Functional Analysis – Exercise sheet 9

Mathematisches Institut der LMU – SS2010
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Due: Monday 5 July 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 5 July tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 33. Consider the Banach spaces

$$\begin{aligned} c &= \big\{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{R}\ \forall n \ \text{ and } \exists \lim_{n\to\infty} x_n\big\} \\ c_0 &= \big\{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{R}\ \forall n \ \text{ and } \lim_{n\to\infty} x_n = 0\big\} \subset c \end{aligned}$$

with the standard norm $\|x\|_\infty = \sup_n |x_n|$.

(i) Prove that c_0 and c are not isomorphic as Banach spaces, i.e., are not isometrically isomorphic. (*Hint*: a point x in a convex set X is called an *extremal point* if one cannot represent x as a non-trivial convex combination

$$x = \alpha_1 x_1 + \alpha_2 x_2, \qquad \alpha_1, \alpha_2 > 0, \ \ \alpha_1 + \alpha_2 = 1,$$

of two other points x_1, x_2 of the set X. In other words, if $x = \alpha_1 x_1 + \alpha_2 x_2$ with $x_1, x_2 \in X$, $\alpha_1, \alpha_2 \in [0, 1]$, and $\alpha_1 + \alpha_2 = 1$, then necessarily either $\alpha_1 = 1$ or $\alpha_2 = 1$. Find the extremal points of the unit ball in each space.)

(ii) Prove that $c^* \simeq c_0^* \simeq \ell^1$. (*Hint*: consider the linear functional $\phi_{\lim}$ on c defined by

$$\phi_{\lim}(x) = \lim_{n\to\infty} x_n, \qquad x = (x_1, x_2, \dots)$$

and prove that $c^* = \phi_{\lim} \oplus c_0^*$.)

Exercise 34.

(i) Let $p \in (1, \infty)$. Let $x_1, x_2 \in \ell^p$ with $\|x_1\|_p = \|x_2\|_p = 1$. Show that

$$\left\| \frac{x_1 + x_2}{2} \right\|_p = 1 \quad \Rightarrow \quad x_1 = x_2.$$

(*Hint:* consider the case when Minkowski's inequality in ℓ^p becomes an equality.)

(ii) Show that the space $C([0,1])$ (with the usual supremum norm) cannot be embedded isometrically into ℓ^p.

(iii) Give an isometric embedding of $C([0,1])$ into ℓ^∞.

Exercise 35. Consider the Banach space ℓ^∞ of *real* bounded sequences, the subspace $c = \{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{R}\ \forall n \text{ and } \exists \lim_{n\to\infty} x_n\}$, and the points

$$\begin{aligned} x_1 &= (0,1,0,1,0,1,0,1,\dots) && \text{(i.e., alternating 0 and 1)} \\ x_2 &= (0,0,0,1,0,1,0,1,\dots) && \text{(i.e., alternating 0 and 1 from the third position)} \\ x_3 &= (1,0,1,0,1,0,1,0,\dots) && \text{(i.e., alternating 1 and 0)}\,. \end{aligned}$$

(i) Show that there exist bounded linear functionals λ and μ in $(\ell^\infty)^*$ such that $\lambda(x) = \mu(x) = \lim_{n\to\infty} x_n\ \forall x \in c$ and $\lambda(x_1) = \frac{1}{2}$, $\mu(x_1) = -2010$.

(ii) Can it happen that the functional λ (resp. μ) considered in (i) satisfies the further condition $\lambda(x_2) = \frac{1}{3}$ (resp. $\mu(x_3) = \frac{1}{3}$)?

(iii) Show that there exists a bounded linear functional $\lambda \in (\ell^\infty)^*$ such that

(•) $\liminf_n x_n \leqslant \lambda(x) \leqslant \limsup_n x_n\ \forall x \in \ell^\infty$ (and therefore $\lambda(x) = \lim_{n\to\infty} x_n\ \forall x \in c$)

(••) $\lambda(Lx) = \lambda(x)\ \forall x \in \ell^\infty$ where $L : \ell^\infty \to \ell^\infty$ is the left-shift operator $L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

(*Hint:* apply the Hahn-Banach theorem to the subspace of the sequences $y = (y_1, y_2, y_3, \dots)$ with $y_n = x_{n+1} - x_n$.)

Exercise 36. Let $f \in L^2(\mathbb{R})$. Show that each of the following sequences converges weakly to 0 in $L^2(\mathbb{R})$:

(i) $\{g_n\}_{n=1}^\infty$ with $g_n(x) = f(x-n)$

(ii) $\{h_n\}_{n=1}^\infty$ with $h_n(x) = \sqrt{n}\, f(nx)$

(iii) $\{k_n\}_{n=1}^\infty$ with $k_n(x) = f(x)e^{inx}$.