Dependent sums and dependent products in Bishop's set theory

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Abstract

According to the standard, non type-theoretic accounts of Bishop's constructivism (BISH), dependent functions are not necessary to BISH. Dependent functions though, are explicitly used by Bishop in his definition of the intersection of a family of subsets, and they are necessary to the definition of arbitrary products. In this paper we present the basic notions and principles of CSFT, a semi-formal constructive theory of sets and functions intended to be a minimal, adequate and faithful, in Feferman's sense, semi-formalisation of Bishop's set theory (BST). We define the notions of dependent sum (or exterior union) and dependent product of set-indexed families of sets within CSFT, and we prove the distributivity of $\prod$ over $\sum$ i.e., the translation of the type-theoretic axiom of choice within CSFT. We also define the notions of dependent sum (or interior union) and dependent product of set-indexed families of subsets within CSFT. For these definitions we need to extend BST with two classes, the universe of sets $\mathbb{V}_0$ and the universe of functions $\mathbb{V}_1$.

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1 Introduction

Bishop's original approach to constructive mathematics, developed in his seminal book *Foundations of Constructive Analysis*, was an important motivation to Martin-Löf's type theory (MLTT). Martin-Löf opened his first published paper on type theory ([22], p. 73) as follows.

> The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book of Bishop.

As Martin-Löf explains in [21], p. 13, he got access to Bishop's book only shortly after his own book on constructive mathematics [21] was finished. A surprising historical fact is that the first who considered a type-theoretic system as a formal system for Bishop's book [4] was Bishop himself. In the unpublished manuscript [5] Bishop developed an extensional dependent type theory with one universe as a formal system for his book. In the also unpublished manuscript [6] Bishop elaborated the implementation of his type theory into Algol. A similar pattern is followed in [7], where, influenced by Gödel's Dialectica interpretation, Bishop introduced Σ, a variant of HA^{ω}, as a formal system for his book, and discussed the implementation of Σ into Algol (see [7], p. 70).

The question Q of finding a formal system suitable for Bishop's system of informal constructive mathematics BISH was a major question in the foundational studies of the 1970's. Myhill's system CST, introduced in [25], and later Aczel's CZF (see [1]), Friedman's

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system B, developed in [16], and Feferman's system of explicit mathematics T_0 (see [14] and [15]), are some of the systems motivated by Q, but soon developed independently from it. These systems were influenced a lot from the classical Zermelo-Fraenkel set theory, and could be described as "top-down" approaches to Q, as they have many "unexpected" features with respect to BISH[1]. Beeson's systems S and S_0 in [2], and Greenleaf's system of liberal constructive set theory LCST in [17] were dedicated to Q. Especially Beeson tried to find a faithful and adequate formalisation of BISH, and by including a serious amount of proof relevance to his systems stands in between the set-theoretic, proof-irrelevant point of view and the type-theoretic, proof-relevant point of view.

All aforementioned systems though, were not really "tested" with respect to BISH. Only very small parts of BISH were actually implemented in them, and their adequacy for BISH was mainly a claim, rather than a shown fact. The implementation of Bishop's constructivism within a formal system for it was taken seriously in the type-theoretic formalisations of BISH, and especially in the work of Coquand (see e.g., [10] and [12]), Palmgren (see e.g., [18] and the collaborative work [11]), the Nuprl research group of Constable (see e.g., [33]), and the Minimalist Foundation of Sambin and Maietti (see [35] and [20]).

Bishop's (informal) set theory (BST), developed in Chapter 3 of [4] (or [8]), is reflected in MLTT through the theory of setoids (see especially the work of Palmgren [26]-[29]). The identity type of MLTT (see [23]) though, has no counterpart in BST, a fact with many consequences, as e.g., the existence in MLTT of a free setoid from a given type (see [28], p. 90), a result crucial to the proof of the presentation axiom in MLTT (see [11], p. 75).

Currently, we revisit question Q, aiming at a minimal, adequate and faithful formalisation of BST. For that we elaborate a semi-formal constructive set and function theory (CSFT), as the first necessary step to an adequate and faithful, full formalisation of BST. Although a universe of sets $\mathbb{V}_0$ and a universe of functions $\mathbb{V}_1$ are included in CSFT, and not explicitly mentioned in BST, in section 5 we explain why these classes are implicit in BST.

The standard, non type-theoretic view regarding dependency within BST is that dependent functions are not necessary. Dependent functions though, do appear explicitly in Bishop's definition of the intersection of $\bigcap_{t\in T}\lambda(t)$, where T is an inhabited set and λ is a family of subsets of some set X indexed by T. In [4], p. 65, Bishop writes that

> ..., an element u of $\bigcap_{t\in T}\lambda(t)$ is a rule that associates an element a_t of $\lambda(t)$ to each element t of T, such that ...,

a definition repeated in [8], p. 70. Dependent functions are also necessary to the definition of products of families of sets indexed by an arbitrary set, and can be avoided, if one is restricted to countable products only. Although Bishop himself considered e.g., only countable products of metric spaces, the constructive development of general algebra (see [24]), or general topology (see e.g., [30] and [31]), require the use of arbitrary products, hence the use of dependent functions. As we noted above, Bishop also defined in [5] a notion of dependent types within his type-theoretic system for BISH.

The somewhat "silent" existence of dependency in BISH is replaced by a central presence in CSFT. This is necessary, if we want to make some very basic definitions in BISH precise enough to be formalised.

[1] Using Feferman's terminology from [15], these formal systems are not, in our view, *faithful* to BISH, as they contain concepts or axioms that do not appear, neither explicitly nor implicitly, in BISH. Feferman also introduced the notion of an adequate formalisation T of a body of informal mathematics M. Namely, T is *adequate* for M, if every concept, argument, and result of M is represented by a concept, proof, and a theorem, respectively, of T (see also [2], p. 153).

2 Basic notions of CSFT

In this section we present in an informal and brief manner those fundamentals of CSFT required to the material presented in the following sections. A complete presentation is planned to be included in [32].

The general logical framework of CSFT is that of an intuitioniostic first-order predicate logic with *definitional* equality. The expression[2] $a := b$ is read as "a is by definition equal to b". Similarly, the expression $P :\Leftrightarrow Q$ is read as "P is by definition equivalent to Q". The basic primitives of CSFT are the set of natural numbers $\mathbb{N}$, equipped with its basic equality $=_{\mathbb{N}}$, operations and order, a primitive notion of n-tuple of given objects, for every natural n larger than 2, an undefined notion of *finite routine*, or *construction*, or *algorithm*, and the *assignment routines* $\mathbf{pr}_i(a_1, \ldots, a_n) := a_i$, for every i between 1 and n, and for every n larger than 2, where an assignment routine is defined as a certain finite routine.

A *defined totality* X is defined by a membership condition $\mathcal{M}_X$ i.e., $x \in X :\Leftrightarrow \mathcal{M}_X(x)$, and $\mathcal{M}_X(x)$ is the *membership formula* for X. If X, Y are defined totalities with membership formulas $\mathcal{M}_X$ and $\mathcal{M}_Y$, respectively, we say that X and Y are *definitionally equal*, $X := Y$, if $\big[\mathcal{M}_X(x) :\Leftrightarrow \mathcal{M}_Y(x)\big]$. A *totality* is either the primitive $\mathbb{N}$ or a defined totality. A totality X is called *inhabited*, if there is $x_0 \in X$. A *defined totality with equality* is a defined totality X equipped with an equality condition $\mathcal{E}_X$ i.e., $x =_X y :\Leftrightarrow \mathcal{E}_X(x, y)$, where the *equality formula* $\mathcal{E}_X(x, y)$ satisfies the defining conditions of an equivalence relation. A *defined set* is a defined totality with equality such that the membership formula $\mathcal{M}_X(x)$ for X represents a construction, or a finite routine. A *set* is the primitive $\mathbb{N}$ or a defined set.

E.g., if X, Y are sets, their *product* $X \times Y$ is the defined totality with equality given by

$$z \in X \times Y :\Leftrightarrow \exists_{x \in X} \exists_{y \in Y} \big(z := (x, y)\big),$$

$$z =_{X \times Y} w :\Leftrightarrow \mathbf{pr}_1(z) =_X \mathbf{pr}_1(w) \;\&\; \mathbf{pr}_2(z) =_Y \mathbf{pr}_2(w).$$

For simplicity, we usually write an equality formula, as that for $X \times Y$, as follows: $(x, y) =_{X \times Y} (x', y') :\Leftrightarrow x =_X x' \;\&\; y =_Y y'$. In contrast to MLTT, we allow the use of definitional equality within membership formulas (only). Clearly, if X, Y are sets, then $X \times Y$ is also a set, since the construction of an element of $X \times Y$ is reduced to the construction of an element of X and of an element of Y.

If X, Y are totalities, an *assignment routine* $f : X \rightsquigarrow Y$ from X to Y is a finite routine assigning an element y of Y i.e., $\mathcal{M}_Y(y)$), to each given element x of X i.e., $\mathcal{M}_X(x)$. In this case we write $f(x) := y$. E.g., the assignment routine $\mathtt{pr}_X$ from $X \times Y$ to X is defined by

$$\mathtt{pr}_X(x, y) := \mathbf{pr}_1(x, y) := x,$$

for every $(x, y) \in X \times Y$. If X, Y, Z are totalities, $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ are assignment routines, the *composition* assignment routine $g \circ f : X \rightsquigarrow Z$ is defined by $(g \circ f)(x) := g(f(x))$, for every $x \in X$. If f and g are assignment routines from X to Y, they are *definitionally equal*, $f := g$, if $\forall_{x \in X}\big(f(x) := g(x)\big)$. E.g., for the assignment routine $\mathrm{id}_X : X \rightsquigarrow X$, defined by $\mathrm{id}_X(x) := x$, for every $x \in X$, we have that $f \circ \mathrm{id}_X := f$. If X, Y are sets, we call an assignment routine from X to Y an *operation*, while a *function* $f : X \to Y$ from a set X to a set Y is an extensional operation from X to Y i.e., $f(x) =_Y f(x')$, for every $x, x' \in X$ such that $x =_X x'$. A function $f : X \to Y$ is an *embedding* of X into Y, if $x =_X x'$, whenever

[2] Bishop's notation for definitional equality is $a \equiv b$.

