition, one can use the equality of $\mathbb{R}$ for the same purpose

Regarding the completeness conditions and the space $\mathcal{M}(X)$, we show the following.

Proposition 7.3.8. *Let* $\mathcal{M}(X) := (X, \delta_0\mathbb{F}(X,\mathbb{2})(\boldsymbol{X}), \mu^*_{x_0})$ *be the measure space of complemented detachable subsets of* X.

(i) $\mathcal{M}(X)$ *satisfies condition* (CM_1).

(ii) *The limited principle of omniscience* (LPO) *implies that* $\mathcal{M}(X)$ *satisfies condition* (CM_2).

(iii) *In general,* $\mathcal{M}(X)$ *does not satisfy condition* (CM_3).

Proof. (i) Let $f \in \mathbb{F}(X,\mathbb{2})$, let $\boldsymbol{B} := (B^1, B^0)$ be a given complemented subset of X with $\boldsymbol{\alpha}_0(0) := \boldsymbol{B}$, and let $\delta_0^1(f) \subseteq B^1$ and $\delta_0^0(f) \subseteq B^0$. Since $X = \delta_0^1(f) \cup \delta_0^0(f) \subseteq (B^1 \cup B^0) \subseteq X$, we get $B^1 \cup B^0 = X$, and hence $\boldsymbol{B} = \boldsymbol{\delta}_0(\chi_{\boldsymbol{B}})$.
(ii) Let $\alpha, \beta : \mathbb{N} \to \mathbb{F}(X,\mathbb{2})$, and $l \in (0,+\infty)$ such that

$$\forall_{m\in\mathbb{N}}\bigg(\bigcup_{n=1}^{m} \boldsymbol{\delta}_0(\alpha_n) = \boldsymbol{\delta}_0(\beta_m)\bigg) \ \& \ \exists \lim_{m\to+\infty} \beta_m(x_0) \ \& \ \lim_{m\to+\infty} \beta_m(x_0) = l.$$

The last conjunct is equivalent to $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\big(\beta_m(x_0) = l\big)$, and since $\beta_m(x_0) \in \mathbb{2}$, we get $l \in \mathbb{2}$. For every $x \in X$ the sequence $n \mapsto \alpha_n(x)$ is in $\mathbb{F}(\mathbb{N},\mathbb{2})$, hence by (LPO) we define the function f from X to $\mathbb{2}$ by the rule

$$f(x) := \begin{cases} 1 & , \exists_{n\in\mathbb{N}}\big(\alpha_n(x) = 1\big) \\ 0 & , \forall_{n\in\mathbb{N}}\big(\alpha_n(x) = 0\big). \end{cases}$$

By the definition of interior union and intersection it is immediate to show that

$$\bigcup_{n\in\mathbb{N}} \boldsymbol{\delta}_0(\alpha_n) = \boldsymbol{\delta}_0(f) \Leftrightarrow \bigcup_{n\in\mathbb{N}} \delta_0^1(\alpha_n) = \delta_0^1(f) \ \& \ \bigcap_{n\in\mathbb{N}} \delta_0^0(\alpha_n) = \delta_0^0(f).$$

It remains to show that $f(x_0) = l$. If $l = 0$, then $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\big(\beta_m(x_0) = 0\big)$, which implies that $\forall_{n\in\mathbb{N}}\big(\alpha_n(x_0) = 0\big) :\Leftrightarrow f(x_0) := 0$. If $l = 1$, then $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\big(\beta_m(x_0) = 1\big)$, which implies that $\exists_{n\in\{1,\ldots,m_0\}}\big(\alpha_n(x_0) = 1\big) \Rightarrow f(x_0) := 1$.
(iii) If $X := \mathbb{3}$, let $f : \mathbb{3} \to \mathbb{2}$ be defined by $f(0) := 1, f(1) := 0 =: f(2)$ and let $g \colon \mathbb{3} \to \mathbb{2}$ be the constant function with value 1. If $\boldsymbol{B} := \big(\{0\},\{1\}\big)$, then $\delta_0^1(f) := \{0\} = B^1 \subseteq \delta_0^1(g)$ and $\emptyset = \delta_0^0(g) \subseteq B^0 \subseteq \delta_0^0(f)$. If $x_0 := 0$, then $\mu_0(f) = \mu_0(g)$, but $\boldsymbol{B}$ cannot "pseudo-belong" to $\mathcal{D}(\mathbb{3})$, since $B^1 \cup B^0$ is a proper subset of $\mathbb{3}$. $\square$

7.4 Real-valued partial functions

We present here all facts on real-valued partial functions necessary to the definition of an integration space within BST (Definition 7.5.1).

Definition 7.4.1. *If* $(X, =_X, \neq_X)$ *is an inhabited set, we denote by* $f_A := (A, i_A^X, f_A^{\mathbb{R}})$ *a real-valued partial function on* X

$$\begin{tikzcd} A \arrow[r, hook, "i_A^X"] \arrow[dr, "f_A^{\mathbb{R}}"'] & X \\ & \mathbb{R}. \end{tikzcd}$$

We say that f_A is strongly extensional, if $f_A^{\mathbb{R}}$ is strongly extensional, where A is equipped with its canonical inequality as a subset of X i.e., for every $a, a' \in A$

$$f_A^{\mathbb{R}}(a) \neq_{\mathbb{R}} f_A^{\mathbb{R}}(a') \Rightarrow i_A^X(a) \neq_X i_A^X(a').$$

Let $\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{R})$ be the class of partial functions from X to $\mathbb{R}$, and $\mathfrak{F}^{\mathrm{se}}(X)$ the class of strongly extensional partial functions from $(X =_X, \neq_X)$ to $(\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}})$.

Definition 7.4.2. *Let $f_A := (A, i_A^X, f_A^{\mathbb{R}}), f_B := (B, i_B^X, f_B^{\mathbb{R}})$ in $\mathfrak{F}(X)$*

$$\begin{tikzcd}
A \arrow[r, hook, "i_A^X"] \arrow[dr, "f_A^{\mathbb{R}}"'] & X & B \arrow[l, hook', "i_B^X"'] \arrow[dl, "f_B^{\mathbb{R}}"] \\
& \mathbb{R}. &
\end{tikzcd}$$

If $\lambda \in \mathbb{R}$, let $\lambda f_A := (A, i_A^X, \lambda f_A^{\mathbb{R}}) \in \mathfrak{F}(X)$, and $f_A \mathbin{\square} f_B := \big(A \cap B, i_{A\cap B}^X, \big(f_A^{\mathbb{R}} \mathbin{\square} f_B^{\mathbb{R}}\big)_{A\cap B}^{\mathbb{R}}\big)$,

$$\big(f_A^{\mathbb{R}} \mathbin{\square} f_B^{\mathbb{R}}\big)_{A\cap B}^{\mathbb{R}} := f_A^{\mathbb{R}}(a) \mathbin{\square} f_B^{\mathbb{R}}(b); \qquad (a, b) \in A \cap B, \quad \square \in \{+, \cdot, \wedge, \vee\}.$$

The operation $\big(f_A^{\mathbb{R}} \mathbin{\square} f_B^{\mathbb{R}}\big)_{A\cap B}^{\mathbb{R}} \colon A \cap B \rightsquigarrow \mathbb{R}$ is a function, as if $(a, b) =_{A\cap B} (a', b') :\Leftrightarrow i_A^X =_X i_A^X(a')$, and since $i_B^X(b) =_X i_A^X(a)$ and $i_B^X(b') =_X i_A^X(a')$, we get $a =_A a'$, hence $f_A^{\mathbb{R}}(a) =_{\mathbb{R}} f_A^{\mathbb{R}}(a')$, and $b =_B b'$, hence $f_B^{\mathbb{R}}(b) =_{\mathbb{R}} f_B^{\mathbb{R}}(b')$. If λ denotes also the constant function $\lambda \in \mathbb{R}$ on X

$$\begin{tikzcd}
A \arrow[r, hook, "i_A^X"] \arrow[dr, "f_A^{\mathbb{R}}"'] & X & X \arrow[l, hook', "\mathrm{id}_X"'] \arrow[dl, "\lambda"] \\
& \mathbb{R}, &
\end{tikzcd}$$

we get as a special case the partial function $f_A \wedge \lambda := \big(A \cap X, i_{A\cap X}^X, \big(f_A^{\mathbb{R}} \wedge \lambda\big)_{A\cap X}^{\mathbb{R}}\big)$, where $A \cap X := \{(a, x) \in A \times X \mid i_A^X(a) =_X x\}$, $i_{A\cap X}(a, x) := i_A^X(a)$, and $\big(f_A^{\mathbb{R}} \wedge \lambda\big)_{A\cap X}^{\mathbb{R}}(a, x) := f_A^{\mathbb{R}}(a) \wedge \lambda(x) := f_A^{\mathbb{R}}(a) \wedge \lambda$, for every $(a, x) \in A \cap X$. By Definition 4.8.1, if $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathtt{Fam}(I, X, \mathbb{R})$, if $(i, j) \in D(I)$, the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, bend left, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i^X"'] \arrow[ddr, bend right=40, "f_i^{\mathbb{R}}"'] & & \lambda_0(j) \arrow[ll, bend left, "\lambda_{ji}"] \arrow[dl, hook', "\mathcal{E}_j^X"] \arrow[ddl, bend left=40, "f_j^{\mathbb{R}}"] \\
& X & \\
& \mathbb{R}, &
\end{tikzcd}$$

$$f_i := \big(\lambda_0(i), \mathcal{E}_i^X, f_i^{\mathbb{R}}\big) \in \mathfrak{F}(X), \qquad f_i^{\mathbb{R}} := \mathcal{P}_i^{\mathbb{R}} \colon \lambda_0(i) \to \mathbb{R}; \qquad i \in I.$$

If $f_i^{\mathbb{R}}$ is strongly extensional, then, for every $u, w \in \lambda_0(i)$, we have that $f_i^{\mathbb{R}}(u) \neq_{\mathbb{R}} f_i^{\mathbb{R}}(w) \Rightarrow \mathcal{E}_i^X(u) \neq_X \mathcal{E}_i^X(w)$. As in Definition 4.1.11, If $\kappa \colon \mathbb{N}^+ \to I$, the family $\Lambda(X, \mathbb{R}) \circ \kappa := \big(\lambda_0 \circ \kappa, \mathcal{E}^X \circ \kappa, \lambda_1 \circ \kappa, \mathcal{P}^{\mathbb{R}} \circ \kappa\big) \in \mathtt{Fam}(\mathbb{N}^+, X, \mathbb{R})$ is the κ-subsequence of $\Lambda(X, \mathbb{R})$, where

$$(\lambda_0 \circ \kappa)(n) := \lambda_0(\kappa(n)), \qquad \big(\mathcal{E}^X \circ \kappa\big)_n := \mathcal{E}_{\kappa(n)}^X, \qquad (\lambda_1 \circ \kappa)(n, n) := \lambda_{\kappa(n)\kappa(n)} := \mathrm{id}_{\lambda_0(\kappa(n))},$$

$$\big(\mathcal{P}^{\mathbb{R}} \circ \kappa\big)_n := \mathcal{P}^{\mathbb{R}}_{\kappa(n)} := f^{\mathbb{R}}_{\kappa(n)}; \quad n \in \mathbb{N}^+.$$

Let $\lambda_0 I(X, \mathbb{R})$ be the totality I, and we write $f_i \in \lambda_0 I(X, \mathbb{R})$, instead of $i \in I$, as we define

$$i =_{\lambda_0 I(X,\mathbb{R})} j :\Leftrightarrow f_i =_{\mathfrak{F}(X)} f_j.$$

If we consider the intersection $\bigcap_{n\in\mathbb{N}^+}(\lambda_0 \circ \kappa)(n) := \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n))$, by Definition 4.3.1

$$\Phi\colon \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) :\Leftrightarrow \Phi\colon \bigcurlywedge_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) \ \ \& \ \ \forall_{n,m\in\mathbb{N}^+}\big(\mathcal{E}^X_{\kappa(n)}(\Phi_n) =_X \mathcal{E}^X_{\kappa(m)}(\Phi_m)\big),$$

$$\Phi =_{\bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n))} \Theta :\Leftrightarrow \mathcal{E}^X_{\kappa(1)}(\Phi_1) =_X \mathcal{E}^X_{\kappa(1)}(\Theta_1),$$

$$e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap}\colon \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) \hookrightarrow X, \quad e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap}(\Phi) := \big(\mathcal{E}^X \circ \kappa\big)_1(\Phi_1) := \mathcal{E}^X_{\kappa(1)}(\Phi_1).$$

Definition 7.4.3. *Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathtt{Fam}(I, X, \mathbb{R})$, $\kappa\colon \mathbb{N}^+ \to I$, and $\Lambda(X,\mathbb{R})\circ\kappa$ the κ-subsequence of $\Lambda(X, \mathbb{R})$. If $(A, i^{\bigcap}_A) \subseteq \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n))$, we define the function*

$$\sum_{n\in\mathbb{N}^+}^{A} f^{\mathbb{R}}_{\kappa(n)}\colon A \to \mathbb{R}, \quad \bigg(\sum_{n\in\mathbb{N}^+}^{A} f^{\mathbb{R}}_{\kappa(n)}\bigg)(a) := \sum_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big); \quad a \in A,$$

under the assumption that the series on the right converge in $\mathbb{R}$, for every $a \in \mathbb{R}$.

