The Role of the Fifth Postulate in the Euclidean Construction of Parallels[1]

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Abstract: We ascribe to the Euclidean Fifth Postulate a genuine constructive role, which makes it absolutely necessary in the parallel construction. In order to do this, we provide a reconstruction of the general principles of a Euclidean construction of a geometric property. As a consequence, the epistemological role of Euclidean constructions is revealed. We also give some first philosophical implications of our interpretation to the relation between Euclidean and non-Euclidean geometries. The Bolyai construction of limiting parallels is shortly discussed from the reconstructed Euclidean point of view.

1 The Standard Interpretation of the Fifth Postulate

From Proclus up to our days a hermeneutic tradition regarding the Fifth Postulate (FP) has been developed, which we call the *Standard Interpretation* (SI). According to it, the Euclidean FP, though differently formulated, actually asserts that through a given point outside a given straight line at most a unique parallel straight line can be drawn to it. This formulation, commonly known as Playfair's Axiom (PA), is logically equivalent to the original FP. Since a parallel line exists independently from PA, addition of PA establishes the existence of exactly one such parallel. Expression of the SI predominance is that PA was made the standard form of the FP in the axiomatic presentations of Euclidean geometry.

In order to describe SI and its shortcomings we give briefly the Euclidean line of presentation of the parallel construction in a formal scheme compatible to our later reconstruction.

If a, b and c are Euclidean coplanar straight lines, we define the following geometric properties:

$$T(a,b,c) \leftrightarrow c \text{ falls on } a \text{ and } b,$$

$$Q_b(a) \leftrightarrow a \text{ is parallel to } b,$$

$$P_{b,c}(a) \leftrightarrow T(a,b,c) \text{ and } c \text{ makes the alternate angles equal to one another.}$$

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The first major step in the Euclidean parallel construction is Proposition 27 of Book I of the Elements.

Proposition I.27 (Criterion of Parallelism): $P_{b,c}(a) \rightarrow Q_b(a)$.

Proposition I.28 contains two more criteria of parallelism reducible to the one of Proposition I.27. In Proposition I.29 the converse implication is established.

Proposition I.29: Let a, b, c, such that $T(a, b, c)$, then

$$Q_b(a) \rightarrow P_{b,c}(a).$$

In Proposition I.29 Euclid uses the FP for the first time. Its original formulation is the following:

Euclidean Fifth Postulate: If $T(a, b, c)$ and c makes the interior angles less than two right angles ($2\llcorner$), then a, b, if produced indefinitely, meet on that side on which are the angles less than $2\llcorner$.

Proposition I.29 is required in the proof of Proposition I.30, a proposition crucial for the development of the SI, since it proves the uniqueness of the parallel line. This result though, is not included in the Elements.

Proposition I.30: If $Q_b(a)$ and $Q_b(c)$, then $Q_c(a)$.

Next proposition is the construction of the parallel line.

Proposition I.31: Construction of a straight line a, through a given point A outside line b, such that $Q_b(a)$.

Its proof consists in the construction of lines c and a, such that $P_{b,c}(a)$. Then, by Proposition I.27, $Q_b(a)$ holds too.

Within SI the construction of Proposition I.31 requires only Proposition I.27 therefore, it is independent from the FP. So, it could be placed right after Proposition I.27 and before Proposition I.29. This accepted independence of the FP from the parallel construction is one of the reasons why mathematicians, before the emergence of non-Euclidean geometries, used to consider the FP as a theorem rather than as a Postulate.

In SI the place of the parallel construction after the first use of the FP is explained, though not with absolute certainty, as an expression of Euclid's need, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn[2]. If it was placed right after Proposition I.27, then only the existence of the parallel line would be established. For the SI the Euclidean line of presentation certifies the existence and the uniqueness of the parallel line. Within SI the "true" meaning of the FP is the expression of uniqueness for the parallel line. It is this emphasis of the SI on the uniqueness of the parallel line, which pushed it forward as a central characteristic of

[2] See Heath, vol. 1, p.316. Actually this is Proclus' argument, as expressed in Proclus Commentary (pp. 295-6).

