Limit Spaces with Approximations

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Correctness by Construction CORCON 2014 Workshop

Genova, 24.03.2014
Intuitive, not enough, but still important.

Limit spaces and related notions capture the “sequential” part of topology.

A constructive theory of limit spaces is not elaborated so far.

How to add convergence in formal topology is still open.

Limit spaces are used in Computability at Higher Types (CHT).

Here we study limit spaces and their relation to CHT mostly within BISH.
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Sequential convergence in Topology

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Limit spaces are used in Computability at Higher Types (CHT).

Here we study limit spaces and their relation to CHT mostly within BISH.
A limit space, or a Kuratowski limit space, or an \( \mathcal{L}^* \)-space, is a pair \( \mathbb{L} = (X, \lim) \), where \( X \) is an inhabited set, and \( \lim \subseteq X \times X^{\mathbb{N}} \) is a relation satisfying the following conditions:

(L1) If \( x \in X \), then \( \lim(x, x) \).

(L2) If \( S \) denotes the strictly monotone elements of the Baire space \( \mathcal{N} \), then

\[
\forall_{\alpha \in S} (\lim(x, x_n) \to \lim(x, x_{\alpha(n)})).
\]

(L3) Urysohn’s axiom: If \( x \in X \) and \( x_n \in X^{\mathbb{N}} \), then

\[
\forall_{\alpha \in S} \exists_{\beta \in S} (\lim(x, x_{\alpha(\beta(n))}) \to \lim(x, x_n)).
\]

\( \mathbb{L} \) satisfies the uniqueness property (sequential Hausdorff), if

(L4)

\[
\forall_{x, y \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(y, x_n) \to x = y).
\]

\( \mathbb{L} \) satisfies the weak uniqueness property, if

(L5)

\[
\forall_{x, x' \in X} (\lim(x', x) \to x' = x).
\]
A limit space, or a Kuratowski limit space, or an $\mathcal{L}^*$-space, is a pair $\mathbb{L} = (X, \text{lim})$, where $X$ is an inhabited set, and $\text{lim} \subseteq X \times X^\mathbb{N}$ is a relation satisfying the following conditions:

(L₁) If $x \in X$, then $\text{lim}(x, x)$.

(L₂) If $S$ denotes the strictly monotone elements of the Baire space $\mathcal{N}$, then

$$\forall_{\alpha \in S} (\text{lim}(x, x_n) \rightarrow \text{lim}(x, x_{\alpha(n)})).$$

(L₃) Urysohn’s axiom: If $x \in X$ and $x_n \in X^\mathbb{N}$, then

$$\forall_{\alpha \in S} \exists_{\beta \in S} (\text{lim}(x, x_{\alpha(\beta(n)))} \rightarrow \text{lim}(x, x_n)).$$

(L₄) $\mathbb{L}$ satisfies the uniqueness property (sequential Hausdorff), if

$$\forall_{x, y \in X} \forall_{x_n \in X^\mathbb{N}} (\text{lim}(x, x_n) \rightarrow \text{lim}(y, x_n) \rightarrow x = y).$$

(L₅) $\mathbb{L}$ satisfies the weak uniqueness property, if

$$\forall_{x, x' \in X} (\text{lim}(x', x) \rightarrow x' = x).$$
A limit space, or a Kuratowski limit space, or an $\mathcal{L}^*$-space, is a pair $\mathbb{L} = (X, \lim)$, where $X$ is an inhabited set, and $\lim \subseteq X \times X^\mathbb{N}$ is a relation satisfying the following conditions:

(L1) If $x \in X$, then $\lim(x, x)$.

(L2) If $S$ denotes the strictly monotone elements of the Baire space $\mathcal{N}$, then

$$\forall_{\alpha \in S} (\lim(x, x_n) \rightarrow \lim(x, x_{\alpha(n)})).$$

(L3) Urysohn’s axiom: If $x \in X$ and $x_n \in X^\mathbb{N}$, then

$$\forall_{\alpha \in S} \exists_{\beta \in S} (\lim(x, x_{\alpha(\beta(n))})) \rightarrow \lim(x, x_n).$$

(L4) $\mathbb{L}$ satisfies the uniqueness property (sequential Hausdorff), if

$$(L_4) \quad \forall_{x, y \in X} \forall_{x_n \in X^\mathbb{N}} (\lim(x, x_n) \rightarrow \lim(y, x_n) \rightarrow x = y).$$

(L5) $\mathbb{L}$ satisfies the weak uniqueness property, if

$$(L_5) \quad \forall_{x, x' \in X} (\lim(x', x) \rightarrow x' = x).$$
Proposition (BISH)

Suppose that $R \subseteq X \times X^\mathbb{N}$. If $\lim(R) \subseteq X \times X^\mathbb{N}$ is defined inductively by the following clauses:

(i) $R \subseteq \lim(R)$ and $\{(x, x) \mid x \in X\} \subseteq \lim(R)$,
(ii) $\lim(R)(x, x_n) \rightarrow \lim(R)(x, x_{\alpha(n)})$, for each $\alpha \in S$,
(iii) $\forall_{\alpha \in S} \exists_{\beta \in S} (\lim(R)(x, x_{\alpha(\beta(n))})) \rightarrow \lim(R)(x, x_n)$,

then $\lim(R)$ is the smallest limit relation including $R$.

Proposition (BISH)

There is a limit space satisfying the weak uniqueness but not the uniqueness property.

Proof.

Take $R = \{(x, x_n), (y, x_n)\}$, where $x, y \in X$ s.t. $x \neq y$ and $x_n$ is a not eventually constant sequence in $X$. Take $\lim(R)$ to be the least limit relation including $R$ defined as above satisfying the additional condition of the weak uniqueness property.
An \( \mathcal{L} \)-space, or a \textbf{pseudo-limit space} is a pair \((X, \lim)\) satisfying \((L_1), (L_2)\) and \((L_4)\).

**Lemma (BISH)**

Suppose that \((X, \lim)\) is a limit space, \(x \in X\) and \(x_n \in X^\mathbb{N}\) such that \(\lim(x, x_n)\). If \(\alpha \in \mathcal{N}\) such that \(\alpha(n) > 0\) and \(x'_n\) is the sequence defined by

\[
x'_\alpha(k) = \ldots = x'_{\alpha(k)+\alpha(k+1)-1} = x_k,
\]

for each \(k \geq 0\), then \(\lim(x, x'_n)\).

An \( \mathcal{L} \)-space which is not a limit space: \((\mathbb{R}, \text{rlim})\), where

\[
\text{rlim}(x, x_n) :\leftrightarrow \sum_{n \in \mathbb{N}} |x - x_n| < \infty.
\]

Clearly, \(\text{rlim}(0, \frac{1}{2^n})\), while

\[
\neg \text{rlim}(0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1, \frac{1}{4}, \frac{1}{4}, \ldots).
\]

If \((\mathbb{R}, \text{rlim})\) was a limit space, we should have that the above sequence of finite repetitions converges to 0 too.
A filter space, or a Choquet space, is a pair $\mathcal{F} = (X, \text{Lim})$, where $X$ is an inhabited set, and $\text{Lim} \subseteq X \times \mathcal{F}(X)$ is a relation satisfying the following conditions:

1. If $x \in X$ and $F_x = \{A \subseteq X \mid x \in A\}$, then Lim$(x, F_x)$.
2. If $F \subseteq G$, then Lim$(x, F) \rightarrow$ Lim$(x, G)$.
3. For all $G \supseteq F$, there exists $H \supseteq G$ such that Lim$(x, H) \rightarrow$ Lim$(x, F)$.