In the special case $\bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)), id_{\bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n))}) \subseteq \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n))$, we get the function

$$\sum_{n\in\mathbb{N}^+}^{\bigcap} f^{\mathbb{R}}_{\kappa(n)}\colon \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) \to \mathbb{R}, \quad \bigg(\sum_{n\in\mathbb{N}^+}^{\bigcap} f^{\mathbb{R}}_{\kappa(n)}\bigg)(\Phi) := \sum_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}(\Phi_n); \quad \Phi \in \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)),$$

under the assumption on the convergence of the corresponding series.

Proposition 7.4.4. *If in Definition 7.4.3 the partial functions $f_{\kappa(n)} := \big(\lambda_0(\kappa(n)), \mathcal{E}^X_{\kappa(n)}, f^{\mathbb{R}}_{\kappa(n)}\big)$ are strongly extensional, for every $n \in \mathbb{N}^+$, then the real-valued partial function*

$$\begin{tikzcd} A \arrow[r, hook, "i^{\bigcap}_A"] \arrow[rrd, bend right, "\sum^{A}_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}"'] & \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) \arrow[r, hook, "e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap}"] & X \\ & & \mathbb{R} \end{tikzcd}$$

$$f_A := \bigg(A,\ e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap} \circ i^{\bigcap}_A,\ \sum_{n\in\mathbb{N}^+}^{A} f^{\mathbb{R}}_{\kappa(n)}\bigg)$$

is strongly extensional.

Proof. Let $a, a' \in A$ such that

$$\bigg(\sum_{n\in\mathbb{N}^+}^{A} f^{\mathbb{R}}_{\kappa(n)}\bigg)(a) := l \neq_{\mathbb{R}} l' := \bigg(\sum_{n\in\mathbb{N}^+}^{A} f^{\mathbb{R}}_{\kappa(n)}\bigg)(a').$$

There is $N \in \mathbb{N}^+$ such that, if $\varepsilon := |l - l'| > 0$, then

$$\begin{aligned}
\varepsilon \leq {} & \left| l - \sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big)\right| + \left|\sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - \sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right| \\
& + \left|\sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big) - l'\right| \\
\leq {} & \frac{\varepsilon}{4} + \left|\sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - \sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right| + \frac{\varepsilon}{4} \Rightarrow
\end{aligned}$$

$$\begin{aligned}
0 & < \left|\sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - \sum_{n=1}^{N} f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right| \\
& = \left|\sum_{n=1}^{N} \left[f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right]\right| \\
& \leq \sum_{n=1}^{N} \left|f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right|.
\end{aligned}$$

By the property of positive real numbers $x + y > 0 \Rightarrow [x > 0 \vee y > 0$ (see [20], p. 13), then

$$0 < \left|f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) - f^{\mathbb{R}}_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big)\right|,$$

for some $n \in \mathbb{N}^+$ with $1 \leq n \leq N$. Since $f^{\mathbb{R}}_{\kappa(n)}$ is strongly extensional, by Definition 7.4.1

$$\mathcal{E}^X_{\kappa(1)}\Big(\big[i^{\bigcap}_A(a)\big]_1\Big) =_X \mathcal{E}^X_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a)\big]_n\Big) \neq_X \mathcal{E}^X_{\kappa(n)}\Big(\big[i^{\bigcap}_A(a')\big]_n\Big) =_X \mathcal{E}^X_{\kappa(1)}\Big(\big[i^{\bigcap}_A(a')\big]_1\Big),$$

which is the required canonical equality of a, a' in A as a subset of X. $\square$

Clearly, we have that (see Definition 2.7.1)

$$\begin{tikzcd}
A \arrow[rr, hook, "i^{\bigcap}_A"] \arrow[dr, hook, bend right, "e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap} \circ i^{\bigcap}_A"] \arrow[ddr, bend right=60, "\sum^{A}_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}"'] & & \bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)) \arrow[dl, hook, bend left, "e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap}"] \arrow[ddl, bend left=60, "\sum^{\bigcap}_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}"'] \\
& X & \\
& \mathbb{R} &
\end{tikzcd}$$

$$\Bigg(A,\ e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap} \circ i^{\bigcap}_A,\ \sum^{A}_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}\Bigg) \leq \Bigg(\bigcap_{n\in\mathbb{N}^+} \lambda_0(\kappa(n)),\ e^{\Lambda(X,\mathbb{R})\circ\kappa}_{\bigcap},\ \sum^{\bigcap}_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}\Bigg).$$

7.5 Integration and pre-integration spaces

Next we reformulate predicatively the Bishop-Cheng definition of an integration space (see Note 7.6.10 to compare it with the original definition).

Definition 7.5.1 (Integration space within BST). *Let $(X, =_X, \neq_X)$ be an inhabited set, $\Lambda(X,\mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathtt{Fam}(I, X, \mathbb{R})$, such that $f_i := \big(\lambda_0(i), \mathcal{E}_i^X, f_i^{\mathbb{R}}\big)$ is strongly extensional, for every $i \in I$, let $\lambda_0 I(X,\mathbb{R})$ be the totality I, equipped with the equality $i =_{\lambda_0 I(X,\mathbb{R})} j :\Leftrightarrow f_i =_{\mathcal{F}(X)} f_j$, for every $i \in I$, and let a mapping*

$$\int : \lambda_0 I(X,\mathbb{R}) \to \mathbb{R}, \qquad f_i \mapsto \int f_i; \qquad i \in I,$$

such that the following conditions hold:

$$\text{(IS}_1\text{)} \qquad \forall_{i\in I}\forall_{a\in\mathbb{R}}\exists_{j\in I}\bigg(af_i =_{\mathfrak{F}(X)} f_j \ \ \& \ \ \int f_j =_{\mathbb{R}} a\int f_i\bigg).$$

$$\text{(IS}_2\text{)} \qquad \forall_{i,j\in I}\exists_{k\in I}\bigg(f_i + f_j =_{\mathfrak{F}(X)} f_k \ \ \& \ \ \int f_k =_{\mathbb{R}} \int f_i + \int f_j\bigg).$$

$$\text{(IS}_3\text{)} \qquad \forall_{i\in I}\exists_{j\in I}\big(|f_i| =_{\mathfrak{F}(X)} f_j\big).$$

$$\text{(IS}_4\text{)} \qquad \forall_{i\in I}\exists_{j\in I}\big(f_i \wedge 1 =_{\mathfrak{F}(X)} f_j\big).$$

$$\text{(IS}_5\text{)} \qquad \forall_{i\in I}\forall_{\kappa\in\mathbb{F}(\mathbb{N}^+,I)}\bigg\{\bigg[\sum_{n\in\mathbb{N}^+}\int f_{\kappa(n)} \in \mathbb{R} \ \ \& \ \sum_{n\in\mathbb{N}^+}\int f_{\kappa(n)} < \int f_i\bigg] \Rightarrow$$

$$\exists_{(\Phi,u)\,\in\,\left(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\right)\,\cap\,\lambda_0(i)}\bigg(\bigg(\sum_{n\in\mathbb{N}^+}^{\cap} f_{\kappa(n)}\bigg)(\Phi) := \sum_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}(\Phi_n) \in \mathbb{R} \ \ \&$$

$$\sum_{n\in\mathbb{N}^+} f^{\mathbb{R}}_{\kappa(n)}(\Phi_n) < f_i^{\mathbb{R}}(u).\bigg)\bigg\}$$

$$\text{(IS}_6\text{)} \qquad \exists_{i\in I}\bigg(\int f_i =_{\mathbb{R}} 1\bigg).$$

$$\text{(IS}_7\text{)} \qquad \forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\bigg(\forall_{n\in\mathbb{N}^+}\bigg(n\bigg(\frac{1}{n}f_i \wedge 1\bigg) =_{\mathfrak{F}(X)} f_{\alpha(n)}\bigg) \Rightarrow$$

$$\lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} \in \mathbb{R} \ \ \& \ \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} =_{\mathbb{R}} \int f_i\bigg).$$

$$\text{(IS}_8\text{)} \qquad \forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\bigg(\forall_{n\in\mathbb{N}^+}\bigg(\frac{1}{n}\big(n|f_i| \wedge 1\big) =_{\mathfrak{F}(X)} f_{\alpha(n)}\bigg) \Rightarrow$$

$$\lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} \in \mathbb{R} \ \ \& \ \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} =_{\mathbb{R}} 0\bigg).$$

We call the triplet $\mathcal{L} := (X, \lambda_0 I(X,\mathbb{R}), \int)$ an integration space.

In the formulation of (IS$_5$) we have that

$$\bigg(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\bigg) \cap \lambda_0(i) := \bigg\{(\Phi,u) \in \bigg(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\bigg) \times \lambda_0(i) \ \Big|\ \mathcal{E}^X_{\kappa(1)}(\Phi) =_X \mathcal{E}^X_i(u)\bigg\},$$

$$\begin{tikzcd}
\lambda_0(\kappa(n)) \arrow[r, hook, "\mathcal{E}^X_{\kappa(n)}"] \arrow[dr, "f^{\mathbb{R}}_{\kappa(n)}"'] & X & \lambda_0(i) \arrow[l, hook', "\mathcal{E}^X_i"'] \arrow[dl, "f^{\mathbb{R}}_i"] \\
& \mathbb{R}. &
\end{tikzcd}$$

If, for every $a \in \mathbb{R}$, such that $a > 0$, and every $i \in I$, we define

$$f_i \ \wedge \ a := a\bigg(\frac{1}{a} f_i \ \wedge \ 1\bigg),$$

the formulation of (IS_7) and (IS_8) becomes, respectively,

$$\forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\bigg(\forall_{n\in\mathbb{N}^+}\big(f_i \ \wedge \ n =_{\mathfrak{F}(X)} f_{\alpha(n)}\big) \Rightarrow \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} =_{\mathbb{R}} \int f_i\bigg),$$

$$\forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\Bigg(\forall_{n\in\mathbb{N}^+}\bigg(|f_i| \ \wedge \ \frac{1}{n}\bigg) =_{\mathfrak{F}(X)} f_{\alpha(n)}\Bigg) \Rightarrow \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)} =_{\mathbb{R}} 0\Bigg),$$

where, for simplicity, we skip to mention he existence of the corresponding limits in $\mathbb{R}$. We also quantify over $\mathbb{F}(\mathbb{N}^+, I)$, in order to avoid the use of countable choice. If we had written in its premise the formula $\forall_{n\in\mathbb{N}^+}\exists_{j\in I}\big(f_i \ \wedge \ n =_{\mathfrak{F}(X)} f_j\big)$, we would need countable choice ($\mathbb{N}$–I) to generate a sequence in I to describe the limit of the corresponding integrals. Moreover, by (IS_1), (IS_3) and (IS_4), and the definition of $f \wedge a$ above we get $\forall_{i\in I}\forall_{a\in\mathbb{R}^+}\exists_{j\in I}\big(f_i \wedge a =_{\mathfrak{F}(X)} f_j\big)$.