Euclidean geometry. Gradually, the difference between Euclidean geometry and non-Euclidean geometries was identified, roughly, with the different number of parallels they permit.

The uniqueness interpretation though, is in our view inadequate. In the first place, there is no explanation within SI why Euclid preferred his formulation of the FP than the uniqueness assumption. Also, study of the Elements shows that Euclid seems indifferent to questions of uniqueness. In the First Postulate (construction of a line segment between two points) there is no mention of the uniqueness of the segment, though it is used in Proposition I.4 in the form: two straight lines cannot enclose a space. The circle of the Third Postulate (construction of a circle of any center and radius) is not mentioned to be unique either. Examination of the perpendicular constructions of Propositions I.11 and I.12 reveals the aforementioned Euclidean attitude too.

2 The basic principles of a Euclidean construction and the constructive role of the Fifth Postulate

The first three Euclidean constructions have a direct constructive role: they provide the fundamental elements for the subsequent line and circle constructions. We believe that the Fourth and the Fifth Postulate have an indirect, though genuine, constructive role. They are less elementary, participating in the less elementary parallel construction.

The constructive role of the Fourth Postulate: It is used in Proposition I.16 (through Proposition I.15), which is necessary in the proof of Proposition I.27. By this line of thought, it participates in the construction of Proposition I.31. Also, by the Fourth Postulate, the right angle is a fixed and universal standard, to which other angles can be compared. The FP, treating the $2\llcorner$ as a fixed quantity, "depends" on the Fourth Postulate.

To reveal the constructive character of the FP, we need to understand the conceptual requirements of ancient Greek mathematics regarding the nature of geometric construction as they are embodied in the Euclidean Elements. These requirements are not explicitly found in Euclid, but we consider them as an accurate reconstruction of the Euclidean constructive spirit.

The Basic Principles of the Euclidean Construction $K(P)$ of a geometric property P:

K1: Construction $K(P)$ is the construction $K(a, P)$ of a geometric object a satisfying geometric property P, i.e.,

$$P(a) \text{ and } K(P) = K(a, P).$$

$K(a, P)$ is a construction establishing an abstract object a, satisfying, as accurately as possible, the definition of P[3].

K2: If an object b, satisfying geometric property R, is used in construction $K(a, P)$, then construction $K(b, R)$ must have already been established.

Thus, K2 guarantees that $K(a, P)$ does not contain constructive gaps, i.e., all geometric concepts used in construction $K(a, P)$ are already constructed[4].

K3: If a is a geometric object satisfying P and Q is another geometric property, such that whenever a satisfies P it satisfies Q, but not the converse, i.e.,

$$P(a) \to Q(a) \text{ and } \neg(Q(a) \to P(a)),$$

then $K(a, Q)$ *cannot* be established through $K(a, P)$.

The principle K3 is the most crucial of our reconstruction. It guarantees that the construction of the abstract object a satisfying property Q cannot be established through the construction of the less general property P, i.e., construction $K(a, P)$ respects the generality hierarchy of geometric concepts. For example, the construction of an isosceles triangle cannot be established through the construction of an equilateral triangle, since there are isosceles triangles which are not equilateral[5].

K4: If a is a geometric object satisfying P and Q is another geometric property, such that whenever a satisfies P it satisfies Q, and the converse, i.e.,

$$P(a) \leftrightarrow Q(a),$$

then $K(a, Q)$ *can* be established through $K(a, P)$, and the converse.

Thus, K4 guarantees that whenever properties P and Q are logically equivalent, having the same generality, they do not differ with respect to construction. K4 is the natural complement to K3 and they form together the core of the Euclidean constructive method.

In order to understand the use of the above set of principles on the parallel construction and their relation to the FP we shall give some useful definitions.