A convergence space is a pair $\mathcal{F} = (X, \text{Lim})$ s.t. (F1), (F2) and

4. Lim$(x, F) \rightarrow$ Lim$(x, G) \rightarrow$ Lim$(x, F \cap G)$.

5. (F3) $\rightarrow$ (F4).
A filter space, or a Choquet space, is a pair $\mathcal{F} = (X, \text{Lim})$, where $X$ is an inhabited set, and $\text{Lim} \subseteq X \times \mathcal{F}(X)$ is a relation satisfying the following conditions:

(F1) If $x \in X$ and $F_x = \{A \subseteq X \mid x \in A\}$, then $\text{Lim}(x, F_x)$.

(F2) $F \subseteq G \Rightarrow \text{Lim}(x, F) \rightarrow \text{Lim}(x, G)$.

(F3) $\forall G \supseteq F \exists x \supseteq H \supseteq G (\text{Lim}(x, H)) \rightarrow \text{Lim}(x, F)$.

A convergence space is a pair $\mathcal{F} = (X, \text{Lim})$ s.t. (F1), (F2) and (F4) $\text{Lim}(x, F) \rightarrow \text{Lim}(x, G) \rightarrow \text{Lim}(x, F \cap G)$.

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A **filter space**, or a Choquet space, is a pair $F = (X, \text{Lim})$, where $X$ is an inhabited set, and $\text{Lim} \subseteq X \times \mathcal{F}(X)$ is a relation satisfying the following conditions:

(F1) If $x \in X$ and $F_x = \{ A \subseteq X \mid x \in A \}$, then $\text{Lim}(x, F_x)$.

(F2) $F \subseteq G \rightarrow \text{Lim}(x, F) \rightarrow \text{Lim}(x, G)$.

(F3) $\forall G \supseteq F \exists X \supseteq H \supseteq G (\text{Lim}(x, H)) \rightarrow \text{Lim}(x, F)$.

**A convergence space** is a pair $F = (X, \text{Lim})$ s.t. (F1), (F2) and

(F4) $\text{Lim}(x, F) \rightarrow \text{Lim}(x, G) \rightarrow \text{Lim}(x, F \cap G)$.

(F3) $\rightarrow$ (F4).
A quasi-topological space is a structure \((X, (K \to X)_{K \in \text{CHTop}})\), where \(\text{CHTop}\) is the category of compact Hausdorff spaces and for each \(K, K', K_1, \ldots, K_n \in \text{CHTop}\) the set of functions \(K \to X \subseteq \mathbb{F}(K, X)\) satisfies the following conditions:

\((QT_1)\) The constant function \(\hat{x} \in K \to X\), for each \(x \in X\).
\((QT_2)\) \(f \in K \to X \to g \in \mathbb{C}(K', K) \to g \circ f \in K' \to X\).
\((QT_3)\) If \(f_1 \in \mathbb{F}(K_1, K), \ldots, f_n \in \mathbb{F}(K_n, K)\) such that

\[
\bigcup_{i=1}^{n} \text{rng}(f_i) = K,
\]

\[
\forall g \in \mathbb{F}(K, X) \forall i (g \circ f_i \in K_i \to X),
\]

then \(g \in K \to X\).
A function space is a pair $(X, F)$, where $F \subseteq F(X, \mathbb{R})$, called the topology, satisfies the following clauses:

1. The constant function $\hat{a} \in F$, for each $a \in \mathbb{R}$.
2. $f, g \in F \rightarrow f + g, fg \in F$.
3. $f \in F \rightarrow g \in C(\mathbb{R}, \mathbb{R}) \rightarrow g \circ f \in F$.
4. $f \in F(X, \mathbb{R}) \rightarrow \forall \epsilon > 0 \exists \delta \in F \forall x \in X (|f(x) - g(x)| \leq \epsilon) \rightarrow f \in F$.

$F(F_0)$ is the least topology including $F_0 \subseteq F(X, \mathbb{R})$, defined like $\lim(R)$.

Bishop-Bridges 1985: This definition “should not be taken seriously. The purpose is merely to list a minimal number of properties that the set of all continuous functions in a topology should be expected to have. Other properties could be added; to find a complete list seems to be a nontrivial and interesting problem”.
A function space is a pair $(X, F)$, where $F \subseteq \mathbb{F}(X, \mathbb{R})$, called the topology, satisfies the following clauses:

1. (FS$_1$) The constant function $\hat{a} \in F$, for each $a \in \mathbb{R}$.
2. (FS$_2$) $f, g \in F \rightarrow f + g, fg \in F$.
3. (FS$_3$) $f \in F \rightarrow g \in C(\mathbb{R}, \mathbb{R}) \rightarrow g \circ f \in F$.
4. (FS$_4$) $f \in \mathbb{F}(X, \mathbb{R}) \rightarrow \forall \epsilon > 0 \exists g \in F \forall x \in X (|f(x) - g(x)| \leq \epsilon) \rightarrow f \in F$.

$F(F_0)$ is the least topology including $F_0 \subseteq \mathbb{F}(X, \mathbb{R})$, defined like lim$(R)$.

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A function space is a pair \((X, F)\), where \(F \subseteq F(X, \mathbb{R})\), called the topology, satisfies the following clauses:

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2. **(FS\(_2\))** \(f, g \in F \rightarrow f + g, fg \in F\).
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Limit Spaces with Approximations
The Birkhoff-Baer topology

1. A set \( O \subseteq X \) is **lim-open**, or \( O \in \mathcal{T}_{\text{lim}} \), if
   \[
   \forall x \in O \forall x_n \in X^\mathbb{N} (\lim(x, x_n) \to \text{ev}_O(x_n)),
   \]
   where if \( A \subseteq X \), we define
   \[
   \text{ev}_A(x_n) := \exists n_0 \forall n \geq n_0 (x_n \in A).
   \]

2. A set \( F \subseteq X \) is called **lim-closed**, if it is the complement of a lim-open set, and in CLASS this is equivalent to
   \[
   \forall x \in X \forall x_n \in X^\mathbb{N} (x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F).
   \]

3. A topological space \((X, \mathcal{T})\) induces a limit space \((X, \lim_{\mathcal{T}})\), where
   \[
   \lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x.
   \]

4. A set \( D \subseteq X \) is called **lim-dense**, if
   \[
   \forall x \in X \exists d_n \in D^\mathbb{N} (\lim(x, d_n)).
   \]

5. (BISH) If \( D \) is lim-dense, then \( D \) is dense in \((X, \mathcal{T}_{\text{lim}})\).

