Borel and Baire sets in Bishop spaces

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Abstract. We study the Borel sets $\mathtt{Borel}(F)$ and the Baire sets $\mathtt{Baire}(F)$ generated by a Bishop topology F on a set X. These are inductively defined sets of F-complemented subsets of X. Because of the constructive definition of $\mathtt{Borel}(F)$, and in contrast to classical topology, we show that $\mathtt{Baire}(F) = \mathtt{Borel}(F)$. We define the uniform version of an F-complemented subset of X and we show the Urysohn lemma for them. We work within Bishop's system BISH* of informal constructive mathematics that includes inductive definitions with rules of countably many premises.

1 Introduction

The set of Borel sets generated by a given family of complemented subsets of a set X, with respect to a set Φ of real-valued functions on X, was introduced in [2], p. 68. This set is inductively defined and plays a crucial role in providing important examples of measure spaces in Bishop's measure theory developed in [2]. As this measure theory was replaced in [4] by the Bishop-Cheng measure theory, an enriched version of [3] that made no use of Borel sets, the Borel sets were somehow "forgotten" in the constructive literature.

In the introduction of [3], Bishop and Cheng explained why they consider their new measure theory "much more natural and powerful theory". They do admit though that some results are harder to prove (see [3], p. v). As it is also noted though, in [20], p. 25, the Bishop-Cheng measure theory is highly impredicative, while, although we cannot explain this here, Bishop's measure theory in [2] is highly predicative. This fact makes the original Bishop-Cheng measure theory hard to implement in some functional-programming language, a serious disadvantage from the computational point of view. This is maybe why, later attempts to develop constructive measure theory were done within an abstract algebraic framework (see [7], [8] and [21].)

Despite the history of measure theory within Bishop-style constructive mathematics, the set of Borel sets is interesting on its own, and, as we try to show here, there are interesting interconnections between the theory of Bishop spaces and the notion of Borel sets. The notion of Bishop space, Bishop used the term function space for it, was also introduced by Bishop in [2], p. 71, as a constructive and function-theoretic alternative to the notion of a topological space. The notion of a least Bishop topology generated by a given set of function from X to $\mathbb{R}$, together with the set of Borel sets generated by a family of complemented subsets of X, are the main two inductively defined concepts found in [2]. The theory of Bishop spaces was not elaborated by Bishop, and it remained in oblivion, until Bridges and Ishihara revived the subject in [5] and [12], respectively. In [14]-[18] we tried to develop their theory.

This paper is the first step towards a systematic study of Borel sets and Baire sets, that we introduce here, in Bishop spaces. A Bishop topology is a set of real-valued functions on X, all elements of which are "a priori" continuous. The study of Borel and Baire sets within Bishop spaces is a constructive counterpart to the study of Borel and Baire algebras within topological spaces.

As it is indicated here, but needs to be elaborated further somewhere else, using complemented subsets to represent pairs of basic open sets and basic closed sets has as a result that some parts of the classical duality between open and closed sets in a topological space are recovered constructively. This reinforces our conviction that the notion of a complemented subset is one of the most important positive notions introduced by Bishop to overcome the difficulties that negatively defined concepts generate in constructive mathematics.

We work within Bishop's informal system of constructive mathematics BISH^*, that is BISH together with inductive definitions with rules of countably many premises. Roughly speaking, [2] is within BISH^*, while [3] and [4] are within BISH. A formal system for BISH^* is Myhill's system CST^* in [13], or CZF with dependence choice and some weak form of Aczel's regular extension axiom (see [1]).

All proofs that are not included here, are omitted as straightforward.

2 *F*-complemented subsets

A Bishop space is a constructive, function-theoretic alternative to the set-theoretic notion of topological space and a Bishop morphism is the corresponding notion of "continuous" function between Bishop spaces. In contrast to topological spaces, continuity of functions is a primitive notion and a concept of open set comes a posteriori. A Bishop topology on a set can be seen as an abstract and constructive approach to the ring of continuous functions $C(X)$ of a topological space X.

Definition 1. *A Bishop space is a couple $\mathcal{F} := (X, F)$, where X is an inhabited set (i.e., a set with a given element in it) and F is a subset of $\mathbb{F}(X)$, the set of all real-valued functions on X, such that the following conditions hold:*

(BS_1) *The set of constant functions* $\mathrm{Const}(X)$ *on X is included in F.*

(BS_2) *If $f, g \in F$, then $f + g \in F$.*

(BS_3) *If $f \in F$ and $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi \circ f \in F$, where* $\mathrm{Bic}(\mathbb{R})$ *is the set of all Bishop-continuous functions from $\mathbb{R}$ to $\mathbb{R}$ i.e., of all functions that are uniformly continuous on every closed interval $[-n, n]$, where $n \geq 1$.*

(BS_4) *If $f \in \mathbb{F}(X)$ and $(g_n)_{n=1}^{\infty}$ such that $U(f, g_n, \frac{1}{n}) :\Leftrightarrow \forall_{x \in X}\big(|f(x) - g_n(x)| \leq \frac{1}{n}\big)$, for every $n \geq 1$, then $f \in F$.*

We call F a Bishop topology on X. If $\mathcal{G} := (Y, G)$ is a Bishop space, a Bishop morphism from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \to Y$ such that $\forall_{g \in G}\big(g \circ h \in F\big)$. We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of Bishop morphisms from $\mathcal{F}$ to $\mathcal{G}$.

It is easy to show (see [2], p. 71) that a Bishop topology F is an algebra and a lattice, where $f \vee g$ and $f \wedge g$ are defined pointwise, and if $a, b \in \mathbb{R}$, then $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Moreover, $\mathrm{Bic}(\mathbb{R})$ is a Bishop topology on $\mathbb{R}$, $\mathrm{Const}(X)$ and $\mathbb{F}(X)$ are Bishop topologies on X. If F is a Bishop topology on X, then $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$, and $F^* := F \cap \mathbb{F}^*(X)$ is a Bishop topology on X, where $\mathbb{F}^*(X)$ denotes the bounded elements of $\mathbb{F}(X)$.

