A density theorem for hierarchies of limit spaces over separable metric spaces

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Abstract. In this paper, we show, almost constructively, a density theorem for hierarchies of limit spaces over separable metric spaces. Our proof is not fully constructive, since it relies on the constructively not acceptable fact that the limit relation induced by a metric space satisfies Urysohn's axiom for limit spaces. By adding the condition of strict positivity to Normann's notion of probabilistic projection we establish a relation between strictly positive general probabilistic selections on a sequential space and general approximation functions on a limit space. Showing that Normann's result, that a (general and strictly positive) probabilistic selection is definable on a separable metric space, admits a constructive proof, and based on the constructively shown in [18] cartesian closure property of the category of limit spaces with general approximations, our quite effective density theorem follows. This work, which is a continuation of [18], is within computability theory at higher types and Normann's Program of Internal Computability.

1 Introduction

Normann introduced the distinction between internal and external computability over a mathematical structure already in [11] and initiated, what can be called, a "Program of Internal Computability" (PIC) in [12]-[16] (see also [10]). As he mentions in [14], p.300, "the internal concepts must grow out of the structure at hand, while external concepts may be inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways". Within PIC the characterization of functionals, like the Kleene-Kreisel functionals, is done without reference to any realizing objects, but via limit spaces. As Longley and Normann mention in [10], p.374, the framework of limit spaces leads "in some cases to sharper results than other approaches; moreover, the limit space approach generalizes well to type structures over other base types such as $\mathbb{R}$".

Limit spaces were introduced in computability theory at higher types by Scarpellini in [19], while Hyland in [7] showed that Scarpellini's hierarchy is identical to Kleene's hierarchy of countable functionals over $\mathbb{N}$. In [12] Normann presented this hierarchy using limit spaces, and the corresponding density theorem using the notion of the nth approximation of a functional, for every $n \in \mathbb{N}$. In [18] we generalized Normann's presentation by defining two new subcategories of limit spaces, the limit spaces with general approximations and the limit spaces with approximations. The constructively shown cartesian closure property for these subcategories enabled us to to prove a constructive density theorem for hierarchies of limit spaces over $\mathbb{N}$ and the Cantor space $\mathcal{C}$. The corresponding density theorem for hierarchies of limit spaces over a compact metric space had an essentially classical proof.

In this paper we prove, almost constructively, a density theorem for hierarchies of limit spaces over an arbitrary separable metric space, generalizing and, in our view, computationally advancing the result of [18]. All main proofs included in this paper are within Bishop's informal system of constructive mathematics BISH (see [1], [2] and [3]). Since the fact that the limit relation on $\mathbb{R}$ induced by its metric satisfies Urysohn's axiom of a limit space implies[1] the limited principle of omniscience (LPO), we cannot say now that our results are fully constructive. We discuss a constructive way out in the last section of this paper.

Nevertheless, our proof seems quite effective, since all the other parts of it are completely constructive. It uses again the cartesian closure property of the category of limit spaces with general approximations, and Normann's result on the existence of a probabilistic selection on a separable metric space. Adding the condition of strict positivity to Normann's notion of probabilistic selection a connection between strictly positive probabilistic selections and general approximation functions is established. This density theorem (Theorem 4) shows that limit spaces with general approximations provide a framework for characterizing hierarchies of functionals over base types maybe even more efficiently than general limit spaces.

2 Basic notions and facts

In order to be self-contained we include some basic definitions and facts necessary to the rest of the paper. For a classical treatment of limit spaces see [8] and [9], while for all general topological notions mentioned here see [6]. If X, Y are sets, $\mathbb{F}(X, Y)$ denotes the set of all functions from X to Y. The third condition of the definition of a limit space is known as Urysohn's axiom.

Definition 1. *A limit space is a structure $\mathcal{L} = (X, \lim_X)$, where X is a set, and $\lim_X \subseteq X \times \mathbb{F}(\mathbb{N}, X)$ is a relation satisfying the following conditions:*

(L_1) If $x \in X$ and (x) denotes the constant sequence x, then $\lim_X(x, (x))$.
(L_2) If $\mathcal{S}$ denotes the set of all strictly monotone elements of the Baire space $\mathbb{F}(\mathbb{N}, \mathbb{N})$, then[2] $\forall_{\alpha \in \mathcal{S}}(\lim_X(x, x_n) \to \lim_X(x, x_{\alpha(n)}))$.
(L_3) If $x \in X$ and $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N}, X)$, then $\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}}(\lim_X(x, x_{\alpha(\beta(n))})) \to \lim_X(x, x_n)$.

If $\forall_{x,y \in X} \forall_{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N}, X)}(\lim(x, x_n) \to \lim(y, x_n) \to x = y)$, then the limit

[1] This is a result of Hannes Diener (personal communication).
[2] If $(x_n)_{n\in\mathbb{N}} \subseteq X$, for simplicity we write $\lim_X(x, x_n)$ instead of $\lim_X(x, (x_n)_{n\in\mathbb{N}})$, and $\lim_X(x, x)$ instead of $\lim_X(x, (x))$.

space has the uniqueness property. A subset D of X is called $\lim_X$*-dense, if* $\forall_{x\in X}\exists_{(d_n)_{n\in\mathbb{N}}\in\mathbb{F}(\mathbb{N},D)}(\lim_X(x,d_n))$*, and $\mathcal{L}$ is called* $\lim_X$*-separable, if there is a countable* $\lim_X$*-dense subset of X. If $(X,\lim_X)$, $(Y,\lim_Y)$ are limit spaces, $f : X \to Y$ is called* $\lim$*-continuous, if* $\forall_{x\in X}\forall_{(x_n)_{n\in\mathbb{N}}\in\mathbb{F}(\mathbb{N},X)}(\lim_X(x,x_n) \to \lim_Y(f(x),f(x_n)))$*. The subset $\mathcal{O}$ of X is in the Birkhoff-Baer topology $\mathcal{T}_{\lim_X}$ on X, or is* $\lim_X$*-open, if* $\forall_{x\in\mathcal{O}}\forall_{(x_n)_{n\in\mathbb{N}}\in\mathbb{F}(\mathbb{N},X)}(\lim_X(x,x_n) \to \mathrm{ev}(x_n,\mathcal{O}))$*, where, if $A \subseteq X$,* $\mathrm{ev}(x_n,A) :\leftrightarrow \exists_{n_0}\forall_{n\geq n_0}(x_n \in A)$*. A topological space $(X,\mathcal{T})$ induces a limit space $(X,\lim_{\mathcal{T}})$, where $\lim_{\mathcal{T}}(x,x_n) :\leftrightarrow (x_n)_n \xrightarrow{\mathcal{T}} x$, and the symbol $(x_n)_n \xrightarrow{\mathcal{T}} x$ denotes the convergence of $(x_n)_{n\in\mathbb{N}}$ to x with respect to $\mathcal{T}$. If (X,d) is a metric space,* $\lim_d$ *denotes the limit relation on X induced by d. A limit space $(X,\lim_X)$ is called topological, if $\lim_X = \lim_{\mathcal{T}_{\lim_X}}$, and a topological space $(X,\mathcal{T})$ is called sequential, if $\mathcal{T} = \mathcal{T}_{\lim_{\mathcal{T}}}$.*

