Functional Analysis – Problems in the class, sheet 4

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 13. Let $(X, \| \ \|)$ be a normed space and let $Y \subset X$ be a closed subspace.

(i) Define the relation $\sim$ on X by $x \sim x'$ iff $x - x' \in Y$. Show that $\sim$ is an equivalence relation.

(ii) Denote by X/Y, or equivalently $X/\!\sim$, the set of equivalence classes w.r.t. $\sim$, i.e., elements $[x] = \{x + y \mid y \in Y\}$. Show that $[x] + [x'] := [x + x']$ and $[\lambda x] := \lambda[x]$ $(x, x' \in X,\ \lambda \in \mathbb{C})$ give a well-defined(*) linear structure in X/Y.

(iii) Define $\|[x]\|_\sim := \inf_{y \in Y} \|x - y\|$. Show that this definition is well-posed(*) and that $\| \ \|_\sim$ is a norm on X/Y.

(iv) Define the projection of X onto the quotient X/Y as the map $\pi : X \to X/Y$ such that $x \mapsto [x]$. Show that π is bounded with $\|\pi\| \leqslant 1$.

(v) Show that $(X/Y, \| \ \|_\sim)$ is complete under the additional assumption that $(X, \| \ \|)$ is a Banach space.

(*) i.e., it does not depend on the choice of the representative x for the equivalence class $[x]$

Problem 14. Define the polynomials $p_{n,k} : [0, 1] \to \mathbb{R}$ by

$$p_{n,k}(x) := \binom{n}{k} x^k (1 - x)^{n-k}\,, \qquad \begin{array}{c} x \in [0,1]\,, \\ n,\ k \text{ integers} \\ \text{with } 0 \leqslant k \leqslant n. \end{array}$$

Prove the following identities:

(i) $$\sum_{k=0}^{n} p_{n,k}(x) \ = \ 1$$

(ii) $$\sum_{k=0}^{n} k\, p_{n,k}(x) \ = \ nx$$

(iii) $$\sum_{k=0}^{n} k(k-1)\, p_{n,k}(x) \ = \ n(n-1)x^2\,.$$

Problem 15. Let d be a positive integer. Define the "*discrete Laplacian*" on $\mathbb{Z}^d$ as the operator $\Delta : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ acting on any element $\phi \in \ell^2(\mathbb{Z}^d)$ as

$$(\Delta\phi)(x) \;:=\; \sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} \big(\,\phi(x) - \phi(y)\,\big)\,, \qquad x \in \mathbb{Z}^d. \tag{$*$}$$

In the notation above $|x - y|$ is the *Euclidean* distance between x and y as points of $\mathbb{R}^d$ with integer coordinates. Prove that Δ is bounded with norm $\|\Delta\| = 4d$.

Problem 16. Let $a = \{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+$. Let

$$S^{(a)} \;:=\; \Big\{ x = (x_1, x_2, x_3, \dots) \text{ such that } |x_n| \leqslant a_n \; \forall n \Big\}\,.$$

Prove that $S^{(a)}$ is compact in ℓ^2 if and only if $a \in \ell^2$.