Definition 7.5.2 (Pre-integration space within BST). *Let $(X, =_X, \neq_X)$ be an inhabited set, and let the set $(I, =_I)$ be equipped with operations $\cdot_a \colon I \rightsquigarrow I$, for every $a \in \mathbb{R}$, $+ \colon I \times I \rightsquigarrow I$, $|.| \colon I \rightsquigarrow I$, and $\wedge_1 \colon I \rightsquigarrow I$, where*

$$\cdot_a(i) := a \cdot i, \qquad +(i,j) := i + j, \qquad |.|(i) := |i|; \qquad i \in I, \ a \in \mathbb{R}.$$

Let also the operation $\wedge_a \colon I \rightsquigarrow I$, defined by the previous operations with the rule

$$\wedge_a := \cdot_a \circ \wedge_1 \circ \cdot_{a^{-1}}; \qquad a \in \mathbb{R} \ \& \ a > 0.$$

Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathtt{Set}(I, X, \mathbb{R})$ i.e., $f_i =_{\mathfrak{F}(X)} f_j \Rightarrow i =_I j$, for every $i, j \in I$, and $f_i := \big(\lambda_0(i), \mathcal{E}^X_i, f^{\mathbb{R}}_i\big)$ is strongly extensional, for every $i \in I$. Let also a mapping

$$\int \colon I \to \mathbb{R}, \qquad i \mapsto \int i; \qquad i \in I,$$

such that the following conditions hold:

(PIS_1) $$\forall_{i\in I}\forall_{a\in\mathbb{R}}\bigg(af_i =_{\mathfrak{F}(X)} f_{a\cdot i} \ \ \& \ \ \int a\cdot i =_{\mathbb{R}} a\int i\bigg).$$

(PIS_2) $$\forall_{i,j\in I}\bigg(f_i + f_j =_{\mathfrak{F}(X)} f_{i+j} \ \ \& \ \ \int (i+j) =_{\mathbb{R}} \int i + \int j\bigg).$$

(PIS_3) $$\forall_{i\in I}\big(|f_i| =_{\mathfrak{F}(X)} f_{|i|}\big).$$

(PIS_4) $$\forall_{i\in I}\big(f_i \wedge 1 =_{\mathfrak{F}(X)} f_{\wedge_1(i)}\big).$$

(PIS$_5$)
$$\forall_{i\in I}\forall_{\kappa\in\mathbb{F}(\mathbb{N}^+,I)}\bigg\{\bigg[\sum_{n\in\mathbb{N}^+}\int\kappa(n)\in\mathbb{R}\ \ \&\ \sum_{n\in\mathbb{N}^+}\int\kappa(n)<\int i\bigg]\Rightarrow$$
$$\exists_{(\Phi,u)\ \in\ \left(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\right)\ \cap\ \lambda_0(i)}\bigg(\bigg(\sum_{n\in\mathbb{N}^+}^{\cap}f_{\kappa(n)}\bigg)(\Phi):=\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)\in\mathbb{R}\ \ \&$$
$$\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)<f^{\mathbb{R}}_i(u).\bigg)\bigg\}$$

(PIS$_6$)
$$\exists_{i\in I}\bigg(\int i=_{\mathbb{R}}1\bigg).$$

(PIS$_7$)
$$\forall_{i\in I}\bigg(\lim_{n\longrightarrow+\infty}\int\wedge_n(i)\in\mathbb{R}\ \ \&\ \lim_{n\longrightarrow+\infty}\int\wedge_n(i)=_{\mathbb{R}}\int i\bigg).$$

(PIS$_8$)
$$\forall_{i\in I}\bigg(\lim_{n\longrightarrow+\infty}\int\wedge_{\frac{1}{n}}(|i|)\in\mathbb{R}\ \ \&\ \lim_{n\longrightarrow+\infty}\int\wedge_{\frac{1}{n}}(|i|)=_{\mathbb{R}}0\bigg).$$

We call the triplet $\mathcal{L}_0 := (X, I, \int)$ a pre-integration space.

All the operations on I defined above are functions. E.g., since $\Lambda(X,\mathbb{R})\in\mathtt{Set}(I,X,\mathbb{R})$,

$$i=_I i'\Rightarrow f_i=_{\mathfrak{F}(X)}f_{i'}\Rightarrow af_i=_{\mathfrak{F}(X)}af_{i'}\Rightarrow f_{a\cdot i}=_{\mathfrak{F}(X)}f_{a'\cdot i}\Rightarrow a\cdot i=_I a'\cdot i.$$

It is immediate to see that a pre-integration space induces an integration space, if

$$\forall_{i\in I}\forall_{a\in\mathbb{R}^+}\big(f_i\ \wedge\ a=_{\mathfrak{F}(X)}f_j\Rightarrow\wedge_a(i)=_I j\big),$$

and hence (PIS$_7$) and (PIS$_8$) imply (IS$_7$) and (IS$_8$), respectively, with the integral

$$\int^* f_i:=\int i;\qquad i\in I.$$

The notion of a pre-integration space is simpler than that of an integration space, and also closer to the Bishop-Cheng notion of an integration space. One could say that a pre-integration space is the "right" notion of integration space within BST. In [18], p. 52 Bishop and Cheng formulate the non-trivial theorem that a measure space induces the integration space of the corresponding simple functions (see also [19], p. 285). In [129] Zeuner interpreting the various constructions of Bishop and Cheng into the framework of pre-measure and pre-integration spaces[6] gave a proof of this theorem within BST. Here we only sketch this construction.

Let $\boldsymbol{\Lambda}(X) := \big(\lambda_0^1, \mathcal{E}^{1,X}, \lambda_1^1, \lambda_0^0, \mathcal{E}^{0,X}, \lambda_1^0\big)\in\mathtt{Fam}(I,\boldsymbol{X})$ and $i_0\in I$. To each $i\in I$ corresponds the real-valued partial function

$$\chi_i := \big(\lambda_0^1(i)\cup\lambda_0^0(i), \mathcal{E}_i^X, \chi_i^{\mathbb{R}}\big)\in\mathfrak{F}(X),$$

[6] The notion of pre-measure space used in [129], which is a bit different from the one included here, is an appropriate copy of Bishop's definition of a measure space given in [9] (see Note 7.6.6).

$$\begin{tikzcd}
& \lambda_0^1(i) \arrow[dl, hook', "\mathrm{id}"'] \arrow[d, hook, "\mathcal{E}_i^{0,X}"] \\
\lambda_0^1(i) \cup \lambda_0^0(i) \arrow[r, hook, "\mathcal{E}_i^X"] \arrow[ddr, "\chi_i^{\mathbb{R}}"'] & X \\
& \lambda_0^0(i) \arrow[ul, hook, "\mathrm{id}"] \arrow[u, hook', "\mathcal{E}_i^{1,X}"'] \\
& \mathbb{R}.
\end{tikzcd}$$

where $\mathcal{E}_i^X$ is the canonical embedding of $\lambda_0^1(i) \cup \lambda_0^0(i)$ into X, and $\chi_i^{\mathbb{R}}$ is given by the rule of the partial function $\chi_{\boldsymbol{\lambda}_0(i)}$. The symbol id in the above diagram denotes the corresponding function defined by the identity-map rule. If $m, n \in \mathbb{N}^+$, $i_1, \ldots, i_n, j_1, \ldots, j_m \in I$, and $a_{1_i}, \ldots a_{i_n}, b_1, \ldots, b_m \in \mathbb{R}$, the equality of the following real-valued partial functions is given by the commutativity of the following diagram

$$\sum_{k=1}^{n} a_{i_k} \chi_{i_k} := \bigg(\bigcap_{k=1}^{n} \big(\lambda_0^1(i_k) \cup \lambda_0^0(i_k)\big), i^X_{\bigcap_{k=1}^{n}(\lambda_0^1(i_k) \cup \lambda_0^0(i_k))}, \sum_{k=1}^{n} a_{i_k} \chi_{i_k}^{\mathbb{R}} \bigg) \in \mathfrak{F}(X),$$

$$\sum_{l=1}^{m} b_{j_l} \chi_{j_l} := \bigg(\bigcap_{l=1}^{m} \big(\lambda_0^1(j_l) \cup \lambda_0^0(j_l)\big), i^X_{\bigcap_{l=1}^{m}(\lambda_0^1(j_l) \cup \lambda_0^0(j_l))}, \sum_{l=1}^{m} b_{j_l} \chi_{j_l}^{\mathbb{R}} \bigg) \in \mathfrak{F}(X),$$

$$\begin{tikzcd}
\bigcap_{k=1}^{n} \big(\lambda_0^1(i_k) \cup \lambda_0^0(i_k)\big) \arrow[rr, bend left, "e"] \arrow[dr, hook, "i^X_{\bigcap_{k=1}^{n}(\lambda_0^1(i_k) \cup \lambda_0^0(i_k))}"'] \arrow[ddr, bend right, "\sum_{k=1}^{n} a_{i_k} \chi_{i_k}^{\mathbb{R}}"'] & & \bigcap_{l=1}^{m} \big(\lambda_0^1(j_l) \cup \lambda_0^0(j_l)\big) \arrow[ll, bend left, "e'"] \arrow[dl, hook', "i^X_{\bigcap_{l=1}^{m}(\lambda_0^1(j_l) \cup \lambda_0^0(j_l))}"] \arrow[ddl, bend left, "\sum_{l=1}^{m} b_{j_l} \chi_{j_l}^{\mathbb{R}}"] \\
& X & \\
& \mathbb{R}, &
\end{tikzcd}$$

for some (unique up to equality) functions e and e'. If $\mu_0 \colon \mathbb{N}^+ \rightsquigarrow \mathbb{V}_0$ is defined by the rule $\mu_0(n) := (\mathbb{R} \times I)^n$, for every $n \in \mathbb{N}^+$, and if the corresponding dependent operation μ_1 is defined in the obvious way, let the totality

$$S(I, \boldsymbol{\Lambda}(X)) := \sum_{n \in \mathbb{N}^+} (\mathbb{R} \times I)^n,$$

$$(n, u) =_{S(I, \boldsymbol{\Lambda}(X))} (m, w) :\Leftrightarrow \sum_{k=1}^{n} a_{i_k} \chi_{i_k} =_{\mathfrak{F}(X)} \sum_{l=1}^{m} b_{j_l} \chi_{j_l};,$$

where $u := \big((a_1, i_1), \ldots, (a_n, i_n)\big)$ and $w := \big((b_1, j_1), \ldots, (b_m, j_m)\big)$. The family of simple functions generated by the family $\boldsymbol{\Lambda}(I, \boldsymbol{X})$ of complemented subsets of X is the structure

$\Delta(X, \mathbb{R}) := \big(\mathrm{dom}_0, \mathcal{Z}^X, \mathrm{dom}_1, \mathcal{P}^{\mathbb{R}}\big) \in \mathtt{Fam}\big(S(I, \mathbf{\Lambda}(X), X, \mathbb{R}\big)$, where $\mathrm{dom}_0 \colon S(I, \mathbf{\Lambda}(X)) \rightsquigarrow \mathbb{V}_0$ is defined by the rule

$$\mathrm{dom}_0(n, u) := \bigcap_{k=1}^{n} \big(\lambda_0^1(i_k) \cup \lambda_0^0(i_k)\big); \quad u := \big((a_1, i_1), \ldots, (a_n, i_n)\big),\ n \in \mathbb{N}^+,$$

the embedding $\mathcal{Z}^X_{n,u} \colon \mathrm{dom}_0(n, u) \hookrightarrow X$ is defined in a canonical way through the embeddings $\mathcal{E}^{1,X}_{i_k}$ and $\mathcal{E}^{0,X}_{i_k}$, where $k \in \{1, \ldots, n\}$. If $(n, u) =_{S(I, \mathbf{\Lambda}(X))} (m, w)$, the mapping $\mathrm{dom}_{(n,u)(m,w)} \colon \mathrm{dom}_0(n, u) \to \mathrm{dom}_0(m, w)$ is defined, in order to avoid choice, as the mapping $E_{(n,u)(m,w)}$, where E is a modulus of equality for $D(S(I, \mathbf{\Lambda}(X)))$ with $E_{(n,u)(n,u)} := \mathrm{id}_{\mathrm{dom}_0(n,u)}$, for every $(n, u) \in S(I, \mathbf{\Lambda}(X))$. The fact that $\Delta(X, \mathbb{R}) \in \mathtt{Set}\big(S(I, \mathbf{\Lambda}(X), X, \mathbb{R}\big)$ is immediate to show. Hence, to every $(n, u) \in S(I, \mathbf{\Lambda}(X))$ corresponds the partial function

$$s_{(n,u)} := \left(\mathrm{dom}_0(n, u), \mathcal{Z}^X_{(n,u)}, \sum_{k=1}^{n} a_{i_k} \chi^{\mathbb{R}}_{i_k}\right) \in \mathfrak{F}(X); \quad u := \big((a_1, i_1), \ldots, (a_n, i_n)\big).$$

If $\mathcal{M}(\mathbf{\Lambda}(X)) := (X, I, \mu)$ is a pre-measure space, then $\mathcal{M}(\mathbf{\Lambda}(X))$ induces the pre-integration space $\mathcal{L}(\mathbf{\Lambda}(X)) := (X, S(I, \mathbf{\Lambda}(X)), \int_\mu)$, where

$$\int_\mu \colon S(I, \mathbf{\Lambda}(X)) \to \mathbb{R}, \quad (n, u) \mapsto \int_\mu (n, u),$$

$$\int_\mu (n, u) := \sum_{k=1}^{n} a_k \mu(i_k); \quad u := \big((a_1, i_1), \ldots, (a_n, i_n)\big).$$

The many steps of this involved proof of Bishop and Cheng, appropriately translated into the predicative framework of BST, are found in [129], pp. 34–45.