It is easy to show constructively that D is dense in $(X,\mathcal{T}_{\lim_X})$, if D is lim-dense in $(X,\lim_X)$. Moreover, classically a metric space is a sequential space. The following proposition is folklore in the classical literature, but one can show that it holds constructively (see [17]).

Proposition 1. *Let $\mathcal{L} = (X,\lim_X)$, $\mathcal{M} = (Y,\lim_Y)$ be limit spaces, and $A \subseteq X$. The relative limit space $\mathcal{L}_A := (A,\lim_A)$ is defined by $\lim_A = (\lim_X)_{|A\times\mathbb{F}(\mathbb{N},A)}$, and the product limit space $\mathcal{L}\times\mathcal{M} := (X\times Y,\lim_{X\times Y})$ is defined by the condition $\lim_{X\times Y}((x,y),(x_n,y_n)) :\leftrightarrow \lim_X(x,x_n) \wedge \lim_Y(y,y_n)$, for every $x \in X, y \in Y$, $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},X)$ and $(y_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},Y)$. The exponential limit space $\mathcal{L} \to \mathcal{M} := (X \to Y,\lim_{X\to Y})$, where $X \to Y$ is the set of all* lim*-continuous functions from $\mathcal{L}$ to $\mathcal{M}$, is defined by the condition* $\lim_{X\to Y}(f,f_n) :\leftrightarrow \forall_{x\in X}\forall_{(x_n)_{n\in\mathbb{N}}\in\mathbb{F}(\mathbb{N},X)}(\lim_X(x,x_n) \to \lim_Y(f(x),f_n(x_n)))$*,*

Definition 2. *A limit space with general approximations is a structure $\mathcal{A} = (X,\lim_X,(\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ such that $(X,\lim_X)$ is a limit space, and, for every $n \in \mathbb{N}$ the approximation functions $\mathrm{XAppr}_n : X \to X$ satisfy the following properties:*

(A_1) If $x \in X$, then $\mathrm{XAppr}_n(\mathrm{XAppr}_n(x)) = \mathrm{XAppr}_n(x)$.
(A_2) $\mathrm{XAppr}_n(X) = \{\mathrm{XAppr}_n(x) \mid x \in X\}$ is an inhabited finite set.
(A_3) If $x \in X$ and $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N},X)$, then

$$\lim_X(x,x_n) \to \lim_X(x,\mathrm{XAppr}_n(x_n)).$$

A limit space with general approximations is a limit spaces with approximations, if the following conditions are satisfied:

$(A_1{}')$ If $x \in X$, then $\mathrm{XAppr}_n(\mathrm{XAppr}_m(x)) = \mathrm{XAppr}_{\min(n,m)}(x)$.
(A_4) XAppr_n is lim*-continuous.*

A limit space $(X,\lim_X)$ admits (general) approximations, if there are functions $(\mathrm{XAppr}_n)_{n\in\mathbb{N}}$ such that $(X,\lim_X,(\mathrm{XAppr}_n)_{n\in\mathbb{N}}))$ is a limit space with (general) approximations

The following two results were proved in [18] constructively.

Proposition 2. *If* $\mathcal{A} = (X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ *is a limit space with general approximations and* $x \in X$*, then* $\lim_X(x, \mathrm{XAppr}_n(x))$*. Moreover, the set* $A = \bigcup_{n\in\mathbb{N}} \mathrm{XAppr}_n(X)$ *is a countable* $\lim_X$*-dense subset of* X*, and therefore dense in* $(X, \mathcal{T}_{\lim_X})$*.*

Theorem 1. *If* $\mathcal{A} = (X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$*,* $\mathcal{B} = (Y, \lim_Y, (\mathrm{YAppr}_n)_{n\in\mathbb{N}})$ *are limit spaces with (general) approximations,* $n \in \mathbb{N}$*,* $x \in X, y \in Y$*, and* $f \in X \to Y$*, we define*

$$(\mathrm{X} \times \mathrm{Y})\mathrm{Appr}_n(x, y) := (\mathrm{XAppr}_n(x), \mathrm{YAppr}_n(y)),$$

$$f \mapsto (\mathrm{X} \to \mathrm{Y})\mathrm{Appr}_n(f),$$

$$(\mathrm{X} \to \mathrm{Y})\mathrm{Appr}_n(f)(x) := \mathrm{YAppr}_n(f(\mathrm{XAppr}_n(x))).$$

The structures $\mathcal{A} \times \mathcal{B} = (X \times Y, \lim_{X\times Y}, ((\mathrm{X} \times \mathrm{Y})\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *and* $\mathcal{A} \to \mathcal{B} = (X \to Y, \lim_{X\to Y}, ((\mathrm{X} \to \mathrm{Y})\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *are limit spaces with (general) approximations.*

From the last two results the following density theorem for a hierarchy of limit spaces over a compact metric space was shown in [18] classically.