A construction $K(a, P)$ is called *direct*, if $K(a, P)$ establishes an object a which satisfies completely the definition of P. In that case we call P a *finite* property. A geometric property Q is called *infinite*, if it is impossible to give

[3] The expression "as accurately as possible" in K1 will be evident in section 3. K1 can also be found, though not as explicitly as here, in the intuitionistic literature on the concept of species (intuitionistic property). A constructive principle such as K1 can be detected in Brouwer's notes. Also, for Griss, a species is defined by a property of mathematical objects, but such a property can only have a clear sense after we have constructed an object which satisfies it (see Heyting 1971, p.126). The role of K1 in Brouwer's concept of species is examined in Petrakis 2007.

[4] Though K2 is very natural to accept, it is not trivial. In a sense, Bolyai's construction of limiting parallels violates it; see section 4.

[5] Euclid uses the concept of an isosceles triangle in Proposition I.5, without providing first a construction of it, because this construction is a simple generalization of the equilateral one (Proposition I.1). Evidently, Euclid found no reason to include this, strictly speaking, different, but expected construction.

a direct construction of Q. This impossibility is not a logical one, but just a result of the definition of Q. A construction $K(a, Q)$ is called *indirect*, if $K(a, Q)$ establishes an object a, which satisfies the definition of Q indirectly, i.e., through a logically equivalent, finite property P. Most of Euclidean constructions are direct. For example, at the end of the perpendicular construction of Proposition I.12, Euclid restates the definition of the perpendicular line, showing that he has constructed an object which satisfies completely that very definition. So, the property of a perpendicular line is a finite property.

On the other hand, the parallel property is an infinite property. Euclid defined parallel lines (Definition 23 of Book 1) as straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. It is impossible to give a direct construction of a line parallel to a given one, since we cannot reproduce the above definition. The infinite character of this definition lies in our mental inability to produce a line indefinitely and act as if this product was a completed object. Each moment we know a finite part of the on going line, from which we cannot infer that every extension of it does not meet the given line. The formation of the parallel line never ends.

Euclidean construction of the infinite parallel property: Euclid gradually established (mainly through the Fourth Postulate and Propositions I.16 and I.27) the geometric property $P_{b,c}(a)$, which is a finite property. Given a line b, we can construct directly lines c and a such that $P_{b,c}(a)$ (actually this is the construction of Proposition I.31), using only the direct construction of Proposition I.23 (construction of a rectilinear angle equal to a given one, on a given straight line and at a point on it).
The implication $P_{b,c}(a) \to Q_b(a)$ is established by Proposition I.27, but it would be a violation of K3 if construction $K(a, P_{b,c}(a))$ was considered as construction $K(a, Q_b(a))$. Construction $K(a, P_{b,c}(a))$ can be considered as construction $K(a, Q_b(a))$ only if the converse implication $Q_b(a) \to P_{b,c}(a)$ is proved. Then, P and Q will have the same generality and we can apply K4. That is why Euclid "postponed" the parallel construction, placing it after Proposition I.29, which establishes the converse implication.

The constructive role of the FP: The FP is this (intuitively true) proposition, through which the implication $Q_b(a) \to P_{b,c}(a)$ is established, and then by K4, construction $K(a, P_{b,c}(a))$ of Proposition I.31 is also construction $K(a, Q_b(a))$ of parallels.

Euclid used the FP in the formulation needed, so that the proof of Proposition I.29 requires one only conceptual step, reaching his goal in the most direct way. So, Euclid does not postpone the use of the FP as long as possible[6], recognizing its "problematic" nature. On the contrary, he uses it exactly the moment he

[6] For a recent reference to this long repeated view see Hartshorne.

needs it, revealing in that way its function.

In Euclid, if P is a finite property then $K(P)$ is always given through P itself and not through an equivalent property Q, i.e., K4 is not used in constructions of finite properties. It is used only when an infinite property Q is to be constructed. Otherwise, its function wouldn't be clear.