7.6 Notes

Note 7.6.1. The set of Borel sets generated by a given family of complemented subsets of a set X, with respect to a set Φ of real-valued functions on X, was introduced in [9], p. 68. This set is inductively defined and plays a crucial role in providing important examples of measure spaces in Bishop's measure theory developed in [9]. As this measure theory was replaced in [19] by the Bishop-Cheng measure theory, an enriched version of [18] that made no use of Borel sets, the Borel sets were somehow "forgotten" in the constructive literature. In the introduction of [18], Bishop and Cheng explained why they consider their new measure theory "much more natural and powerful theory". They do admit though, that some results are harder to prove (see [18], p. v). As it is also noted in [120], p. 25, the Bishop-Cheng measure theory is highly impredicative, while Bishop's measure theory in [9] is highly predicative. This fact makes the original Bishop-Cheng measure theory hard to implement in some functional-programming language, a serious disandvantage from the computational point of view. This is maybe why, later attempts to develop constructive measure theory were done within an abstract algebraic framework (see [38], [41] and [121]). Despite the above history of measure theory within Bishop-style constructive mathematics the set of Borel sets is interestingly connected to the theory of Bishop spaces.

Note 7.6.2. The definition of $\mathtt{Borel}(\mathbf{\Lambda}(X))$ is given by Bishop in [9], p. 68, although a rough notion of a family of complemented subsets is used, condition ($\mathtt{Borel}_3$) is not mentioned, and F is an arbitrary subset of $\mathbb{F}(X)$, and not necessarily a Bishop topology. If we want to avoid the extensionality of $\mathtt{Borel}(\mathbf{\Lambda}(X))$, we need to introduce a "pseudo"-membership condition

$$\boldsymbol{A} \mathrel{\dot{\in}} \mathtt{Borel}(\mathbf{\Lambda}(X)) :\Leftrightarrow \exists_{\boldsymbol{B} \in \mathtt{Borel}(\mathbf{\Lambda}(X))}\big(\boldsymbol{A} =_{\mathcal{P}^{\rrbracket\llbracket_F}(X)} \boldsymbol{B}\big).$$

A similar condition is necessary, if we want to avoid extensionality in the definition the least Bishop topology $\bigvee F_0$. Such an approach though, is not practical, and not compatible to the standard practice to study extensional subsets of sets. The quantification over $\mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$ is not equivalent to the quantification over the class $\mathcal{P}^{\rrbracket\llbracket_F}(X)$, as in order to define a family in $\mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$, we need to have *already constructed* an F-complemented subset of X. I.e., an element of $\mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$ is generated by an already constructed, or given element of $\mathcal{P}^{\rrbracket\llbracket_F}(X)$, and not from an abstract element of it. Recall that we never define an assignment routine from a class, like $\mathcal{P}^{\rrbracket\llbracket_F}(X)$, to a set like $\mathtt{Fam}(\mathbb{1}, F, \boldsymbol{X})$.

Note 7.6.3. The notion of a least Bishop topology generated by a given set of function from X to $\mathbb{R}$, together with the set of Borel sets generated by a family of complemented subsets of X, are the main two inductively defined concepts found in [9]. The difference between the two inductive definitions is non-trivial. The first is the inductive definition of a subset of $\mathbb{F}(X)$, while the second is the inductive definition of a subset of the class $\mathcal{P}^{\rrbracket\llbracket_F}(X)$.

Note 7.6.4. As Bishop remarks in [9], p. 69, the proof of Proposition 7.1.5(iii) rests on the property of F that $\big(\frac{1}{n} - f\big) \in F$, for every $f \in F$ and $n \geq 1$. If we define similarly the Borel sets generated by any set of real-valued functions Θ on X, then we can find Θ such that $\mathtt{Borel}(\Theta)$ is closed under complements without satisfying the condition $f \in \Theta \Rightarrow \big(\frac{1}{n} - f\big) \in \Theta$. Such a set is $\mathbb{F}(X, \mathbb{2})$. In this case we have that

$$\boldsymbol{o}_{\mathbb{F}(X,\mathbb{2})}(f) := \big([f = 1], [f = 0]\big) \quad \& \quad -\boldsymbol{o}_{\mathbb{F}(X,\mathbb{2})}(f) = \boldsymbol{o}_{\mathbb{F}(X,\mathbb{2})}(1 - f).$$

Hence, the property mentioned by Bishop is sufficient, but not necessary.

Note 7.6.5. A measure space is defined in [19], p. 282, and a complete measure space in [19], p. 289. These definitions appeared first in [18] p. 47 and p. 55, respectively[7].

Bishop-Cheng definition of a measure space. A measure space is a triplet (X, M, μ) consisting of a nonvoid set X with an inequality $\neq$, a set M of complemented sets in X, and a mapping μ of M into $\mathbb{R}^{0+}$, such that the following properties hold.

(BCMS_1) If A and B belong to M, then so do $A \vee B$ and $A \wedge B$, and $\mu(A) + \mu(B) = \mu(A \vee B) + \mu(A \wedge B)$.

(BCMS_2) If A and $A \wedge B$ belong to M, then so does $A - B$, and $\mu(A) = \mu(A \wedge B) + \mu(A - B)$.

(BCMS_3) There exists A in M such that $\mu(A) > 0$.

(BCMS_4) If (A_n) is a sequence of elements of M such that $\lim_{k\to\infty} \mu\big(\bigwedge_{n=1}^{k} A_n\big)$ exists and is positive, then $\bigcap_n A_n^1$ is nonvoid.

We then call μ the *measure*, and the elements of M the *integrable sets*, of the measure space (X, M, μ). For each A in M the nonnegative number $\mu(A)$ is called the *measure* of A.

[7] In [18], p. 55, condition (BCM_1) appears in the equivalent form: if B is an element of M such that $B^1 \subseteq A^1$ and $B^0 \subseteq A^0$, then $A \in M$, where we have used the terminology that corresponds to the formulation of (BCM_1) in the definition of Bishop-Cheng.

Bishop-Cheng definition of a complete measure space. A measure space (X, M, μ) is *complete* if the following three conditions hold.

(BCCMS_1) If A is a complemented set, and B is an element of M such that $\chi_A = \chi_B$ on $B^1 \cup B^0$, then $A \in M$.

(BCCMS_2) If (A_n) is a sequence of elements of M such that

$$l := \lim_{N \to \infty} \mu\bigg(\bigvee_{n=1}^{N} A_n \bigg)$$

exists, then $\bigvee_n A_n$ belongs to M and has measure l.

(BCCMS_3) If A is a complemented set, and if B, C are elements of M such that $B < A < C$ and $\mu(B) = \mu(C)$, then $A \in M$.

As there is no indication of indexing in the description of M, the Bishop-Cheng definition of a measure space seems to employ the powerset axiom in the formulation of M. The powerset axiom is clearly used in (BMS_1) and BCM_3.

Note 7.6.6. The following definition of Bishop is given in [9], p. 183.

Bishop definition of a measure space. Let F be a nonvoid family of real-valued functions on a set X, such that $\epsilon - f \in F$ whenever $\epsilon > 0$ and $f \in F$. Let $\mathfrak{F}$ be any family of complemented subsets of X (relative to F), closed with respect to countable unions, countable intersections, and complementation. Let $\mathfrak{M}$ be a subfamily of $\mathfrak{F}$ closed under finite unions, intersections, and differences. Let the function $\mu : \mathfrak{M} \to \mathbb{R}^{0+}$ satisfy the following conditions:

(BMS_1) There exists a sequence $S_1 \subset S_2 \subset \ldots$ of elements of $\mathfrak{M}$ such that[8] $\bigcup_{n=1}^{\infty} S_n = X_0$ and $\lim_{n \to \infty} \mu(A \cap S_n) = \mu(A)$ for all A in $\mathfrak{M}$.

(BMS_2) If $A \in \mathfrak{F}$, and if there exist B and N in $\mathfrak{M}$ such that (i) $\mu(N) = 0$, (ii) $x \in A$ whenever $x \in B - N$, and (iii) $x \in -A$ whenever $x \in -B - N$, then $A \in \mathfrak{M}$ and $\mu(A) = \mu(B)$.

(BMS_3) If $A \in \mathfrak{M}$, if $B \in \mathfrak{F}$, and if $A \cap B \in \mathfrak{M}$, then $A - B \in \mathfrak{M}$ and $\mu(A) = \mu(A - B) + \mu(A \cap B)$.

(BMS_4) We have $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ for all A and B in $\mathcal{M}$.

(BMS_5) For each sequence $\{An\}$ of sets in $\mathfrak{M}$ such that $c := \lim_{n \to \infty} \mu\big(\bigcup_{k=1}^{n} A_k\big)$ [respectively, $c := \lim_{n \to \infty} \mu\big(\bigcap_{k=1}^{n} A_k\big)$] exists, the set $A := \bigcup_{k=1}^{\infty} A_k$ (respectively, $A := \bigcap_{k=1}^{\infty} A_k$) is in $\mathcal{M}$, and $\mu(A) = c$.

(BMS_6) Each A in $\mathfrak{M}$ with $\mu(A) > 0$ is nonvoid.

Then the quintuple $(X, F, \mathfrak{F}, \mathfrak{M}, \mu)$ is called a *measure space*, μ is the *measure*, $\mathfrak{F}$ is the class of *Borel sets*, and $\mathfrak{M}$ is the class of *integrable sets*.

If in Bishop's definition we understand the families of complemented subsets $\mathfrak{F}$ and $\mathfrak{M}$ as indexed families $(\boldsymbol{A})_{i \in I}, (\boldsymbol{A})_{j \in J}$ over some sets I and J, respectively, with $J \subseteq I$, then the quantifications involved in the clauses of Bishop's definition are over I and J, and not over some class. Since in [9], p. 65, a family of subsets of X is defined as an appropriate set-indexed family of sets, Bishop's first definition of measure space is predicative.

Note 7.6.7. Regarding the exact definition of a measure space within the formal system Σ introduced by Bishop in [12], Bishop writes in [12], p. 67, the following:

[8] Bishop here means $\bigcup_{n=1}^{\infty} S_n = X$.

> To formalize in Σ the notion of an abstract measure space, definition 1 of chapter 7 of [9] must be rewritten as follows. A *measure space* is a family $\mathcal{M} \equiv \{A_t\}_{t \in T}$ of complemented subsets of a set X relative to a certain family $\mathcal{F}$ of real-valued function on X, a map $\mu : T \to \mathbb{R}^{0+}$, and an additional structure as follows: The void set $\emptyset$ is an element A_{t_0} of $\mathcal{M}$, and $\mu(t_0) = 0$. If s and t are in T, there exists an element $s \vee t$ of T such that $A_{s \vee t} < A_s \cup A_t$. Similarly, there exist operations $\wedge$ and $\sim$ on T, corresponding to the set theoretic operations $\cap$ and $-$. The usual algebraic axioms are assumed, such as $\sim(s \vee t) = \sim s \wedge \sim t$. Certain measure-theoretic axioms, such as $\mu(s \vee t) + \mu(s \wedge t) = \mu(s) + \mu(t)$, are also assumed. Finally, there exist operations $\vee$ and $\wedge$. If, for example, $\{t_n\}$ is a sequence such that $C \equiv \lim_{k \to \infty} \mu(t_1 \vee \ldots \vee t_k)$ exists, then $\vee\{t_n\}$ is an element of T with measure C. Certain axioms for $\vee$ and $\wedge$ are assumed. If T is the family of measurable sets of a compact space relative to a measure μ, and the set-theoretic function $\mu : T \to \mathbb{R}^{0+}$ and the associated operations are defined as indicated above, the result is a measure space in the sense just described.
>
> Considerations such as the above indicate that essentially all of the material in [9], appropriately modified, can be comfortably formalised in Σ.

The expression $A_{s \vee t} < A_s \cup A_t$ is probably a typo (it is the writing $A_{s \vee t} = A_s \cup A_t$, which expresses the “weak belongs to” relation for $\lambda_0 I$). Bishop does not mention that $\mathcal{M}$ is a set of complemented subsets of X, he only says that it is a family of such sets. This is not the case in [19], p. 282. This explanation given by Bishop regarding the explicit and unfolded writing of many of the definitions in constructive mathematics refer to [9]. I have found no similar comment of Bishop with respect to his later measure theory, developed with Cheng. Moreover, I have found no such comment in the extensive work of Chan on Bishop-Cheng measure and probability theory.

Note 7.6.8. In [80], p. 354, Myhill criticised Bishop for using a set of subsets M in the definition of a measure space, hence, according to Myhill, Bishop used the powerset axiom. Since M is an I-set of subsets of X, in the sense described in section 4.6, Myhill's critique is not correct. Bishop's exaplanation in the previous extract is also a clear reply to a critique like Myhill's. Notice that Myhill's paper [80] refers only to [9], and it does not mention [12], which includes Bishop's clear explanation. This is quite surprising, as Myhill's paper, received in January 1974, was surely written after the publication of [66], in which Bishop's paper [12] is included and Myhill is one of its three editors! Myhill's critique would be correct, if he was referring to the Bishop-Cheng measure space defined in [18], a work published quite some time before Myhill submit [80]. Myhill though, does not refer to [18] in [80].