Theorem 2. *Let* (X, d) *be a compact metric space. If* $\iota = X \mid \rho \to \sigma$ *is an inductively defined type system* $\mathbb{T}$ *over the base type* X*, then in the* $\mathbb{T}$*-typed hierarchy of limit spaces over* X*, defined by*

$$\mathcal{X}(\iota) := (X(\iota), \lim_{\iota}) := (X, \lim_{d}),$$

$$\mathcal{X}(\rho \to \sigma) := (X(\rho) \to X(\sigma), \lim_{\rho\to\sigma}),$$

the limit space $\mathcal{X}(\tau)$ *admits general approximations* $(\tau\mathrm{Appr}_n)_{n\in\mathbb{N}}$*, for every type* τ *in* $\mathbb{T}$*. Moreover, there is a countable subset* D_τ *of* $X(\tau)$*, which is dense in* $(X(\tau), \mathcal{T}_{\lim_\tau})$*, for every type* τ *in* $\mathbb{T}$*.*

A similar density theorem was shown constructively for $\iota = \mathbb{N}$ and $\iota = \mathcal{C}$, where $\mathcal{C}$ denotes the Cantor space. In section 4 we show a density theorem for a hierarchy of limit spaces over an arbitrary separable space (Theorem 4), based again on Proposition 2 and Theorem 1. In this case though we use appropriately Normann's notion of a probabilistic selection on a sequential space to define general approximation functions on a separable metric space.

3 Positive and strictly positive probabilistic projections

The use of probability distributions in the study of hierarchies of functionals over $\mathbb{R}$ appeared first in Normann's work [12], following the work of DeJaeger in [5]. The next definition includes a slight variation[3] of Normann's definition of

[3] Namely, the continuity condition used by Normann is different from the condition (P_3) used here, but one can show that they are equivalent. Since no continuity condition affects the main density theorem, we do not include here the proof of their equivalence.

a probabilistic projection found in [14]. Moreover, the notions of general, positive and strictly positive probabilistic projections are introduced. Note that in the following definition we use Normann's starting point of a sequential space X, but what we only need for the proof of Theorem 4, and this is how one should read Definition 3 constructively, is that X is a metric space (recall that we need classical reasoning to show that a metric space is sequential).

Definition 3. *A structure $\mathcal{P} = (X, \mathcal{T}, Y, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is called a sequential space with a general probabilistic projection from X to Y, if $(X, \mathcal{T})$ is a sequential topological space, $(A_n)_{n\in\mathbb{N}}$ is a sequence of inhabited finite subsets of X, which is called the support of $\mathcal{P}$, Y is a subset of X such that*

$$A := \bigcup_{n\in\mathbb{N}} A_n \subseteq Y,$$

and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of functions of type

$$\mu_n : X \to \mathbb{F}(A_n, [0,1])$$

$$x \mapsto \mu_n(x),$$

that satisfies the following properties:

(P_1) For every $n \in \mathbb{N}$ the function $\mu_n(x) : A_n \to [0,1]$ is a probability distribution on A_n i.e., it satisfies the condition

$$\sum_{a\in A_n} \mu_n(x)(a) = 1.$$

(P_2) If $y \in Y$, $(y_n)_{n\in\mathbb{N}} \subseteq Y$ such that $\lim_{\mathcal{T}_{|Y}}(y, y_n)$, where $\lim_{\mathcal{T}_{|Y}}$ is the limit relation on Y induced by the limit relation $\lim_{\mathcal{T}}$ on X, and if $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $a_n \in A_n$, for every $n \in \mathbb{N}$, the following implication holds:

$$\forall_{n\in\mathbb{N}}(\mu_n(y_n)(a_n) > 0) \to \lim_{\mathcal{T}_{|Y}}(y, a_n).$$

The sequence of functions $(\mu_n)_{n\in\mathbb{N}}$ is called a general probabilistic projection from X to Y. A sequential space $(X, \mathcal{T})$ admits a general probabilistic projection from X to Y, if there is a general probabilistic projection from X to Y. A structure $\mathcal{P} = (X, \mathcal{T}, Y, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a sequential space with a probabilistic projection from X to Y, if $(\mu_n)_{n\in\mathbb{N}}$ satisfies also the following condition:

(P_3) If $a \in A_n$, for some $n \in \mathbb{N}$, the function $\hat{a} : X \to [0,1]$, defined by

$$x \mapsto \mu_n(x)(a),$$

for every $x \in X$, is continuous.

A general probabilistic projection $(\mu_n)_{n\in\mathbb{N}}$ from X to Y is called positive, if the following conditions are satisfied:

(P_4) If $a \in A_n$, for some $n \in \mathbb{N}$, then

$$\mu_n(a)(a) > 0,$$

$$\forall_{b\in A_n}(b \neq a \to \mu_n(a)(b) < \mu_n(a)(a)).$$

A positive probabilistic projection from X to Y is called strictly positive, if the following condition is satisfied:

(P_5) *If $a \in A_n$, for some $n \in \mathbb{N}$, then*

$$\mu_n(a)(a) = 1.$$

A (general) probabilistic projection $(\mu_n)_{n\in\mathbb{N}}$ from X to X is called a (general) probabilistic selection on X, and the structure $\mathcal{S} = (X, \mathcal{T}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$, or simpler $\mathcal{S} = (X, \mathcal{T}, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$, is a sequential space with a (general) probabilistic selection.

By condition (P_1), if $(\mu_n)_{n\in\mathbb{N}}$ is a strictly positive probabilistic projection from X to Y, then

$$\forall_{b\in A_n}(b \neq a \to \mu_n(a)(b) = 0),$$

since, if $\mu_n(a)(b) > 0$, for some $b \in A_n$ such that $b \neq a$, then $\sum_{b\in A_n} \mu_n(a)(b) > 1$, which is a contradiction. Hence, $\mu_n(a)(b) \leq 0$, which together with the assumed condition $\mu_n(a)(b) \geq 0$ gives $\mu_n(a)(b) = 0$. A first constructive reading of condition (P_1) gives that $\neg\neg[\exists_{a\in A_n}(\mu_n(x)(a) > 0)]$; if $\neg[\exists_{a\in A_n}(\mu_n(x)(a) > 0)]$, then $\forall_{a\in A_n}(\mu_n(x)(a) \leq 0)$, since if $a \in A_n$ such that $\mu_n(x)(a) > 0$, then we get a contradiction, hence $\mu_n(x)(a) \leq 0$. Since $\forall_{a\in A_n}(\mu_n(x)(a) \geq 0)$, we get $\forall_{a\in A_n}(\mu_n(x)(a) = 0)$, hence $\sum_{a\in A_n} \mu_n(x)(a) = 0 = 1$. Next we show constructively how to shift double negation.