$f(x) =_Y f(x')$. We denote such an embedding by $f : X \hookrightarrow Y$. If X, Y are sets, the defined totality with equality $\mathbb{F}(X, Y)$ of functions from X to Y, defined by

$$z \in \mathbb{F}(X, Y) :\Leftrightarrow z := f : X \to Y,$$

$$f =_{\mathbb{F}(X,Y)} g :\Leftrightarrow \forall_{x \in X}\big(f(x) =_Y f(y)\big),$$

is a set, as $\mathcal{M}_{\mathbb{F}(X,Y)}(z)$ represents a construction. A *subset* of a set X is a couple (A, i_A), where A is a set and $i_A : A \hookrightarrow X$. The *powerset* of X is the defined totality $\mathcal{P}(X)$ of subsets of X with equality defined by

$$(A, i_A) =_{\mathcal{P}(X)} (B, i_B) :\Leftrightarrow \exists_{f:A \to B}\exists_{g:B \to A}\big(i_A \circ g =_{\mathbb{F}(B,X)} i_B \ \& \ i_B \circ f =_{\mathbb{F}(A,X)} i_A\big)$$

$$\begin{tikzcd}
B \arrow[d, "g"'] \arrow[dr, hook, "i_B"] & \\
A \arrow[r, hook, "i_A"] \arrow[dr, "f"'] & X \\
& B. \arrow[u, hook, "i_B"']
\end{tikzcd}$$

If f and g realize the equality between (A, i_A) and (B, i_B) in $\mathcal{P}(X)$, we write $(f, g) : (A, i_A) =_{\mathcal{P}(X)} (B, i_B)$. For simplicity, we may write $A =_{\mathcal{P}(X)} B$ instead of $(A, i_A) =_{\mathcal{P}(X)} (B, i_B)$. To construct an element of $\mathcal{P}(X)$ one needs to construct a set A and an embedding from A to X. This membership condition does not express a construction that can be carried out in a finite time, since there is no known finite algorithm to construct a set. A *class* is a defined totality with equality C such that for the membership formula of C it cannot be accepted constructively that it reflects a construction. Consequently, $\mathcal{P}(X)$ is a class. If $P(x)$ is an *extensional property* on X i.e., a formula satisfying $\forall_{x,y \in X}\big(x =_X y \ \& \ P(x) \Rightarrow P(y)\big)$, the totality with equality X_P is defined by

$$x \in X_P :\Leftrightarrow x \in X \ \& \ P(x),$$

and $x =_{X_P} x' :\Leftrightarrow x =_X x'$. We may also use the notation $\{x \in X \mid P(x)\}$ for X_P. If X is a set, then X_P is a set, and the couple (X_P, i_{X_P}), where $i_{X_P} : X_P \hookrightarrow X$ is defined by $i_{X_P}(x) := x$, for every $x \in X_P$, is in $\mathcal{P}(X)$. We call X_P the *extensional subset* of X generated by $P(x)$. If X is a set, the *diagonal* of X is the set

$$D(X) := \{(x, y) \in X \times X \mid x =_X y\}$$

i.e., the extensional subset of $X \times X$ generated by $P(x, y) :\Leftrightarrow x =_X y$ on $X \times X$. If (A, i_A) and (B, i_B) are subsets of X, their *intersection* $A \cap B$ is defined by

$$A \cap B := \{(a, b) \in A \times B \mid i_A(a) =_X i_B(b)\}.$$

Let $i : A \cap B \rightsquigarrow X$ the assignment routine defined by $i(a, b) := i_A(\mathbf{pr}_1(a, b)) := i_A(a)$, for every $(a, b) \in A \cap B$. The equality on $A \cap B$ is defined by $(a, b) =_{A \cap B} (a', b') :\Leftrightarrow i(a, b) =_X i(a', b')$. It is immediate to show that $=_{A \cap B}$ satisfies the conditions of an equivalence relation and that $A \cap B$ is a set. Moreover, the assignment routine i is an embedding of $A \cap B$ into X, hence the couple $(A \cap B, i)$ is a subset of X. The *union* $A \cup B$ of A and B is the totality defined by $z \in A \cup B :\Leftrightarrow z \in A$ or $z \in B$. If $j : A \cup B \rightsquigarrow X$ is defined by

$$j(z) := \begin{cases} i_A(z) & , z \in A \\ i_B(z) & , z \in B, \end{cases}$$

for every $z \in A \cup B$, we define $z =_{A\cup B} w :\Leftrightarrow j(z) =_X j(w)$. It is immediate to show that $=_{A\cup B}$ satisfies the conditions of an equivalence relation and that $A \cup B$ is a set. Moreover, the assignment routine j is an embedding of $A \cup B$ into X, hence the couple $(A \cap B, j)$ is a subset of X.

The *universe of sets* $\mathbb{V}_0$ is the defined totality with equality defined by

$$X \in \mathbb{V}_0 :\Leftrightarrow X \text{ is a set},$$

$$X =_{\mathbb{V}_0} Y :\Leftrightarrow \exists_{f:X\to Y}\exists_{g:Y\to X}\big(g \circ f =_{\mathrm{id}_X} \ \& \ f \circ g = \mathrm{id}_Y\big)$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "\mathrm{id}_X"] & \\
Y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_Y"'] & X \arrow[d, "f"] \\
& Y.
\end{tikzcd}$$

If the functions f, g realize the equality between X and Y in $\mathbb{V}_0$, we write $(f, g) : X =_{\mathbb{V}_0} Y$. It is easy to show that $X =_{\mathbb{V}_0} Y$ satisfies the conditions of an equivalence relation[3]. The defined totality with equality $\mathbb{V}_0$ is a class, since its membership condition does not reflect a construction. It is also easy to see that if $(f, g) : (A, i_A) =_{\mathcal{P}(X)} (B, i_B)$, then $(f, g) : A =_{\mathbb{V}_0} B$. Since sets and functions in BST are objects that are not reduced to one another, the next defined totality complements naturally the universe of sets $\mathbb{V}_0$ and it is proven instrumental to the formulation of dependency within CSFT. The *universe of functions* $\mathbb{V}_1$ is the defined totality with equality defined by

$$z \in \mathbb{V}_1 :\Leftrightarrow \exists_{X,Y\in\mathbb{V}_0}\exists_{f\in\mathbb{F}(X,Y)}\big(z := (X, Y, f)\big),$$

$$(X, Y, f) =_{\mathbb{V}_1} (Z, W, g) :\Leftrightarrow \exists_{e_{XZ}\in\mathbb{F}(X,Z)}\exists_{e_{ZX}\in\mathbb{F}(Z,X)}\exists_{e_{YW}\in\mathbb{F}(Y,W)}\exists_{e_{WY}\in\mathbb{F}(W,Y)}$$
$$\big((e_{XZ}, e_{ZX}) : X =_{\mathbb{V}_0} Z, \ \& \ (e_{YW}, e_{WY}) : Y =_{\mathbb{V}_0} W \ \& \ e_{YW} \circ f = g \circ e_{XZ}\big)$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e_{XZ}"'] & Y \arrow[d, "e_{YW}"] \\
Z \arrow[r, "g"'] & W.
\end{tikzcd}$$

If the functions e_{XZ}, e_{ZX}, e_{YW} and e_{WY} realize the equality between (X, Y, f) and (Z, W, g) in $\mathbb{V}_1$ we write $(e_{XZ}, e_{ZX}, e_{YW}, e_{WY}) : (X, Y, f) =_{\mathbb{V}_1} (Z, W, g)$. Clearly, $\mathbb{V}_1$ is a class. It is straightforward to show that $(X, Y, f) =_{\mathbb{V}_1} (Z, W, g)$ satisfies the conditions of an equivalence relation. It is also easy to see that if $(f, g) : (A, i_A) =_{\mathcal{P}(X)} (B, i_B)$, then $(f, g, \mathrm{id}_X, \mathrm{id}_X) : (A, X, i_A) =_{\mathbb{V}_1} (B, X, i_B)$.

[3] The defined equality on the universe $\mathbb{V}_0$ expresses that $\mathbb{V}_0$ is *univalent*, as isomorphic sets are equal in $\mathbb{V}_0$. In univalent type theory, which is MLTT extended with Voevodsky's axiom of univalence (see [36]), the existence of a pair of quasi-inverses between types A and B implies that they are equivalent in Voevodsky's sense, and by the univalence axiom, also propositionally equal. The univalence of $\mathbb{V}_0$ in CSFT is not a surprise. Already in BST the type-theoretic axiom of function extensionality is just the defined equality on the function space.

3 Exterior union and dependent products in CSFT

The concept of a family of sets indexed by a (discrete) set was asked to be defined in [4] (Exercise 2, p. 72), and the required definition, attributed to Richman, is included in [8], (Exercise 2, p. 78), where the discreteness-hypothesis is omitted. The definition has a strong type-theoretic flavor, although Richman's motivation was categorical [4]. The concept of a (discrete) set-indexed family of sets is tacitly used in [4] in the definition of a countable product of metric spaces (see also the related comment in [8], p. 125.). We reformulate Richman's definition using the universes $\mathbb{V}_0$, $\mathbb{V}_1$ and the notion of assignment routine.

