Functional Analysis – Exercise sheet 8

Mathematisches Institut der LMU – SS2010
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Due: Monday 28 June 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 28 June tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 29. Let $f_0 \in L^2(S^1)$.

(i) Find the Fourier series of the solution in $C^1(\mathbb{R}_t^+, C^4(S_x^1))$ of the partial differential equation

$$\partial_t f \;=\; -\Delta_x(\Delta_x f) - 20\Delta_x f + 630 f$$

that satisfies the condition

$$\lim_{t\to 0^+} f(t,\cdot) \;=\; f_0$$

where the limit is in $L^2(S^1)$). (*Hint:* the same as in Problems in class no. 29 and 30.)

(ii) Find all the f_0's for which $\lim\limits_{t\to+\infty} f(t,\cdot)$ exists in $L^2(S^1)$ and compute this limit for such f_0's.

Exercise 30. Consider the space $\mathcal{S}(\mathbb{R}^n)$ of functions of rapid decrease on $\mathbb{R}^n$ with its natural topology (this space was introduced in the Problem in class no. 32).

(i) Show that for every $x \in \mathbb{R}^n$ the linear map $\delta_x : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ with $\delta_x(f) := f(x)$ (the "*Dirac delta*") is continuous.

(ii) Let $h \in L^1(\mathbb{R}^n)$ with $\int h(x)\mathrm{d}x = 1$ and let $h_N(x) := N^n h(Nx)$ $\forall x \in \mathbb{R}^n$. Show that the linear map $\delta^{(N)} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ with $\delta^{(N)}(f) := \displaystyle\int_{\mathbb{R}^d} h_N(x) f(x)\,\mathrm{d}x$ is continuous and

$$\delta^{(N)}(f) \xrightarrow{N\to\infty} \delta(f) \qquad \forall f \in \mathcal{S}(\mathbb{R}^n)\,.$$

(iii) Show that for any $f \in \mathcal{S}(\mathbb{R})$ the limit $\displaystyle\lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} \frac{1}{x} f(x)\,\mathrm{d}x$ exists and is finite.

(iv) Show that the linear map $P\dfrac{1}{x} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ with

$$\Big(P\frac{1}{x}\Big)(f) \;:=\; \lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} \frac{1}{x} f(x)\,\mathrm{d}x$$

(the "*Cauchy principal value*") is continuous.

Exercise 31. Let $(X, \| \, \|_X)$ be a normed space and let $(X^*, \| \, \|_{X^*})$ be its dual. Let $\phi \in X^* \backslash \{0\}$. Prove that for any $x \in X$

$$\operatorname{dist}(x, \ker \phi) \;=\; \frac{|\phi(x)|}{\|\phi\|_{X^*}} \,.$$

Exercise 32. Compute the norm of the following functionals:

(i) $\quad \phi \in (C([-1,1]), \| \, \|_\infty)^* \quad$ with $\quad \phi(f) := \displaystyle\int_{-1}^{1} x f(x) \, \mathrm{d}x$

(ii) $\quad \phi \in (L^1[-1,1], \| \, \|_1)^* \quad$ with $\quad \phi(f) := \displaystyle\int_{-1}^{1} x f(x) \, \mathrm{d}x$

(iii) $\quad \phi \in (L^2[-1,1], \| \, \|_2)^* \quad$ with $\quad \phi(f) := \displaystyle\int_{-1}^{1} x f(x) \, \mathrm{d}x$

(iv) $\quad \phi \in (\ell^1, \| \, \|_{\ell^1})^* \quad$ with $\quad \phi(x) := \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n} \qquad (x = \{x_n\}_{n=1}^{\infty})$

(v) $\quad \phi \in (\ell^2, \| \, \|_{\ell^2})^* \quad$ with $\quad \phi(x) := \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n} \qquad (x = \{x_n\}_{n=1}^{\infty})$.