Proposition 3. *If $n \in \mathbb{N}$, $a_1, \ldots, a_n \geq 0$, and $l > 0$, then*

$$\sum_{i=1}^{n} a_i = l \to \exists_{j\in\{1,\ldots,n\}}(a_j > 0).$$

Proof. We show $\forall_{n\in\mathbb{N}} P(n)$, where

$$P(n) := \forall_{a_1,\ldots,a_n \geq 0}\forall_{l>0}\left(\sum_{i=1}^{n} a_i = l \to \exists_{j\in\{1,\ldots,n\}}(a_j > 0)\right).$$

If $n = 1$, then $j = 1$. To show $P(n+1)$ from $P(n)$ let $a_1, \ldots, a_{n+1} \geq 0$, and $l > 0$ such that $\sum_{i=1}^{n+1} a_i = l$. If $b := \sum_{i=1}^{n} a_i \geq 0$, then $b + a_{n+1} = l$. By the constructive version of trichotomy of reals (see [2], p.26) we have that $a_{n+1} > 0$ or $a_{n+1} < \frac{l}{2}$. In the first case we get that the required $j = n + 1$. If $a_{n+1} < \frac{l}{2}$, then $b = l - a_{n+1} > l - \frac{l}{2} = \frac{l}{2}$. Consequently, $\sum_{i=1}^{n} a_i = b > 0$, and by condition $P(n)$ on $a_1, \ldots, a_n$ and b we get some $j \in \{1, \ldots, n\}$ such that $a_j > 0$.

Hence, if $(\mu_n)_{n\in\mathbb{N}}$ is a general probabilistic projection from X to Y, the set

$$I_n(x) := \{a \in A_n \mid \mu_n(x)(a) > 0\}$$

is inhabited. The intuition behind the notion of a probabilistic projection from X to Y can be described as follows. The fact $\mu_n(x)(a) > 0$ expresses that a is "close" to x, and moreover, the closer to 1 the positive value $\mu_n(x)(a)$ is, the closer to x a is. The fact $\mu_n(x)(a) = 0$ expresses that a is "not close" to x. With this interpretation conditions (P_2) and (P_4) are quite natural. Note that the notion of a general probabilistic projection from X to Y corresponds to the notion of a limit space with general approximations, since in both cases a continuity condition is not necessarily satisfied. As in the case of limit spaces with (general) approximations, a dense subset is (classically) generated from a general probability projection.

Proposition 4. *(i) If $\mathcal{P} = (X, \mathcal{T}, Y, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a sequential space with a general probability projection from X to Y, and Y is a closed, or open, subspace of X, then A is dense in Y.*
(ii) If $\mathcal{P} = (X, \mathcal{T}, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a sequential space with a general probability selection, then A is dense in X.

Proof. We show (i), and (ii) follows immediately from (i). If $y \in Y$, let $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $a_n \in A_n$ and $\mu_n(y)(a_n) > 0$. The existence of such an element a_n of A_n follows from condition (P_1). Since $\lim_{\mathcal{T}}(y, y)$, by condition (P_2) we get $\lim_{\mathcal{T}_{|Y}}(y, a_n)$ i.e., A is $\lim_{\mathcal{T}_{|Y}}$-dense in X. Since a closed, or open, subspace of a sequential space is sequential, and since a $\lim_{\mathcal{S}}$-dense subset of a sequential space $(Z, \mathcal{S})$ is also dense in Z, we conclude that A is dense in Y.

Since A is countable, the relative space Y is separable. Consequently, if Y is not a separable subspace of X, there can be no probabilistic projection from X to Y. As in the density theorem for limit spaces with general approximations the continuity condition (P_3) plays no role in the above proof. Next follows the lim-version of Definition 3.

Definition 4. *A structure $\mathcal{N} = (X, \lim_X, Y, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a limit space with a general* lim*-probabilistic projection from X to Y, if $(X, \lim_X)$ is a limit space and $Y, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}$ are as in Definition 3, though the limit relation in (P_2) is the limit relation on Y inherited from* $\lim_X$*. A limit space with a* lim*-probabilistic projection from X to Y is a limit space with a general* lim*-probabilistic projection from X to Y such that the following condition is satisfied:*

$(P_3{}')$ If $a \in A_n$, for some $n \in \mathbb{N}$, the function $\hat{a} : X \to [0, 1]$, defined by $x \mapsto \mu_n(x)(a)$, for every $x \in X$, is lim*-continuous i.e.,*

$$\lim_X(x, x_m) \to \lim_{[0,1]}(\mu_n(x)(a), \mu_n(x_m)(a)),$$

for every $x \in X$ and $(x_m)_{m\in\mathbb{N}} \subseteq X$, where $\lim_{[0,1]}$ *is the limit relation on $[0, 1]$ generated by its Euclidean metric. A limit space with a (general)* lim*-probabilistic selection, and the notions of a (strictly) positive (general)* lim*-probabilistic projection (selection) are defined as in Definition 3.*

In the next classically shown proposition the hypothesis of positivity is used.

Proposition 5. *If* $(X, \lim_X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ *is a limit space with a positive, general* lim*-probabilistic selection, then there are approximation functions* $(\mathrm{XAppr}_n)_{n\in\mathbb{N}}$ *on* X *such that* $(X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ *is a limit space with general approximations, and* $\mathrm{XAppr}_n(X) = A_n$*, for every* $n \in \mathbb{N}$.