Note 7.6.9. Definition 7.4.3 is the explicit writing within BST of the corresponding definition in [19], pp. 216–217.

Note 7.6.10. The following definition is given in [18], p. 2, and it is repeated in [19], p. 217.

Bishop-Cheng definition of an integration space. A triplet (X, L, I) is an integration space if X is a nonvoid set with an inequality $\neq$, L is a subset of $\mathfrak{F}(X)$ (this set is $\mathfrak{F}^{se}(X)$ in our terminology), and I is a mapping of L into $\mathbb{R}$ such that the following properties hold.

(BCIS_1) If $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$, $|f|$, and $f \wedge 1$ belong to L, and $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.

(BCIS_2) If $f \in L$ and (f_n) is a sequence of nonnegative functions in L such that $\sum_n I(f_n)$ converges and $\sum_n I(f_n) < I(f)$, then there exists $x \in X$ such that $\sum_n f_n(x)$ converges and $\sum_n f_n(x) < f(x)$.

(BCIS_3) There exists a function p in L with $I(p) = 1$.

(BCIS_4) For each f in L, $\lim_{n\to\infty} I(f \wedge n) = I(f)$ and $\lim_{n\to\infty} I(|f| \wedge n^{-1}) = 0$.

The notion of an integration space is a constructive version of the Daniell integral, introduced in [43]. The Bishop-Cheng definition of an integration space is impredicative, as the class $\mathfrak{F}(X)$ is treated as a set. The notion of a subset is defined only for sets, and L is considered a subset of $\mathfrak{F}(X)$. The extensional character of L is also not addressed. This impredicative approach to L is behind the simplicity of the Bishop-Cheng definition. E.g., in (BCIS_4) the formulation of the limit is immediate as the terms $f \wedge n \in L$ and I is defined on L. In Definition 7.5.1 though, we need to use an element $\alpha(n)$ of the index-set I such that $f \wedge n =_{\mathfrak{F}(X)} f_{\alpha(n)}$, in order to express the limit.

Note 7.6.11. The Bishop-Cheng definition of the "set" L^1 (or L^p, where $p \geq 1$) of *integrable functions* is also impredicative, as it rests on the use of the totality $\mathfrak{F}^{se}(X)$ as a set (see Definition (2.1) in [19], p. 222). In [129], pp. 49–60, the pre-integration space L^1 of *canonically* integrable functions is studied instead within BST, as the completion of an integration space. The set L^1 is predicatively defined in [9], p. 190, as an integrable function is an appropriate measurable function, which is defined using quantification over the set-indexed family $\mathfrak{M}$ of integrable sets in a Bishop measure space (see Note 7.6.6).

Chapter 8

Epilogue

8.1 BST between dependent type theory and category theory

In this Thesis we tried to show how the elaboration of the notion of a set-indexed family of sets within BST expands the range of BISH both in its foundation and its practice. Chapters 2-5 are concerned with the foundations of BISH, and chapters 6 and 7 with the practice of BISH.

Chapter 2 presents the set-like objects, the families of which are studied later: sets, subsets, partial functions, and complemented subsets. Operations between these objects generate corresponding operations between their families and family-maps. Chapter 3 includes the fundamental notions and results about set-indexed families of sets. A family of sets $\Lambda \in \mathtt{Fam}(I)$, together with its $\sum$- and $\prod$-set, and a family map $\Psi \colon \Lambda \Rightarrow M$, are examples of notions with a strong type-theoretic, or categorical flavour, depending on the point of observation view. This is not accident, as MLTT was motivated by Bishop's book [9]. Moreover, BST can roughly be described as a fundamental informal theory of totalities and assignment routines, and (informal) category theory as a fundamental (informal) theory of objects and arrows. A fundamental similarity between BST and MLTT is the explicit use of dependency, which is suppressed in category theory. The fundamental categorical concepts of a functor and a natural transformation, which are translated within BST as an I-family of sets and a family-map between I-families of sets, have an immediate and explicit formulation within dependent type theory or within BST (see Note 3.11.13). The formulation of dependency though, within category theory is much more involved (see e.g., [85]). On the other hand, a fundamental similarity between BST and category theory is the use of definitions that do not "force" facts and results, as in the case of MLTT and its recent extension HoTT. While the language of MLTT is clearly closer to BST, a large part of pure category theory, the size of the totalities involved excluded, follows the "pattern" of doing constructive mathematics in the style of BISH: all notions are defined, no powerful axioms are used, and despite the generality in the categorical formulations, most results have a concrete algorithmic[1] meaning.

The interconnections between category theory and dependent type theory is a standard theme behind foundational studies on mathematics and theoretical computer science the last forty years. The recent explosion of univalent foundations, spearheaded by the Fields

[1] The question of the constructive character of general category theory is addressed in [75]. There constructivism in mathematics is identified with Brouwer's intuitionism. The inclusion of Bishop-style constructivism and of type-theoretic constructivism in the interpretation of mathematical constructivism is necessary and sheds more light on the original question.

medalist Vladimir Voevodsky, regenerated the study of these interconnections. The appropriate categorical understanding of the univalence axiom brought category-theorists and type-theorists even closer. BST seems to be in some kind of common territory between dependent type theory and category theory. It also features simultaneously the proof-irrelevance of category theory and classical mathematics, and the proof-relevance of MLTT. In contrast to HoTT, where a type A has a rich space-structure due to the induction principle corresponding to the introduction of the identity family $=_A : A \to A \to \mathcal{U}$ on A, the notion of space in BISH, as in classical set-based mathematics, is not identical to that of a set. This is also captured in category theory, where the category of sets behaves differently from the category of topological spaces. We need to add, by definition, extra structure to a set X, in order to acquire a non-trivial space structure. In this Thesis the concept of space considered was that of a Bishop space. This is one option, which is shown to be very fruitful, if we work within BISH*, but it is not the only one.

As in the case of MLTT or HoTT, a non-trivial part of category theory can be studied *within* BST. We gave a glimpse of that in Note 3.11.13. Working in a similar fashion, most of the theory of small categories can appropriately be translated into BST. This modelling of pure category theory "suffers", as any modelling, from the inclusion of features, like conditions $(\mathrm{Cat}_3), (\mathrm{Cat}_4)$, and (Funct_3), that depend on the system BST itself and are not part of the original theory. In any event, such a translation is not meant to be an attempt to replace pure category theory, but to embed into BISH concepts and facts from category theory useful to the practice of BISH. For example, all categorical notions and facts of constructive algebra presented in [76], within a category theory irrelevant to the version of Bishop's theory of sets underlying [76], can, in principle, be approached within BST and the corresponding category theory within BST. Unfolding proof-relevance in BISH through BST, categorical facts, like the Yoneda lemma for $\mathtt{Fam}(\widehat{I})$ in section 5.4, are translated from $\mathrm{MLTT} + \mathrm{FunExt}$ to BISH. It remains to find though, interesting applications of such results to BISH.

Inductive definitions bring the language of BISH* closer to dependent type theory. The induction principles that accommodate inductive definitions in the latter correspond to universal properties in category theory. The formalisation of BST, and its possible extension BST* with inductive definitions with rules of countably many premises, is an important open problem. The natural requirement for a faithful and adequate formal system for BST and BST* makes the choice of the formal framework even more difficult. It seems that a version of a formal version of extensional Martin-Löf type theory, and the corresponding theory of setoids within it, is a formal system very close to the informal system BST. As we have explained in Note 1.3.2, a formal version of intensional MLTT does not seem to be a faithful formal system for the informal theory BISH. The logical framework of an extensional version of dependent type theory though, and the identification of propositions with types, is quite far from the usual practice of BISH, which is, in this respect, close to the standard practice of classical mathematics. It is natural to search for a formal system of BST where logic is not built in, as in MLTT, and which reflects the way sets are defined in BST. We hope that the presentation of BST in this Thesis will be helpful to the construction of such a formal counterpart.

Category theory can also be very helpful to the formulation of the properties of Bishop sets and functions. The work of Palmgren [83] on the categorical properties of the category of setoids and setoid maps within intensional MLTT is expected to be very useful to this. A similar formulation of the categorical properties of the theory of setoids and setoid maps within extensional MLTT could be even closer to the formulation of the categorical properties of Bishop sets and functions.

8.2 Further open questions and future tasks

We collect here some further open questions and future tasks stemming form this Thesis.

1. To develop the theory of neighbourhood spaces using the notion of a neighbourhood family of subsets of a set X that covers X (see Note 4.11.2).
2. Is it possible to use families of complemented subsets to describe a neighbourhood space? The starting idea is to assign to each $i \in I$ a complemented subset $\boldsymbol{\nu}_0(i) := \big(\nu_0^1(i), \nu_0^0(i)\big)$ of X, such that $\nu_0^1(i)$ is open and $\nu_0^0(i)$ is closed. The benefit of such an approach to constructive topology is that the classical duality between open and closed sets is captured constructively. E.g., the 1-component of the complement $-\boldsymbol{\nu}_0(i) := \big(\nu_0^0(i), \nu_0^1(i)\big)$ of $\boldsymbol{\nu}_0(i)$ is a closed set and its 0-component is an open set.
3. Can we use complemented subsets of $\mathbb{N}$ in a constructive reconstruction of recursion theory, instead of just subsets of $\mathbb{N}$? This question is inspired from the work of Nemoto on recursion theory within intuitionistic logic.
4. To explore further the notion of an impredicative set, and the hierarchy mentioned in Note 3.11.12.
5. To find interesting purely mathematical applications of set-relevant families of sets and of families of families of sets.
6. To investigate the possibility of a BHK-interpretation of a negated formula (see Note 5.7.7).
7. To develop a (predicative) theory of ordinals within BST.
8. To study families of sets with a proof-relevant equality over an index-set with a proof-relevant equality.
9. To translate more notions and results from MLTT and HoTT to BISH through BST. As a special case, to translate higher inductive types (HITs) into BISH, other than the truncation $||A||$ of A. If we work directly with a space in BST i.e., with a Bishop space $\mathcal{F} := (X, F)$, and not with an arbitrary type, as in HoTT, we can define within BST notions like the cone and the suspension of $\mathcal{F}$. If $\mathbb{I}_{01} := [0,1]$, we call $\mathbb{I}_{01}^{\cup} := \{0\} \cup (0,1) \cup \{1\}$ the *pseudo-interval* $[0,1]$. To $\mathbb{I}_{01}^{\cup}$ we can associate the least Bishop topology generated by the restriction of the identity map to it. The relation $\sim_X^{\tau}$ on $X \times \mathbb{I}_{01}^{\cup}$, defined by
$$(x,i) \sim_X^{\tau} (x',i') :\Leftrightarrow \big(i, i' \in \{0\} \cup (0,1) \ \&\ i =_{\mathbb{I}_{01}^{\cup}} i' \ \&\ x =_X x'\big) \ \text{ or } \ i =_{\mathbb{I}_{01}^{\cup}} 1 =_{\mathbb{I}_{01}^{\cup}} i',$$
is an extensional equivalence relation. If $Y := X \times \mathbb{I}_{01}^{\cup}$ and $\tau_0^X : Y \to \tau_0^X Y$ is the function that maps (x,i) to its equivalence class (see section 4.7), then $\tau_0^X Y(Y)$, equipped with an an appropriate Bishop topology, is the cone of $\mathcal{F}$. For the suspension of $\mathcal{F}$ we work similarly.
10. To find interesting mathematical applications of $(-2), (-1)$- and 0-sets in BISH.
11. To elaborate the study of category theory within BST. So far we have formulated within BST most of the category theory formulated within the Calculus of Inductive Constructions in [61].
12. To develop along the lines of Chapter 6 the theory of spectra of other structures, like groups, rings, modules etc.

13. To develop further the theory of Borel sets of a Bishop topology. E.g., to find the exact relation between the Borel sets $\mathtt{Borel}(\mathcal{F})$ and $\mathtt{Borel}(\mathcal{G})$ and the Borel sets $\mathtt{Borel}(\mathcal{F} \times \mathcal{G})$ of the product Bishop space $\mathcal{F} \times \mathcal{G}$. And similarly for all important constructions of new Bishop spaces from given ones.
14. To formulate various parts of the constructive algebra developed in [76] and [68] within BST.
15. To elaborate the theory of (pre-)measure spaces and (pre-)integrations spaces. The past work [129] and the forthcoming work [102] are in this direction.
16. To approach Chan's probability theory in [35], which is within BCMT, through a predicative reconstruction of BCMT within BST.

Chapter 9

Appendix

9.1 Bishop spaces

We present the basic notions and facts on Bishop spaces that are used in the previous sections. For all concepts and results from constructive real analysis that we use here without further explanation we refer to [19]. For all proofs that are not included in this section we refer to [88]. We work within the extension BISH^* of BISH with inductive definitions with rules of countably many premises. A Bishop space is a constructive, function-theoretic alternative to the classical notion of a topological space, and a Bishop morphism is the corresponding function-theoretic notion of "continuous function" between Bishop spaces.