6. (CLASS) \( \mathbb{I} \) is dense in \((\mathbb{R}, \mathcal{T}_{\text{coc}})\), but it is not lim-dense in \((\mathbb{R}, \lim_{\mathcal{T}_{\text{coc}}})\).
A set $\mathcal{O} \subseteq X$ is **lim-open**, or $\mathcal{O} \in \mathcal{T}_{\text{lim}}$, if
\[
\forall x \in \mathcal{O} \forall x_n \in X^\mathbb{N} (\lim(x, x_n) \rightarrow \text{ev}_\mathcal{O}(x_n)),
\]
where if $A \subseteq X$, we define
\[
\text{ev}_A(x_n) := \exists n_0 \forall n \geq n_0 (x_n \in A).
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A set $F \subseteq X$ is called **lim-closed**, if it is the complement of a lim-open set, and in CLASS this is equivalent to
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\forall x \in X \forall x_n \in X^\mathbb{N} (x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F).
\]
A topological space $(X, \mathcal{T})$ induces a limit space $(X, \lim_{\mathcal{T}})$, where
\[
\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x.
\]
A set $D \subseteq X$ is called **lim-dense**, if
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\forall x \in X \exists d_n \in D^\mathbb{N} (\lim(x, d_n)).
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(BISH) If $D$ is lim-dense, then $D$ is dense in $(X, \mathcal{T}_{\text{lim}})$.
(CLASS) $I$ is dense in $(\mathbb{R}, \mathcal{T}_{\text{coc}})$, but it is not lim-dense in $(\mathbb{R}, \lim_{\mathcal{T}_{\text{coc}}})$.  

Iosif Petrakis  Limit Spaces with Approximations
The Birkhoff-Baer topology

1. A set $\mathcal{O} \subseteq X$ is lim-open, or $\mathcal{O} \in \mathcal{T}_{\text{lim}}$, if

$$\forall x \in \mathcal{O} \forall x_n \in X^\infty (\lim(x, x_n) \rightarrow \text{ev}_\mathcal{O}(x_n)),$$

where if $A \subseteq X$, we define

$$\text{ev}_A(x_n) := \exists n_0 \forall n \geq n_0 (x_n \in A).$$

2. A set $F \subseteq X$ is called lim-closed, if it is the complement of a lim-open set, and in CLASS this is equivalent to

$$\forall x \in X \forall x_n \in X^\infty (x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F).$$

3. A topological space $(X, \mathcal{T})$ induces a limit space $(X, \lim_{\mathcal{T}})$, where

$$\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x.$$

4. A set $D \subseteq X$ is called lim-dense, if

$$\forall x \in X \exists d_n \in D^\infty (\lim(x, d_n)).$$

5. (BISH) If $D$ is lim-dense, then $D$ is dense in $(X, \mathcal{T}_{\text{lim}})$.

6. (CLASS) $I$ is dense in $(\mathbb{R}, \mathcal{T}_{\text{coc}})$, but it is not lim-dense in $(\mathbb{R}, \lim_{\mathcal{T}_{\text{coc}}})$. 
The Birkhoff-Baer topology

1. A set $\mathcal{O} \subseteq X$ is \textbf{lim-open}, or $\mathcal{O} \in \mathcal{T}_{\text{lim}}$, if

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where if $A \subseteq X$, we define

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2. A set $F \subseteq X$ is called \textbf{lim-closed}, if it is the complement of a lim-open set, and in \text{CLASS} this is equivalent to

$$\forall x \in X \forall x_n \in X^\mathbb{N} (x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F).$$

3. A topological space $(X, \mathcal{T})$ induces a limit space $(X, \lim_{\mathcal{T}})$, where

$$\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x.$$

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The Birkhoff-Baer topology

1. A set \( O \subseteq X \) is **lim-open**, or \( O \in \mathcal{T}_{\text{lim}} \), if
\[
\forall x \in O \forall x_n \in X^N (\lim(x, x_n) \to \text{ev}_O(x_n)),
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where if \( A \subseteq X \), we define
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\text{ev}_A(x_n) := \exists n_0 \forall n \geq n_0 (x_n \in A).
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2. A set \( F \subseteq X \) is called **lim-closed**, if it is the complement of a lim-open set, and in CLASS this is equivalent to
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\forall x \in X \forall x_n \in X^N (x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F).
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3. A topological space \((X, \mathcal{T})\) induces a limit space \((X, \lim_{\mathcal{T}})\), where
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\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \underset{\mathcal{T}}{\rightarrow} x.
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5. (BISH) If \( D \) is lim-dense, then \( D \) is dense in \((X, \mathcal{T}_{\text{lim}})\).

6. (CLASS) \( \mathbb{I} \) is dense in \((\mathbb{R}, \mathcal{T}_{\text{coc}})\), but it is not lim-dense in \((\mathbb{R}, \lim_{\mathcal{T}_{\text{coc}}})\).
1. Trivially, \( \lim \subseteq \lim_{\mathcal{T}_{\text{lim}}} \). A limit space is called **topological**, if

\[
\lim \supseteq \lim_{\mathcal{T}_{\text{lim}}}.
\]

2. Trivially, \( \mathcal{T} \subseteq \mathcal{T}_{\text{lim}} \). A topological space is called **sequential**, if

\[
\mathcal{T} \supseteq \mathcal{T}_{\text{lim}}.
\]

3. If \((X, \mathcal{T})\) is a sequential space and \(D\) is a \(\lim_{\mathcal{T}}\)-dense subset of it, then \(D\) is dense in \(X\).

4. An open (closed) subset of a sequential space is sequential.
1. Trivially, $\lim \subseteq \lim_{\mathcal{T}_{\text{lim}}}$. A limit space is called **topological**, if

\[
\lim \supseteq \lim_{\mathcal{T}_{\text{lim}}}.
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\mathcal{T} \supseteq \mathcal{T}_{\text{lim}_{\mathcal{T}}}.
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1. Trivially, $\lim \subseteq \lim_{\mathcal{T}_{\text{lim}}}$. A limit space is called topological, if

$$\lim \supseteq \lim_{\mathcal{T}_{\text{lim}}}.$$

2. Trivially, $\mathcal{T} \subseteq \mathcal{T}_{\text{lim}}_T$. A topological space is called sequential, if

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Trivially, \( \mathcal{T} \subseteq \mathcal{T}_{\lim_{\mathcal{T}}} \). A topological space is called **sequential**, if

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If \((X, \mathcal{T})\) is a sequential space and \(D\) is a \(\lim_{\mathcal{T}}\)-dense subset of it, then \(D\) is dense in \(X\).

An open (closed) subset of a sequential space is sequential.
Suppose that \((X, \lim)\) and \((Y, \lim)\) are limit spaces.

1. \((X \times Y, \lim)\) is the **product** limit space, where
   \[
   \lim((x, y), (x_n, y_n)) :\leftrightarrow \lim(x, x_n) \land \lim(y, y_n).
   \]

2. \((X \to Y, \lim)\) is the **function** limit space, where
   
   \[
   f \in X \to Y :\leftrightarrow \forall x \in X \forall x_n \in X^N (\lim(x, x_n) \to \lim(f(x), f(x_n))),
   \]
   
   \[
   \lim(f, f_n) :\leftrightarrow \forall x \in X \forall x_n \in X^N (\lim(x, x_n) \to \lim(f(x), f_n(x_n))).
   \]

3. If \(A \subseteq X\), then \((A, \lim_A)\) is the **relative** limit space, where
   \[
   \lim_A = \lim \cap (A \times A^N).
   \]

4. If \(f : (X, \lim) \to (Y, \lim)\) is lim-continuous, then \(f : (X, \lim) \to (f(X), \lim_{f(X)})\) is lim-continuous.
Suppose that \((X, \lim)\) and \((Y, \lim)\) are limit spaces.