Proof. If $n \in \mathbb{N}$, suppose that $A_n = \{a_1^{(n)}, \ldots, a_{m(n)}^{(n)}\}$. If $x \in X$, let

$$i_{0,n}(x) := \Big\{ i \in \{1, \ldots, m(n)\} \mid \mu_n(x)(a_i^{(n)}) > 0, \text{ and}$$

$$\forall_{j\in\{1,\ldots,m(n)\}}(\mu_n(x)(a_j^{(n)}) \leq \mu_n(x)(a_i^{(n)})) \Big\}.$$

By the properties of the order on classical real numbers $i_{0,n}(x)$ is well-defined. For every $x \in X$ and every $n \in \mathbb{N}$ we define

$$\mathrm{XAppr}_n(x) := a_{i_{0,n}(x)}^{(n)}.$$

Since $(\mu_n)_{n\in\mathbb{N}}$ is positive, if $i \in \{1, \ldots, m(n)\}$, then

$$i_{0,n}(a_i^{(n)}) = \{i\},$$

and $\mathrm{XAppr}_n(a_i^{(n)}) = a_{i_{0,n}(a_i^{(n)})}^{(n)} = a_i^{(n)}$. The conditions $\mathrm{XAppr}_n(\mathrm{XAppr}_n(x)) = \mathrm{XAppr}_n(x)$ and $\mathrm{XAppr}_n(X) = A_n$ are then immediately satisfied. By the definition of $i_{0,n}(x)$ we have that

$$\mu_n(x)(\mathrm{XAppr}_n(x)) > 0.$$

If $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\lim_X(x, x_n)$, then since $\mu_n(x_n)(\mathrm{XAppr}_n(x_n)) > 0$, for every $n \in \mathbb{N}$, by condition (P_2) of Definition 4 we get $\lim_X(x, \mathrm{XAppr}_n(x_n))$.

Note that constructively we can't find an algorithm providing an element of $i_{0,n}(x)$. We overcome this difficulty in Proposition 7, where the hypothesis of a strictly positive probabilistic selection is used. The next proposition is also shown classically.

Proposition 6. *(i) A limit space* $(X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ *with general approximations admits a strictly positive, general* lim*-probabilistic selection.*
(ii) A limit space $(X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ *with approximations, where* $(X, \lim_X)$ *has the uniqueness property, admits a strictly positive general* lim*-probabilistic selection.*

Proof. (i) We define $A_n = \mathrm{XAppr}_n(X)$, and for every $x \in X$ the function $x \mapsto \mu_n(x)$ is defined by

$$\mu_n(x)(a) = \begin{cases} 1 \text{ , if } a = \mathrm{XAppr}_n(x) \\ 0 \text{ , ow.} \end{cases}$$

Clearly, $\mu_n(x)$ is a probability distribution on A_n. Since $\mu_n(x_n)(a_n) > 0 \leftrightarrow a_n = \mathrm{XAppr}_n(x_n)$, we get $\lim_X(x, x_n) \to \lim_X(x, a_n)$. If $a \in \mathrm{XAppr}_n(X)$, there is some $x \in X$ such that $a = \mathrm{XAppr}_n(x)$, hence

$$\mu_n(a)(a) = \mu_n(\mathrm{XAppr}_n(x))(\mathrm{XAppr}_n(x)) = 1 > 0,$$

since $\mathrm{XAppr}_n(x) = \mathrm{XAppr}_n(\mathrm{XAppr}_n(x))$.
(ii) Suppose that $\lim_X(x, x_m)$ and that $\mu_n(x)(a) = 1 \leftrightarrow a = \mathrm{XAppr}_n(x)$. By the classical proof of Proposition 21(i) in [18], pp.749-750, the sequence $(\mathrm{XAppr}_n(x_m))_{m\in\mathbb{N}}$ is eventually constant with value a. Thus, $(\mu_n(x_m)(a))_{m\in\mathbb{N}}$ is eventually constant 1. The case $a \neq \mathrm{XAppr}_n(x)$ is treated similarly.

The above proof corroborates the aforementioned intuition behind the existence of a probabilistic projection, that is $\mu_n(x)(a) > 0$ expresses a proximity of a to x, while $\mu_n(x)(a) = 0$ expresses a non-proximity of a to x. Regarding the proof of Proposition 6(ii), the lim-continuity of the approximation functions XAppr_n entails the lim-continuity of the function $\hat{a}$, where $a \in A$. Next follows the constructive version of Proposition 5, which is essential to the proof of Theorem 4. One needs to replace the condition of positivity by the condition of strict positivity.

Proposition 7. *If $(X, \lim_X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a limit space with a strictly positive, general* lim*-probabilistic selection, then there are approximation functions $(\mathrm{XAppr}_n)_{n\in\mathbb{N}}$ on X such that $(X, \lim_X, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ is a limit space with general approximations, and $\mathrm{XAppr}_n(X) = A_n$, for every $n \in \mathbb{N}$.*

Proof. If $n \in \mathbb{N}$, suppose that $A_n = \{a_1^{(n)}, \ldots, a_{m(n)}^{(n)}\}$. If $x \in X$, the set

$$I_{0,n}(x) := \{i \in \{1, \ldots, m(n)\} \mid \mu_n(x)(a_i^{(n)}) > 0\}$$

is inhabited, i.e., for every $n \in \mathbb{N}$ there exists $i \in I_{0,n}(x)$. If $S_x \subseteq \mathbb{N} \times \bigcup_{n=1}^{\infty} I_{0,n}(x)$ is defined by $S_x(n, i) := i \in I_{0,n}(x)$, then by the principle of countable choice[4] there is a function $f_x : \mathbb{N} \to \bigcup_{n=1}^{\infty} I_{0,n}(x)$ such that $f_x(n) \in I_{0,n}(x)$, for every $n \in \mathbb{N}$. We define

$$\mathrm{XAppr}_n(x) := a_{f_x(n)}^{(n)},$$

for every $x \in X$ and every $n \in \mathbb{N}$. Since $(\mu_n)_{n\in\mathbb{N}}$ is a strictly positive probabilistic selection on X, if $i \in \{1, \ldots, m(n)\}$, then $I_{0,n}(a_i^{(n)}) = \{i\}$, hence

$$f_{a_i^{(n)}}(n) = i,$$

and $\mathrm{XAppr}_n(a_i^{(n)}) = a_{f_{a_i^{(n)}}(n)}^{(n)} = a_i^{(n)}$. The conditions $\mathrm{XAppr}_n(\mathrm{XAppr}_n(x)) = \mathrm{XAppr}_n(x)$ and $\mathrm{XAppr}_n(X) = A_n$ are then immediately satisfied. By the definition of $I_{0,n}(x)$ we have that

$$\mu_n(x)(\mathrm{XAppr}_n(x)) > 0.$$

If $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\lim_X(x, x_n)$, then since $\mu_n(x_n)(\mathrm{XAppr}_n(x_n)) > 0$, for every $n \in \mathbb{N}$, by condition (P_2) of Definition 4 we get $\lim_X(x, \mathrm{XAppr}_n(x_n))$.