▶ **Definition 1.** Let I be a set and $D(I)$ its diagonal. A *family of sets* indexed by I, or an I-family of sets, is a couple $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ and $\lambda_1 : D(I) \rightsquigarrow \mathbb{V}_1$ are assignment routines such that for every $(i, j) \in D(I)$

$$\lambda_1(i, j) := (\lambda_0(i), \lambda_0(j), \lambda_{ij})$$

such that for every $i \in I$ we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$, and for every $i, j, k \in I$, satisfying $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\lambda_{ij}"'] \arrow[dr, "\lambda_{ik}"] & \\
\lambda_0(j) \arrow[r, "\lambda_{jk}"'] & \lambda_0(k).
\end{tikzcd}$$

We call I the *index set* of the family Λ, the function λ_{ij} the *transport* function[5] from $\lambda_0(i)$ to $\lambda_0(j)$, and the assignment routine λ_1 the *modulus of function-likeness* of λ_0. If Y is a set and $\lambda_0(i) := Y$, for every $i \in I$, and $\lambda_1(i, j) := (Y, Y, \mathrm{id}_Y)$, for every $(i, j) \in D(I)$, we call Λ the *constant* I-family Y.

Next why see why we used the term modulus of function-likeness for the routine λ_1.

▶ Remark. If $\Lambda = (\lambda_0, \lambda_1)$ is an I-family of sets and $i =_I j$, then $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$.

Proof. By Definition 1 we have that $\lambda_{ii} = \lambda_{ji} \circ \lambda_{ij}$ and $\lambda_{jj} = \lambda_{ij} \circ \lambda_{ji}$. ◀

▶ **Definition 2.** Let $\Lambda^2 := (\lambda_0^2, \lambda_1^2)$, where $\lambda_0^2 : \mathbb{2} \rightsquigarrow \mathbb{V}_0$ with $\lambda_0^2(0) := X$ and $\lambda_0^2(1) := Y$, and $\lambda_1^2 : \{(0,0), (1,1)\} \rightsquigarrow \mathbb{V}_1$ is defined by $\lambda_1^2(0,0) := (X, X, \mathrm{id}_X)$ and $\lambda_1^2(1,1) := (Y, Y, \mathrm{id}_Y)$. We call Λ^2 the $\mathbb{2}$-family of X and Y. The $\mathbb{n}$-family of the sets $X_1, \ldots X_n$, for every $n \geq 1$, is defined similarly. Let $\Lambda^{\mathbb{N}} := (\lambda_0^{\mathbb{N}}, \lambda_1^{\mathbb{N}})$, where $\lambda_0^{\mathbb{N}} : \mathbb{N} \rightsquigarrow \mathbb{V}_0$ with $\lambda_0^{\mathbb{N}}(n) := X_n$, and $\lambda_1^{\mathbb{N}} : \{(n, n) \mid n \in \mathbb{N}\} \rightsquigarrow \mathbb{V}_0$ is defined by $\lambda_1^{\mathbb{N}}(n, n) := (X_n, X_n, \mathrm{id}_{X_n})$, for every $n \in \mathbb{N}$. We call $\Lambda^{\mathbb{N}}$ the $\mathbb{N}$-family of $(X_n)_n$.

Following Beeson's notation in [3], p. 44, we use the type-theoretic notation of $\sum$-types for the exterior union of a set-indexed family of sets.

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[4] In a personal communication Richman referred to the definition of a set-indexed family of objects of a category, given in [24], p.18, as the source of the definition attributed to him in [8], p. 78.

[5] We draw this term from MLTT.

▶ **Definition 3.** Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The *exterior union*, or *disjoint union*, $\sum_{i\in I} \lambda_0(i)$ of Λ is defined by

$$w \in \sum_{i\in I} \lambda_0(i) :\Leftrightarrow \exists_{i\in I}\exists_{x\in\lambda_0(i)}\big(w := (i,x)\big).$$

$$(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \&\ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

▶ Remark. The equality on $\sum_{i\in I} \lambda_0(i)$ satisfies the conditions of an equivalence relation, and $\sum_{i\in I} \lambda_0(i)$ is a set.

Proof. Let $(i,x),(j,y)$ and $(k,z) \in \sum_{i\in I} \lambda_0(i)$. Since $i =_I i$ and $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$, we get $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$, then $j =_I i$ and $\lambda_{ji}(y) = \lambda_{ji}(\lambda_{ij}(x)) = \lambda_{ii}(x) := \mathrm{id}_{\lambda_0(i)}(x) := x$, hence $(j,y) =_{\sum_{i\in I}\lambda_0(i)} (i,x)$. If $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$ and $(j,y) =_{\sum_{i\in I}\lambda_0(i)} (k,z)$, then from the hypotheses $i =_I j$ and $j =_I k$, we get $i =_I k$, and $\lambda_{ik}(x) = (\lambda_{jk} \circ \lambda_{ij})(x) := \lambda_{jk}(\lambda_{ij}(x)) = \lambda_{jk}(y) = z$. Clearly, the membership condition of $\sum_{i\in I} \lambda_0(i)$ reflects a construction. ◀

▶ **Definition 4.** Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The *first projection* on $\sum_{i\in I} \lambda_0(i)$ is the assignment routine $\mathtt{pr}_1(\Lambda) : \sum_{i\in I} \lambda_0(i) \rightsquigarrow I$, defined by, for every $(i,x) \in \sum_{i\in I} \lambda_0(i)$,

$$\mathtt{pr}_1(\Lambda)(i,x) := \mathbf{pr}_1(i,x) := i.$$

We may only write $\mathtt{pr}_1$, when the family of sets Λ is clearly understood from the context.

By the definition of equality on $\sum_{i\in I} \lambda_0(i)$ we get immediately that $\mathtt{pr}_1 : \sum_{i\in I} \lambda_0(i) \to I$. At the moment, for the second projection rule $\mathtt{pr}_2(i,x) := x$, for every $(i,x) \in \sum_{i\in I} \lambda_0(i)$, we do not have a way to describe its codomain. If $\Lambda^{\mathbb{N}}$ is the $\mathbb{N}$-family of $(X_n)_n$ (Definition 2), its exterior union is by definition

$$\sum_{n\in\mathbb{N}} X_n =: \{(n,x) \mid n \in \mathbb{N} \ \&\ x \in X_n\},$$

$$(n,x) =_{\sum_{n\in\mathbb{N}} X_n} (m,y) :\Leftrightarrow n =_{\mathbb{N}} m \ \&\ x =_{X_n} y.$$

Traditionally, the countable product of this sequence of sets is defined by

$$\prod_{n\in\mathbb{N}} X_n := \bigg\{\phi : \mathbb{N} \to \sum_{n\in\mathbb{N}} X_n \mid \forall_{n\in\mathbb{N}}\big(\phi(n) \in X_n\big)\bigg\},$$

which is a rough writing of the following

$$\prod_{n\in\mathbb{N}} X_n := \bigg\{\phi : \mathbb{N} \to \sum_{n\in\mathbb{N}} X_n \mid \forall_{n\in\mathbb{N}}\big(\mathtt{pr}_1(\phi(n)) =_{\mathbb{N}} n\big)\bigg\}.$$

In the second writing the condition $\mathtt{pr}_1(\phi(n)) =_{\mathbb{N}} n$ implies that $\mathtt{pr}_1(\phi(n)) := n$, hence, if $\phi(n) := (m,y)$, then $m := n$ and $y \in X_n$. When the equality of I though, is not like that of $\mathbb{N}$, we cannot solve this problem in a satisfying way. One could define

$$\phi \in \prod_{i\in I} \lambda_0(i) :\Leftrightarrow \phi \in \mathbb{F}\bigg(I, \sum_{i\in I} \lambda_0(i)\bigg) \ \&\ \forall_{i\in I}\big(\mathtt{pr}_1(\phi(i)) := i\big).$$

This approach has the problem that the property

$$Q(\phi) :\Leftrightarrow \forall_{i\in I}\big(\mathtt{pr}_1(\phi(i)) := i\big)$$

is not necessarily extensional; let $\phi =_{\mathbb{F}\left(I, \sum_{i \in I} \lambda_0(i)\right)} \theta$ i.e., $\forall_{i \in I}\big(\phi(i) =_{\sum_{i \in I} \lambda_0(i)} \theta(i)\big)$, and suppose that $Q(\phi)$. If we fix some $i \in I$, and $\phi(i) := (i, x)$ and $\theta(i) := (j, y)$, we only get that $j =_I i$. The universe of functions $\mathbb{V}_1$ allows us to take a different approach to the definition of an arbitrary product, which, in our view, reflects accurately Bishop's formulation of dependent functions in [4], p. 65.

▶ **Definition 5.** Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets, and let $\mathbb{1} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0\} =: \{0\}$. A *dependent function* over Λ is an assignment routine $\Phi : I \rightsquigarrow \mathbb{V}_1$, where, for every $i \in I$,

$$\Phi(i) := (\mathbb{1}, \lambda_0(i), \phi_i)$$

such that, for every $(i, j) \in D(I)$, the following diagram commutes

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\phi_i"] \arrow[d, "\mathrm{id}_{\mathbb{1}}"'] & \lambda_0(i) \arrow[d, "\lambda_{ij}"] \\ \mathbb{1} \arrow[r, "\phi_j"'] & \lambda_0(j) \end{tikzcd}$$

Since $\phi_i : \mathbb{1} \to \lambda_0(i)$, the triple $\Phi(i)$ determines the element $\phi_i(0) \in \lambda_0(i)$. If $i =_I j$, the commutativity of the above diagram gives that $\phi_j(0) =_{\lambda_0(j)} \lambda_{ij}(\phi_i(0))$. A dependent function Φ is a function-like object i.e., $i =_I j \Rightarrow \Phi(i) =_{\mathbb{V}_1} \Phi(j)$, since $(\mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}, \lambda_{ij}, \lambda_{ji}) : (\mathbb{1}, \lambda_0(i), \phi_i) =_{\mathbb{V}_1} (\mathbb{1}, \lambda_0(j), \phi_j)$. Since $\mathrm{id}_{\mathbb{1}}$ is the only function from $\mathbb{1}$ to $\mathbb{1}$, from now on we avoid mentioning it in commutative diagrams.