Definition 9.1.1. *If X is a set and $\mathbb{R}$ is the set of real numbers, we denote by $\mathbb{F}(X)$ the set of functions from X to $\mathbb{R}$, and by $\mathrm{Const}(X)$ the subset of $\mathbb{F}(X)$ of all constant functions on X. If $a \in \mathbb{R}$, we denote by $\overline{a}^X$ the constant function on X with value a. We denote by $\mathbb{N}^+$ the set of non-zero natural numbers. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called Bishop continuous, or simply continuous, if for every $n \in \mathbb{N}^+$ there is a function $\omega_{\phi,n} : \mathbb{R}^+ \to \mathbb{R}^+$, $\epsilon \mapsto \omega_{\phi,n}(\epsilon)$, which is called a modulus of continuity of ϕ on $[-n, n]$, such that the following condition is satisfied*

$$\forall_{x,y\in[-n,n]}(|x - y| < \omega_{\phi,n}(\epsilon) \Rightarrow |\phi(x) - \phi(y)| \leq \epsilon),$$

for every $\epsilon > 0$ and every $n \in \mathbb{N}^+$. We denote by $\mathrm{Bic}(\mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, which is equipped with the equality inherited from $\mathbb{F}(\mathbb{R})$.

Note that we could have defined the modulus of continuity $\omega_{\phi,n}$ as a function from $\mathbb{N}^+$ to $\mathbb{N}^+$. Clearly, a continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous on every bounded subset of $\mathbb{R}$. The latter is an impredicative formulation of uniform continuity, since it requires quantification over the class of all subsets of $\mathbb{R}$. The formulation of uniform continuity in the Definition 9.1.1 though, is predicative, since it requires quantification over the sets $\mathbb{N}^+, \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ and $[-n, n]$.

Definition 9.1.2. *If X is a set, $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, let*

$$U(X; f, g, \epsilon) :\Leftrightarrow \forall_{x\in X}\big(|g(x) - f(x)| \leq \epsilon\big),$$

$$U(X; \Phi, f) :\Leftrightarrow \forall_{\epsilon>0}\exists_{g\in\Phi}\big(U(f, g, \epsilon)\big).$$

If the set X is clear from the context, we write simpler $U(f, g, \epsilon)$ and $U(\Phi, f)$, respectively. We denote by Φ^* the bounded elements of Φ, and its uniform closure $\overline{\Phi}$ is defined by

$$\overline{\Phi} := \{f \in \mathbb{F}(X) \mid U(\Phi, f)\}.$$

A Bishop topology on X is a certain subset of $\mathbb{F}(X)$. Since the Bishop topologies considered here are all extensional subsets of $\mathbb{F}(X)$, we do not mention the embedding $i_F^{\mathbb{F}(X)} \colon F \hookrightarrow \mathbb{F}(X)$, which is given in all cases by the identity map-rule.

Definition 9.1.3. *A Bishop space is a pair $\mathcal{F} := (X, F)$, where F is an extensional subset of $\mathbb{F}(X)$, which is called a Bishop topology, or simply a topology of functions on X, that satisfies the following conditions:*

(BS_1) *If $a \in \mathbb{R}$, then $\overline{a}^X \in F$.*

(BS_2) *If $f, g \in F$, then $f + g \in F$.*

(BS_3) *If $f \in F$ and $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi \circ f \in F$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F \ni \phi \circ f"'] & \mathbb{R} \arrow[d, "\phi \in \mathrm{Bic}(\mathbb{R})"] \\ & \mathbb{R}. \end{tikzcd}$$

(BS_4) *$\overline{F} = F$.*

If $\mathcal{F} := (X, F)$ is a Bishop space, then $\mathcal{F}^* := (X, F^*)$ is the Bishop space of bounded elements of F. The constant functions $\mathrm{Const}(X)$ is the *trivial* topology on X, while $\mathbb{F}(X)$ is the *discrete* topology on X. Clearly, if F is a topology on X, then $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$, and the set of its bounded elements F^* is also a topology on X. It is straightforward to see that the pair $\mathcal{R} := (\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$ is a Bishop space, which we call the *Bishop space of reals*. A Bishop topology F is a ring and a lattice; since $|\mathrm{id}_{\mathbb{R}}| \in \mathrm{Bic}(\mathbb{R})$, where $\mathrm{id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$, by BS_3 we get that if $f \in F$ then $|f| \in F$. By BS_2 and BS_3, and using the following equalities

$$f \cdot g = \frac{(f+g)^2 - f^2 - g^2}{2} \in F,$$

$$f \vee g = \max\{f, g\} = \frac{f + g + |f - g|}{2} \in F,$$

$$f \wedge g = \min\{f, g\} = \frac{f + g - |f - g|}{2} \in F,$$

we get similarly that if $f, g \in F$, then $f \cdot g, f \vee g, f \wedge g \in F$. Turning the definitional clauses of a Bishop topology into inductive rules, Bishop defined in [9], p. 72, the least topology including a given subbase F_0. This inductive definition, which is also found in [19], p. 78, is crucial to the definition of new Bishop topologies from given ones.

Definition 9.1.4. *The Bishop closure of F_0, or the least topology $\bigvee F_0$ generated by some $F_0 \subseteq \mathbb{F}(X)$, is defined by the following inductive rules:*

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}, \quad \frac{a \in \mathbb{R}}{\overline{a}^X \in \bigvee F_0}, \quad \frac{f, g \in \bigvee F_0}{f + g \in \bigvee F_0}, \quad \frac{f \in \bigvee F_0, \ g =_{\mathbb{F}(X)} f}{g \in \bigvee F_0},$$

$$\frac{f \in \bigvee F_0, \ \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \bigvee F_0}, \qquad \frac{\big(g \in \bigvee F_0, \ U(f, g, \epsilon)\big)_{\epsilon > 0}}{f \in \bigvee F_0}.$$

We call $\bigvee F_0$ the Bishop closure of F_0, and F_0 a subbase of $\bigvee F_0$.

If F_0 is inhabited, then (BS_1) is provable by (BS_3). The last, most complex rule above can be replaced by the rule

$$\frac{g_1 \in \bigvee F_0 \ \wedge \ U\big(f, g_1, \frac{1}{2}\big), \quad g_2 \in \bigvee F_0 \ \wedge \ U\big(f, g_2, \frac{1}{2^2}\big), \ \ldots}{f \in \bigvee F_0},$$

a rule with countably many premisses. The corresponding induction principle $\mathrm{Ind}_{\bigvee F_0}$ is

$$\Big[\forall_{f_0 \in F_0}\big(P(f_0)\big) \ \& \ \forall_{a \in \mathbb{R}}\big(P(\overline{a}^X)\big) \ \& \ \forall_{f,g \in \bigvee F_0}\big(P(f) \ \& \ P(g) \Rightarrow P(f+g)\big)$$

$$\& \ \forall_{f \in \bigvee F_0} \forall_{g \in \mathbb{F}(X)}\big(g =_{\mathbb{F}(X)} f \Rightarrow P(g)\big)$$
$$\& \ \forall_{f \in \bigvee F_0} \forall_{\phi \in \mathrm{Bic}(\mathbb{R})}\big(P(f) \Rightarrow P(\phi \circ f)\big)$$

$$\& \ \forall_{f \in \bigvee F_0}\big(\forall_{\epsilon > 0} \exists_{g \in \bigvee F_0}(P(g) \ \& \ U(f, g, \epsilon)) \Rightarrow P(f)\big)\Big]$$

$$\Rightarrow \forall_{f \in \bigvee F_0}\big(P(f)\big),$$

where P is any bounded formula. Next we define the notion of a Bishop morphism between Bishop spaces. The Bishop morphisms are the arrows in the category of Bishop spaces **Bis**.

Definition 9.1.5. *If $\mathcal{F} := (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a function $h : X \to Y$ is called a Bishop morphism, if $\forall_{g \in G}(g \circ h \in F)$*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g \circ h"'] & Y \arrow[d, "g \in G"] \\ & \mathbb{R}. \end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of Bishop morphisms from $\mathcal{F}$ to $\mathcal{G}$. As F is an extensional subset of $\mathbb{F}(X)$, $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is an extensional subset of $\mathbb{F}(X, Y)$. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, the induced mapping $h^* \colon G \to F$ from h is defined by the rule

$$h^*(g) := g \circ h; \qquad g \in G.$$

If $\mathcal{F} := (X, F)$ is a Bishop space, then $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$, and one can show inductively that if $\mathcal{G} := (Y, \bigvee G_0)$, then $h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ if and only if $\forall_{g_0 \in G_0}(g_0 \circ h \in F)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "F \ni g_0 \circ h"'] & Y \arrow[d, "g_0 \in G_0"] \\ & \mathbb{R}. \end{tikzcd}$$

We call this fundamental fact the $\bigvee$-*lifting of morphisms*. A Bishop morphism is a *Bishop isomorphism*, if it is an isomorphism in the category **Bis**. We write $\mathcal{F} \simeq \mathcal{G}$ to denote that $\mathcal{F}$ and $\mathcal{G}$ are Bishop isomorphic. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is a bijection, then h is a Bishop isomorphism if and only if it is *open* i.e., $\forall_{f \in F} \exists_{g \in G}\big(f = g \circ h\big)$.

Definition 9.1.6. *Let $\mathcal{F} := (X, F), \mathcal{G} := (Y, G)$ be Bishop spaces, and $(A, i_A) \subseteq X$ inhabited. The product Bishop space $\mathcal{F} \times \mathcal{G} := (X \times Y, F \times G)$ of $\mathcal{F}$ and $\mathcal{G}$, the relative Bishop space $\mathbb{F}_{|A} := (A, F_{|A})$ on A, and the pointwise exponential Bishop space $\mathcal{F} \to \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \to G)$ are defined, respectively, by*

$$F \times G := \bigvee [\{f \circ \mathtt{pr}_X, \mid f \in F\} \cup \{g \circ \mathtt{pr}_Y \mid g \in G\}] =: \bigvee_{f \in F}^{g \in G} f \circ \mathtt{pr}_X, g \circ \mathtt{pr}_Y,$$

$$F_{|A} = \bigvee \{f_{|A} \mid f \in F\} =: \bigvee_{f \in F} f_{|A}$$

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[rr, bend right, "f_{|A}"'] & X \arrow[r, "f"] & \mathbb{R}, \end{tikzcd}$$

$$F \to G := \bigvee \big\{\phi_{x,g} \mid x \in X, g \in G\big\} := \bigvee_{x \in X}^{g \in G} \phi_{x,g},$$

$$\phi_{x,g} : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R}, \quad \phi_{x,g}(h) = g(h(x)); \quad x \in X, \ g \in G.$$

One can show inductively the following $\bigvee$-liftings

$$\begin{aligned} \bigvee F_0 \times \bigvee G_0 &:= \bigvee [\{f_0 \circ \mathtt{pr}_X, \mid f_0 \in F_0\} \cup \{g_0 \circ \mathtt{pr}_Y \mid g_0 \in G_0\}] \\ &=: \bigvee_{f_0 \in F_0}^{g_0 \in G_0} f_0 \circ \mathtt{pr}_X, g_0 \circ \mathtt{pr}_Y, \end{aligned}$$

$$\big(\bigvee F_0\big)_{|A} = \bigvee \{f_{0|A} \mid f_0 \in F_0\} =: \bigvee_{f_0 \in F_0} f_{0|A},$$

$$F \to \bigvee G_0 = \bigvee \big\{\phi_{x,g_0} \mid x \in X, g_0 \in G_0\big\} := \bigvee_{x \in X}^{g_0 \in G_0} \phi_{x,g_0}.$$

The relative topology F_A is the least topology on A that makes i_A a Bishop morphism, and the product topology $F \times G$ is the least topology on $X \times Y$ that makes the projections $\mathtt{pr}_X$ and $\mathtt{pr}_Y$ Bishop morphisms. The term pointwise exponential Bishop topology is due to the fact that $F \to G$ behaves like the the classical topology of the pointwise convergence on $C(X, Y)$, the set of continuous functions from the topological space X to the topological space Y.