[4] This principle is generally accepted within BISH (see [3], p.12).

4 The density theorem

In [14] Normann proved that a complete and separable metric space X admits a probabilistic projection from X to a closed subspace Y of X of the form $Y = \overline{\bigcup_n A_n}$, where $A_n \subseteq A_{n+1} \subseteq X$, for every $n \in \mathbb{N}$. In [13] Normann defined a probabilistic selection on a separable metric space. The proof is not given in [13], although it is actually in [14], which appeared later, but was written before [13]. In between Normann realized that completeness played no role in his original proof.

Here we show that the probabilistic selections defined by Normann differ in a crucial way. The one given in [14] is shown here to be positive, while the one given in [13] is shown to be strictly positive, a property crucial to the proof of Theorem 4. Next we give a new constructive treatment of Normann's result adding the properties of positivity and strict positivity, respectively. Note that Normann included his equivalent to (P_3) continuity condition to his results, but since the proof of continuity requires classical reasoning and does not play a role in our proof of Theorem 4, it is avoided here. The only non-effective element in the formulation of the following theorem (and not in its proof) is that $(X, \lim_d)$ is a limit space, hence that $\lim_d$ satisfies Urysohn's axiom.

Theorem 3 (Normann (BISH)). *Suppose that (X, d) is a separable metric space and $A = \{a_n \mid n \in \mathbb{N}\}$ is a countable dense subset of X, where $d(a_n, a_m) > 0$, if $n \neq m$. If $A_n = \{a_1, \ldots, a_n\}$, for every $n \in \mathbb{N}$ and, for every $1 \leq j \leq n$, we define*[5]

$$\mu_n(x)(a_j) := \frac{N_{n,x}(a_j)}{D_{n,x}},$$

$$N_{n,x}(a_j) := (d(x, A_n) + 2^{-n}) \dot{-} d(x, a_j),$$

$$D_{n,x} := \sum_{i=1}^{n} [(d(x, A_n) + 2^{-n}) \dot{-} d(x, a_i)],$$

$$\mu'_n(x)(a_j) := \frac{N'_{n,x}(a_j)}{D'_{n,x}},$$

$$N'_{n,x}(a_j) := (d(x, A_n) + \delta_n) \dot{-} d(x, a_j),$$

$$D'_{n,x} := \sum_{i=1}^{n} [(d(x, A_n) + \delta_n) \dot{-} d(x, a_i)],$$

where

$$d(x, A_n) := \min\{d(x, a_i) \mid 1 \leq i \leq n\},$$

$$\delta_n := \min\{2^{-n}, d(a_i, a_j) \mid i \neq j, i, j \in \{1, \ldots, n\}\},$$

$$a \dot{-} b := (a - b) \vee 0.$$

[5] If $c, d \in \mathbb{R}$, we use the notations $c \vee d := \max\{c, d\}$, and $c \wedge d := \min\{c, d\}$.

(i) The structure $(X, \lim_d, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ *is a limit space space with a positive, general* lim*-probabilistic selection on* X*.*
(ii) The structure $(X, \lim_d, (A_n)_{n\in\mathbb{N}}, ({\mu'}_n)_{n\in\mathbb{N}})$ *is a limit space with a strictly positive, general* lim*-probabilistic selection on* X*.*

Proof. (i) The fact that $D_{n,x} > 0$ and conditions (P_1) and (P_2) are shown as in case (ii). For the positivity condition we have first that

$$\mu_n(a_j)(a_j) = \frac{2^{-n}}{D_{n,a_j}} > 0,$$

for every $j \in \{1, \ldots, n\}$. If $i \neq j$, then $N_{n,a_j}(a_i) = 2^{-n} \dot{-} d(a_j, a_i) = (2^{-n} - d(a_j, a_i)) \vee 0$. Since $2^{-n} - d(a_j, a_i) < 2^{-n}$ and $0 < 2^{-n}$, we get $(2^{-n} - d(a_j, a_i)) \vee 0 < 2^{-n}$ (here we used the following property of real numbers: $a \vee b < c \leftrightarrow a < c$ and $b < c$, see [4], p.57, Ex.3). Hence, $\mu_n(a_j)(a_i) < \mu_n(a_j)(a_j)$.
(ii) If $c_1, \ldots, c_n > 0$, then one shows[6] that their minimum $\bigwedge_{i=1}^n c_i > 0$, hence, since there are no repetitions in the sequence of A, we have that $\delta_n > 0$. Next we show[7] that ${D'}_{n,x} > 0$. The subspace A_n is totally bounded, since for every $\epsilon > 0$ it is an ϵ-approximation of itself, and since the distance d_x at x, defined by $a_j \mapsto d(x, a_j)$, is uniformly continuous on A_n, there exists $\inf d_x(A_n)$ (see [2], p.94). It is immediate to see that $\inf d_x(A_n) = d(x, A_n)$ is the greatest lower bound of $\{d(x, a_j) \mid j \in \{1, \ldots, n\}\}$, and hence equal to $\bigwedge_{i=1}^n d(x, a_i)$, since $\bigwedge_{i=1}^n d(x, a_i)$ can be shown[8] to be the greatest lower bound of $\{d(x, a_j) \mid j \in \{1, \ldots, n\}\}$ too. By the definition of the infimum of a bounded below set of real numbers for $\frac{\delta_n}{2} > 0$ we get that the existence of some $j \in \{1, \ldots, n\}$ such that

$$\begin{aligned} d(x, a_j) < d(x, A_n) + \frac{\delta_n}{2} \to -\frac{\delta_n}{2} < d(x, A_n) - d(x, a_j) \to \\ 0 < \delta_n - \frac{\delta_n}{2} < d(x, A_n) + \delta_n - d(x, a_j) \to \\ 0 < \frac{\delta_n}{2} < (d(x, A_n) + \delta_n) \dot{-} d(x, a_j) \to \\ 0 < {D'}_{n,x}. \end{aligned}$$