▶ **Definition 6.** Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The *I-product* of the family Λ is the totality $\prod_{i \in I} \lambda_0(i)$ of dependent functions over Λ equipped with the equality

$$\Phi =_{\prod_{i \in I} \lambda_0(i)} \Theta :\Leftrightarrow \forall_{i \in I}\big(\phi_i(0) =_{\lambda_0(i)} \theta_i(0)\big)$$

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\phi_i"] \arrow[d] & \lambda_0(i) \arrow[d, "\lambda_{ii}"] \\ \mathbb{1} \arrow[r, "\theta_i"'] & \lambda_0(i) \end{tikzcd}$$

If Y is a set and Λ is the constant I-family Y, we use the notation $Y^I := \prod_{i \in I} Y$.

Clearly, the equality on $\prod_{i \in I} \lambda_0(i)$ satisfies the conditions of an equivalence relation, and $\prod_{i \in I} \lambda_0(i)$ is a set. As expected, the dependent product generalises the cartesian product.

▶ Proposition 7. If $\Lambda^{\mathbb{2}}$ is the $\mathbb{2}$-family of the sets X and Y, then

$$\prod_{i \in \mathbb{2}} \lambda_0^{\mathbb{2}}(i) =_{\mathbb{V}_0} X \times Y.$$

Proof. If $\Phi \in \prod_{i \in I} \lambda_0^{\mathbb{2}}(i)$, then $\Phi : \mathbb{2} \rightsquigarrow \mathbb{V}_1$, where $\Phi(0) := (\mathbb{1}, X, \phi_0)$ with $\phi_0 : \mathbb{1} \to X$, and $\Phi(1) := (\mathbb{1}, X, \phi_1)$ with $\phi_1 : \mathbb{1} \to Y$, such that the following diagrams commute

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\phi_0"] \arrow[d] & X \arrow[d, "\mathrm{id}_X"] \\ \mathbb{1} \arrow[r, "\phi_0"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\phi_1"] \arrow[d] & Y \arrow[d, "\mathrm{id}_Y"] \\ \mathbb{1} \arrow[r, "\phi_1"'] & Y. \end{tikzcd}$$

Since this is always the case, ϕ_0, ϕ_1 are arbitrary. If $\Phi, \Theta \in \prod_{i\in I} \lambda_0^2(i)$, then $\Phi =_{\prod_{i\in I} \lambda_0^2(i)} \Theta$, if the following diagrams commute

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\phi_0"] \arrow[d] & X \arrow[d, "\mathrm{id}_X"] \\ \mathbb{1} \arrow[r, "\theta_0"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\phi_1"] \arrow[d] & Y \arrow[d, "\mathrm{id}_Y"] \\ \mathbb{1} \arrow[r, "\theta_1"'] & Y \end{tikzcd}$$

i.e., if $\theta_0(0) =_X \phi_0(0)$ and $\theta_1(0) =_Y \phi_1(0)$. If we define $f : \prod_{i\in I} \lambda_0^2(i) \to X \times Y$ by $f(\Phi) := (\phi_0(0), \phi_1(0))$, and $g : X \times Y \to \prod_{i\in I} \lambda_0^2$ by $g(x, y) := \Phi_{x,y}$ with $\phi_0(0) := x$ and $\phi_1(0) := y$, it is immediate to show that $(f, g) : \prod_{i\in I} \lambda_0^2(i) =_{\mathbb{V}_0} X \times Y$. ◀

If $\Lambda^{\mathbb{N}} := (\lambda_0^{\mathbb{N}}, \lambda_1^{\mathbb{N}})$ is the $\mathbb{N}$-family of $(X_n)_n$, and if $\Phi \in \prod_{n\in\mathbb{N}} X_n$, then, for every $n \in \mathbb{N}$, we have that $\Phi(n) := (\mathbb{1}, X_n, \phi_n)$ and the required diagram is commutative. If (X_n, ρ_n) is a metric space, for every $n \in \mathbb{N}$, Bishop's definition in [4], p. 79, of the *countable product metric* on $\prod_{n\in\mathbb{N}} X_n$ takes the form

$$\rho(\Phi, \Theta) := \sum_{n=1}^{\infty} \frac{\rho_n\big(\phi_n(0), \theta_n(0)\big)}{2^n}.$$

▶ Proposition 8. If $\Lambda := (\lambda_0, \lambda_1)$ is the constant I-family Y, then $Y^I =_{\mathbb{V}_0} \mathbb{F}(I, Y)$.

Proof. Let the assignment routine $e : Y^I \rightsquigarrow \mathbb{F}(I, Y)$ be defined by $\Phi \mapsto e(\Phi)$, and $e(\Phi)(i) := \phi_i(0)$, where $\Phi(i) := (\mathbb{1}, \lambda_0(i), \phi_i)$, for every $i \in I$. This routine is well-defined, since, if $i =_I j$, and using the equality $\lambda_{ij}(\phi_i(0)) =_{\lambda_0(j)} \phi_j(0)$, we get $e(\Phi)(i) := \phi_i(0) =_{\lambda_0(j)} \phi_j(0) := e(\Phi)(j)$, hence $e(\Phi)$ is in $\mathbb{F}(I, Y)$. The assignment routine e is also a function i..e, $\Phi =_{Y^I} \Theta \Rightarrow e(\Phi) =_{\mathbb{F}(I,Y)} e(\Theta)$, since for every $i \in I$, we have that $e(\Phi)(i) := \phi_i(0) =_{\lambda_0(i)} \theta_i(0) := e(\Theta)(i)$. Let the assignment routine $e' : \mathbb{F}(I, Y) \rightsquigarrow Y^I$ be defined by $f \mapsto e'(f)$, and $e'(f)(i) := (\mathbb{1}, Y, f_i)$, where $f_i : \mathbb{1} \to Y$ is defined by $f_i(0) := f(i)$. The assignment routine e' is a function i.e., $f =_{\mathbb{F}(I,Y)} g \Rightarrow e'(f) =_{Y^I} e'(g)$, by the equalities $f_i(0) := f(i) =_Y g(i) := g_i(0)$ and the resulting commutativity of the following diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "g_i"] \arrow[d] & Y \arrow[d, "\mathrm{id}_Y"] \\ \mathbb{1} \arrow[r, "f_i"'] & Y \end{tikzcd}$$

for every $i \in I$. Since $e'(f)(i) := (\mathbb{1}, Y, f_i)$, we get $e(e'(f))(i) := f_i(0) := f(i)$, hence $e \circ e' := f$. Since $e'(e(\Phi))(i) := (\mathbb{1}, Y, e(\Phi)_i)$, where $e(\Phi)_i : \mathbb{1} \to Y$ is defined by $e(\Phi)_i(0) := e(\Phi)(i) := \phi_i(0)$, we get $e(\Phi)_i := \phi_i$, and since $\Phi(i) := (\mathbb{1}, Y, \phi_i)$, for every $i \in I$, we conclude that $e'(e(\Phi)) := \Phi$. Consequently, $(e, e') : Y^I =_{\mathbb{V}_0} \mathbb{F}(I, Y)$. ◀

▶ **Definition 9.** Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The $\sum_{i\in I}\lambda_0(i)$-family $M_\Lambda := (\mu_0, \mu_1)$ of sets is defined by

$$\mu_0(i,x) := \lambda_0(i),$$

$$\mu_1\big((i,x),(j,y)\big) := \big(\mu_0(i,x), \mu_0(j,y), \mu_{(i,x)(j,y)}\big) := (\lambda_0(i), \lambda_0(j), \lambda_{ij}),$$

for every $(i,x) \in \sum_{i\in I}\lambda_0(i)$ and $\big((i,x),(j,y)\big)$ in the diagonal of $\sum_{i\in I}\lambda_0(i)$. The *second projection* on $\sum_{i\in I}\lambda_0(i)$ is the assignment routine $\mathtt{pr}_2(\Lambda) : \sum_{i\in I}\lambda_0(i) \rightsquigarrow \mathbb{V}_1$, defined, for every $(i,x) \in \sum_{i\in I}\lambda_0(i)$, by

$$\mathtt{pr}_2(\Lambda)(i,x) := (\mathbb{1}, \lambda_0(i), \phi_{(i,x)}),$$

where $\phi_{(i,x)} : \mathbb{1} \to \lambda_0(i)$ is defined by $\phi_{(i,x)}(0) := x$. We may only write $\mathtt{pr}_1$, when the family of sets Λ is clearly understood from the context.

▶ Proposition 10. If Λ and M_Λ are as in Definition 9, then

$$\mathtt{pr}_2(\Lambda) \in \prod_{w\in\sum_{i\in I}\lambda_0(i)} \mu_0(w) := \prod_{w\in\sum_{i\in I}\lambda_0(i)} \lambda_0(\mathtt{pr}_1(w)).$$

Proof. It suffices to show that if $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$, the following diagram commutes

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi_{(i,x)}"] \arrow[d] & \lambda_0(i) \arrow[d, "\lambda_{ij}"] \\
\mathbb{1} \arrow[r, "\phi_{(j,y)}"'] & \lambda_0(j)
\end{tikzcd}$$

By the related definitions we get $\lambda_{ij}(\phi_{(i,x)}(0)) := \lambda_{ij}(x) =_{\lambda_0(j)} y := \phi_{(j,y)}(0)$. ◀

3.1 The distributivity of $\prod$ over $\sum$

Next we prove the translation of the type-theoretic axiom of choice within CSFT (Theorem 16), or, as it was suggested to us by M. Maietti, the distributivity of $\prod$ over $\sum$[6]. For the proof of Theorem 16 we need some preparation.