The indirect construction of an infinite geometric property is not the only way ancient Greeks used to handle an infinite property. If an infinite property Q has no finite equivalent, it may have a special case F with a strong finite character accompanying the infinite one. We call F a *finite-infinite property*. Infinite anthyphairesis (infinite continued fraction) Q is an infinite property studied in Book X of the Elements, which does not have a finite equivalent. Periodic anthyphairesis (periodic continued fraction) F is a special case of Q, which possesses a strong finite character beside its infinity. Although the sequence of the quotients forming the periodic continued fraction never ends (infinity of F), its finite period expresses our knowledge of this sequence (finite character of F)[7].

3 The epistemological role of Euclidean constructions

Our description of the Euclidean constructive principles reveals also the difference between Euclidean construction and Euclidean existence. We use the following notation:

$$\exists a Q(a): \text{ there exists geometric object } a \text{ satisfying geometric property } Q.$$

In Euclid $\exists a Q(a)$ is established either by $K(a, Q)$ or by $K(a, P)$, where $P(a) \rightarrow Q(a)$ but not the converse. Euclidean geometry is (except, e.g., Eudoxus' theory of ratios) the basic paradigm of a constructive mathematical theory, since existence of a mathematical object or concept is constructively established. For example, if the construction of Proposition I.31 was placed right after Proposition I.27, that would only show the existence of a parallel line. This proof of existence though, does not constitute construction of the parallel line.

The traditionally accepted independence between the FP and the construction of Proposition I.31 is based on the identification between $\exists a Q(a)$ and $K(a, Q)$[8]. For Euclid though, construction of property Q is generally an enterprise larger than the exhibition-construction of a single object satisfying Q.

[7] Ancient Greeks had also found a necessary and sufficient condition for an infinite anthyphairesis to be periodic (logos criterion). Its knowledge and its importance in Plato's system have been developed in recent times in Negrepontis' program on Plato. See, for example, Negrepontis 2006. In Negrepontis' reconstruction of Plato, the concept of a finite-infinite property is of central importance.

[8] According to Zeuthen 1896, the main purpose of a geometric construction is to provide a proof of existence, so the purpose of the FP is to ensure the existence of the intersection point of the non parallel lines. This approach fails to see though, the difference between existence and construction.

Parallel construction shows this fact very clearly. We safely reach the following conclusions:

$\exists a Q(a)$ shows that property Q is not void, that it possesses, in modern terms, an extension. Therefore, it is meaningful to study it. On the other hand, $K(a, Q)$ shows that we have found a way to grasp mentally property Q, fully if Q is finite, as much as possible if Q is infinite.

Traditionally, the Elements are considered as the original model of the axiomatic method and logical deduction. In our view, they are also, and even more, the model of the constructive method. It is this combination of the axiomatic and the constructive method that reflects the philosophical importance of the Elements. For the first time in the history of mathematics a mathematical theory answers simultaneously the ontological and the epistemological problem of the mathematical concepts involved. The ontology of Euclidean geometric objects and concepts is of mental (and not empirical) nature. Almost certainly Euclidean ontology is Platonic ontology[9]. This mental ontology of mathematical concepts imposes the constructive method. It is the construction of mathematical concepts which provides their study with a firm epistemology.

Euclid does not only care about the logical relations between geometric concepts and objects. He also needs to answer the main epistemological question: *how do we understand the concepts that we employ in our deductions*? And his answer is: *we understand them because we construct them.* So, *geometric constructions form the indispensable epistemology of Euclidean geometry*[10].

4 The relation between Euclidean and non-Euclidean Geometries

It is impossible here to study fully the relation between Euclidean geometry (EG) and non-Euclidean geometries (n-EG). We shall only stress some points which derive directly from our previous analysis.

There is here too a traditional view regarding the above relation. According to it, EG and n-EG can be seen as mathematical structures of the same kind, differing only in the number of parallels. One such common mathematical framework is the Hilbert plane concept[11]. A Hilbert plane (HP) is a system of

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[9] Euclid was a Platonist and his definitions are closely related to the Platonic ones (see Heath p.168). The most accurate description of the Elements would be: Platonic Euclidean geometry. A Kantian ontological foundation of geometrical objects and concepts would transform the same corpus of results and constructions into Kantian Euclidean geometry.