Condition (P_1) is immediately satisfied. For the proof of condition (P_2) we fix $x \in X$, $(x_n)_{n\in\mathbb{N}} \subseteq X$, such that $\lim_d(x, x_n)$, and $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A_n$ and ${\mu'}_n(x_n)(a_n) > 0$, for every $n \in \mathbb{N}$. We need to show that $\lim_d(x, a_n) \leftrightarrow \forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}(d(x, a_n) \leq \epsilon)$. Let $\epsilon > 0$. By the hypothesis $\lim_d(x, x_n)$ there

[6] The argument for the case of two positive numbers is the one used in the inductive step of the induction on n. If $c_1, c_2 > 0$, there are rationals q_1, q_2 such that $0 < q_1 < c_1$ and $0 < q_2 < c_2$ (see [2], p.25). Since $q_1 \wedge q_2$ is either q_1 or q_2, we get that $q_1 \wedge q_2 < c_1$ and $q_1 \wedge q_2 < c_2$, hence $0 < q_1 \wedge q_2 \leq c_1 \wedge c_2$.

[7] Classically, this is trivial, since there is some $j \in \{1, \ldots, n\}$ such that $d(x, A_n) = d(x, a_j)$, hence ${D'}_{n,x} \geq (d(x, A_n) + \delta_n) \dot{-} d(x, a_i) = \delta_n \vee 0 = \delta_n > 0$.

[8] The proof is based on the fact that if $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$, since if $c > a \wedge b$, then $c > a$ or $c > b$ (this is the dual of a property of the maximum of real numbers included in [4], p.57, Ex.3).

is $n_1 \in \mathbb{N}$ such that $\forall_{n \geq n_1}(d(x, x_n) \leq \frac{\epsilon}{4})$. By the density of A in X there exists $a \in A$ such that $d(x, a) \leq \frac{\epsilon}{4}$. If $a = a_{n_2}$, for some $n_2 \in \mathbb{N}$, we get that $\exists_{a \in A_{n_2}}(d(x, a) \leq \frac{\epsilon}{4})$. Clearly, there exists $n_3 \in \mathbb{N}$ such that $2^{-n_3} \leq \frac{\epsilon}{4}$. If we define $n_0 := \max(n_1, n_2, n_3)$, for every $n \in \mathbb{N}$ such that $n \geq n_0$ we get $A_n \supseteq A_{n_0}$ and $d(x, A_n) \leq d(x, a) \leq \frac{\epsilon}{4}$. Moreover, if $n \geq n_0$, then

$$d(x_n, A_n) \leq d(x_n, x) + d(x, A_n) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

The first inequality above is shown as follows: If $b \in A_n$, then using some basic properties of $\leq$ on $\mathbb{R}$ (see [2], p.23) we get

$$\begin{aligned}
&d(x_n, A_n) \leq d(x_n, b) \leq d(x_n, x) + d(x, b) \rightarrow \\
&d(x_n, A_n) - d(x_n, x) \leq d(x, b) \rightarrow \\
&d(x_n, A_n) - d(x_n, x) \leq \min\{d(x, b) \mid b \in A_n\} = d(x, A_n) \rightarrow \\
&d(x_n, A_n) \leq d(x_n, x) + d(x, A_n).
\end{aligned}$$

Moreover, if $n \geq n_0$, then

$$\mu'_n(x_n)(a_n) > 0 \rightarrow d(x_n, a_n) \leq \frac{3\epsilon}{4},$$

since, using the property[9] $\forall_{c \in \mathbb{R}}(c \vee 0 > 0 \rightarrow c \vee 0 = c)$ we have that

$$\begin{aligned}
\mu'_n(x_n)(a_n) > 0 \rightarrow\ & N'_{n,x_n}(a_n) > 0 \\
&\leftrightarrow (d(x_n, A_n) + \delta_n) \dot{-} d(x_n, a_n) > 0 \\
&\rightarrow (d(x_n, A_n) + \delta_n) - d(x_n, a_n) > 0 \\
&\rightarrow d(x_n, a_n) < d(x_n, A_n) + \delta_n \leq \frac{\epsilon}{2} + 2^{-n} \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} = \frac{3\epsilon}{4}.
\end{aligned}$$

Hence, if $n \geq n_0$, we get

$$d(x, a_n) \leq d(x, x_n) + d(x_n, a_n) \leq \frac{\epsilon}{4} + \frac{3\epsilon}{4} = \epsilon.$$

Next we show the strict positivity of $(\mu'_n)_{n \in \mathbb{N}}$. If $n \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$, then $N'_{n,a_j}(a_j) = \delta_n$, since $d(a_j, A_n) = d(a_j, a_j) = 0$. Moreover, $D'_{n,a_j}(a_j) = \sum_{i=1}^{n}[(d(a_j, A_n) + \delta_n) \dot{-} d(a_j, a_i)] = \sum_{i=1}^{n}(\delta_n \dot{-} d(a_j, a_i)) = \delta_n \dot{-} d(a_j, a_j) = \delta_n$, since for every $i \neq j$, we have that $\delta_n \dot{-} d(a_j, a_i) = 0$, since $\delta_n \leq d(a_j, a_i) \leftrightarrow \delta_n - d(a_j, a_i) \leq 0$. Consequently, $\mu'_n(a_j)(a_j) = 1$. Similarly, if $i \neq j$, we have that $N'_{n,a_j}(a_i) = \delta_n \dot{-} d(a_j, a_i) = 0$ i.e., $\mu'_n(a_j)(a_i) = 0$.

A "geometric" interpretation of the probabilistic selection $(\mu_n)_{n \in \mathbb{N}}$ of Theorem 3 goes as follows. By its definition $N_{n,x}(a_j) \geq 0$, while $\mu_n(x)(a_j) = 0 \leftrightarrow N_{n,x}(a_j) = 0 \leftrightarrow d(x, a_j) \geq d(x, A_n) + 2^{-n}$. If $x \notin A_n$ that can happen if a_j is sufficiently far from the point of A_n at which x attains its minimum distance from

[9] If $c \vee 0 > 0$, then $c > 0 \vee 0 > 0$ (see [4], p.57). Hence, $c > 0$ is the case, and then we get immediately that $c \vee 0 = c$.