▶ **Definition 11.** Let X, Y be sets, and $R := (\rho_0, \rho_1)$ a family of sets indexed by $X \times Y$. If $x \in X$ let $\Lambda^x := (\lambda_0^x, \lambda_1^x)$, where $\lambda_0^x : Y \rightsquigarrow \mathbb{V}_0$ and $\lambda_1^x : D(Y) \rightsquigarrow \mathbb{V}_1$ are defined by

$$\lambda_0^x(y) := \rho_0(x,y),$$

$$\lambda_1^x(y,y') := \big(\lambda_0^x(y), \lambda_0^x(y'), \lambda_{yy'}^x\big) := \big(\rho_0(x,y), \rho_0(x,y'), \rho_{(x,y)(x,y')}\big),$$

for every $y \in Y$ and every $(y,y') \in D(Y)$, respectively. Let also $M := (\mu_0, \mu_1)$, where $\mu_0 : X \rightsquigarrow \mathbb{V}_0$ and $\mu_1 : D(X) \rightsquigarrow \mathbb{V}_1$ are defined by

$$\mu_0(x) := \sum_{y\in Y} \rho_0(x,y),$$

[6] We would like to E. Palmgren for pointing to us that such a distributivity holds in every locally cartesian closed category. In [37] it is mentioned that this fact is generally attributed to Martin-Löf and his work [23]. It is easy to show that sets in CSFT form a cartesian closed category already, but we include the proof of the type-theoretic axiom of choice in CSFT in order to be self-contained.

$$\mu_1(x,x') := \big(\mu_0(x), \mu_0(x'), \mu_{xx'}\big) := \bigg(\sum_{y\in Y}\rho_0(x,y), \sum_{y\in Y}\rho_0(x',y), \mu_{xx'}\bigg),$$

for every $x \in X$ and every $(x,x') \in D(X)$, respectively, where, for every $(y,u) \in \sum_{y\in Y}\rho_0(x,y)$,

$$\mu_{xx'} : \sum_{y\in Y}\rho_0(x,y) \to \sum_{y\in Y}\rho_0(x',y)$$

$$\mu_{xx'}(y,u) := \big(y, \rho_{(x,y)(x',y)}(u)\big).$$

▶ **Lemma 12.** *The couples $\Lambda^x := (\lambda_0^x, \lambda_1^x)$ and $M := (\mu_0, \mu_1)$ in Definition 11 are families of sets indexed by Y and X, respectively.*

Proof. Since by hypothesis R is an $X \times Y$-family of sets, we get

$$\lambda_1^x(y,y) := \big(\rho_0(x,y), \rho_0(x,y), \rho_{(x,y)(x,y)}\big) := \big(\rho_0(x,y), \rho_0(x,y), \mathrm{id}_{\rho_0(x,y)}\big),$$

and the commutativity of the left diagram

$$\begin{tikzcd} \lambda_0^x(y) \arrow[d, "\lambda^x_{yy'}"'] \arrow[dr, "\lambda^x_{yy''}"] & \\ \lambda_0^x(y') \arrow[r, "\lambda^x_{y'y''}"'] & \lambda_0^x(y'') \end{tikzcd} \qquad \begin{tikzcd} \rho_0(x,y) \arrow[d, "\rho_{(x,y)(x,y')}"'] \arrow[dr, "\rho_{(x,y)(x,y'')}"] & \\ \rho_0(x,y') \arrow[r, "\rho_{(x,y')(x,y'')}"'] & \rho_0(x,y'') \end{tikzcd}$$

is by definition the known commutativity of the right diagram. Similarly,

$$\mu_1(x,x) := \big(\mu_0(x), \mu_0(x), \mu_{xx}\big) := \bigg(\sum_{y\in Y}\rho_0(x,y), \sum_{y\in Y}\rho_0(x,y), \mu_{xx}\bigg),$$

where $\mu_{xx} : \sum_{y\in Y}\rho_0(x,y) \to \sum_{y\in Y}\rho_0(x,y)$ is defined by

$$\mu_{xx}(y,u) := \big(y, \rho_{(x,y)(x,y)}(u)\big) := \big(y, \mathrm{id}_{\rho_0(x,y)}(u)\big) := (y,u),$$

for every $(y,u) \in \sum_{y\in Y}\rho_0(x,y)$. For the commutativity of the left diagram

$$\begin{tikzcd} \mu_0(x) \arrow[d, "\mu_{xx'}"'] \arrow[dr, "\mu_{xx''}"] & \\ \mu_0(x') \arrow[r, "\mu_{x'x''}"'] & \mu_0(x'') \end{tikzcd} \qquad \begin{tikzcd} \rho_0(x,y) \arrow[d, "\rho_{(x,y)(x',y)}"'] \arrow[dr, "\rho_{(x,y)(x'',y)}"] & \\ \rho_0(x',y) \arrow[r, "\rho_{(x',y)(x'',y)}"'] & \rho_0(x'',y) \end{tikzcd}$$

we use the known commutativity of the right diagram, since

$$\begin{aligned} \mu_{x'x''}\big(\mu_{xx'}(y,u)\big) &:= \mu_{x'x''}\big(y, \rho_{(x,y)(x',y)}(u)\big) \\ &:= \big(y, \rho_{(x',y)(x'',y)}\big(\rho_{(x,y)(x',y)}(u)\big)\big) \\ &:= \big(y, \rho_{(x,y)(x'',y)}(u)\big) \\ &:= \mu_{xx''}(y,u), \end{aligned}$$

for every $(y,u) \in \sum_{y\in Y}\rho(x,y)$. ◀

▶ **Lemma 13.** *Let $R := (\rho_0, \rho_1)$, $\Lambda^x := (\lambda_0^x, \lambda_1^x)$ and $M := (\mu_0, \mu_1)$ be the families of sets of Definition 11. If $\Phi \in \prod_{x\in X}\mu_0(x)$, then Φ generates a function $f_\Phi : X \to Y$.*

Proof. By definition, $\Phi : X \rightsquigarrow \mathbb{V}_1$, where, for every $x \in X$,

$$\Phi(x) := (\mathbb{1}, \mu_0(x), \phi_x) := \Big(\mathbb{1}, \sum_{y \in Y} \rho_0(x,y), \phi_x\Big),$$

where $\phi_x : \mathbb{1} \to \sum_{y \in Y} \rho_0(x,y)$. We define the assignment routine $f_\Phi : X \rightsquigarrow Y$ by the rule $f_\Phi(x) := \mathtt{pr}_1(\phi_x(0))$, for every $x \in X$. Next we show that the routine f_Φ is a function. Let $x =_X x'$. By the commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi_x"] \arrow[d] & \mu_0(x) \arrow[d, "\mu_{xx'}"] \\
\mathbb{1} \arrow[r, "\phi_{x'}"'] & \mu_0(x')
\end{tikzcd}$$

we have that, if $\phi_x(0) := (y, u)$, for some $y \in Y$ and $u \in \rho_0(x,y)$, then

$$\mu_{xx'}(\phi_x(0)) := \mu_{xx'}(y,u) := \big(y, \rho_{(x,y)(x',y)}(u)\big) =_{\sum_{y \in Y} \rho_0(x',y)} \phi_{x'}(0),$$

hence, since $\mathtt{pr}_1$ is a function, we get

$$f(x') := \mathtt{pr}_1(\phi_{x'}(0)) =_Y \mathtt{pr}_1\big(y, \rho_{(x,y)(x',y)}(u)\big) := y := \mathtt{pr}_1(\phi_x(0)) := f(x).$$

◀

▶ **Lemma 14.** *Let $R := (\rho_0, \rho_1)$, $\Lambda^x := (\lambda_0^x, \lambda_1^x)$ and $M := (\mu_0, \mu_1)$ be the families of sets of Definition 11. If $f : X \to Y$, let $N^f := (\nu_0^f, \nu_1^f)$, where $\nu_0^f : X \rightsquigarrow \mathbb{V}_0$ and $\nu_1^f : D(X) \rightsquigarrow \mathbb{V}_1$ are defined by*

$$\nu_0^f(x) := \rho_0(x, f(x)),$$

$$\nu_1^f(x,x') := \big(\nu_0^f(x), \nu_0^f(x'), \nu_{xx'}^f\big) := \big(\rho_0(x, f(x)), \rho_0(x', f(x')), \rho_{(x,f(x))(x',f(x'))}\big),$$

for every $x \in X$ and every $(x,x') \in D(X)$, respectively, then N^f is an X-family of sets.

Proof. Since by hypothesis R is an $X \times Y$-family of sets, we get

$$\nu_1^f(x,x) := \big(\rho_0(x, f(x)), \rho_0(x, f(x)), \rho_{(x,f(x))(x,f(x))}\big) := \big(\rho_0(x, f(x)), \rho_0(x, f(x)), \mathrm{id}_{\rho_0(x,f(x))}\big).$$

Since by Lemma 13 f_Φ is a function, the commutativity of the left diagram

$$\begin{tikzcd}
\nu_0^f(x) \arrow[d, "\nu_{xx'}^f"'] \arrow[dr, "\nu_{xx''}^f"] & \\
\nu_0^f(x') \arrow[r, "\nu_{x'x''}^f"'] & \nu_0^f(x'')
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho_0(x, f(x)) \arrow[d, "\rho_{(x,f(x))(x',f(x'))}"'] \arrow[dr, "\rho_{(x,f(x))(x'',f(x''))}"] & \\
\rho_0(x', f(x')) \arrow[r, "\rho_{(x',f(x'))(x'',f(x''))}"'] & \rho_0(x'', f(x''))
\end{tikzcd}$$

is by definition the known commutativity of the right diagram. ◀

▶ **Lemma 15.** *Let $R := (\rho_0, \rho_1)$ be the family of sets in Definition 11, and $N^f := (\nu_0^f, \nu_1^f)$ the family of sets defined in Lemma 14. If $\Xi := (\xi_0, \xi_1)$, where $\xi_0 : \mathbb{F}(X,Y) \rightsquigarrow \mathbb{V}_0$ and $\xi_1 : D(\mathbb{F}(X,Y)) \rightsquigarrow \mathbb{V}_1$ are defined by*

$$\xi_0(f) := \prod_{x \in X} \nu_0^f(x) := \prod_{x \in X} \rho_0(x, f(x))$$

$$\xi_1(f, f') := \big(\xi_0(f), \xi_0(f'), \xi_{ff'}\big),$$

where

$$\xi_{ff'} : \prod_{x \in X} \rho_0(x, f(x)) \to \prod_{x \in X} \rho_0(x, f'(x))$$

is defined by

$$\xi_{ff'}(H)(x) := \big(\mathbb{1}, \rho_0(x, f'(x)), h'_x\big),$$
$$h'_x(0) := \rho_{(x,f(x))(x,f'(x))}\big(h_x(0)\big),$$

and

$$H(x) := \big(\mathbb{1}, \nu_0^f(x), h_x\big) := \big(\mathbb{1}, \rho_0(x, f(x)), h_x\big)$$

for every $H \in \prod_{x \in X} \rho_0(x, f(x))$ and $x \in X$, then Ξ is a family of sets indexed by $\mathbb{F}(X, Y)$.