Definition 2. *Turning the definitional clauses $(\mathrm{BS}_1) - (\mathrm{BS}_4)$ into inductive rules one can define the least Bishop topology $\bigvee F_0$ on X that includes a given subset F_0 of $\mathbb{F}(X)$. In this case F_0 is called a subbase of F. A base of F is a subset B of F such that for every $f \in F$ there is a sequence $(g_n)_{n=1}^{\infty} \subseteq B$ such that $\forall_{n \geq 1}\big(U(f, g_n, \frac{1}{n})\big)$.*

From now on, F denotes a Bishop topology on an inhabited set X and G a Bishop topology on an inhabited set Y. For simiplicity, we denote the constant function on X with value $a \in \mathbb{R}$ also by a.

A complemented subset of X is a couple (A^1, A^0) of subsets of X such that every element of A^1 is "apart" from every element of A^0, where the apartness relation $x \neq y$ on a set X is a positive and stronger version of the negatively defined inequality $\neg(x =_X y)$. Here $x \neq y$ is defined through a given set of functions from X to $\mathbb{R}$. The induced apartness between A^1 and A^0 is a positive and stronger version of the negatively defined disjointness $A^1 \cap A^0 = \emptyset$.

Definition 3. *An inequality on X is a relation $x \neq y$ such that the following conditions are satisfied:*

(Ap_1) $\forall_{x,y \in X}\big(x =_X y \ \& \ x \neq y \Rightarrow \bot\big)$.

(Ap_2) $\forall_{x,y \in X}\big(x \neq y \Rightarrow y \neq x\big)$.

(Ap_3) $\forall_{x,y \in X}\big(x \neq y \Rightarrow \forall_{z \in X}(z \neq x \ \vee \ z \neq y)\big)$.

If $a, b \in \mathbb{R}$, we define $a \neq_{\mathbb{R}} b :\Leftrightarrow |a - b| > 0$. Usually, we write $a \neq b$ instead of $a \neq_{\mathbb{R}} b$. The inequality $x \neq_F y$ on X generated by F is defined by

$$x \neq_F y :\Leftrightarrow \exists_{f \in F}\big(f(x) \neq_{\mathbb{R}} f(y)\big).$$

A complemented subset of X with respect to $\neq_F$, or an F-complemented subset of X, is a pair $\boldsymbol{A} := (A^1, A^0)$ such that $\forall_{x \in A^1}\forall_{y \in A^0}\big(x \neq_F y\big)$. In this case we write $A^1][_F A^0$, and we denote their totality by $\mathcal{P}^{][_F}(X)$. The characteristic function of $\boldsymbol{A}$ is the map $\chi_{\boldsymbol{A}} : A^0 \cup A^1 \to 2$ defined by

$$\chi_{\boldsymbol{A}}(x) := \begin{cases} 1 \ , \ x \in A^1 \\ 0 \ , \ x \in A^0. \end{cases}$$

If $\boldsymbol{A}, \boldsymbol{B}$ are in $\mathcal{P}^{][_F}(X)$, then $\boldsymbol{A} = \boldsymbol{B} :\Leftrightarrow A^1 = B^1 \ \& \ A^0 = B^0$, and $\boldsymbol{A} \subseteq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0$.

Clearly, $a \neq b \Leftrightarrow a > b \vee a < b$ and $A^1][_F A^0 \Rightarrow A^1 \cap A^0 = \emptyset$. If $A, B \subseteq \mathbb{R}$, the implication $A \cap B = \emptyset \Rightarrow A \;][_{\mathrm{Bic}(\mathbb{R})} \; B$ implies Markov's principle, hence it cannot be accepted in BISH*. To see this, take $A := \{x \in \mathbb{R} \mid \neg(x = 0)\}$ and $B := \{x \in \mathbb{R} \mid x = 0\}$. If $A \;][_{\mathrm{Bic}(\mathbb{R})} \; B$, then for every $x \in A$, there is some $\phi_x \in \mathrm{Bic}(\mathbb{R})$ such that $\phi_x(x) \neq \phi_x(0)$. Every element of $\mathrm{Bic}(\mathbb{R})$ though, is strongly extensional i.e., $\phi_x(x) \neq \phi_x(0) \Rightarrow x \neq 0$ (see Proposition 5.1.2 in [14], p. 102). Actually, we have that $\forall_{x,y \in \mathbb{R}}\big(x \neq_{\mathrm{Bic}(\mathbb{R})} y \Leftrightarrow x \neq_{\mathbb{R}} y\big)$. In this way we get $\forall_{x \in \mathbb{R}}\big(\neg(x = 0) \Rightarrow x \neq 0\big)$, which is equivalent to Markov's principle (see [6], p. 20).

Corollary 1. *If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}^{][_F}(X)$, then*

$$\boldsymbol{A} \cup \boldsymbol{B} := (A^1 \cup B^1, A^0 \cap B^0) \;\&\; \boldsymbol{A} \cap \boldsymbol{B} := (A^1 \cap B^1, A^0 \cup B^0),$$

$$\boldsymbol{A} - \boldsymbol{B} := (A^1 \cap B^0, A^0 \cup B^1) \;\;\&\;\; -\boldsymbol{A} := (A^0, A^1),$$

are F-complemented subsets of X.

Clearly, $-(-\boldsymbol{A})) = \boldsymbol{A}$, and $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} \cap (-\boldsymbol{B})$. In [3], p. 16, and in [4], p. 73, the "union" and the "intersection" of $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined in a more complex way, so that their corresponding characteristic functions are given through the characteristic functions of $\boldsymbol{A}$ and $\boldsymbol{B}$. Since here we do not use the characteristic functions of the complemented subsets, we keep the above simpler definitions given in [2], p. 66. If $\boldsymbol{A}_{2n} := \boldsymbol{A}$ and $\boldsymbol{A}_{2n+1} := \boldsymbol{B}$, for every $n \geq 1$, the definitions of $\boldsymbol{A} \cup \boldsymbol{B}$ and $\boldsymbol{A} \cap \boldsymbol{B}$ are special cases of the following definitions.