A_n, or if $x \in A_n$ and $d(x, a_j) \geq 2^{-n}$. Moreover, $\mu_n(x)(a_j) > 0 \leftrightarrow N_{n,x}(a_j) > 0 \leftrightarrow d(x, a_j) < d(x, A_n) + 2^{-n}$ i.e., either x attains its minimum distance from A_n at a_j or, otherwise, the distance $d(x, a_j)$ is less than 2^{-n}-close to the minimum distance $d(x, A_n)$ i.e., a_j is very close to the point of A_n at which x attains its minimum distance from A_n. A similar interpretation can be given for Normann's probabilistic selection $({\mu'}_n)_{n\in\mathbb{N}}$. Note that a simpler definition, like

$$\nu_n(x)(a_j) = \frac{d(x, a_j)}{\sum_{i=1}^{n} d(x, a_i)}$$

gives rise to a probability distribution on A_n, which trivially satisfies the continuity condition, but it is not positive, and the hypothesis $\nu_n(x_n)(a_n) > 0$ is equivalent to $x_n \neq a_n$, which is far from satisfying condition (P_2) of a probabilistic selection.

Note that the constructive proof of Theorem 3 works for dense subsets A of X with a decidable equality, like $\mathbb{Q}$ in $\mathbb{R}$. Next follows a density theorem for hierarchies of limit spaces over separable metric spaces, the countable dense subsets of which are appropriately enumerated, or have a decidable equality.

Theorem 4 (density theorem). *Let (X, d) be a separable metric space, and let $A = \{a_n \mid n \in \mathbb{N}\}$ be a dense subset of X, where $d(a_n, a_m) > 0$, if $n \neq m$. If $\iota = X \mid \rho \to \sigma$ is an inductively defined type system $\mathbb{T}$ over the base type X, then in the $\mathbb{T}$-typed hierarchy of limit spaces over X, defined by*

$$\mathcal{X}(\iota) := (X(\iota), \lim_{\iota}) := (X, \lim_{d}),$$

$$\mathcal{X}(\rho \to \sigma) := (X(\rho) \to X(\sigma), \lim_{\rho \to \sigma}),$$

the limit space $\mathcal{X}(\tau)$ admits general approximations $(\tau\mathrm{Appr}_n)_{n\in\mathbb{N}}$, for every type τ in $\mathbb{T}$. Moreover, there is a countable subset D_τ of $X(\tau)$, which is $\lim_\tau$-dense in X_τ, therefore dense in $(X(\tau), \mathcal{T}_{\lim_\tau})$, for every type τ in $\mathbb{T}$.

Proof. If $\tau = \iota$, then by Theorem 3(ii) $(X, \lim_d, (A_n)_{n\in\mathbb{N}}, ({\mu'}_n)_{n\in\mathbb{N}})$ is a limit space with a strictly positive, general lim-probabilistic selection on X. By Proposition 7 there exist approximation functions $(\mathrm{XAppr}_n)_{n\in\mathbb{N}}$ on X such that the structure $(X, \lim_d, (\mathrm{XAppr}_n)_{n\in\mathbb{N}})$ is a limit space with general approximations, and $\mathrm{XAppr}_n(X) = A_n$, for every $n \in \mathbb{N}$. We define $\iota\mathrm{Appr}_n := \mathrm{XAppr}_n$, for every $n \in \mathbb{N}$. By Theorem 1, if $f \in X(\rho) \to X(\sigma)$, and $n \in \mathbb{N}$, then the function $(\rho \to \sigma)\mathrm{Appr}_n$, defined by

$$[(\rho \to \sigma)\mathrm{Appr}_n](f)(x) = \sigma\mathrm{Appr}_n(f(\rho\mathrm{Appr}_n(x))),$$

for every $x \in X(\rho)$, is the n-th approximation function that the limit space $\mathcal{X}(\rho \to \sigma)$ admits. The existence of the countable subset D_τ of $X(\tau)$ that is $\lim_\tau$-dense in X_τ, therefore dense in $(X(\tau), \mathcal{T}_{\lim_\tau})$, for every type τ in $\mathbb{T}$, follows from Proposition 2.

Note that constructively we only have that $\mathcal{T}_d \subseteq \mathcal{T}_{\lim_d}$, where $\mathcal{T}_d$ is the topology on X induced by its metric. Thus, what we determine through Theorem 4 are countable $\lim_\tau$-dense subsets of each limit space X_τ. Of course, classically, these are exactly the subsets one needs to find. Clearly, a density theorem for a hierarchy of limit spaces over more than one separable metric spaces can be shown similarly.

5 Concluding remarks

The proof of the main density theorem presented in this paper reveals, in our view, the merits of the generalization of Normann's notion of the nth approximation of a functional in the typed hierarchy over $\mathbb{N}$ through the notion of a limit space with general approximations. The quite effective character of its proof is also worth noticing. As Normann writes in [14], p.305,

> [We would like to claim that an internal approach to computability in analysis will result in easy-to-use, high level, programming languages for computing in analysis, but the development cannot support this claim yet. The possibility of finding support for such a claim, together with basic curiosity, is nevertheless the motivation behind trying to find out what internally based algorithms might look like.]

The application of limit spaces with approximations to the (classical) study of limit spaces over other base types looks also promising. Moreover, it is interesting to see if the general idea behind the theory of limit spaces with approximations can be extended to other notions of space. Namely, to find a cartesian closed category $\mathcal{A}$, such that if X is an object of $\mathcal{A}$, general approximation functions XAppr_n of type $X \to X$ can be defined[10], for every $n \in \mathbb{N}$, such that the objects of $\mathcal{A}$ with general approximations form a cartesian closed subcategory of $\mathcal{A}$.

A plan to provide a fully constructive proof of Theorem 4 is the following. We expect that abstracting from the constructive properties of $\lim_d$ we can define a notion of a constructive limit space (X, clim_X) that preserves the cartesian closure property of limit spaces (with the same definition of the limit relation on the function space). In this case the proof of Theorem 4 goes through completely constructively, since the proof of Theorem 1 does not depend on Urysohn's axiom. We hope to realize this plan in future work.

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[10] Where the notion of approximation, as it is expressed in condition (A_3) of Definition 2, will depend on the structure of X.

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