Functional Analysis – Exercise sheet 3

Mathematisches Institut der LMU – SS2010
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Due: Monday 17 May 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Monday 17 tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

> *The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.*

Exercise 9. Let S be any set and let (X, d_X) be a metric space. Define

$$\mathcal{B}(S,X) \;:=\; \{f : S \to X \mid f \text{ is bounded}\}.$$

(Recall that $f : S \to X$ is bounded means that $\sup\limits_{s,s' \in S} d_X(f(s_1), f(s_2)) < \infty$.) For every $f, g \in \mathcal{B}(S,X)$ define

$$d_\infty(f,g) \;:=\; \sup_{s \in S} d_X(f(s), g(s)).$$

(i) Show that d_∞ is a metric on $\mathcal{B}(S,X)$.

(ii) Assume that (X, d_X) is *complete*. Show that then $(\mathcal{B}(S,X), d_\infty)$ is also complete.

Exercise 10. Let $a = \{a_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^+$. Let

$$S^{(a)} \;:=\; \left\{ x = (x_1, x_2, x_3, \dots) \text{ such that } \sum_{n=1}^{\infty} |x_n|^2 < \infty \text{ and } |x_n| < a_n \; \forall n \right\} \;\subset\; \ell^2.$$

Prove that $S^{(a)}$ is open in ℓ^2 if and only if $\inf_n a_n > 0$.

Exercise 11. Let (X, μ) be a measure space and let $\{f_n\}_{n=1}^\infty$ be a sequence of integrable functions on (X, μ). Suppose that there exists an integrable function f such that

(*a*) $f_n \to f$ pointwise almost everywhere,

(*b*) $\int_X |f_n| \, \mathrm{d}\mu \to \int_X |f| \, \mathrm{d}\mu$ as $n \to \infty$.

Prove that $\int_X |f - f_n| \, \mathrm{d}\mu \to 0$ as $n \to \infty$.

Exercise 12. Recall that the distance $d(x, W)$ of a point x in the normed space $(X, \| \ \|_X)$ from a subspace $W \subset X$ is defined as $d(x, W) := \inf_{w \in W} \|x - w\|_X$ (see also the discussion in Exercise 3, part (ii)).

(i) Consider the normed space ℓ^∞ with the natural supremum norm. Let $a = (a_1, a_2, a_3 \dots) \in \ell^\infty$ with $a_n = \sum_{k=1}^n \frac{1}{k^2}$. Compute the distance of a from the subspace

$$c_0 := \{x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{C} \text{ and } \lim_{n \to \infty} x_n = 0\}.$$

(ii) Consider the normed space $C([-1, 1])$ equipped with the supremum norm. Compute the distance of the function $f \in C([-1, 1])$, $f(x) = |x|$, from the subspace $C^1([-1, 1])$.

(iii) Compute the distance, in the L^1-norm, of the function $f \in L^1(\mathcal{B}_2)$, $f(x) = |x|^{-1}$, where

$$\mathcal{B}_R = \{x \in \mathbb{R}^3 \mid |x| \leqslant R\} \qquad \text{(the ball of radus } R\text{)},$$

from the subspace $\{g \mid g \in L^1(\mathcal{B}_2) \text{ and } g \equiv 0 \text{ as } x \in \mathcal{B}_1\}$.