Functional Analysis – Exercise sheet 6

Mathematisches Institut der LMU – SS2010
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Due: Monday 14 June 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 14 June tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 21. Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and let $\{\psi_n\}_{n=1}^{\infty}$ be an orthonormal system. Prove that if

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 \;<\; \infty$$

then $\{\psi_n\}_{n=1}^{\infty}$ too is an orthonormal basis of $\mathcal{H}$.

Exercise 22. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ such that $\|\psi_n\| = 1$ $\forall n$ and $\langle \psi_n, \psi_m \rangle = \frac{1}{10}$ whenever $m \neq n$. Prove that there exists $\psi \in \mathcal{H}$ such that

$$\langle \psi_n, \varphi \rangle \xrightarrow{n\to\infty} \langle \psi, \varphi \rangle \qquad \forall \varphi \in \mathcal{H}\,.$$

Exercise 23. Consider the collection $\{e_n\}_{n\in\mathbb{Z}}$ in $L^2[a,b]$ with $e_n(x) = e^{2\pi i n x}$. Prove that the orthogonal complement of such a collection

(i) is only $\{0\}$ if $b - a \leqslant 1$

(ii) is different from $\{0\}$ if $b - a > 1$.

Exercise 24. Consider the following subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of the Hilbert space $\mathcal{H} = L^2[-1,1]$:

$$\begin{aligned} \mathcal{M}_1 &= \Big\{ f \in \mathcal{H} \mid \int_{-1}^{1} f(t)\,\mathrm{d}t = 0 \Big\} \\ \mathcal{M}_2 &= \big\{ f \in \mathcal{H} \mid f(-t) = f(t) \big\}\,. \end{aligned}$$

(i) Show that $\mathcal{M}_1$ is closed and find $\mathcal{M}_1^{\perp}$.

(ii) Show that $\mathcal{M}_2$ is closed and find $\mathcal{M}_2^{\perp}$.

(iii) Compute the distance of f from $\mathcal{M}_1$ with $f(x) = x^2$ and find the decomposition of f in $\mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}$.

(iv) Compute the distance of f from $\mathcal{M}_2$ with $f(x) = e^x$ and find the decomposition of f in $\mathcal{M}_2 \oplus \mathcal{M}_2^{\perp}$.