1. \((X \times Y, \lim)\) is the **product** limit space, where

   \[ \lim((x, y), (x_n, y_n)) :\leftrightarrow \lim(x, x_n) \land \lim(y, y_n). \]

2. \((X \to Y, \lim)\) is the **function** limit space, where

   \[ f \in X \to Y :\leftrightarrow \forall x \in X \forall x_n \in X^\mathbb{N} (\lim(x, x_n) \to \lim(f(x), f(x_n))), \]

   \[ \lim(f, f_n) :\leftrightarrow \forall x \in X \forall x_n \in X^\mathbb{N} (\lim(x, x_n) \to \lim(f(x), f_n(x_n))). \]

3. If \(A \subseteq X\), then \((A, \lim_A)\) is the **relative** limit space, where

   \[ \lim_A = \lim \cap (A \times A^\mathbb{N}). \]

4. If \(f : (X, \lim) \to (Y, \lim)\) is \(\lim\)-continuous, then \(f : (X, \lim) \to (f(X), \lim_{f(X)})\) is \(\lim\)-continuous.
Suppose that \((X, \lim)\) and \((Y, \lim)\) are limit spaces.

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4. If \(f : (X, \lim) \to (Y, \lim)\) is \(\lim\)-continuous, then \(f : (X, \lim) \to (f(X), \lim_{f(X)})\) is \(\lim\)-continuous.
(BISH) If \((X, \mathcal{T}_{\text{lim}})\) is a \(T_2\)-space, then \((X, \text{lim})\) has the uniqueness property. The converse doesn’t hold in general (Dudley 1964).

**Proposition**

(CLASS) Suppose that \((X, \text{lim})\) is a limit space, \((Y, \text{lim})\) is a limit space with the uniqueness property, \(D\) is a lim-dense subset of \(X\), and \(f, g : X \to Y\) are \(\text{lim-continuous}\) functions. Then the following hold:

(i) If \(f|_D = g|_D\), then \(f = g\).
(ii) If \(f : (D, \text{lim}_D) \to (Y, \text{lim})\) is \(\text{lim-continuous}\), then it has at most one \(\text{lim-continuous}\) extension to \(X\).
(iii) The set \(Z(f, g) = \{x \in X \mid f(x) = g(x)\}\) is \(\text{lim-closed}\).
(iv) The graph \(G_f\) of \(f\) is \(\text{lim-closed}\) in \((X \times Y, \text{lim})\).
(v) If \(f\) is 1-1, then \((X, \text{lim})\) has the uniqueness property.
Useful continuity facts

1. A ⊆ X is called a **lim-retract** of X, if there is a lim-continuous function \( r : X \to A \) such that \( r(a) = a \), for each \( a \in A \).

2. (BISH) If \((X, \text{lim}), (Y, \text{lim})\) are limit spaces, \( A \) is a lim-retract of \( X \) and \( f : (A, \text{lim}_A) \to (Y, \text{lim}) \) is lim-continuous, then \( f \) has a lim-continuous extension \( F : (X, \text{lim}) \to (Y, \text{lim}) \).

3. If \( F = f \circ r \), then \( F \) is lim-continuous as a composition of lim-continuous functions, and \( F(a) = f(r(a)) = f(a) \), for each \( a \in A \).

4. (BISH) If \((X, \text{lim})\) and \((Y, \text{lim})\) are limit spaces and \( f : X \to Y \) is lim-continuous, then \( f : (X, \mathcal{T}_\text{lim}) \to (Y, \mathcal{T}_\text{lim}) \) is continuous.

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5. **Kisyński**’s theorem 1960: a limit space with the uniqueness property is topological. The classical proof of: “a limit space inducing a Hausdorff topology is topological” is much easier.
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3. Under the influence of the work of Ershov and Scott, CHT was connected in the 80's and 90's mainly with the development of domain theory.

4. The interplay of notions and methods between domain theory and the theory of (sequential or filter) limit spaces was from the beginning evident e.g., general ideas of the proof of an effective density theorem of Hyland 1979 have their domain-theoretic counterpart in the later work of U. Berger.

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The category \( \text{Lim} \) and its relation to \( \text{Seq}, \text{Equ}, \omega\text{Equ} \) (Scott, Simpson, Rosolini, Bauer).

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So far use of classical logic.

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1. **Normann 1982**: “the internal concepts [of computability] must grow out of the structure at hand, while external concepts maybe inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways”

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3. **Motivation**:
   1. “the weaker tools we use to obtain a result, the more extra knowledge can be obtained from the process of obtaining the result”
   2. “proofs are simpler”
   3. “the internally computable functions are defined from elements, relations and functions present in the structure, using acceptable operators that form new functions. The problem will be to decide what the acceptable operators are”
   4. “on the one hand one does not have to translate everything to the set of representatives, and on the other hand an internally defined object will always be well defined”

4. **Soundness criterion**: an internally computable object is externally computable.

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1. **Normann 1982**: “the internal concepts [of computability] must grow out of the structure at hand, while external concepts maybe inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways”

2. **Normann 2000-**: series of papers elaborating the main ideas of IC.

3. **Motivation**:
   1. “the weaker tools we use to obtain a result, the more extra knowledge can be obtained from the process of obtaining the result”
   2. “proofs are simpler”
   3. “the internally computable functions are defined from elements, relations and functions present in the structure, using acceptable operators that form new functions. The problem will be to decide what the acceptable operators are”
   4. “on the one hand one does not have to translate everything to the set of representatives, and on the other hand an internally defined object will always be well defined”

4. **Soundness criterion**: an internally computable object is externally computable.

5. **There are externally computable objects which are not internally computable** [the Fan functional is Kleene-computable (it has a representative with a recursive associate), but it is not S1-S9 definable over the total Kleene-Kreisel functionals].
Mainly limit spaces are used.

It is “useful to see how far we can get towards constructing an effective infrastructure on such spaces without introducing superstructures and imposing external notions of computability on the given structures ... One way to create a useful part of an infrastructure will be to isolate a dense subset that in some way is effectively dense.”.

Although Normann is working in CLASS, we work mostly in BISH.

Normann 2008 presented Kleene’s countable functionals over $\mathbb{N}$ using limit spaces and the corresponding density theorem using the notion of the $n$th approximation of a functional, for each $n \in \mathbb{N}$.

Here we generalize Normann’s presentation by defining two new subcategories of limit spaces, $\text{Appr}$ and $\text{Gappr}$ and connect them to later work of Normann.

Dense sets are very direct to find in $\text{Appr}$ and $\text{Gappr}$.