Proof. First we show that if $f =_{\mathbb{F}(X,Y)} f'$, then $\xi_{ff'}(H) \in \prod_{x \in X} \rho_0(x, f'(x)) := \prod_{x \in X} \nu_0^{f'}(x)$, by showing that if $x =_X x'$, then the commutativity of the left diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "h_x"] \arrow[d] & \rho_0(x, f(x)) \arrow[d, "\nu^f_{xx'}"] \\
\mathbb{1} \arrow[r, "h_{x'}"'] & \rho_0(x', f(x'))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "h'_x"] \arrow[d] & \rho_0(x, f'(x)) \arrow[d, "\nu^{f'}_{xx'}"] \\
\mathbb{1} \arrow[r, "h'_{x'}"'] & \rho_0(x', f'(x'))
\end{tikzcd}$$

implies the commutativity of the right one. By definition we have that

$$\begin{aligned}
\nu^{f'}_{xx'}\big(h'_x(0)\big) &:= \nu^{f'}_{xx'}\Big(\rho_{(x,f(x))(x,f'(x))}\big(h_x(0)\big)\Big) \\
&:= \rho_{(x,f'(x))(x',f'(x'))}\Big(\rho_{(x,f(x))(x,f'(x))}\big(h_x(0)\big)\Big) \\
&=_{\rho_0(x',f'(x'))} \quad \rho_{(x,f(x)(x',f'(x'))}\big(h_x(0)\big).
\end{aligned}$$

since the couples $(x, f(x)), (x, f'(x))$ and $(x', f'(x'))$ are equal in $X \times Y$, by the hypotheses $x =_X x'$ and $f =_{\mathbb{F}(X,Y)} f'$. Moreover, by the commutativity of the left diagram above we get

$$h_{x'}(0) =_{\rho_0(x',f(x'))} \quad \nu^f_{xx'}\big(h_x(0)\big) =_{\rho_0(x',f(x'))} \quad \rho_{(x,f(x))(x',f(x'))}\big(h_x(0)\big),$$

hence,

$$\begin{aligned}
h'_{x'}(0) &:= \rho_{(x',f(x')(x',f'(x')}\big(h_{x'}(0)\big) \\
&=_{\rho_0(x',f(x'))} \quad \rho_{(x',f(x')(x',f'(x')}\Big(\rho_{(x,f(x))(x',f(x'))}\big(h_x(0)\big)\Big) \\
&=_{\rho_0(x',f(x'))} \quad \rho_{(x,f(x)(x',f'(x'))}\big(h_x(0)\big),
\end{aligned}$$

and consequently, $\nu^{f'}_{xx'}\big(h'_x(0)\big) =_{\rho_0(x',f(x'))} \quad h'_{x'}(0)$. Next we show that ξ_1 satisfies the properties of Definition 1. By definition $\xi_{ff}(H)(x) := \big(\mathbb{1}, \rho_0(x, f(x)), h'_x\big)$, where

$$h'_x(0) := \rho_{(x,f(x))(x,f(x))}\big(h_x(0)\big) := \mathrm{id}_{\rho_0(x,f(x))}\big(h_x(0)\big) := h_x(0),$$

hence $\xi_{ff}(H) := H$, and since H is arbitrary, we get $\xi_{ff} := \mathrm{id}_{\xi_0(f)}$. Finally, if $f =_{\mathbb{F}(X,Y)} f' =_{\mathbb{F}(X,Y)} f''$, we show the commutativity of the following diagram

$$\begin{tikzcd}
\xi_0(f) \arrow[d, "\xi_{ff'}"'] \arrow[dr, "\xi_{ff''}"] & \\
\xi_0(f') \arrow[r, "\xi_{f'f''}"'] & \xi_0(f'').
\end{tikzcd}$$

If $H \in \xi_0(f)$, we show $\xi_{ff''}(H) =_{\xi_0(f'')} \xi_{f'f''}(\xi_{ff'}(H))$ i.e.,

$$\xi_{ff''}(H) \quad =_{\prod_{x\in X} \rho_0(x,f''(x))} \quad \xi_{f'f''}(\xi_{ff'}(H)).$$

By definition we have that $[\xi_{f'f''}(\xi_{ff'}(H))](x) := (\mathbb{1}, \rho_0(x, f''(x)), {h''}_x)$, where

$${h''}_x(0) := \rho_{(x,f'(x))(x,f''(x))}\big({h'}_x(0)\big).$$

Since $\xi_{ff'}(H)(x) := (\mathbb{1}, \rho_0(x, f'(x)), {h'}_x)$, where ${h'}_x(0) := \rho_{(x,f(x))(x,f'(x))}\big(h_x(0)\big)$, we get

$${h''}_x(0) := \rho_{(x,f'(x))(x,f''(x))}\bigg(\rho_{(x,f(x))(x,f'(x))}\big(h_x(0)\big)\bigg) = \rho_{(x,f(x))(x,f''(x))}\big(h_x(0)\big) := \tau_x(0),$$

where $\xi_{ff''}(H)(x) := (\mathbb{1}, \rho_0(x, f''(x)), \tau_x)$, with $\tau_x(0) := \rho_{(x,f(x))(x,f''(x))}\big(h_x(0)\big)$, and since $x \in X$ is arbitrary, the required commutativity is shown. ◀

▶ **Theorem 16** (Distributivity of $\prod$ over $\sum$)**.** *Let X, Y be sets, and $R := (\rho_0, \rho_1)$, $\Lambda^x := (\lambda_0^x, \lambda_1^x)$, $M := (\mu_0, \mu_1)$ the families of sets of Definition 11. If*

$$\Phi \in \prod_{x\in X} \mu_0(x) := \prod_{x\in X} \sum_{y\in Y} \rho_0(x,y),$$

there is $\Theta_\Phi \in \prod_{x\in X} \nu_0^{f_\Phi}(x)$, where $f_\Phi : X \to Y$ is defined in Lemma 13, and

$$\big(f_\Phi, \Theta_\Phi\big) \in \sum_{f\in\mathbb{F}(X,Y)} \prod_{x\in X} \nu_0^f(x) := \sum_{f\in\mathbb{F}(X,Y)} \prod_{x\in X} \rho_0(x, f(x)).$$

Moreover, the assignment routine

$$\mathtt{ac} : \prod_{x\in X} \sum_{y\in Y} \rho_0(x,y) \;\rightsquigarrow\; \sum_{f\in\mathbb{F}(X,Y)} \prod_{x\in X} \rho_0(x, f(x))$$

$$\mathtt{ac}(\Phi) := \big(f_\Phi, \Theta_\Phi\big)$$

is a function.

Proof. By Proposition 10 we have that

$$\mathtt{pr}_2(\Lambda^x) \in \prod_{w\in\sum_{y\in Y}\lambda_0^x(y)} \lambda_0^x(\mathtt{pr}_1(w)) := \prod_{w\in\sum_{y\in Y}\rho_0(x,y)} \rho_0(x, \mathtt{pr}_1(w)),$$

where, if $(y,u) \in \sum_{y\in Y} \rho_0(x,y)$, then $\mathtt{pr}_2(\Lambda^x)(y,u) := \big(\mathbb{1}, \rho_0(x,y), \sigma_{(y,u)}\big)$, and $\sigma_{(y,u)} : \mathbb{1} \to \rho_0(x,y)$ is defined by $\sigma_{(y,u)}(0) := u$. We define the assignment routine $\Theta_\Phi : X \rightsquigarrow \mathbb{V}_1$ by

$$\Theta_\Phi(x) := \big(\mathbb{1}, \nu_0^{f_\Phi}(x), \theta_x\big) := \big(\mathbb{1}, \rho_0(x, f_\Phi(x)), \theta_x\big),$$

where $\theta_x : \mathbb{1} \to \rho_0(x, f_\Phi(x))$ is defined by $\theta_x(0) := \sigma_{(y,u)}(0) := u$, and $\phi_x(0) := (y,u) := (f_\Phi(x), u)$. Since $(y,u) := \phi_x(0) \in \sum_{y\in Y} \rho_0(x,y)$, we have that $u \in \rho_0(x,y) := \rho_0(x, f_\Phi(x))$. In order to show that $\Theta_\Phi \in \prod_{x\in A} \nu_0^{f_\Phi}(x) := \prod_{x\in A} \rho_0(x, f_\Phi(x))$, we need to show, for $x =_X x'$, the commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\theta_x"] \arrow[d] & \rho_0(x, f_\Phi(x)) \arrow[d, "\nu^{f_\Phi}_{xx'}"] \\
\mathbb{1} \arrow[r, "\theta_{x'}"'] & \rho_0(x', f_\Phi(x')).
\end{tikzcd}$$