Corollary 2. *If $(\boldsymbol{A}_n)_{n=1}^{\infty} \subseteq \mathcal{P}^{][_F}(X)$, then*

$$\bigcup_{n=1}^{\infty} \boldsymbol{A}_n := \left(\bigcup_{n=1}^{\infty} A_n^1, \bigcap_{n=1}^{\infty} A_n^0\right) \;\;\&\;\; \bigcap_{n=1}^{\infty} \boldsymbol{A}_n := \left(\bigcap_{n=1}^{\infty} A_n^1, \bigcup_{n=1}^{\infty} A_n^0\right),$$

are F-complemented subsets of X. Moreover,

$$-\bigcap_{n=1}^{\infty} \boldsymbol{A}_n = \bigcup_{n=1}^{\infty}(-\boldsymbol{A}_n) \;\;\&\;\; -\bigcup_{n=1}^{\infty} \boldsymbol{A}_n = \bigcap_{n=1}^{\infty}(-\boldsymbol{A}_n).$$

Proposition 1. *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}^{][_G}(Y)$, $(\boldsymbol{A}_n)_{n=1}^{\infty} \subseteq \mathcal{P}^{][_F}(Y)$, let*

$$h^{-1}(\boldsymbol{A}) := \big(h^{-1}(A^1), h^{-1}(A^0)\big).$$

(i) $h^{-1}(\boldsymbol{A}) \in \mathcal{P}^{][_F}(X)$.
(ii) $h^{-1}(\boldsymbol{A} \cup \boldsymbol{B}) = h^{-1}(\boldsymbol{A}) \cup h^{-1}(\boldsymbol{B})$, and $h^{-1}(\boldsymbol{A} \cap \boldsymbol{B}) = h^{-1}(\boldsymbol{A}) \cap h^{-1}(\boldsymbol{B})$.
(iii) $h^{-1}(-\boldsymbol{A}) = -h^{-1}(\boldsymbol{A})$ and $h^{-1}(\boldsymbol{A} - \boldsymbol{B}) = h^{-1}(\boldsymbol{A}) - h^{-1}(\boldsymbol{B})$.
(iv) $h^{-1}\big(\bigcup_{n=1}^{\infty} \boldsymbol{A}_n\big) = \bigcup_{n=1}^{\infty} h^{-1}(\boldsymbol{A}_n)$ and $h^{-1}\big(\bigcap_{n=1}^{\infty} \boldsymbol{A}_n\big) = \bigcap_{n=1}^{\infty} h^{-1}(\boldsymbol{A}_n)$.

Proof. (i) Let $x \in h^{-1}(A^1)$ and $y \in h^{-1}(A^0)$ i.e., $h(x) \in A^1$ and $h(y) \in A^0$. Let $g \in G$ such that $g(h(x)) \neq g(h(y))$. Hence, $g \circ h \in F$ and $(g \circ h)(x) \neq (g \circ h)(y)$. The rest of the proof is omitted as straightforward.

3 Borel sets

The Borel sets in a topological space $(X, \mathcal{T})$ is the least set of subsets of X that includes the open (or, equivalently the closed) sets in X and it is closed under countable unions, countable intersections and relative complements. The Borel sets in a Bishop space (X, F) is the least set of complemented subsets of X that includes the *basic* F-complemented (open-closed) subsets of X that are generated by F, and it is closed under countable unions and countable intersections (the closure under relative complements is redundant in the case of Bishop spaces). The next definition is Bishop's definition, given in [2], p. 68, restricted though, to Bishop topologies.

Definition 4. *An I-family of F-complemented subsets of X is an assignment routine $\boldsymbol{\lambda}$ that assigns to every $i \in I$ an F-complemented subset $\boldsymbol{\lambda}(i)$ of X such that $\forall_{i,j \in I}\big(i =_I j \Rightarrow \boldsymbol{\lambda}(i) =_{\mathcal{P}^{][}_{F}(X)} \boldsymbol{\lambda}(j)\big)$. An I-family of F-complemented subsets of X is called an I-set of complemented subsets of X, if $\forall_{i,j \in I}\big(\boldsymbol{\lambda}(i) =_{\mathcal{P}^{][}_{F}(X)} \boldsymbol{\lambda}(j) \Rightarrow i =_I j\big)$. The set $\mathtt{Borel}(\boldsymbol{\lambda})$ of Borel sets generated by $\boldsymbol{\lambda}$ is defined inductively by the following rules:*

$$(\mathtt{Borel}_1) \qquad \frac{i \in I}{\boldsymbol{\lambda}(i) \in \mathtt{Borel}(\boldsymbol{\lambda})}$$

$$(\mathtt{Borel}_2) \qquad \frac{\boldsymbol{B}(1) \in \mathtt{Borel}(\boldsymbol{\lambda}), \boldsymbol{B}(2) \in \mathtt{Borel}(\boldsymbol{\lambda}), \ldots}{\bigcup_{n=1}^{\infty} \boldsymbol{B}(n) \in \mathtt{Borel}(\boldsymbol{\lambda}) \ \ \& \ \ \bigcap_{n=1}^{\infty} \boldsymbol{B}(n) \in \mathtt{Borel}(\boldsymbol{\lambda})}.$$

In the induction principle $\mathrm{Ind}_{\mathtt{Borel}(\boldsymbol{\lambda})}$ *associated to the definition of* $\mathtt{Borel}(\boldsymbol{\lambda})$ *we take P to be any formula in which the set* $\mathtt{Borel}(F)$ *does not occur.*

$$\forall_{i \in I}\big(P(\boldsymbol{\lambda}(i))\big) \ \& \ \forall_{\boldsymbol{\alpha}: \mathbb{N} \to \mathcal{P}^{][}_{F}(X)}\Bigg[\forall_{n \geq 1}\big(\boldsymbol{\alpha}(n) \in \mathtt{Borel}(\boldsymbol{\lambda}) \ \& \ P(\boldsymbol{\alpha}(n))\big) \Rightarrow$$

$$P\bigg(\bigcup_{n=1}^{\infty} \boldsymbol{\alpha}(n)\bigg) \ \& \ P\bigg(\bigcap_{n=1}^{\infty} \boldsymbol{\alpha}(n)\bigg)\Bigg] \Rightarrow \forall_{\boldsymbol{B} \in \mathtt{Borel}(\boldsymbol{\lambda})}\big(P(\boldsymbol{B})\big).$$

Let $\boldsymbol{o}_F$, or simply $\boldsymbol{o}$, be the F-family of the basic F-complemented subsets of X:

$$\boldsymbol{o}_F(f) := \big([f > 0], [f \leq 0]\big),$$

$$[f > 0] := \{x \in X \mid f(x) > 0\}, \quad [f \leq 0] := \{x \in X \mid f(x) \leq 0\}.$$

We write $\mathtt{Borel}(F) := \mathtt{Borel}(\boldsymbol{o}_F)$ *and we call its elements the Borel sets in F.*

Proposition 2. *(i) If we keep the pointwise equality of functions as the equality of F, then the F-family $\boldsymbol{o}$ is not a set of F-complemented subsets of X.*

(ii) $\boldsymbol{o}(1) = (X, \emptyset)$ and $\boldsymbol{o}(-1) = (\emptyset, X)$.