[10] For a recent discussion on the role of Euclidean constructions see Harari 2003. Unfortunately, the interpretation proposed there is, in our opinion, unsatisfactory. Also, Knorr's arguments on the subject (see Knorr 1983) are not, in our view, satisfactory too.

[11] This framework is not as absolute as it is often named, since it does not contain

points, lines and planes satisfying the well known Hilbert axioms of incidence, betweenness and congruence. In a HP the parallel line (as any other geometric property) is not constructed, only its existence is established. A HP is neutral with respect to the uniqueness of the parallel line. A Euclidean plane is a HP permitting one only parallel and a hyperbolic plane is a HP permitting more than one parallels. The consequences of this "coexistence" of EG and n-EG were very serious. Foundations of mathematics and mathematics itself were influenced immensely from the loss of the a priori character of EG. EG became just one possible geometry. Kantian a priori suffered a serious blow and especially the a priori of space. As a result of this, all major foundational programs rested either on a Kantian a priori of discrete nature or on a purely logical substratum[12].

Our reconstruction of the parallel construction suggests a strong rejection of the traditional view. In our opinion, EG has a certain constructive character, which n-EG lack. Of course, this opinion echoes Kant. In 1995 Webb remarks[13]:

> [It was a commonplace of older Kantian scholarship that the discovery of non-euclidean geometry undermined his theory of the synthetic a priori status of geometry. It is commonplace of newer Kant scholarship that he already knew about non-euclidean geometry from his friend Lambert, one of the early pioneers of this geometry, and that in fact its very possibility only reinforces Kant's doctrine that euclidean geometry is synthetic a priori because only its concepts are constructible in intuition.]

The common HP language (or any other common mathematical framework) ignores the role and the necessity of the FP in the parallel construction just as the epistemological role of constructions. Modern geometry generally, seems quite indifferent to epistemological questions.

We can only indicate here that EG and n-EG are not directly comparable, from the constructive point of view. Therefore, EG has not lost its a priori character. To show that the Euclidean concepts are the only (mentally) constructible ones is a big enterprise. We shall only describe here why Bolyai's construction of limiting parallels is unacceptable from the Euclidean point of view. A hyperbolic plane (LP) is a HP satisfying the following axiom:

Lobachevsky's axiom (L): If a is a line and A is a point outside a, there exist rays Ab, Ac, not on the same line, which do not intersect a, and each ray Ad in the angle bAc intersects a.

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the elliptic plane, in which there exist no parallels at all, and every line through the pole of a given line is perpendicular to it. Hilbert's classic work is still the best introduction to Hilbert planes (Hilbert 1971). A more absolute framework, which contains elliptic geometry, is the concept of a Bachmann plane, or metric plane (see Bachmann 1973).

[12] Putnam's assessment (Putnam 1975, p.x) is characteristic: "...the overthrow of EG is the most important event in the history of science for the epistemologist".

[13] See Webb 1999, p.1.

For the Bolyai's construction we need the following propositions:

Proposition 4.1: A triangle in a hyperbolic plane has angle sum less than $2\llcorner$.

A quadrilateral $PQRS$ is a *Lambert quadrilateral*, if it has right angles at P, Q and S.

Proposition 4.2: In a hyperbolic plane the fourth angle (the angle at R) of a Lambert quadrilateral $PQRS$ is acute, and a side adjacent to it is greater than its opposite side ($QR > PS$ and $SR > PQ$).

Proposition 4.3: Suppose we are given a line a and a point P not on a, in a hyperbolic plane. Let PQ be the perpendicular to a. Let m be a line through P, perpendicular to PQ. Choose any point R on a, and let RS be the perpendicular to m. If Pc is a limiting parallel ray intersecting RS at X, then $PX = QR$.

Elementary Continuity Principle (ECP): If one endpoint of a line segment is inside a circle and the other outside, then the segment intersects the circle.