Since $\Phi \in \prod_{x\in X} \mu_0(x) := \prod_{x\in X} \sum_{y\in Y} \rho_0(x, y)$, we have the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi_x"] \arrow[d] & \sum_{y\in Y} \rho_0(x, y) \arrow[d, "\mu_{xx'}"] \\
\mathbb{1} \arrow[r, "\phi_{x'}"'] & \sum_{y\in Y} \rho_0(x', y),
\end{tikzcd}$$

where by Definition 11 this commutativity becomes

$$\mu_{xx'}(\phi_x(0)) := \mu_{xx'}(y, u) := \big(y, \rho_{(x,y)(x',y)}(u)\big) := \big(y, \rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big)$$

$$=_{\sum_{y\in Y} \rho_0(x',y)} \quad \phi_{x'}(0) := (y', u') := (f_\Phi(x'), u').$$

Since the equality

$$\big(y, \rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big) \quad =_{\sum_{y\in Y} \rho_0(x',y)} \quad (y', u')$$

is by definition the equality

$$\big(y, \rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big) \quad =_{\sum_{y\in Y} \lambda_0^{x'}(y)} \quad (y', u'),$$

we have that $y =_Y y'$ and

$$\lambda^{x'}_{yy'}\big(\rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big) =_{\lambda_0^{x'}(y')} \quad u',$$

while by the definition of $\lambda^{x'}_{yy'}$ and since $\lambda_0^{x'}(y') := \rho_0(x', y')$ we get

$$\rho_{(x',y)(x',y')}\big(\rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big) =_{\rho_0(x',y')} \quad u'$$

i.e.,

$$\rho_{(x',f_\Phi(x))(x',f_\Phi(x')}\big(\rho_{(x,f_\Phi(x))(x',f_\Phi(x))}(u)\big) =_{\rho_0(x',y')} \quad u'.$$

By the commutativity of the following diagram

$$\begin{tikzcd}
\rho_0(x, f_\Phi(x)) \arrow[d, "\rho_{(x,f_\Phi(x))(x',f_\Phi(x))}"'] \arrow[dr, "\rho_{(x,f_\Phi(x))(x',f_\Phi(x'))}"] & \\
\rho_0(x', f_\Phi(x)) \arrow[r, "\rho_{(x',f_\Phi(x))(x',f_\Phi(x'))}"'] & \rho_0(x', f_\Phi(x'))
\end{tikzcd}$$

we get

$$\rho_{(x,f_\Phi(x))(x',f_\Phi(x')}(u) =_{\rho_0(x',y')} \quad u',$$

and the required commutativity of the diagram for Θ_Φ is shown as follows:

$$\nu^{f_\Phi}_{xx'}(\theta_x(0)) := \nu^{f_\Phi}_{xx'}(\sigma_{(y,u)}(0)) := \nu^{f_\Phi}_{xx'}(u_0) := \rho_{(x,f_\Phi(x))(x',f_\Phi(x'))}(u) =_{\rho_0(x',y')} \quad u' := \theta_{x'}(0).$$

Next we show that ac is a function i.e., $\Phi =_{\prod_{x\in X}\mu_0(x)} \Phi' \Rightarrow \mathtt{ac}(\Phi) =_{\sum_{f\in\mathbb{F}(X,Y)}\xi_0(f)} \mathtt{ac}(\Phi')$. If

$$\Phi(x) := (\mathbb{1}, \mu_0(x), \phi_x) := (\mathbb{1}, \sum_{y\in Y}\rho_0(x,y), \phi_x),$$

$$\Phi'(x) := (\mathbb{1}, \mu_0(x), {\phi'}_x) := (\mathbb{1}, \sum_{y\in Y}\rho_0(x,y), {\phi'}_x),$$

the hypothesis $\Phi =_{\prod_{x\in X}\mu_0(x)} \Phi'$ is reduced to $\phi_x(0) =_{\mu_0(x)} {\phi'}_x(0)$, for every $x \in X$. By definition the equality

$$\big(f_\Phi, \Theta_\Phi\big) \ =_{\sum_{f\in\mathbb{F}(X,Y)}\xi_0(f)} \ \big(f_{\Phi'}, \Theta_{\Phi'}\big)$$

is reduced to $f_\Phi =_{\mathbb{F}(X,Y)} f_{\Phi'}$ and

$$\xi_{f_\Phi f_{\Phi'}}(\Theta_\Phi) \ =_{\xi_0(f_{\Phi'})} \ \Theta_{\Phi'} \ :\Leftrightarrow \xi_{f_\Phi f_{\Phi'}}(\Theta_\Phi) \ =_{\prod_{x\in X}\rho_0(x,f_{\Phi'}(x))} \ \Theta_{\Phi'}.$$

If $x \in X$, then

$$f_\Phi(x) := \mathtt{pr}_1(\phi_x(0)) =_Y \mathtt{pr}_1({\phi'}_x(0)) := f_{\Phi'}(x),$$

hence, since $x \in X$ is arbitrary, $f_\Phi =_{\mathbb{F}(X,Y)} f_{\Phi'}$. By definition $\Phi(x) := (\mathbb{1}, \sum_{y\in Y}\rho_0(x,y), \phi_x)$ and $\Theta_\Phi(x) := (\mathbb{1}, \rho_0(x, f_\Phi(x)), \theta_x)$, where $\theta_x(0) := \sigma_{(y,u)}(0) := u$, and $\phi_x(0) := (y,u) := (f_\Phi(x), u)$. Similarly, $\Phi'(x) := (\mathbb{1}, \sum_{y\in Y}\rho_0(x,y), {\phi'}_x)$ and $\Theta_{\Phi'}(x) := (\mathbb{1}, \rho_0(x, f_{\Phi'}(x)), {\theta'}_x)$, where ${\theta'}_x(0) := \sigma_{(y',u')}(0) := u'$, and ${\phi'}_x(0) := (y', u') := (f_{\Phi'}(x), u')$. Moreover,

$$\xi_{f_\Phi f_{\Phi'}}(\Theta_\Phi)(x) := (\mathbb{1}, \rho_0(x, f_{\Phi'}(x)), {h'}_x),$$

$${h'}_x(0) := \rho_{(x,f_\Phi(x))(x,f_{\Phi'}(x))}\big(\theta_x(0)\big) := \rho_{(x,f_\Phi(x))(x,f_{\Phi'}(x))}(u).$$

By definition, we need to show that, for every $x \in X$,

$${\theta'}_x(0) \ =_{\rho_0(x,f_{\Phi'}(x))} \ {h'}_x(0) :\Leftrightarrow u' \ =_{\rho_0(x,f_{\Phi'}(x))} \ \rho_{(x,f_\Phi(x))(x,f_{\Phi'}(x))}(u).$$

Since

$$\phi_x(0) =_{\mu_0(x)} {\phi'}_x(0) :\Leftrightarrow \phi_x(0) =_{\sum_{y\in Y}\rho_0(x,y)} {\phi'}_x(0) :\Leftrightarrow (f_\Phi(x), u) =_{\sum_{y\in Y}\lambda_0^x(y)} (f_{\Phi'}(x), u'),$$

we get

$$\lambda^x_{yy'}(u) =_{\rho_0(x,y')} u' :\Leftrightarrow \rho_{(x,f_\Phi(x))(x,f_{\Phi'}(x))}(u) =_{\rho_0(x,y')} u',$$

which is exactly what we need to show. ◀

4 Interior union and dependent products in CSFT

Next we formulate Bishop's definition of a set-indexed family of subsets, given in [4], p. 65, in analogy to our definition of a set-indexed family of sets.

▶ **Definition 17.** Let X and I be sets. A *family of subsets* of X indexed by I is a triple $\lambda := (\lambda_0, \sigma_1, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and $\sigma_1 : I \rightsquigarrow \mathbb{V}_1$, such that, for every $i \in I$, we have that $\sigma_1(i) := (\lambda_0(i), X, e_i)$ and e_i is an embedding of $\lambda_0(i)$ into X. Moreover, $\lambda_1 : D(I) \rightsquigarrow \mathbb{V}_1$ is called the *modulus of function-likeness* of λ_0, and for every $i \in I$ it satisfies $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$, while for every $(i,j) \in D(I)$ it satisfies $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$ i.e., the following inner diagrams commute

$$\begin{tikzcd}
\lambda_0(j) \arrow[d, "\lambda_{ji}"'] \arrow[dr, hook, "e_j"] & \\
\lambda_0(i) \arrow[r, hook, "e_i"] \arrow[dr, "\lambda_{ij}"'] & X \\
 & \lambda_0(j). \arrow[u, hook, "e_j"']
\end{tikzcd}$$

▶ Remark. If $\lambda := (\lambda_0, \sigma_1, \lambda_1)$ is an I-family of subsets of X, then $\Lambda_\lambda := (\lambda_0, \lambda_1)$ is an I-family of sets.