(iii) If $f, g \in F$, then $\boldsymbol{o}(f) \cup \boldsymbol{o}(g) = \boldsymbol{o}(f \vee g)$.

(iv) If $\boldsymbol{B} \in \mathtt{Borel}(F)$, then $-\boldsymbol{B} \in \mathtt{Borel}(F)$.

(v) There is a Bishop space (X, F) and some $f \in F$ such that $-\boldsymbol{o}(f)$ is not equal to $\boldsymbol{o}(g)$ for some $g \in F$.

(vi) $\boldsymbol{o}(f) = \boldsymbol{o}([f \vee 0] \wedge 1)$.

Proof. (i) and (ii) If $f \in F$, then $\boldsymbol{o}(f) = \boldsymbol{o}(2f)$, but $\neg(f = 2f)$. (ii) is trivial.
(iii) This equality is implied from the following property for reals

$$a \vee b > 0 \Leftrightarrow a > 0 \vee b > 0 \ \ \& \ \ a \vee b \leq 0 \Leftrightarrow a \leq 0 \wedge b \leq 0.$$

(iv) If $a \in \mathbb{R}$, then $a \leq 0 \Leftrightarrow \forall_{n \geq 1}\big(a < \frac{1}{n}\big)$ and $a > 0 \Leftrightarrow \exists_{n \geq 1}\big(a \geq \frac{1}{n}\big)$, hence

$$\begin{aligned} -\boldsymbol{o}(f) &:= \big([f \leq 0], [f > 0]\big) \\ &= \bigg(\bigcap_{n=1}^{\infty}\big[\big(\frac{1}{n} - f\big) > 0\big], \bigcup_{n=1}^{\infty}\big[\big(\frac{1}{n} - f\big) \leq 0\big]\bigg) \\ &:= \bigcap_{n=1}^{\infty}\boldsymbol{o}\big(\frac{1}{n} - f\big) \in \texttt{Borel}(F). \end{aligned}$$

If $P(\boldsymbol{B}) := -\boldsymbol{B} \in \texttt{Borel}(F)$, the above equality proves the first step of the corresponding induction on $\texttt{Borel}(F)$. The rest of the inductive proof is easy.
(v) Let the Bishop space $(\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$. If we take $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) := \big([x > 0], [x \leq 0]\big)$, and if we suppose that $-\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) := \big([x \leq 0], [x > 0]\big) = \big([\phi > 0], [\phi \leq 0]\big) =: \boldsymbol{o}(\phi)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi(0) > 0$ and ϕ is not continuous at 0, which contradicts the fact that ϕ is uniformly continuous on every bounded subset of $\mathbb{R}$.
(vi) The proof is based on basic properties of $\mathbb{R}$, like $a \wedge 1 = 0 \Rightarrow a = 0$.

Since $\texttt{Borel}(F)$ is closed under intersections and complements, if $\boldsymbol{A}, \boldsymbol{B} \in \texttt{Borel}(F)$, then $\boldsymbol{A} - \boldsymbol{B} \in \texttt{Borel}(F)$. As Bishop remarks in [2], p. 69, the proof of Proposition 2(iv) rests on the property of F that $\big(\frac{1}{n} - f\big) \in F$, for every $f \in F$ and $n \geq 1$. If we define similarly the Borel sets generated by any family of real-valued functions Θ on X, then we can find Θ such that $\texttt{Borel}(\Theta)$ is closed under complements without satisfying the condition $f \in \Theta \Rightarrow \big(\frac{1}{n} - f\big) \in \Theta$. Such a family is the set $\mathbb{F}(X, 2)$ of all functions from X to $2 := \{0, 1\}$. In this case we have that

$$\boldsymbol{o}_{\mathbb{F}(X,2)}(f) := \big([f = 1], [f = 0]\big) \ \ \& \ \ -\boldsymbol{o}_{\mathbb{F}(X,2)}(f) = \boldsymbol{o}_{\mathbb{F}(X,2)}(1 - f).$$

Hence, the property mentioned by Bishop is sufficient, but not necessary. Constructively, we cannot show, in general, that $\boldsymbol{o}(f) \cap \boldsymbol{o}(g) = \boldsymbol{o}(f \wedge g)$. If $f := \mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$ and $g := -\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, then $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}}) \cap \boldsymbol{o}(-\mathrm{id}_{\mathbb{R}}) = \big([x > 0] \cap [x < 0], [x \leq 0] \cup [-x \leq 0]\big) = \big(\emptyset, [x \leq 0] \cup [x \geq 0]\big)$ Since $x \wedge (-x) = -|x|$, we get $\boldsymbol{o}(\mathrm{id}_{\mathbb{R}} \wedge (-\mathrm{id}_{\mathbb{R}})) = \boldsymbol{o}(-|x|) = \big(\emptyset, [|x| \geq 0]\big)$. The supposed equality implies that $|x| \geq 0 \Leftrightarrow x \leq 0 \vee x \geq 0$. Since $|x| \geq 0$ is always the case, we get $\forall_{x \in \mathbb{R}}\big(x \leq 0 \vee x \geq 0\big)$, which implies LLPO (see [6], p. 20). If one add the condition $|f| + |g| > 0$, then $\boldsymbol{o}(f) \cap \boldsymbol{o}(g) = \boldsymbol{o}(f \wedge g)$ follows constructively. The condition BS_4 in the definition of a Bishop space is crucial to the next proof.