9.2 Directed sets

Definition 9.2.1. *Let I be a set and $i \preccurlyeq_I j$ a binary extensional relation on I i.e.,*

$$\forall_{i,j,i',j' \in I}\big(i =_I i' \ \& \ j =_I j' \ \& \ i \preccurlyeq_I j \Rightarrow i' \preccurlyeq_I j'\big).$$

If $i \preccurlyeq_I j$ is reflexive and transitive, then $(I, \preccurlyeq_I)$ is called a a preorder. We call a preorder $(I, \preccurlyeq_I)$ a directed set, and inverse-directed, respectively, if

$$\forall_{i,j \in I} \exists_{k \in I}\big(i \preccurlyeq_I k \ \& \ j \preccurlyeq_I k\big),$$

$$\forall_{i,j \in I} \exists_{k \in I} \big(i \succcurlyeq_I k \ \& \ j \succcurlyeq_I k\big).$$

The covariant covariant diagonal $D^{\preccurlyeq}(I)$ *of* $\preccurlyeq_I$*, the contravariant diagonal* $D^{\succcurlyeq}(I)$ *of* $\preccurlyeq_I$*, and the* $\preccurlyeq_I$*-upper set* $I_{ij}^{\preccurlyeq}$ *of* $i, j \in I$ *are defined, respectively, by*

$$D^{\preccurlyeq}(I) := \big\{(i,j) \in I \times I \mid i \preccurlyeq_I j\big\},$$

$$D^{\succcurlyeq}(I) := \big\{(j,i) \in I \times I \mid j \succcurlyeq_I i\big\},$$

$$I_{ij}^{\preccurlyeq} := \{k \in I \mid i \preccurlyeq_I k \ \& \ j \preccurlyeq_I k\}.$$

Since $i \preccurlyeq_I j$ is extensional, $D^{\preccurlyeq}(I), D^{\succcurlyeq}(I)$, and $I_{ij}^{\preccurlyeq}$ are extensional subsets of $I \times I$.

Definition 9.2.2. *Let* $(I, \preccurlyeq_I)$ *be a poset i.e., a preorder such that* $\big[i \preccurlyeq_I j \ \& \ j \preccurlyeq_I i\big] \Rightarrow i =_I j$*, for every* $i, j, \in I$*. A modulus of directedness for* I *is a function* $\delta \colon I \times I \to I$*, such that for every* $i, j, k \in I$ *the following conditions are satisfied:*

(δ_1) $i \preccurlyeq_I \delta(i,j)$ *and* $j \preccurlyeq_I \delta(i,j)$*.*

(δ_2) *If* $i \preccurlyeq_I j$*, then* $\delta(i,j) =_I \delta(j,i) =_I j$*.*

(δ_3) $\delta\big(\delta(i,j),k\big) =_I \delta\big(i, \delta(j,k)\big)$*.*

In what follows we avoid for simplicity the use of subscripts on the relation symbols. If $(I, \preccurlyeq)$ is a preordered set and $(J, e) \subseteq I$, where $e : J \hookrightarrow I$, and using for simplicity the same symbol $\preccurlyeq$, if we define $j \preccurlyeq j' :\Leftrightarrow e(j) \preccurlyeq e(j')$, for every $j, j' \in J$, then $(J, \preccurlyeq)$ is only a preordered set. If J is a cofinal subset of I, which classically it is defined by the condition $\forall_{i \in I} \exists_{j \in J} \big(i \preccurlyeq j\big)$, then $(J, \preccurlyeq)$ becomes a directed set. To avoid the use of dependent choice, we add in the definition of a cofinal subset J of I a modulus of cofinality for J.

Definition 9.2.3. *Let* $(I, \preccurlyeq)$ *be a directed set and* $(J, e) \subseteq I$*, and let* $j \preccurlyeq j' :\Leftrightarrow e(j) \preccurlyeq e(j')$*, for every* $j, j' \in J$*. We say that* J *is cofinal in* I*, if there is a function* $\texttt{cof}_J : I \to J$*, which we call a modulus of cofinality of* J *in* I*, that satisfies the following conditions:*

$(\texttt{Cof}_1)$ $\forall_{j \in J} \big(\texttt{cof}_J(e(j)) =_J j\big)$*.*

$$\begin{tikzcd} J \arrow[r, hook, "e"'] \arrow[rr, bend left, "\mathrm{id}_J"] & I \arrow[r, "\texttt{cof}_J"'] & J. \end{tikzcd}$$

$(\texttt{Cof}_2)$ $\forall_{i,i' \in I} \big(i \preccurlyeq i' \Rightarrow \texttt{cof}_J(i) \preccurlyeq \texttt{cof}_J(i')\big)$*.*

$(\texttt{Cof}_3)$ $\forall_{i \in I} \big(i \preccurlyeq e(\texttt{cof}_J(i))\big)$*.*

We denote the fact that J *is cofinal in* I *by* $(J, e, \texttt{cof}_J) \subseteq^{\texttt{cof}} I$*, or, simpler, by*$J \subseteq^{\texttt{cof}} I$*.*

Taking into account the embedding e of J into I, the condition (iii) is the exact writing of the classical defining condition $\forall_{i \in I} \exists_{j \in J} \big(i \preccurlyeq j\big)$. To add the condition (i) is harmless, since $\preccurlyeq$ is reflexive. If we consider the condition (iii) on $e(j)$, for some $j \in J$, then by the condition (i) we get the transitivity $e(j) \preccurlyeq e(\texttt{cof}_J(e(j))) = e(j)$. The condition (ii) is also harmless to add. In the classical setting if $i \preccurlyeq i'$, and $j, j' \in J$ such that $i \preccurlyeq j$ and $i' \preccurlyeq j'$, then there is some $i'' \in I$ such that $j' \preccurlyeq i''$ and $j \preccurlyeq i''$. If $i'' \preccurlyeq j''$, for some $j'' \in J$,

$$\begin{tikzcd}
 & j'' & \\
 & i'' \arrow[u] & \\
j' \arrow[ur] & & \\
i' \arrow[u] & & j \arrow[uul] \\
 & i \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

then $j \preccurlyeq j''$. Since $i' \preccurlyeq j''$ too, the condition (ii) is justified. The added conditions (i) and (ii) are used in the proofs of Theorem 6.5.12 and Lemma 6.5.11(ii), respectively. Moreover, they are used in the proof of Theorem 6.6.5. The extensionality of $\preccurlyeq$ is also used in the proofs of Theorem 6.5.12 and Theorem 6.6.5.

E.g., if $\texttt{Even}$ and $\texttt{Odd}$ denote the sets of even and odd natural numbers, respectively, let $e \colon \texttt{Even} \hookrightarrow \mathbb{N}$, defined by the identity map-rule, and $\texttt{cof}_{\texttt{Even}} \colon \mathbb{N} \to 2\mathbb{N}$, defined by the rule

$$\texttt{cof}_{2\mathbb{N}}(n) := \begin{cases} n & , n \in \texttt{Even} \\ n+1 & , n \in \texttt{Odd}. \end{cases}$$

Then $(\texttt{Even}, e, \texttt{cof}_{\texttt{Even}}) \subseteq \mathbb{N}$.

Remark 9.2.4. *If $(I, \preccurlyeq)$ is a directed set and $(J, e, \texttt{cof}_J) \subseteq^{\texttt{cof}} I$, then $(J, \preccurlyeq)$ is directed.*

Proof. Let $j, j' \in J$ and let $i \in I$ such that $e(j) \preccurlyeq i$ and $e(j') \preccurlyeq i$. Since $i \preccurlyeq e(\texttt{cof}_J(i))$, we get $e(j) \preccurlyeq e(\texttt{cof}_J(i))$ and $e(j') \preccurlyeq e(\texttt{cof}_J(i))$ i.e., $j \preccurlyeq \texttt{cof}_J(i)$ and $j' \preccurlyeq \texttt{cof}_J(i)$. $\square$

Bibliography

[1] P. Aczel, M. Rathjen: *Constructive Set Theory*, book draft, 2010.

[2] S. Awodey: Axiom of Choice and Excluded Middle in Categorical Logic, Bulletin of Symbolic Logic, volume 1, 1995, 344.

[3] S. Awodey: *Category Theory*, Oxford University Press, 2010.

[4] A. Bauer, A. Swan: Every metric space is separable in function realizability, arXiv:1804.00427, 2018.

[5] M. J. Beeson: Formalizing constructive mathematics: why and how, in [105], 1981, 146–190.

[6] M. J. Beeson: Problematic principles in constructive mathematics, in D. van Dalen, D. Lascar, J. Smiley (Eds.) *Logic Colloquium '80*, North-Holland, 1982, 11–55.

[7] M. J. Beeson: *Foundations of Constructive Mathematics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag, 1985.

[8] S. Bhat, A. Agarwal, R. Vuduc: A type theory for probability density functions, POPL'12, ACM, 2012, 545–556.

[9] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[10] E. Bishop: A General Language, unpublished manuscript, 1968(9)?

[11] E. Bishop: How to Compile Mathematics into Algol, unpublished manuscript, 1968(9)?

[12] E. Bishop: Mathematics as a Numerical Language, in [66], 1970, 53–71.

[13] E. Bishop: The Neat Category of Stratified Spaces, unpublished manuscript, University of California, San Diego, 1971.

[14] E. Bishop: Aspects of Constructivism, Notes on the lectures delivered at the Tenth Holiday Mathematics Symposium held at New Mexico State University, Las Cruses, during the period December 27-31, 1972.

[15] E. Bishop: Schizophrenia in Contemporary Mathematics, American Mathematical Society Colloquium Lectures, Missoula University of Montana 1973, and in [108].

[16] E. Bishop: The crisis in contemporary mathematics, Historia Mathematics 2, 1975, 507–517.

[17] *Selected Papers, Errett Bishop*, edited by J. Wermer, World Scientific Publishing, Singapore and Philadelphia, 1986.

[18] E. Bishop and H. Cheng: *Constructive Measure Theory*, Mem. Amer. Math. Soc. 116, 1972.

[19] E. Bishop and D. S. Bridges: *Constructive Analysis*, Grundlehren der math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[20] D. S. Bridges and F. Richman: *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.

[21] D. Bridges, O. Demuth: On the Lebesgue measurability of continuous functions in constructive analysis, Bulletin of the American Mathematical Society, Vol. 24, No. 2, 1991, 259–276.

[22] D. S. Bridges and S. Reeves: Constructive Mathematics in Theory and Programming Practice, Philosophia Mathematica (3), 1999, 65–104.

[23] D. S. Bridges: Constructive mathematics; a foundation for computable analysis, Theoretical Computer Science 219, 1999, 95–109.

[24] D. S. Bridges and L. S. Vîţă: *Techniques of Constructive Analysis*, in: Universitext, Springer, New York, 2006.

[25] D. S. Bridges, L. S. Vîţă: *Apartness and Uniformity: A Constructive Development*, in: CiE series "Theory and Applications of Computability", Springer Verlag, Berlin Heidelberg, 2011.

[26] D. S. Bridges: Reflections on function spaces, Annals of Pure and Applied Logic 163, 2012, 101–110.

[27] G. Cantor: Über unendliche, lineare Punktmannichfaltigkeiten, Nummer 3. Mathematische Annalen, 20, 1882, 113–121.

[28] D. S. Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg (Eds.): *Handbook of Bishop Constructive Mathematics*, in preparation, expected year of publication 2021.

[29] C. Carathéodory: *Mass und Integral und ihre algebraisierung*, Springer Basel AG, 1956.

[30] Y. K. Chan: A constructive approach to the theory of stochastic processes, Transactions of the American Mathematical Society, Vol. 65, 1972, 37–44.

[31] Y. K. Chan: A constructive study of measure theory, Pacific Journal of Mathematics, Vol. 41, No. 1, 1972, 63–79.

[32] Y. K. Chan: A constructive approach to the theory of stochastic processes, Transactions of the American Mathematical Society, Vol. 165, 1972, 37–44.

[33] Y. K. Chan: Notes on constructive probability theory, The Annals of Probability, Vol. 2, No. 1, 1974, 51–75.

[34] Y. K. Chan: A short proof of an existence theorem in constructive measure theory, Proceedings of the American Mathematical Society, Vol. 48, No. 2, 1975, 435–436.

[35] Y. K. Chan: *Foundations of Constructive Probability Theory*, arXiv:1906.01803v2, 2019.

[36] R. L. Constable et al. *Implementing Mathematics with the Nuprl Proof Development 373 System*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1986.

[37] T. Coquand, H. Persson: Integrated Development of Algebra in Type Theory, preprint, 1998.

[38] T. Coquand, E. Palmgren: Metric Boolean algebras and constructive measure theory, Arch. Math. Logic 41, 2002, 687–704.

[39] T. Coquand, P. Dybjer, E. Palmgren, A. Setzer: *Type-theoretic Foundations of Constructive Mathematics*, book-draft, 2005.

[40] T. Coquand, A. Spiwack: Towards Constructive Homological Algebra in Type Theory, in LNCS 4573, 2007, 40–54.

[41] T. Coquand, B. Spitters: Integrals and valuations, Journal of Logic & Analysis, 1:3, 2009, 1–22.

[42] T. Coquand: A remark on singleton types, manuscript, 2014, available at http://www.cse.chalmers.se/∼coquand/singl.pdf, 2014.