As Scott’s information systems have the approximation objects (tokens and formal neighborhoods) as primitive notions, forming a constructive counterpart to abstract algebraic domains, the approximation functions in a limit space with approximations are given beforehand too.
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As Scott’s information systems have the approximation objects (tokens and formal neighborhoods) as primitive notions, forming a constructive counterpart to abstract algebraic domains, the approximation functions in a limit space with approximations are given beforehand too.
A limit space with approximations is a structure $\mathbb{A} = (X, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})$ such that $(X, \lim)$ is a limit space, and, for each $n \in \mathbb{N}$ the approximation functions $\text{Appr}_n : X \to X$ satisfy the following properties:

(A1) $\text{Appr}_n$ is lim-continuous.
(A2) $\text{Appr}_n(\text{Appr}_m(x)) = \text{Appr}_{\min(n,m)}(x)$, for each $x \in X$.
(A3) $D_n = \text{Appr}_n(X) = \{\text{Appr}_n(x) \mid x \in X\}$ is an inhabited finite set.
(A4) $\lim(x, x_n) \to \lim(x, \text{Appr}_n(x_n))$, for each $x \in X$ and $x_n \in X^\mathbb{N}$.

Corollary 1: (A5) $\text{Appr}_n(\text{Appr}_n(x)) = \text{Appr}_n(x)$.

Corollary 2: $n < m \to D_n \subseteq D_m$.

Corollary 3: $\mathcal{B} = \{D_n \mid n \in \mathbb{N}\}$ is a countable filter base on $X$.

A structure $\mathbb{A}$ satisfying (A5), (A3) and (A4) is a limit space with general approximations.
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A structure $\mathbb{A}$ satisfying (A5), (A3) and (A4) is a limit space with general approximations.
(BISH) If $A$ is a limit space with (general) approximations and $x \in X$, then
\[ \lim(x, \text{Appr}_n(x)), \] and the set
\[ D = \bigcup_{n \in \mathbb{N}} D_n \]
is an enumerable dense subset of $(X, \mathcal{T}_\text{lim})$.

Proof.
By $(A_4)$, considering the constant sequence $x$, we get
\[ \lim(x, x) \to \lim(x, \text{Appr}_n(x)) \]
i.e., $D$ is a lim-dense subset of $X$. Therefore, $D$ is a dense subset of $(X, \mathcal{T}_\text{lim})$. $D$ is enumerable.
Extension theorem

For each function $f$ defined on $D$ there is a sequence of lim-continuous functions which extend uniformly arbitrary big “parts” of $f$.

**Proposition**

(BISH) If $\mathbb{A}$ is a limit space with approximations, then

(i) Each set $D_n$ is a lim-retract of $X$.

(ii) If $(Y, \lim)$ is a limit space, any lim-continuous function $f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a lim-continuous extension $F_n : (X, \lim) \to (Y, \lim)$.

(iii) If $f : (D, \lim_D) \to (Y, \lim)$ is lim-continuous, then there is a sequence $(F_n)_n$ of lim-continuous functions $F_n : X \to Y$ such that, for each $n$,

$$F_n|_{D_n} = f|_{D_n} \text{ and } F_{n+1}|_{D_n} = F_n|_{D_n}.$$ 

**Proof.**

(i) Since each $\text{Appr}_n : (X, \lim) \to (X, \lim)$ is lim-continuous, each $\text{Appr}_n : (X, \lim) \to (D_n, \lim_{D_n})$ is lim-continuous too. Since any $a \in \text{Appr}_n(X)$ has the form $\text{Appr}_n(x)$, for some $x \in X$, we get that $\text{Appr}_n(a) = \text{Appr}_n(\text{Appr}_n(x)) = a$.

(ii) Then a lim-continuous function $f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a lim-continuous extension $F$.

(iii) $f_n = f|_{D_n}$ is extended to a lim-continuous function $F_n : X \to Y$, and by $D_n \subseteq D_{n+1}$ we get that $F_{n+1}|_{D_n} = F_n|_{D_n}$, for each $n$. 

\[\square\]
Proposition

(BISH) If \((X, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})\) and \((Y, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})\) are limit spaces with (general) approximations, and if we define on \(X \times Y\)

\[
\text{Appr}_n(x, y) := (\text{Appr}_n(x), \text{Appr}_n(y)),
\]

for each \(n\), then \((X \times Y, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})\) is a limit space with (general) approximations, where \(\lim\) is the already defined lim-relation on \(X \times Y\).
Theorem

(BISH) If $(X, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})$ and $(Y, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})$ are limit spaces with (general) approximations, and if we define, for each $n$ and $f \in X \to Y$,

$$f \mapsto \text{Appr}_n(f),$$

$$\text{Appr}_n(f)(x) := \text{Appr}_n(f(\text{Appr}_n(x))),$$

for each $x \in X$, then $(X \to Y, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})$ is a limit space with (general) approximations, where $\lim$ is the already defined $\lim$-relation on $X \to Y$.
On the proof of \((A_3)\)

If \(\text{Appr}_n(Y)\) \textit{in inhabited}, then \(\text{Appr}_n(X \to Y)\) \textit{is inhabited}: If \(y \in Y\), then for the constant function \(\hat{y}\) we have that

\[
\text{Appr}_n(\hat{y})(x) = \text{Appr}_n(\hat{y}(\text{Appr}_n(x))) = \text{Appr}_n(y).
\]

Hence,

\[
\text{Appr}_n(\hat{y}) = \text{Appr}_n(y)
\]

If \(y\) inhabits \(\text{Appr}_n(Y)\), then the constant function \(\text{Appr}_n(\hat{y}) = \text{Appr}_n(y) = \hat{y}\) inhabits \(\text{Appr}_n(X \to Y)\). I.e., in this case the \(n\)th approximation of \(\hat{y}\) is identical to it.

To prove the \textbf{finiteness of} \(\text{Appr}_n(X \to Y)\) we show that the \(n\)th-approximation of a function in the function limit space acts equally on its input and on the \(n\)th-approximation of it, since

\[
\text{Appr}_n(f)(\text{Appr}_n(x)) = \text{Appr}_n(f(\text{Appr}_n(\text{Appr}_n(x)))) \\
\stackrel{A_5}{=} \text{Appr}_n(f(\text{Appr}_n(x))) = \text{Appr}_n(f)(x).
\]

Then \(\text{Appr}_n(f) : X \to Y\) is determined by its restriction

\[
\text{Appr}_n(f)|_{\text{Appr}_n(X)} : \text{Appr}_n(X) \to \text{Appr}_n(Y)
\]

\[
\text{Appr}_n(f)|_{\text{Appr}_n(X)} = \text{Appr}_n(g)|_{\text{Appr}_n(X)} \to \text{Appr}_n(f) = \text{Appr}_n(g).
\]

\[
|\text{Appr}_n(X \to Y)| \leq |\text{Appr}_n(Y)^{\text{Appr}_n(X)}|.
\]
We fix \( x \in X \) and \( x_n \in X^\mathbb{N} \) such that \( \lim(x, x_n) \). By \((A_4)\) on \( X \) we get that

\[
\lim(x, x_n) \to \lim(x, \text{Appr}_n(x_n)),
\]

while by the definition of \( \lim(f, f_n) \) on \( x \) and the sequence \( \text{Appr}_n(x_n) \) we have that \( \lim(f(x), f_n(\text{Appr}_n(x_n))) \). By \((A_4)\) on \( Y \) we get that

\[
\lim(f(x), \text{Appr}_n(f_n(\text{Appr}_n(x_n)))) \leftrightarrow \lim(f(x), \text{Appr}_n(f_n)(x_n)).
\]
\[ \iota = \mathbb{N} | \rho \to \sigma, \]
\[ \text{Ct}(\iota) := (\mathbb{N}, \lim_{T_{di}}), \]
\[ \text{Ct}(\rho \to \sigma) := (\text{Ct}(\rho) \to \text{Ct}(\sigma), \lim_{\rho \to \sigma}), \]