Proposition 3. *If* $(f_n)_{n=1}^{\infty} \subseteq F$, *then* $f := \sum_{n=1}^{\infty}(f_n \vee 0) \wedge 2^{-n} \in F$ *and*

$$\boldsymbol{o}(f) = \bigcup_{n=1}^{\infty}\boldsymbol{o}(f_n) = \bigg(\bigcup_{n=1}^{\infty}[f_n > 0], \bigcap_{n=1}^{\infty}[f_n \leq 0]\bigg).$$

Proof. The function f is well-defined by the comparison test (see [4], p. 32). If $g_n := (f_n \vee 0) \wedge 2^{-n}$, for every $n \geq 1$, then

$$\left|\sum_{n=1}^{\infty} g_n - \sum_{n=1}^{N} g_n\right| = \left|\sum_{n=N+1}^{\infty} g_n\right| \leq \sum_{n=N+1}^{\infty} |g_n| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \xrightarrow{N} 0,$$

the sequence of the partial sums $\sum_{n=1}^{N} g_n \in F$ converges uniformly to f, hence by BS_4 we get $f \in F$. Next we show that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[f_n > 0]$. If $x \in X$ such that $f(x) > 0$, there is $N \geq 1$ such that $\sum_{n=1}^{N} g_n(x) > 0$. By Proposition (2.16) in [4], p. 26, there is $n \geq 1$ and $n \leq N$ with $g_n(x) > 0$, hence $(f_n(x) \vee 0) \geq g_n(x) > 0$, which implies $f_n(x) > 0$. For the converse inclusion, if $f_n(x) > 0$, for some $n \geq 1$, then $g_n(x) > 0$, hence $f(x) > 0$. To show $[f \leq 0] \subseteq \bigcup_{n=1}^{\infty}[f_n \leq 0]$, let $x \in X$ such that $f(x) \leq 0$, and suppose that $f_n(x) > 0$, for some $n \geq 1$. By the previous argument we get $f(x) > 0$, which contradicts our hypothesis $f(x) \leq 0$. For the converse inclusion, let $f_n(x) \leq 0$, for every $n \geq 1$, hence $f_n(x) \vee 0 = 0$ and $g_n(x) = 0$, for every $n \geq 1$. Consequently, $f(x) = 0$.

Proposition 4. *If* $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ *and* $\boldsymbol{B} \in \mathtt{Borel}(G)$*, then* $h^{-1}(\boldsymbol{B}) \in \mathtt{Borel}(F)$*.*

Proof. By the definition of $h^{-1}(\boldsymbol{B})$ in Proposition 1, if $g \in G$, then

$$\begin{aligned} h^{-1}(\boldsymbol{o}_G(g)) &:= h^{-1}\big([g > 0], [g \leq 0]\big) \\ &:= \big(h^{-1}[g > 0], h^{-1}[g \leq 0]\big) \\ &= \big([(g \circ h) > 0], [(g \circ h) \leq 0]\big) \\ &:= \boldsymbol{o}_F(g \circ h) \in \mathtt{Borel}(F). \end{aligned}$$

If $P(\boldsymbol{B}) := h^{-1}(\boldsymbol{B}) \in \mathtt{Borel}(F)$, the above equality is the first step of the corresponding inductive proof on $\mathtt{Borel}(G)$. The rest of the inductive proof follows immediately from Proposition 1(iv).

Definition 5. *If* $B \subseteq F$*, let* $\boldsymbol{o}_B$ *be the* B*-family of* F*-complemented subsets of* X *defined by* $\boldsymbol{o}_B(f) := \boldsymbol{o}_F(f)$*, for every* $f \in B$*. We denote by* $\mathtt{Borel}(B)$ *the set of Borel sets generated by* $\boldsymbol{o}_B$*.*

If F_0 is a subbase of F, then, $\mathtt{Borel}(F_0) \subseteq \mathtt{Borel}(F)$. More can be said on the relation between $\mathtt{Borel}(B)$ and $\mathtt{Borel}(F)$, when B is a base of F.

Proposition 5. *Let* B *be a base of* F*.*

(i) If for every $f \in F$*,* $\boldsymbol{o}_F(f) \in \mathtt{Borel}(B)$*, then* $\mathtt{Borel}(F) = \mathtt{Borel}(B)$*.*

(ii) If for every $g \in B$ *and* $f \in F$*,* $f \wedge g \in B$*, then* $\mathtt{Borel}(F) = \mathtt{Borel}(B)$*.*

(iii) If for every $g \in B$ *and every* $n \geq 1$*,* $g - \frac{1}{n} \in B$*, then* $\mathtt{Borel}(F) = \mathtt{Borel}(B)$*.*

Proof. (i) It follows by a straightforward induction on $\mathtt{Borel}(F)$.
(ii) and (iii) Let $(g_n)_{n=1}^{\infty} \subseteq B$ such that $\forall_{n \geq 1}\big(U(f, g_n, \frac{1}{n})\big)$. We have that

$$\boldsymbol{o}_F(f) \subseteq \bigcup_{n=1}^{\infty} \boldsymbol{o}_B(g_n) := \left(\bigcup_{n=1}^{\infty}[g_n > 0], \bigcap_{n=1}^{\infty}[g_n \leq 0]\right)$$

i.e., by Definition 3, $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[g_n > 0]$ and $\bigcap_{n=1}^{\infty}[g_n \leq 0] \subseteq [f \leq 0]$; if $x \in X$ with $f(x) > 0$ there is $n \geq 1$ with $g_n(x) > 0$, and if $\forall_{n\geq 1}\big(g_n(x) \leq 0\big)$, then for the same reason $f(x)$ cannot be > 0, hence $f(x) \leq 0$.
Because of (i), for (ii) it suffices to show that $\boldsymbol{o}_F(f) \in \mathtt{Borel}(B)$. We show that

$$\boldsymbol{o}_F(f) = \bigcup_{n=1}^{\infty} \boldsymbol{o}_B(f \wedge g_n) := \bigg(\bigcup_{n=1}^{\infty}[(f \wedge g_n) > 0], \bigcap_{n=1}^{\infty}[(f \wedge g_n) \leq 0]\bigg) \in \mathtt{Borel}(B).$$

If $f(x) > 0$, then we can find $n \geq 1$ such that $g_n(x) > 0$, hence $f(x) \wedge g_n(x) > 0$. Hence we showed that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}[(f \wedge g_n) > 0]$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $(f \wedge g_n)(x) > 0$. Then $f(x) > 0$ and $x \in [f > 0]$. If $f(x) \leq 0$, then $\forall_{n\geq 1}\big(f(x) \wedge g_n(x) \leq 0\big)$. Suppose next that $\forall_{n\geq 1}\big(f(x) \wedge g_n(x) \leq 0\big)$. If $f(x) > 0$, there is $n \geq 1$ with $g_n(x) > 0$, hence $f(x) \wedge g_n(x) > 0$, which contradict the hypothesis $f(x) \wedge g_n(x) \leq 0$. Hence $f(x) \leq 0$.
Because of (i), for (iii) it suffices to show that $\boldsymbol{o}_F(f) \in \mathtt{Borel}(B)$. We show that

$$\boldsymbol{o}_F(f) = \bigcup_{n=1}^{\infty} \boldsymbol{o}_B\big(g_n - \frac{1}{n}\big) := \bigg(\bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big], \bigcap_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) \leq 0\big]\bigg) \in \mathtt{Borel}(B).$$

