Functional Analysis – Problems in the class, sheet 3

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 9. Decide for which $p \in [1, \infty]$ the following sequences $\{f_n\}_{n=1}^{\infty}$ converge in $L^p([0,1])$:

(i) $f_n(x) = x^n - x^{n-1}$

(ii) $f_n(x) = x^n - x^{2n}$.

Problem 10. Consider the space $C^1([0,1])$ and define the following functions on it:

(i) $F_1(f) := \max_{x\in[0,1]} |f'(x)|$,

(ii) $F_2(f) := |f(0) - f(1)| + \max_{x\in[0,1]} |f'(x)|$,

(iii) $F_3(f) := |f(0)| + \max_{x\in[0,1]} |f'(x)|$,

(iv) $F_4(f) := \int_0^1 |f(x)|\,\mathrm{d}x + \max_{x\in[0,1]} |f'(x)|$.

Which of the F_j's are norms? Among the norms, which are equivalent?

Problem 11.

(i) Show that the space $c_0 := \{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{C} \text{ and } \lim x_n = 0\}$ is closed in ℓ^∞.

(ii) Show that the space

$$s_0 := \{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{C} \text{ and } x_n = 0 \text{ for all but a finite number of } n\text{'s}\}$$

is dense in ℓ^p for any $p \in [1, \infty)$, but not in ℓ^∞.

Problem 12. Let Ω be a measure space. Show that when $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some $q \in [1, \infty)$ then $f \in L^p(\Omega)$ for all $p > q$ and

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p .$$