Bolyai's construction of limiting parallel: Consider a hyperbolic plane satisfying ECP. Suppose we are given a line a and a point P not on a. Let PQ be the perpendicular to a. Let m be a line through P, perpendicular to PQ. Choose any point R on a, and let RS be the perpendicular to m. Then the circle of radius QR around P will meet the segment RS at a point X, and the ray PX will be the limiting parallel ray to a through P.

Proof. Since $Q = \llcorner$, $PR > QR$, and from Proposition 4.1 the angle at Q is the largest angle in triangle PQR. Also, $PS < QR$, since $PQRS$ is a Lambert quadrilateral satisfying Proposition 4.2. Therefore, endpoints R and S of segment RS are outside and inside circle (P, QR) and, by ECP, segment RS intersects (P, QR) at a (unique) point X. PX is the limiting parallel ray to a through P, since L guarantees its existence and by Proposition 4.3 we know that it satisfies $PX = QR$.

The curious feature of the above proof, namely that we prove that this construction works only by first assuming (via L) that the object we wish to construct already exists, is common knowledge[14]. But the presupposed existence of the limiting parallel is axiomatic and not constructive; therefore, Bolyai's construction violates the Euclidean Principle K2.

Another aspect of the problematic character of Bolyai's construction is related to constructive principles K3 and K4. Proposition 4.3 is in analogy to Proposition I.29, since it can be written in the form

$$\mathrm{L} \rightarrow PX = QR.$$

In our language, L is an infinite property and $PX = QR$ is a finite one. In order to consider, from the Euclidean point of view, the direct construction of X as

[14] See, for example, Hartshorne, p.398.

the construction of the limiting ray, we have to prove directly, in a hyperbolic plane satisfying ECP, the analogue to Proposition I.27:

$$PX = QR \to L.$$

Such a direct proof has not yet been found. Therefore, although the above line and circle construction of the most important concept of hyperbolic geometry shows Bolyai's constructive sensitivity, it does not satisfy the constructive principles of the Euclidean parallel construction.

The usual proof of the existence of limiting parallel is based on Dedekind's continuity axiom[15].

Dedekind's Continuity Axiom (D): Any (set theoretical) separation of points on a line (i.e., a Dedekind cut) is produced by a unique point.

Axiom D is highly problematic from the Euclidean point of view. Its set theoretical nature is highly non constructive. So, the question, whether Bolyai's construction could be used to prove the existence of limiting parallel for a system of axioms that includes ECP but does not include D, was naturally raised by Greenberg[16].

Pejas, working in the framework of Bachmann plane geometry, a geometry without betweenness and continuity axioms, succeeded to classify all Hilbert planes[17]. Greenberg, using Pejas' classification of Hilbert planes succeeded in answering his question positively[18].

Proposition 4.4 (Pejas-Greenberg): If the ECP holds and the fourth angle of a Lambert quadrilateral is acute, then Bolyai's construction gives the two lines through P that have a "common perpendicular at infinity" with a through the ideal points at which they meet a. Among Hilbert planes satisfying the ECP, the Klein models are the only ones which are hyperbolic, and Bolyai's construction gives the asymptotic parallels for them.

An important corollary is the following proposition.

Proposition 4.5: Every Archimedean, non-Euclidean[19] HP in which the ECP holds is hyperbolic.

Though Pejas-Greenberg managed to show that the Bolyai construction does yield the limiting parallel replacing D with more elementary axioms, their proof is indirect, since it is based on a classification theorem.

[15] See, for example, Greenberg 1980, p.156.
[16] See Greenberg 1979a.
[17] A Hilbert plane corresponds to an ordered Bachmann plane with free mobility. As Greenberg puts it (see Greenberg 1979b), Hilbert's approach is thus incorporated into Klein's Erlangen program, whereby the group of motions becomes the primordial object of interest. For Pejas classification theorem see Pejas 1961.
[18] In Greenberg 1979a.
[19] A HP is called non-Euclidean if PA axiom fails.

So, from the (Euclidean) constructive point of view, there is still no direct constructive proof of the concept of limiting parallel[20].

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[20] We conjecture, on philosophical grounds, that such a proof cannot be found.