First we show that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big]$. If $f(x) > 0$, there is $n \geq 1$ with $f(x) > \frac{1}{n}$, hence, since $-\frac{1}{2n} \leq g_{2n}(x) - f(x) \leq \frac{1}{2n}$, we get

$$g_{2n}(x) - \frac{1}{2n} \geq \big(f(x) - \frac{1}{2n}\big) - \frac{1}{2n} = f(x) - \frac{1}{n} > 0$$

i.e., $x \in \big[\big(g_{2n} - \frac{1}{2n}\big) > 0\big]$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $g_n(x) - \frac{1}{n} > 0$. Since $0 < g_n(x) - \frac{1}{n} \leq f(x)$, we get $x \in [f > 0]$. Next we show that $[f \leq 0] \subseteq \bigcap_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) \leq 0\big]$. Let $x \in X$ with $f(x) \leq 0$, and suppose that $n \geq 1$ with $g_n(x) - \frac{1}{n} > 0$. Then $0 \geq f(x) > 0$. By this contradiction we get $g_n(x) - \frac{1}{n} \leq 0$. For the converse inclusion let $x \in X$ such that $g_n(x) - \frac{1}{n} \leq 0$, for every $n \geq 1$, and suppose that $f(x) > 0$. Since we have already shown that $[f > 0] \subseteq \bigcup_{n=1}^{\infty}\big[\big(g_n - \frac{1}{n}\big) > 0\big]$, there is some $n \geq 1$ with $g_n(x) - \frac{1}{n} > 0$, which contradicts our hypothesis, hence $f(x) \leq 0$.

4 Baire sets

One of the definitions[1] of the set of Baire sets in a topological space $(X, \mathcal{T})$, which was given by Hewitt in [11], is that it is the least σ-algebra of subsets of X that includes the zero sets of X i.e., the sets of the form $f^{-1}(\{0\})$, where $f \in C(X)$. Clearly, a Baire set in $(X, \mathcal{T})$ is a Borel set in $(X, \mathcal{T})$, and for many topological spaces, like the metrizable ones, the two classes coincide. In this section we adopt Hewitt's notion in Bishop spaces and the framework of F-complemented subsets.

[1] A different definition is given in [10]. See [19] for the relations between these two definitions.

Definition 6. *Let $\boldsymbol{\zeta}_F$, or simply $\boldsymbol{\zeta}$, be the F-family of the F-zero complemented subsets of X:*

$$\boldsymbol{\zeta}_F(f) := \big([f = 0], [f \neq 0]\big),$$

$$[f = 0] := \{x \in X \mid f(x) = 0\}, \quad [f \neq 0] := \{x \in X \mid f(x) \neq 0\}.$$

We write $\texttt{Baire}(F) := \texttt{Borel}(\boldsymbol{\zeta}_F)$ *and we call its elements the Baire sets in F.*

Since $a \neq 0 :\Leftrightarrow |a| > 0 \Leftrightarrow a < 0 \vee a > 0$, for every $a \in \mathbb{R}$, we get

$$\boldsymbol{\zeta}_F(f) = \big([f = 0], [|f| > 0]\big) = \big([f = 0], [f > 0] \cup [f < 0]\big).$$

Proposition 6. (i) *If we keep the pointwise equality of functions as the equality of F, then the F-family $\boldsymbol{\zeta}$ is not a set of F-complemented subsets of X.*
(ii) $\boldsymbol{\zeta}(0) = (X, \emptyset)$ *and* $\boldsymbol{\zeta}(1) = (\emptyset, X)$.
(iii) *If $f, g \in F$, then* $\boldsymbol{\zeta}(f) \cap \boldsymbol{\zeta}(g) = \boldsymbol{\zeta}(|f| \vee |g|)$.
(iv) *If* $\boldsymbol{B} \in \texttt{Baire}(F)$, *then* $-\boldsymbol{B} \in \texttt{Baire}(F)$.
(v) *There is a Bishop space (X, F) and some $f \in F$ such that $-\boldsymbol{\zeta}(f)$ is not equal to $\boldsymbol{\zeta}(g)$ for some $g \in F$.*
(vi) $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(|f| \wedge 1)$.

Proof. (i) and (ii) If $f \in F$, then $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(2f)$, but $\neg(f = 2f)$. (ii) is trivial.
(iii) This equality is implied from the following property for reals

$$|a| \vee |b| = 0 \Leftrightarrow |a| = 0 \wedge |b| = 0 \;\;\&\;\; |a| \vee |b| \neq 0 \Leftrightarrow |a| > 0 \vee |b| > 0.$$

(iv) If $f \in F$, then $-\boldsymbol{\zeta}(f) := \big([f \neq 0], [f = 0]\big)$. For every $n \geq 1$, let

$$g_n := \big(|f| \wedge \frac{1}{n}\big) - \frac{1}{n} \in F.$$

We show that

$$\bigcup_{n=1}^{\infty} \boldsymbol{\zeta}(g_n) := \bigg(\bigcup_{n=1}^{\infty} [g_n = 0], \bigcap_{n=1}^{\infty} [g_n \neq 0]\bigg) = -\boldsymbol{\zeta}(f) \in \texttt{Baire}(F).$$

First we show that $[f \neq 0] = \bigcup_{n=1}^{\infty} [g_n = 0]$. If $|f(x)| > 0$, there is $n \geq 1$ such that $|f(x)| > \frac{1}{n}$, hence $|f(x)| \wedge \frac{1}{n} = \frac{1}{n}$, and $g_n(x) = 0$. For the converse inclusion, let $x \in X$ and $n \geq 1$ such that $g_n(x) = 0 \Leftrightarrow |f(x)| \wedge \frac{1}{n} = \frac{1}{n}$, hence $|f(x)| \geq \frac{1}{n} > 0$. Next we show that $[f = 0] = \bigcap_{n=1}^{\infty} [g_n \neq 0]$. If $x \in X$ such that $f(x) = 0$, and $n \geq 1$, then $g_n(x) = -\frac{1}{n} < 0$. For the converse inclusion, let $x \in X$ such that for all $n \geq 1$ we have that $g_n(x) \neq 0$. If $|f(x)| > 0$, there is $n \geq 1$ such that $|f(x)| > \frac{1}{n}$, hence $g_n(x) = 0$, which contradicts our hypothesis. Hence, $|f(x)| \leq 0$, which implies that $|f(x)| = 0 \Leftrightarrow f(x) = 0$. If $P(\boldsymbol{B}) := -\boldsymbol{B} \in \texttt{Baire}(F)$, the above equality proves the first step of the corresponding induction on $\texttt{Baire}(F)$. The rest of the inductive proof is immediate[2].