To each limit space \((\text{Ct}(\rho), \lim_{\rho})\) the following approximation functions are added:

\[ \text{Appr}_{n,\iota}(m) = \min(n, m), \]

while if \(F \in \text{Ct}(\rho \to \sigma)\) and \(f \in \text{Ct}(\rho)\) we define

\[ F \mapsto \text{Appr}_{n,\rho \to \sigma}(F), \]
\[ \text{Appr}_{n,\rho \to \sigma}(F)(f) = \text{Appr}_{n,\sigma}(F(\text{Appr}_{n,\rho}(f))). \]
Corollary

(BISH) The structure $A_\rho = (Ct(\rho), \lim_{\rho}, (\text{Appr}_{n,\rho})_{n \in \mathbb{N}})$ is a limit space with approximations, for each $\rho$. Moreover, there exists an enumerable dense subset $D_\rho$ in $(Ct(\rho), \mathcal{T}_{\lim_\rho})$, for each $\rho$.

Proof.

$\rho = \nu$: each $\text{Appr}_n$ is $\lim_{\mathcal{T}_{\text{di}}}$-continuous i.e., $\lim_{\mathcal{T}_{\text{di}}} (m, m_l)$ implies that $\lim_{\mathcal{T}_{\text{di}}} (\text{Appr}_n(m), \text{Appr}_n(m_l))$, since the hypothesis amounts to the sequence $m_l$ being eventually the constant sequence $m$, therefore the sequence $\text{Appr}_n(m_l)$ is eventually the constant sequence $\text{Appr}_n(m)$.

$$\text{Appr}_n(\mathbb{N}) = \{0, 1, \ldots, n\}.$$  

Condition (iv) is written as $\lim_{\mathcal{T}_{\text{di}}} (m, m_l) \rightarrow \lim_{\mathcal{T}_{\text{di}}} (m, \text{Appr}_l(m_l))$. Since the premiss says that the sequence $m_l$ is after some index $l_0$ constantly $m$, then for $l \geq \max(l_0, m)$ we get that the sequence $\text{Appr}_l(m_l)$ is constantly $m$.

The fact that $(Ct(\rho \rightarrow \sigma), \lim_{\rho \rightarrow \sigma}, (\text{Appr}_{n,\rho \rightarrow \sigma})_{n \in \mathbb{N}})$ is a limit space with approximations is a direct consequence of our Theorem. Moreover, by density theorem $D_\rho = \bigcup_{n \in \mathbb{N}} \text{Appr}_n(Ct(\rho))$ is an enumerable dense subset of $(Ct(\rho), \mathcal{T}_{\lim_\rho})$, for each $\rho$.  

Iosif Petrakis  
Limit Spaces with Approximations
is a countable base of a topology $T$ on $C$. The space $(C, T)$ is a $T_1$, compact space with a countable base of clopen sets, and without isolated points. Consequently,

$$\lim_{T} (\alpha, \alpha_n) \leftrightarrow \forall_k \exists n_0 \forall_{n \geq n_0} (\alpha_n(k) = \alpha(k))$$

$$\leftrightarrow \forall_k \exists n_0 \forall_{n \geq n_0} (\overline{\alpha_n}(k) = \overline{\alpha}(k)),$$

for each $\alpha \in C$ and $\alpha_n \in C^\mathbb{N}$. We define the approximation functions $\text{Appr}_n : C \to C$ by

$$\alpha \mapsto \text{Appr}_n(\alpha),$$

$$\text{Appr}_n(\alpha) = \overline{\alpha}(n + 1) \ast \overline{0}.$$
The functionals over the Cantor space $C$

\[ \nu = C | \rho \to \sigma, \]
\[ C(\nu) := (C, \lim_{\tau}), \]
\[ C(\rho \to \sigma) := (C(\rho) \to C(\sigma), \lim_{\rho \to \sigma}), \]

and supply these spaces with the approximation functions $\text{Appr}_{n,t}$ as defined above, and the arrow functions $\text{Appr}_{n,\rho \to \sigma}$, we get the following corollary:

**Corollary**

(BISH) The structure $A_{\rho} = (C(\rho), \lim_{\rho}, (\text{Appr}_{n,\rho})_{n \in \mathbb{N}})$ is a limit space with approximations, for each $\rho$. Moreover, there exists an enumerable dense subset $D_{\rho}$ in $(C(\rho), T_{\lim_{\rho}})$, for each $\rho$. 
Compact metric spaces in BISH

If \((X, d)\) is a metric space, a set \(Y \subseteq X\) is called an \(\epsilon\)-approximation to \(X\), if

\[
\forall x \in X \exists y \in Y (d(x, y) < \epsilon).
\]

A metric space \((X, d)\) is totally bounded, if for each \(\epsilon > 0\) there exists some \(Y \subseteq X\) s.t. \(Y\) is a finite \(\epsilon\)-approximation to \(X\), and it is compact, if it is complete and totally bounded.

**Lemma**

(BISH) If \((X, d)\) is an inhabited compact metric space and \(r \in (0, \frac{1}{2}]\), there exist sequences \((x_u)_{u \in 2^{<\mathbb{N}}}\) and \(\gamma \in \mathcal{S}\) such that, for each \(n \geq 1\), we have that

(i) \(\{x_u \mid |u| = \gamma(n)\}\) is an \(r^n\)-approximation to \(X\).

(ii) \(|u| = \gamma(n) \rightarrow \forall w \in 2^{<\mathbb{N}} (d(x_u, x_{u \cdot w}) < \frac{r^{n-1}}{1-r})\).

(iii) \(|u| = \gamma(n) \rightarrow d(x, x_u) < r^{n-1} - r^{n+1} \rightarrow \\
\rightarrow \exists w \in 2^{<\mathbb{N}} (|u \cdot w| = \gamma(n + 1) \land d(x, x_{u \cdot w}) < r^{n+1})\).

(iv) \(|u| = \gamma(n) \rightarrow |u \cdot w| < \gamma(n + 1) \rightarrow x_{u \cdot w} = x_u\).

Note that \(u \cdot w\) denotes the concatenation of the finite sequences \(u, w\), and that the proof of the above lemma uses for the definition of \(\gamma\) the principle of dependent choices on \(\mathbb{N}\).
Proposition

If \((X, d)\) is an inhabited compact metric space, \(\lim_d\) is the limit relation induced by its metric \(d\), and \(\text{Appr}_n : X \to X\) is defined, for each \(n\), by

\[
\text{Appr}_n(x) = \begin{cases} 
  x_{\min \{u \in 2^{<\mathbb{N}} | x_u \in \text{Appr}_n(X) \wedge d(x, x_u) < n\}} & \text{if } x \notin \text{Appr}_n(X) \\
  x & \text{if } x \in \text{Appr}_n(X),
\end{cases}
\]

where \(<\) is any fixed total ordering on \(2^{<\mathbb{N}}\), and

\[
\text{Appr}_n(X) = \{x_u \mid |u| = \gamma(n)\}
\]

and the sequences \((x_u)_{u \in 2^{<\mathbb{N}}}\) and \(\gamma \in S\) are determined in the Lemma, then the structure \(A = (X, \lim_d, (\text{Appr}_n)_{n \in \mathbb{N}})\) is a limit space with general approximations.
Proof

The property $\text{Appr}_n(\text{Appr}_n(x)) = \text{Appr}_n(x)$ follows automatically by the definition of $\text{Appr}_n(x)$. The fact that $\text{Appr}_n(X)$ is finite follows by the finiteness of the set of nodes in $2^{<\mathbb{N}}$ of fixed length $\gamma(n)$. Finally we show that $\lim d(x, x_n) \rightarrow \lim d(x, \text{Appr}_n(x_n))$.

