Functional Analysis – Exercise sheet 2

Mathematisches Institut der LMU – SS2010
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Handout: Monday 3 May 2010
Due: Monday 10 May 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Monday 10 tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 5. Let X be a topological space.

(i) Prove that the complement of an open set, $X \setminus O$, is a closed set. (Recall that in class a closed set was defined to be a set containing all its limit points: see the handout "Analysis and topology: an overview".)

(ii) Let $E \subset X$. Prove that the closure $\overline{E}$ of E, defined as $\overline{E} = E \cup \{\text{limit points of } E\}$, is closed in X.

(iii) Let $E \subset X$. Prove that the interior of E, defined as $\mathring{E} = \{\text{interior points of } E\}$, is open in X.

Exercise 6.

(i) Prove that the topology of the pointwise convergence as given in class in the space $C([0,1])$ does not originate from any metric. (Recall that the pointwise convergence in the space $C([0,1])$ was defined in class by declaring an explicit basis of neighbourhoods – see the handout "Topologies on continuous functions". This basis is of course not unique.)

(ii) Give an example of a sequence of functions in $C([0,1])$ that do not converge in a fixed d_p metric but they converge pointwise. Give an example to the opposite as well: a sequence of functions that converge in d_p but not pointwise. Recall that $d_p(f,g) = (\int_0^1 |f-g)|^p)^{1/p}$.

Exercise 7. Let $P^{(N)} := \{\text{polynomials } [0,1] \to \mathbb{R} \text{ of degree} \leqslant N\}$ with $N \in \mathbb{N}$.

(i) For fixed $N \in \mathbb{N}$ and $K > 0$ prove that $\{p \in P^{(N)} \,|\, \|p\|_\infty \leqslant K\}$ is compact in $C([0,1])$ with the $\| \;\|_\infty$-norm topology.

(ii) Let $f \in C([0,1])$ be the uniform limit of a sequence in $P^{(N)}$ for some fixed N. Prove that f is a polynomial.

Exercise 8. Which of these subsets of $C([0,1])$ are pre-compact (i.e., their closure is compact) with respect to the $\| \; \|_\infty$-norm topology?

(i) $\{ f \mid f \in C^1([0,1]) \,, f(0) = 0 \,, \text{ and } |f'(x)| \leqslant 1 \, \forall x \, \}$

(ii) $\{ f \mid f \in C^1([0,1]) \,, f(0) = 0 \,, \text{ and } |f(x)| \leqslant 1 \, \forall x \, \}$

(iii) $\{ f \mid f \in C^2([0,1]) \,, f(0) = 0 \,, \text{ and } |f''(x)| \leqslant 1 \, \forall x \, \}$

(iv) $\{ f \mid f \in C^1([0,1]) \,, |f(x)| \leqslant 1 \, \forall x \,, \text{ and } \int_0^1 |f'(x)|^2 \mathrm{d}x \leqslant 1 \, \}$