[2] Hence, if we define the set of Baire sets over an arbitrary family Θ of functions from X to $\mathbb{R}$, a sufficient condition so that $\texttt{Baire}(\Theta)$ is closed under complements is that Θ is closed under $|.|$, under wedge with $\frac{1}{n}$ and under subtraction with $\frac{1}{n}$, for every $n \geq 1$. If $\Theta := \mathbb{F}(X, 2)$, then $-\boldsymbol{o}_{\mathbb{F}(X,2)}(f) = \boldsymbol{o}_{\mathbb{F}(X,2)}(1 - f) = \boldsymbol{\zeta}_{\mathbb{F}(X,2)}(f)$, hence by Proposition 4(ii) we get $\texttt{Borel}(\mathbb{F}(X, 2)) = \texttt{Baire}(\mathbb{F}(X, 2))$.

(v) Let the Bishop space $(\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$. If we take $\boldsymbol{\zeta}(\mathrm{id}_{\mathbb{R}}) := \big([x = 0], [x \neq 0]\big)$, and if we suppose that $-\boldsymbol{\zeta}(\mathrm{id}_{\mathbb{R}}) := \big([x \neq 0], [x = 0]\big) = \big([\phi = 0], [\phi \neq 0]\big) =: \boldsymbol{\zeta}(\phi)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi(0) > 0 \vee \phi(0) < 0$ and $\phi(x) = 0$, if $x < 0$ or $x > 0$. Hence ϕ is not continuous at 0, which contradicts the fact that ϕ is uniformly continuous on every bounded subset of $\mathbb{R}$.
(vi) This proof is straightforward.

As in the case of basic Borel sets in F, we cannot show constructively that $\boldsymbol{\zeta}(f) \cup \boldsymbol{\zeta}(g) = \boldsymbol{\zeta}(|f| \wedge |g|)$. If we add the condition $|f| + |g| > 0$ though, this equality is constructively provable.

Proposition 7. *If $(f_n)_{n=1}^{\infty} \subseteq F$, then $f := \sum_{n=1}^{\infty} |f_n| \wedge 2^{-n} \in F$ and*

$$\boldsymbol{\zeta}(f) = \bigcap_{n=1}^{\infty} \boldsymbol{\zeta}(f_n) = \bigg(\bigcap_{n=1}^{\infty} [f_n = 0], \bigcup_{n=1}^{\infty} [f_n \neq 0]\bigg).$$

Proof. Working as in the proof of Proposition 3, f is well-defined and if $g_n := |f_n| \wedge 2^{-n}$, for every $n \geq 1$, then the sequence of the partial sums $\sum_{n=1}^{N} g_n \in F$ converges uniformly to f, and by BS_4 we get $f \in F$. Since $f(x) = 0 \Leftrightarrow \forall_{\geq 1}(g_n(x) = 0) \Leftrightarrow \forall_{\geq 1}(f_n(x) = 0)$, we get $[f = 0] = \bigcap_{n=1}^{\infty}[f_n = 0]$. Next we show that $[f \neq 0] \subseteq \bigcup_{n=1}^{\infty}[f_n \neq 0]$. If $|f(x)| > 0$, then there is $N \geq 1$ such that $\sum_{n=1}^{N} g_n(x) > 0$. By Proposition (2.16) in [4], p. 26, there is some $n \geq 1$ and $n \geq N$ such that $g_n(x) > 0$, hence $|f_n(x)| \geq g_n(x) > 0$. The converse inclusion follows trivially.

Theorem 1. *(i) If $\boldsymbol{B} \in \mathtt{Baire}(F)$, then $\boldsymbol{B} \in \mathtt{Borel}(F)$.*
(ii) If $\boldsymbol{o}(f) \in \mathtt{Baire}(F)$, for every $f \in F$, then $\mathtt{Baire}(F) = \mathtt{Borel}(F)$.
(iii) If $f \in F$, then $\boldsymbol{o}(f) = -\boldsymbol{\zeta}\big((-f) \wedge 0\big)$.
(iv) $\mathtt{Baire}(F^*) = \mathtt{Baire}(F) = \mathtt{Borel}(F) = \mathtt{Borel}(F^*)$.

Proof. (i) By Proposition 2(iv) $-\boldsymbol{o}(f) = \big([f \leq 0], [f > 0]\big) \in \mathtt{Borel}(F)$, for every $f \in F$, hence $-\boldsymbol{o}(-f) = \big([f \geq 0], [f < 0] \in \mathtt{Borel}(F)$ too. Consequently

$$-\boldsymbol{o}(f) \cap -\boldsymbol{o}(-f) = \big([f \leq 0] \cap [f \geq 0], [f > 0] \cup [f < 0]\big) = \boldsymbol{\zeta}(f) \in \mathtt{Borel}(F).$$

If $P(\boldsymbol{B}) := \boldsymbol{B} \in \mathtt{Borel}(F)$, the above equality is the first step of the corresponding inductive proof on $\mathtt{Baire}(F)$. The rest of the inductive proof is trivial.
(ii) The hypothesis is the first step of the obvious inductive proof on $\mathtt{Borel}(F)$, which shows that $\mathtt{Borel}(F) \subseteq \mathtt{Baire}(F)$. By (i) we get $\mathtt{Baire}(F) \subseteq \mathtt{Borel}(F)$.
(iii) We show that

$$\big([f > 0], [f \leq 0]\big) = \big([(-f) \wedge 0 \neq 0], [(-f) \wedge 0 = 0]\big).$$