[43] P. J. Daniell: A general form of integral, Annals of mathematics, Second Series, 19 (4), 1918, 279–294.

[44] R. Dedekind: *Was sind und was sollen die Zahlen?*, 1. Auflage, Vieweg, Braunschweig, 1888.

[45] J. Dugundji: *Topology*, Allyn and Bacon, 1966.

[46] A. Edalat: A computable approach to measure and integration theory, Information and Computation 207, 2009, 642–659.

[47] F. Faissole, B. Spitters: Synthetic topology in Homotopy Type Theory for probabilistic programming, preprint, 2016.

[48] S. Feferman: A language and axioms for explicit mathematics, in J. N. Crossley (Ed.) *Algebra and Logic*, Springer Lecture Notes 450, 1975, 87–139.

[49] S. Feferman: Constructive theories of functions and classes, in Boffa et al. (Eds.) *Logic Colloquium 78*, North-Holland, 1979, 159–224.

[50] H. Friedman: The consistency of classical set theory relative to a set theory with intuitionistic logic, J. Symbolic Logic, 38, 1973, 315–319.

[51] H. Friedman: Set theoretic foundations for constructive analysis, Annals of Math. 105, 1977, 1–28.

[52] L. Gillman, M. Jerison: *Rings of Continuous Functions*, Van Nostrand, 1960.

[53] R. Godement: *Topologie algébrique et théorie de faisceaux*, Hermann, Paris, 1973.

[54] R. Goldblatt: *Topoi, The Categorical Analysis of Logic*, North-Holland, 1984.

[55] N. Greenleaf: Liberal constructive set theory, in [105], 1981, 213–240.

[56] N. D. Goodman, R. E. Vesley: Obituary: John R. Myhill (1923-1987), History and Philosophy of Logic, 8, 1987, 243–244.

[57] P. R. Halmos: *Measure theory*, Springer-Verlag New York Inc. 1974.

[58] F. Hausdorff: Grundzüge der Mengenlehre, Veit & Comp., Leipzig 1914 (reproduced in Srishti D. Chatterji et al. (Eds.): Felix Hausdorff. Gesammelte Werke. Band II: Grundzüge der Mengenlehre, Springer, Berlin 2002).

[59] A. Heyting: *Intuitionism, An introduction*, North-Holland, 1956.

[60] E. Hewitt: Linear functionals on spaces of continuous functions, Fund. Math. 37, 1950, 161–189.

[61] G. Huet, A. Saïbi: Constructive Category Theory, in Proceedings of the joint CLICS-TYPES workshop on categories and type theory, Goteborg, MIT Press, 1998.

[62] H. Ishihara, E. Palmgren: Quotient topologies in constructive set theory and type theory, Annals of Pure and Applied Logic 141, 2006, 257–265.

[63] H. Ishihara: Relating Bishop's function spaces to neighborhood spaces, Annals of Pure and Applied Logic 164, 2013, 482–490.

[64] L. Jaun: *Category Theory in Explicit Mathematics*, PhD Thesis, University of Bern, 2019.

[65] P. T. Johnstone: *Sketches of an Elefant A Topos Theory Compendium*, Volume 1, Oxford University Press, 2002.

[66] A. Kino, J. Myhill, R. E. Vesley (Eds.): *Intuitionism and Proof Theory*, North-Holland, 1970.

[67] A. N. Kolmogoroff: Algèbres de Boole métrique complètes, VI Zjazd Mathematyków Polskich, Warwaw, 1948, 21–30.

[68] H. Lombardi, C. Quitté: *Commutative Algebra: Constructive Methods*, Springer, 2015.

[69] R. S. Lubarsky, M. Rathjen: On the regular extension axiom and its variants, Mathematical Logic Quarterly 49 (5), 511–518, 2003.

[70] M. E. Maietti: A minimalist two-level foundation for constructive mathematics, Annals of Pure and Applied Logic 160(3), 2009, 319–354.

[71] P. Martin-Löf: *Notes on Constructive Mathematics*, Almqvist and Wiksell, 1970.

[72] P. Martin-Löf: An intuitionistic theory of types: predicative part, in H. E. Rose and J. C. Shepherdson (Eds.) *Logic Colloquium'73, Proceedings of the Logic Colloquium*, volume 80 of Studies in Logic and the Foundations of Mathematics,North-Holland, 1975, 73–118.

[73] P. Martin-Löf: *Intuitionistic type theory: Notes by Giovanni Sambin on a series of lectures given in Padua, June 1980*, Napoli: Bibliopolis, 1984.

[74] P. Martin-Löf: An intuitionistic theory of types, in [110], 127–172.

[75] C. McLarty: Two Constructicist Aspects of Category Theory, Philosophia Scientiœ, Cahier spécial 6, 2006, 95–114.

[76] R. Mines, F. Richman, W. Ruitenburg: *A course in constructive algebra*, Springer Science+Business Media New York, 1988.

[77] J. Myhill: Formal systems of intuitionistic analysis I, in van Rootselaar and Stall (Eds.) *Logic, methodology and philsosphy of science III*, North-Holland, Amsterdam, 1968, 161–178.

[78] J. Myhill: Formal systems of intuitionistic analysis II, in [66], 1970, 151–162.

[79] J. Myhill: Some properties of intuitionistic Zermelo-Fraenkel set theory, in A. Matthias , H. Rogers (Eds.) *Cambridge Summer School in Mathematical Logic*, LNM 337, Springer, 1972, 206–231.

[80] J. Myhill: Constructive Set Theory, J. Symbolic Logic 40, 1975, 347-382.

[81] E. Palmgren: Bishop's set theory, Slides from TYPES Summer School 2005, Gothenburg, in http://staff.math.su.se/palmgren/, 2005.

[82] E. Palmgren: Proof-relevance of families of setoids and identity in type theory, Arch. Math. Logic, 51, 2012, 35-47.

[83] E. Palmgren: Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets, Annals of Pure and Applied Logic 163, 2012, 1384–1399.

[84] E. Palmgren: Bishop-style constructive mathematics in type theory - A tutorial, Slides, in http://staff.math.su.se/palmgren/, 2013.

[85] E. Palmgren: *Lecture Notes on Type Theory*, 2014.

[86] E. Palmgren: On Equality of Objects in Categories in Constructive Type Theory, TYPES 2017, A. Abel et. al. (Eds.), Article No. 7; pp. 7:1-7:7.

[87] E. Palmgren, O. Wilander: Constructing categories and setoids of setoids in type theory, Logical Methods in Computer Science. 10 (2014), Issue 3, paper 25.

[88] I. Petrakis: *Constructive Topology of Bishop Spaces*, PhD Thesis, Ludwig-Maximilians-Universität, München, 2015.

[89] I. Petrakis: Completely Regular Bishop Spaces, in A. Beckmann, V. Mitrana and M. Soskova (Eds.): *Evolving Computability*, CiE 2015, LNCS 9136, 2015, 302–312.

[90] I. Petrakis: The Urysohn Extension Theorem for Bishop Spaces, in S. Artemov and A. Nerode (Eds.) *Symposium on Logical Foundations of Computer Science 2016*, LNCS 9537, Springer, 2016, 299–316.

[91] I. Petrakis: A constructive function-theoretic approach to topological compactness, Proceedings of the 31st Annual ACM-IEEEE Symposium on Logic in Computer Science (LICS 2016), July 5-8, 2016, NYC, USA, 605–614.

[92] I. Petrakis: Borel and Baire sets in Bishop Spaces, in F. Manea et. al. (Eds): Computing with Foresight and Industry, CiE 2019, LNCS 11558, Springer, 2019, 240–252.

[93] I. Petrakis: Constructive uniformities of pseudometrics and Bishop topologies, Journal of Logic and Analysis 11:FT2, 2019, 1–44.

[94] I. Petrakis: Direct spectra of Bishop spaces and their limits, https://arxiv.org/abs/1907.03273, 2019, 50 pages.

[95] I. Petrakis: Dependent Sums and Dependent Products in Bishop's Set Theory, in P. Dybjer et. al. (Eds) TYPES 2018, LIPIcs, Vol. 130, Article No. 3, 2019.

[96] I. Petrakis: A Yoneda lemma-formulation of the univalence axiom, manuscript, 2019, available at http://www.mathematik.uni-muenchen.de/∼petrakis/content/publications.php.

[97] I. Petrakis: McShane-Whitney extensions in constructive analysis, Logical Methods in Computer Science, Volume 16, Issue 2020, 18:1–18:23

[98] I. Petrakis: Embeddings of Bishop spaces, Journal of Logic and Computation, exaa015, 2020, https://doi.org/10.1093/logcom/exaa015.

[99] I. Petrakis: Functions of Baire-class one over a Bishop topology, in M. Anselmo et. al. (Eds.) *Beyond the Horizon of Computability* CiE 2020, Springer, LNCS 12098, 2020, 215-227.

[100] I. Petrakis: Bases of pseudocompact Bishop spaces, submitted to [28], 2021.

[101] I. Petrakis: Proof-relevance in Bishop-style constructive mathematics, submitted to Mathematical Structures in Computer Science, 2020.

[102] I. Petrakis, M. Zeuner: Pre-measure spaces and pre-integration spaces in predicative Bishop-Cheng measure theory, to be submitted to Archive for Mathematical Logic, 2020.

[103] M. Rathjen: Choice principles in constructive and classical set theories, Lecture Notes in Logic 27, 2006, 299–326.

[104] M. Rathjen: The constructive Hilbert program and the limits of Martin-Löf Type Theory, in S. Lindström et al. (Eds.) *Logicism, Intuitionism, and Formalism: What has become of them?* Synthese Library Volume 341, Springer, 2009, 397–433.

[105] F. Richman: *Constructive Mathematics*, LNM 873, Springer-Verlag, 1981.

[106] F. Richman: Constructive mathematics without choice, in [114], pp.199–205.

[107] E. Rijke: *Homotopy Type Theory*, Master Thesis, Utrecht University, 2012.

[108] M. Rosenblatt (Ed.): *Errett Bishop: Reflections on Him and His Research*, Contemporary Mathematics Volume 39, American Mathematical Society, 1985.

[109] K. A. Ross, K. Stromberg: Baire sets and Baire measures, Arkiv för Matematik Band 6 nr 8, 1965, 151–160.

[110] G. Sambin, J. M. Smith (Eds.): *Twenty-five years of constructive type theory*, Oxford University Press, 1998.

[111] G. Sambin: The Basic Picture: Structures for Constructive Topology, Oxford University Press, 2020.

[112] N. Šanin: *Constructive Real Numbers and Function Spaces*, Translations of Mathematical Monographs, vol. 21, AMS, Providence Rhode Island, 1968.

[113] M. Shulman: Linear Logic for Constructive Mathematics, arXiv:1805.07518v1, 2018.

[114] P. Schuster, U. Berger, H. Osswald (Eds.): *Reuniting the Antipodes Constructive and Nonstandard Views of the Continuum*, Proc. 1999 Venice Symposium, Dordrecht: Kluwer, 2001.

[115] P. Schuster: Countable Choice as a Questionable Uniformity Principle, Philosophia Mathematica (3) Vol. 12, pp. 106–134, 2004.

[116] H. Schwichtenberg, S. Wainer: *Proofs and Computations*, Perspectives in Logic, Association for Symbolic Logic and Cambridge University Press, 2012.

[117] H. Schwichtenberg: *Constructive analysis with witnesses*, Lecture notes, LMU, 2019.

[118] I. Segal: Abstract probability spaces and a theorem of Kolmogoroff, Amer. J. Math. 76, 1954, 721–732.

[119] I. Segal: Algebraic integration theory, Bull. Amer. Math. Soc. 71, 1965, 419–489.

[120] B. Spitters: Constructive and intuitionistic integration theory and functional analysis. PhD Thesis, University of Nijmegen, 2002.

[121] B. Spitters: Constructive algebraic integration theory. Ann. Pure Appl. Logic 137(1-3), 2006, 380–390.

[122] T. Streicher: *Realizability*, Lecture Notes, TU Darmstadt, 2018.

[123] S. J. Taylor: *Introduction to Measure and Integration*, Cambridge University Press, 1973.

[124] The Univalent Foundations Program: *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, Princeton, 2013.

[125] A. S. Troelstra and D. van Dalen: *Constructivism in Mathematics*, Volume I, North-Holland, 1988.

[126] A. S. Troelstra and D. van Dalen: *Constructivism in Mathematics*, Volume II, North-Holland, 1988.

[127] A. Weil: Calcul des probabilités, méthode aximatique, intégration, Revue Sci. (Rev. Rose Illus.) 78, 1940, 201–208.

[128] T. Wiklund: Locally cartesian closed categories, coalgebras, and containers, Uppsala Universitet, U.U.D.M Project Report, 2013:5.

[129] M. Zeuner: *Families of Sets in Constructive Measure Theory*, Master's Thesis, LMU, 2019.

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