Functional Analysis – Problems in the class, sheet 1

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 1. Prove in detail that a uniform limit of continuous functions is continuous. More precisely: let $C([a,b])$ be the space of complex-valued continuous functions on $[a,b]$ with the norm

$$\|f\|_\infty \;=\; \sup_{x\in[a,b]} |f(x)|$$

and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $C([a,b])$ and let $f:[a,b]\to\mathbb{C}$ such that

$$\|f_n - f\|_\infty \;\xrightarrow{n\to\infty}\; 0\,.$$

Prove that $f\in C([a,b])$. As a consequence, show that the normed space $\big(C([a,b]),\|\cdot\|_\infty\big)$ is complete.

Problem 2. Discuss a concrete example of an infinite-dimensional subspace of the space $C([0,1])$ of continuous functions on $[0,1]$ that is not complete with respect to the norm $\|\cdot\|_\infty$.

Problem 3. Let $h:\mathbb{R}\to\mathbb{R}$ with

$$h(x) \;=\; \begin{cases} \dfrac{e^{-\frac{1}{1-x^2}}}{\int_{-1}^{1} e^{-\frac{1}{1-y^2}}\,\mathrm{d}y} & \text{if } |x|<1 \\[2em] 0 & \text{if } |x|\geqslant 1\,. \end{cases}$$

Define $h_n(x) := nh(nx)$ for every $x\in\mathbb{R}$ and positive integer n. (The sequence $\{h_n\}_{n=1}^\infty$ is called an "approximate identity" on $\mathbb{R}$.)

(i) Show that each h_n is a positive $C_0^\infty(\mathbb{R})$ function.

(ii) Prove that $\displaystyle\int_{-\infty}^{+\infty} h_n(x)\,\mathrm{d}x = 1 \qquad \forall n.$

(iii) Say in which sense $h_n \to 0$ as $n \to \infty$.

Problem 4. Let $P_k := \{\text{polynomials } [-1,1] \to \mathbb{R} \text{ of degree} \leqslant k\}$ and let $\{1, x, \dots, x^k\}$ be the standard basis of P_k. Make P_k an inner product space with the usual L^2 inner product, i.e., $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x$.

(i) Construct an orthonormal basis for P_2 and P_3.

(ii) Let $D : P_3 \to P_2$ be the differentiation operator, i.e., $(Df)(x) = f'(x)$. Write the matrix M_D associated with D with respect of the standard bases of P_2 and P_3.

(iii) Compute the singular value decomposition of M_D. In other words, find a 4×3 matrix U, a 3×3 matrix V, and a 3×3 diagonal matrix Σ such that $U^T U = V^T V = \mathbb{1}$ and $(M_D)^T = U \Sigma V^T$. (For an overview of the singular value decomposition of a matrix you may refer to paragraph 7.4 of the lecture notes at

`http://www.mathematik.uni-muenchen.de/~lerdos/Notes/crash.pdf`.)