First we show that $[f > 0] \subseteq [(-f) \wedge 0 \neq 0]$; if $f(x) > 0$, then $-f(x) \wedge 0 = -f(x) < 0$. For the converse inclusion, let $-f(x) \wedge 0 \neq 0 \Leftrightarrow -f(x) \wedge 0 > 0$ or $-f(x) \wedge 0 < 0$. Since $0 \geq -f(x) \wedge 0$, the first option is impossible. If $-f(x) \wedge 0 < 0$, then $-f(x) < 0$ or $0 < 0$, hence $f(x) > 0$. Next we show that

$[f \leq 0] = [(-f) \wedge 0 = 0]$; since $f(x) \leq 0 \Leftrightarrow -f(x) \geq 0 \Leftrightarrow -f(x) \wedge 0 = 0$ (see [6], p. 52), the equality follows.
(iv) Clearly, $\texttt{Baire}(F^*) \subseteq \texttt{Baire}(F)$. By Proposition 6(vi) $\boldsymbol{\zeta}(f) = \boldsymbol{\zeta}(|f| \wedge 1)$, where $|f| \wedge 1 \in F^*$. Continuing with the obvious induction we get $\texttt{Baire}(F) \subseteq \texttt{Baire}(F^*)$. By case (iii) and Proposition 6(iv) we get $\boldsymbol{o}(f) \in \texttt{Baire}(F)$, hence by case (ii) we conclude that $\texttt{Baire}(F) = \texttt{Borel}(F)$. Clearly, $\texttt{Borel}(F^*) \subseteq \texttt{Borel}(F)$. By Proposition 2(vi) $\boldsymbol{o}(f) = \boldsymbol{o}((f \vee 0) \wedge 1)$, where $(f \vee 0) \wedge 1 \in F^*$. Continuing with the obvious induction we get $\texttt{Borel}(F) \subseteq \texttt{Borel}(F^*)$.

Either by definition, as in the proof of Proposition 4, or by Theorem 1(iii) and Proposition 4, if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $\boldsymbol{B} \in \texttt{Baire}(G)$, then $h^{-1}(\boldsymbol{B}) \in \texttt{Baire}(F)$.

5 Uniformly $\boldsymbol{F}$-complemented subsets

Next follows the uniform version of the notion of an F-complemented subset.

Definition 7. *If $\boldsymbol{A} := (A^1, A^0) \in \mathcal{P}^{][_F}(X)$, we say that $\boldsymbol{A}$ is uniformly F-complemented, and we write $A^1 \neq_F A^0$, if*

$$\exists_{f \in F} \forall_{x \in A^1} \forall_{y \in A^0} \big(f(x) = 1 \ \& \ f(y) = 0\big).$$

Taking $(f \vee 0) \wedge 1$ we get $A^1 \neq_F A^0 \Leftrightarrow \exists_{f \in F} \big[0 \leq f \leq 1 \ \& \ \forall_{x \in A^1} \forall_{y \in A^0} \big(f(x) = 1 \ \& \ f(y) = 0\big)\big]$. In [3], p. 55, the following relation is defined:

$$\boldsymbol{A} \leq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \ \& \ \ A^0 \subseteq B^0.$$

If $A^1 \neq_F A^0$, then $\boldsymbol{A} \leq \boldsymbol{o}(f)$. According to the classical Urysohn lemma for $C(X)$-zero sets, the disjoint zero sets of a topological space X are separated by some $f \in C(X)$ (see [9], p. 17). We show a constructive version of this, where disjointness is replaced by a stronger, but positively defined form of it.

Theorem 2 (Urysohn lemma). *If $\boldsymbol{A} := (A^1, A^0) \in \mathcal{P}^{][_F}(X)$, then*

$$A^1 \neq_F A^0 \Leftrightarrow \exists_{f,g \in F} \exists_{c > 0} \big(\boldsymbol{A} \leq \boldsymbol{\zeta}(f) \ \& \ -\boldsymbol{A} \leq \boldsymbol{\zeta}(g) \ \& \ |f| + |g| \geq c\big).$$

Proof. ($\Rightarrow$) Let $h \in F$ such that $0 \leq h \leq 1$, $A^1 \subseteq [h = 1]$ and $A^0 \subseteq [h = 0]$. We take $f := 1 - h \in F, g := h$ and $c := 1$. First we show that $\boldsymbol{A} \leq \boldsymbol{\zeta}(f)$. If $x \in A^1$, then $h(x) = 1$, and $f(x) = 0$. If $y \in A^0$, then $h(y) = 0$, hence $f(y) = 1$ and $y \in [f \neq 0]$. Next we show that $-\boldsymbol{A} \leq \boldsymbol{\zeta}(g)$. If $y \in A^0$, then $h(y) = 0 = g(y)$. If $x \in A^1$, then $h(x) = 1 = g(y)$ i.e., $x \in [g \neq 0]$. If $x \in X$, then $1 = |1 - h(x) + h(x)| \leq |1 - h(x)| + |h(x)|$.
($\Leftarrow$) Let $h := 1 - \big(\frac{1}{c}|f| \wedge 1\big) \in F$. If $x \in A^1$, then $f(x) = 0$, and hence $h(x) = 1$. If $y \in A^0$, then $g(y) = 0$, hence $|f(y)| \geq c$, and consequently $h(y) = 0$.

The condition BS_3 is crucial to the next proof.

Corollary 3. *If $\boldsymbol{A} := (A^1, A^0) \in \mathcal{P}^{][_F}(X)$ and $f \in F$, then*

$$f(A^1) \neq_{\mathrm{Bic}(\mathbb{R})} f(A^0) \Rightarrow A^1 \neq_F A^0.$$

Proof. If $f(\boldsymbol{A}) := \big(f(A^1), f(A^0)\big)$ is uniformly $\mathrm{Bic}(\mathbb{R})$-complemented, then by Urysohn lemma there are $\phi, \theta \in \mathrm{Bic}(\mathbb{R})$ and $c > 0$ with $f(\boldsymbol{A}) \leq \boldsymbol{\zeta}(\phi), -f(\boldsymbol{A}) \leq \boldsymbol{\zeta}(\theta)$ and $|\phi| + |\theta| \geq c$. Consequently, $\boldsymbol{A} \leq \boldsymbol{\zeta}(\phi \circ f), -\boldsymbol{A} \leq \boldsymbol{\zeta}(\theta \circ f)$ and $|\phi \circ f| + |\theta \circ f| \geq c$. Since by BS_3 we have that $\phi \circ f$ and $\theta \circ f \in F$, by the other implication of the Urysohn lemma we get $A^1 \neq_F A^0$.

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