The premiss is

$$\forall \epsilon > 0 \exists n_0 \forall n \geq n_0 (d(x, x_n) < \epsilon),$$

while the conclusion amounts to

$$\forall \epsilon > 0 \exists n_0 \forall n \geq n_0 (d(x, \text{Appr}_n(x_n)) < \epsilon).$$

We fix some $\epsilon > 0$, and by the unfolding of the premiss we find $n_0(\frac{\epsilon}{2})$ such that $d(x, x_n) < \frac{\epsilon}{2}$, for each $n \geq n_0(\frac{\epsilon}{2})$. Also, there is some $n_1$ such that $r^n < \frac{\epsilon}{2}$, for each $n \geq n_1$. For each $n \geq \max(n_0(\frac{\epsilon}{2}), n_1)$ we have that

$$d(x, \text{Appr}_n(x_n)) \leq d(x, x_n) + d(x_n, \text{Appr}_n(x_n))$$

$$< \frac{\epsilon}{2} + r^n$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$
The functionals over a compact metric space \( \mathcal{X} \)

\[ \iota = \mathcal{X} | \rho \rightarrow \sigma, \]

\[ X(\iota) := (\mathcal{X}, \lim \rho \downarrow), \]

\[ X(\rho \rightarrow \sigma) := (X(\rho) \rightarrow X(\sigma), \lim \rho \rightarrow \sigma), \]

and add to these spaces the approximation functions \( \text{Appr}_{n, \iota} \) as defined above, and the arrow functions \( \text{Appr}_{n, \rho \rightarrow \sigma} \), we get directly by the fact that \( \text{Gappr} \) is cartesian closed the following corollary.

**Corollary**

*The structure \( \mathcal{A}_\rho = (X(\rho), \lim \rho, (\text{Appr}_{n, \rho})_{n \in \mathbb{N}}) \) is a limit space with general approximations, for each \( \rho \). Moreover, there exists an enumerable dense subset \( D_\rho \) in \((X(\rho), T_{\lim \rho})\), for each \( \rho \).*

Of course, we could use a type system where the base types are determined by more than one compact metric spaces and have a similar result similar.

**Remark**

*(CLASS) A metric space \((X, d)\) is a sequential space.*
A corollary of Kisyński’s theorem

Kisyński’s theorem suffices to prove classically that all limit spaces in the above hierarchies are topological, since all of them satisfy the uniqueness property.

**Corollary**

(CLASS) (i) If \( f : (X, \mathcal{T}_{\text{lim}X}) \rightarrow (Y, \mathcal{T}_{\text{lim}Y}) \) is continuous and \( (Y, \text{lim}Y) \) has the uniqueness property, then \( f : (X, \text{lim}X) \rightarrow (Y, \text{lim}Y) \) is lim-continuous.

(ii) If \( (X, \text{lim}) \) is a limit space and \( (Y, \text{lim}Y) \) has the uniqueness property, then

\[
\mathbb{C}(X, Y) = X \rightarrow Y,
\]

where \( \mathbb{C}(X, Y) \) denotes the set of continuous functions from \( X \) to \( Y \) w.r.t. the topologies induced by the corresponding limits.
The use of probability distributions first in the study of hierarchies of functionals over $\mathbb{R}$ in Normann 2008 following the work of DeJaeger 2003.

We study the notion of a positive probabilistic projection adding the property of positivity to Normann’s notion of probabilistic projection.

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The probabilistic projections proved to exist by Normann are actually positive ones.

Through positivity the notion of a probabilistic projection is connected to general approximation limit spaces.
Suppose that \((Y, \mathcal{T})\) is a **sequential** topological space, \(A_n\) is an inhabited finite subset of \(Y\), for each \(n \in \mathbb{N}\), and 
\[
A = \bigcup_{n \in \mathbb{N}} A_n \subseteq X \subseteq Y.
\]

A **probabilistic projection** from \(Y\) to \(X\) is a sequence of functions 
\[
\mu_n : Y \to \mathbb{R}(A_n, [0, 1]) \quad y \mapsto \mu_n(y),
\]

(P_1) \(\mu_n(y) : A_n \to [0, 1]\) is a probability distribution on \(A_n\), for each \(n \in \mathbb{N}\) i.e., it satisfies the condition 
\[
\sum_{a \in A_n} \mu_n(y)(a) = 1.
\]

(P_2) The function \(\hat{a} : Y \to [0, 1]\) defined by 
\[
y \mapsto \mu_n(y)(a)
\]

is **continuous**, for each \(a \in A_n\) and for each \(n \in \mathbb{N}\).

(iii) For each \(x \in X\), \(x_n \subseteq X\) such that \(\lim(x, x_n)\) and for each \(a_n \subseteq A\) such that \(a_n \in A_n\), for each \(n \in \mathbb{N}\), we have that 
\[
\forall_n (\mu_n(x_n)(a_n) > 0) \to \lim_{\mathcal{T}}(x, a_n),
\]

where \(\lim_{\mathcal{T}}\) is the limit relation on \(X\) induced by the limit relation \(\lim_{\mathcal{T}}\) on \(Y\).
We denote a probability projection by $\mathcal{P} = (Y, \mathcal{T}, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$, while the sequence of sets $(A_n)_{n \in \mathbb{N}}$ is called the **support** of $\mathcal{P}$.

We call a probabilistic projection from $Y$ to $X$ **general**, if conditions $(P_1)$ and $(P_3)$ are satisfied but not necessarily the continuity condition $(P_2)$.

We call a (general) probabilistic projection from $Y$ to $X$ **positive**, if

$$\mu_n(a)(a) > 0,$$

for each $a \in A_n$ and for each $n \in \mathbb{N}$. 
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for each $a \in A_n$ and for each $n \in \mathbb{N}$. 
Density theorem

Next natural density theorem explains why $Y$ is considered sequential. Without this hypothesis we can only conclude that $A$ is lim-dense in $(X, \lim\mathcal{T})$.

**Proposition**

Suppose that $Y$ is a sequential space, $X$ is a closed (or open) subspace of $Y$ and $\mathcal{P} = (Y, \mathcal{T}, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a (general) probability projection from $Y$ to $X$. Then $A$ is dense in $X$ with the relative topology.