Proof. Let $i =_I j =_I k$. If $a \in \lambda_0(i)$, by the commutativity of the following inner diagrams

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "e_i"] \arrow[dr, "\lambda_{ij}"'] & X & \lambda_0(i) \arrow[l, hook', "e_i"'] \arrow[d, "\lambda_{ik}"] \\
 & \lambda_0(j) \arrow[u, hook, "e_j"'] \arrow[r, "\lambda_{jk}"'] & \lambda_0(k) \arrow[ul, hook, "e_k"']
\end{tikzcd}$$

and omitting the subscripts in the following equalities, we have that

$$e_k\big(\lambda_{jk}(\lambda_{ij}(a))\big) = e_j(\lambda_{ij}(a)) = e_i(a) = e_k\big(\lambda_{ik}(a)\big),$$

hence $\lambda_{jk}(\lambda_{ij}(a)) = \lambda_{ik}(a)$, and since $a \in \lambda_0(i)$ is arbitrary, we get $\lambda_{jk} \circ \lambda_{ij} = \lambda_{ik}$. ◀

▶ **Definition 18.** Let $\lambda := (\lambda_0, \sigma_0, \lambda_1)$ be an I-family of subsets of X. The *interior union* of λ is the totality $\bigcup_{i\in I} \lambda_0(i)$ defined by

$$z \in \bigcup_{i\in I} \lambda_0(i) :\Leftrightarrow \exists_{i\in I}\exists_{x\in\lambda_0(i)}\big(z := (i, x)\big).$$

Let the assignment routine $\varepsilon : \bigcup_{i\in I} \lambda_0(i) \rightsquigarrow X$ be defined by $\varepsilon(i, x) := e_i(x)$, for every $(i, x) \in \bigcup_{i\in I} \lambda_0(i)$, where $e_i : \lambda_0(i) \hookrightarrow X$ is the embedding of $\lambda_0(i)$ into X, for every $i \in I$. The equality on $\bigcup_{i\in I} \lambda_0(i)$ is defined by

$$(i, x) =_{\bigcup_{i\in I} \lambda_0(i)} (j, y) :\Leftrightarrow \varepsilon(i, x) =_X \varepsilon(j, y).$$

It is immediate to show that $(i, x) =_{\bigcup_{i\in I} \lambda_0(i)} (j, y)$ satisfies the conditions of an equivalence relation, and $\bigcup_{i\in I} \lambda_0(i)$ is a set. Moreover, the assignment routine ε is an embedding of $\bigcup_{i\in I} \lambda_0(i)$ into X, hence the couple $\big(\bigcup_{i\in I} \lambda_0(i), \varepsilon\big)$ is a subset of X. Note that the defined totalities $\sum_{i\in I} \lambda_0(i)$ and $\bigcup_{i\in I} \lambda_0(i)$ have the same membership formula, but their equalities are different. The equality of $\sum_{i\in I} \lambda_0(i)$ is determined *externally* by the transport function λ_{ij} and the equalities of $\lambda_0(j)$, while the equality of $\bigcup_{i\in I} \lambda_0(i)$ is determined *internally* by the embeddings e_i, e_j and the equality of X.

▶ **Definition 19.** Let $\lambda := (\lambda_0, \sigma_0, \lambda_1)$ be an I-family of subsets of X. A *dependent function over* λ is[7] a dependent function over the I-family of sets Λ_λ. Based on Definition 5, and

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[7] The definition of $\prod_{i\in I} \lambda_0(i)$, given in [8], p. 70, as the set $\big\{f : I \to \bigcup_{i\in I} \lambda_0(i) \mid \forall_{i\in I}\big(f(i) \in \lambda_0(i)\big)\big\}$ is not compatible with the precise definition of $\bigcup_{i\in I} \lambda_0(i)$, given previously in the same page, and it is not included in [4].

using a superscript to emphasize that we deal with a family of subsets, we denote their set by $\prod_{i\in I}^{X}\lambda_0(i)$.

Next we formulate precisely Bishop's definition of the intersection of a set-indexed family of subsets, given in [4], p. 65.

▶ **Definition 20.** Let $\lambda := (\lambda_0, \sigma_1, \lambda_1)$ be an I-family of subsets of X, where I is inhabited by some element i_0. The intersection $\bigcap_{i\in I}\lambda_0(i)$ of λ is the totality defined by

$$\Phi \in \bigcap_{i\in I}\lambda_0(i) :\Leftrightarrow \Phi \in \prod_{i\in I}^{X}\lambda_0(i) \ \&\ \forall_{i,i'\in I}\big(e_i(\phi_i(0)) =_X e_{i'}(\phi_{i'}(0))\big),$$

where, for every $i \in I$, $\Phi(i) := (\mathbb{1}, \lambda_0(i), \phi_i)$ and $\sigma_1(i) := (\lambda_0(i), X, e_i)$. Let the assignment routine $e : \bigcap_{i\in I}\lambda_0(i) \rightsquigarrow X$ be defined by $e(\Phi) := e_{i_0}\big(\phi_{i_0}(0)\big)$. If $\Phi, \Theta \in \bigcap_{i\in I}\lambda_0(i)$, we define

$$\Phi =_{\bigcap_{i\in I}\lambda_0(i)} \Theta :\Leftrightarrow e(\Phi) =_X e(\Theta).$$

It is immediate to show that $\Phi =_{\bigcap_{i\in I}\lambda_0(i)} \Theta$ satisfies the conditions of an equivalence relation, and $\bigcap_{i\in I}\lambda_0(i)$ is a set. Moreover, the assignment routine e is an embedding of $\bigcap_{i\in I}\lambda_0(i)$ into X, hence the couple $\big(\bigcap_{i\in I}\lambda_0(i), e\big)$ is a subset of X. As expected, Definition 18 is the family-version of the definition of $A \cup B$, and Definition 20 is the family-version of the definition of $A \cap B$.

▶ Remark. Let $A, B \in \mathcal{P}(X)$, and let $\lambda^2 := (\lambda_0^2, \sigma_1^2, \lambda_1^2)$ be a $\mathbb{2}$-family of subsets of X, where λ_0^2, λ_1^2 are defined as in Definition 2, $\sigma_1^2(0) := (A, X, i_A)$, and $\sigma_1^2(1) := (B, X, i_B)$. Then

$$\bigcup_{i\in\mathbb{2}}\lambda_0(i) =_{\mathcal{P}(X)} A \cup B \quad \& \quad \bigcap_{i\in\mathbb{2}}\lambda_0(i) =_{\mathcal{P}(X)} A \cap B.$$

Proof. We work similarly to the proof of Proposition 7. ◀

5 Concluding remarks

There are many issues regarding the relation between BST and CSFT that, due to lack of space, cannot be elaborated here. E.g., in the literature of constructive mathematics (see e.g., [9]) the powerset $\mathcal{P}(X)$ of a set X is treated as a set. Bishop's comment in [4], p. 68, on the existence of a map (i.e., a function) from the complemented subsets of X to $\mathcal{P}(X)$ seems to support such a view. In his definition though, of a set-indexed family of subsets in [4], p. 65, Bishop is careful to use the notion of a rule (an assignment routine) which only behaves like a function. As Bishop himself explains in [7], p. 67, on the occasion of the precise definition of a measure space, one must rewrite appropriately the material in [4], in order to be "comfortably" formalised. Such an appropriate rewriting explains our use of the universes $\mathbb{V}_0$ and $\mathbb{V}_1$. In our view, the totality $\mathbb{V}_0$ is implicit in Bishop's formulation in [4], p. 72, regarding the definition of a set-indexed family of sets. There he writes about

> ... a rule which assigns to each t in a discrete set T a set $\lambda(t)$.

Similarly, the universe $\mathbb{V}_1$ is just a way to rewrite appropriately notions of rules that assign elements of an index set to sets and functions between them with certain properties.

A variation of Definition 1 is the constructive version of the direct spectrum over a directed set (see [13], p. 420). If I is a set and $i \prec j$ an extensional and transitive relation on $I \times I$, let $\prec(I) := \{(i,j) \in I \times I \mid i \prec j\}$. An I-*transitive family of sets with respect to* $\prec$ is a couple $\Lambda^{\prec} := (\lambda_0, \lambda_1^{\prec})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\lambda_1^{\prec} : \prec(I) \rightsquigarrow \mathbb{V}_1$ where $\lambda_1^{\prec}(i,j) := (\lambda_0(i), \lambda_0(j), \lambda_{ij}^{\prec})$, such that for every $i, j, k \in I$ with $i \prec j$ and $j \prec k$ the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\lambda_{ij}^{\prec}"'] \arrow[dr, "\lambda_{ik}^{\prec}"] & \\
\lambda_0(j) \arrow[r, "\lambda_{jk}^{\prec}"'] & \lambda_0(k).
\end{tikzcd}$$

If $(I, \prec)$ is a directed preorder i.e., $i \prec j$ is irreflexive, transitive, and directed i.e., $\forall_{i,j \in I} \exists_{k \in I}\big(i \prec k \ \& \ j \prec k\big)$, we call $\Lambda^{\prec}$ a *direct family of sets* over I. One can define on $\sum_{i \in I} \lambda_0(i)$ the following equality

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow \exists_{k \in I}\big(i \prec k \ \& \ j \prec k \ \& \ \lambda_{ik}^{\prec}(x) =_{\lambda_0(k)} \lambda_{jk}^{\prec}(y)\big).$$

In [4], p. 65, Bishop defined an I-*set* of subsets of a set X as an I-family $\lambda := (\lambda_0, \sigma_0, \lambda_1)$ of subsets of X such that $\forall_{i,j \in I}\big(\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j) \Rightarrow i =_I j\big)$ i.e., the converse to $i =_I j \Rightarrow \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$ also holds. A basic property of such a family is that functions on the index set I generate functions on the set $\lambda_0 I$ defined by $z \in \lambda_0 I :\Leftrightarrow \exists_{i \in I}\big(z := \lambda_0(i)\big)$, equipped with the equality $\lambda_0(i) =_{\lambda_0 I} \lambda_0(j) :\Leftrightarrow \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. This property is crucial to the definition of measure space, given in [8], p. 282 (see Bishop's comment in [7], p. 67).

A general feature of BST is its harmonious relationship with the topology of Bishop spaces (see [30]). If F_i is a Bishop topology on $\lambda_0(i)$, for every $i \in I$, one can define (using the notion of a least Bishop topology) a canonical Bishop topology on the exterior union $\sum_{i \in I} \lambda_0(i)$ and the dependent product $\prod_{i \in I} \lambda_0(i)$. A precise formulation of this concept relies on the study of inductively defined sets within Bishop's system BISH^* and its expected reconstruction within an appropriate extension CSFT^* of CSFT. The development of CSFT^*, the extension of CSFT with inductive definitions of sets using rules with countably many premisses, is, hopefully, future work.

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