**Proof.**

Since $\lim(x, x)$, by $(P_3)$ we get $\lim(x, a_n)$, for some $a_n$ such that $\mu_n(x)(a_n) > 0$, for each $n \in \mathbb{N}$. There is always such an $a_n$, since $\mu_n(x)$ is a probability distribution on $A_n$. Thus, $A$ is $\lim_{\mathcal{T}'}$-dense in $X$, where $\mathcal{T}'$ is the relative topology of $Y$ on $X$. Since a closed (or open) subspace of a sequential space is also sequential, $A$ is $\mathcal{T}'$-dense. □
A **lim-probabilistic projection** $\mathcal{P} = (Y, \lim, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ where $(Y, \lim)$ is a limit space, $X, A$ are as above and the functions $(\mu_n)_{n \in \mathbb{N}}$ satisfy $(P_1), (P_3)$ and $(P_4)$. The function $\hat{a} : Y \to [0, 1]$ defined by $y \mapsto \mu_n(y)(a)$ is lim-continuous, for each $a \in A_n$ and for each $n \in \mathbb{N}$, i.e.,

$$\lim(y, y_m) \to \lim(\mu_n(y)(a), \mu_n(y_m)(a)),$$

**Proposition**

(BISH) (i) If $(Y, \mathcal{T}, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a (positive) probabilistic projection, then $(Y, \lim_{\mathcal{T}}, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a (positive) lim-probabilistic projection.

(BISH) (ii) If $(Y, \lim)$ is a topological limit space and $(Y, \lim, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a (positive) lim-probabilistic projection, then $(Y, \lim_{\mathcal{T}}, X, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a (positive) probabilistic projection.
A **probabilistic selection** on a sequential space $Y$ is a probabilistic projection from $Y$ to $Y$, and we denote it by $(Y, \mathcal{T}, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$.

Normann: “a probabilistic selection from a dense subset may replace the use of a continuous or even effective selection of a sequence from a dense subset converging to a given point, when such topological selections are impossible”.

A **lim-probabilistic selection** on a limit space $Y$ is a lim-probabilistic projection from $Y$ to $Y$, and we denote it by $(Y, \text{lim}, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$.

Next proposition shows the connection of positive probabilistic selections to limit spaces with general approximations.
A probabilistic selection on a sequential space $Y$ is a probabilistic projection from $Y$ to $Y$, and we denote it by $(Y, \mathcal{T}, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$.

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Probabilistic selections

1. A **probabilistic selection** on a sequential space $Y$ is a probabilistic projection from $Y$ to $Y$, and we denote it by $(Y, \mathcal{T}, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$.

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3. A **lim-probabilistic selection** on a limit space $Y$ is a lim-probabilistic projection from $Y$ to $Y$, and we denote it by $(Y, \lim, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$.

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Next proposition shows the connection of positive probabilistic selections to limit spaces with general approximations.
Proposition

If $(Y, \lim_{\mathcal{T}}, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a positive lim-probabilistic selection on $Y$, there are approximation functions $\text{Appr}_n$ such that $(Y, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})$ is a limit space with general approximations and $\text{Appr}_n(X) = A_n$, for each $n$.

Proof.

Suppose that each $A_n$ is given with a fixed modulus of finiteness $e_n$. We define

$$\text{Appr}_n(x) = \begin{cases} a_{i_0}, & \text{if } x \notin A_n \\ x, & \text{if } x \in A_n, \end{cases}$$

where

$$i_0 = \min\{j \in \mathbb{N} \mid e_n(a) = j \land \mu_n(x)(a) > 0\}.$$  

Clearly, $\text{Appr}_n(X) = A_n$ by the second case of the above definition. The condition $\text{Appr}_n(\text{Appr}_n(x)) = \text{Appr}_n(x)$ is also satisfied by definition. Suppose next that $x \in X, x_n \subseteq X$ and $\lim(x, x_n)$. We also have that

$$\mu_n(x)(\text{Appr}_n(x)) > 0,$$

since, if $x \notin A_n$, then by definition $\mu_n(x)(a_{i_0}) > 0$, while if $x \in A_n$, we have $\mu_n(x)(x) > 0$ by the positivity condition. Hence, $\mu_n(x_n)(\text{Appr}_n(x_n)) > 0$ for each $n$. By (P3) we conclude $\lim(x, \text{Appr}_n(x_n))$. 

Iosif Petrakis
Limit Spaces with Approximations
Conversely

Proposition

(i) (BISH) A limit space with general approximations \((X, \lim, (\text{Appr}_n)_{n \in \mathbb{N}})\) induces a positive, general lim-probabilistic selection \((X, \lim, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})\) on \(X\).

(ii) (CLASS) A limit space with approximations \((X, \lim!, (\text{Appr}_n)_{n \in \mathbb{N}})\) that satisfies the uniqueness property induces a positive lim-probabilistic selection \((X, \lim, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})\) on \(X\).

Proof.

(i) We define \(A_n = D_n = \text{Appr}_n(X)\) and \(x \mapsto \mu_n(x)\), where

\[
\mu_n(x)(a) = \begin{cases} 
1 & \text{if } a = \text{Appr}_n(x) \\
0 & \text{ow.}
\end{cases}
\]

Clearly \(\mu_n(x)\) is a probability distribution on \(A_n\), and, since \(\mu_n(x_n)(a_n) > 0 \iff a_n = \text{Appr}_n(x_n)\), we get \(\lim(x, x_n) \rightarrow \lim(x, a_n)\). Also, \(\mu_n(a)(a) = 1 > 0\), since \(a = \text{Appr}_n(x)\), for some \(x \in X\), therefore, \(\text{Appr}_n(a) = \text{Appr}_n(\text{Appr}_n(x)) = a\).

(ii) Suppose that \(\lim(x, x_m)\) and \(\mu_n(x)(a) = 1 \iff a = \text{Appr}_n(x)\). The sequence \((\text{Appr}_n(x_m))_m\) is eventually constant \(a\). Thus, \(\mu_n(x_m)(a)\) is eventually constant 1. The case \(a \neq \text{Appr}_n(x)\) is treated similarly.
Existence of a probabilistic projection

**Proposition**

(CLASS) Suppose that $(X, d)$ is a separable metric space where $A = \{a_1, a_2, \ldots, \}$ is a countable dense subset of $X$. If we define $A_n = \{a_1, \ldots, a_n\}$ and, for each $1 \leq j \leq n$,

$$\mu_n(x)(a_j) := \frac{(d(x, A_n) + 2^{-n}) \div d(x, a_j)}{\sum_{i=1}^{n}[(d(x, A_n) + 2^{-n}) \div d(x, a_i)]},$$

where $d(x, A_n) = \min\{d(x, a_i) \mid 1 \leq i \leq n\}$ is the distance of $x$ from $A_n$ and $a \div b := \max(a - b, 0)$, then $(X, T_d, (A_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}})$ is a positive probabilistic selection on $X$. 

Iosif Petrakis

Limit Spaces with Approximations
The product of lim-probabilistic selections does not preserve the continuity condition.

Normann 2009 defined $Q$-spaces: sequential Hausdorff spaces with a countable pseudo-base of closed sets, to show that for semi-convex $Y$, and $X, Y$ are $Q$-spaces with probabilistic selection, then $X \to Y$ is a $Q$-space with a probabilistic selection.

The existence of dense subsets in the product $X \times Y$ and the function spaces $X \to Y$ is direct by the fact that $X, Y$ are limit spaces with general approximations. It suffices that $X, Y$ are sequential spaces admitting positive lim-probabilistic selections.

Of course, the results of Normann on $Q$-spaces are of independent interest and value.

The limit spaces with approximations are useful in the separable non-compact case too.

There are more related results but even more open questions.
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