Functional Analysis – Exercise sheet 1

Mathematisches Institut der LMU – SS2010
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Due: Monday 3 May 2010 by 10 a.m. in the designated "Funktionalanalysis" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 1. Let

$$A \;=\; \begin{pmatrix} 14 & -2 & -4 \\ -2 & 17 & -2 \\ -4 & -2 & 14 \end{pmatrix}.$$

(i) Find an explicit 3×3 matrix V, with $VV^T = V^TV = \mathbb{1}$, and a diagonal 3×3 matrix Λ, such that

$$A \;=\; V\Lambda V^T \tag{1}$$

(V^T is the transpose of V). Show that (1) can be re-written in the form

$$A \;= \sum_{i=1}^{3} \lambda_i \,\mathbf{v}_i\mathbf{v}_i^T \tag{2}$$

where the λ_i's are the diagonal entries of Λ and the $\mathbf{v}_i$'s are 3×1 (column) vectors.

(ii) Are the $\mathbf{v}_i$'s unique (apart from trivial sign flips $\mathbf{v}_i \to -\mathbf{v}_i$)?

(iii) Prove that

$$A^k \;=\; \sum_{i=1}^{3} \lambda_i^k \,\mathbf{v}_i\mathbf{v}_i^T \qquad k = 1, 2, \dots \tag{3}$$

(iv) Define

$$\sin A \;:=\; \sum_{i=1}^{3} (\sin\lambda_i)\,\mathbf{v}_i\mathbf{v}_i^T\,. \tag{4}$$

Prove that

$$\sin A \;=\; \sum_{j=0}^{\infty} (-1)^j\,\frac{A^{2j+1}}{(2j+1)!}\,. \tag{5}$$

(Note: you need to justify the convergence of the series in the right-hand side. Use Cauchy sequences in the Euclidean matrix norm $\|A\| := \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$.)

Exercise 2.

(i) Consider
$$C([0,1]) \;=\; \{\, f : [0,1] \to \mathbb{R},\ f \text{ is continuous} \,\}$$
with the natural structure of vector space given by the pointwise sum between functions and pointwise multiplication by a scalar. Prove that $\dim C([0,1]) = \infty$ by constructing n linearly independent elements of $C([0,1])$ for each n.

(ii) Show that the map $M : C([0,1]) \to C([0,1])$ such that $(Mf)(x) := xf(x)$ is injective but not surjective.

(iii) Give an example of an infinite-dimensional subspace $V \subset C([0,1])$ such that $M\big|_V$, the restriction of M to V, is a bijection $M\big|_V : V \to M(V)$.

Exercise 3. Let V be a subspace of $\mathbb{R}^n$ (n being a positive integer). Denote by $|\cdot|$ the absolute value in $\mathbb{R}^n$. Denote by $V^\perp$ the orthogonal complement of V, i.e., the space
$$V^\perp \;:=\; \{w \in \mathbb{R}^n \ \text{ such that } \ w \cdot v = 0 \ \forall v \in V\}$$
where $w \cdot v$ is the scalar product in $\mathbb{R}^n$.

(i) Prove that $(V^\perp)^\perp = V$.

(ii) Prove that for any $x \in \mathbb{R}^n$ there exists a unique element $v_{(x)} \in V$ "closest to x", i.e., such that $|x - v_{(x)}| \leqslant |x - v| \ \forall v \in V$. (*Hint:* for the existence use a sequence $\{v_j\}_{j=1}^\infty$ in V such that
$$|x - v_j| \xrightarrow{j\to\infty} \inf_{v \in V} |x - v|$$
and show that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence...).

(iii) As a consequence, prove that any $x \in \mathbb{R}^n$ can be uniquely written as $x = v + v'$ with $v \in V$ and $v' \in V^\perp$.

(iv) Let
$$V \;=\; \mathrm{Span}\left\{ \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}, \begin{pmatrix} 4\\3\\2\\1 \end{pmatrix} \right\} \subset \mathbb{R}^4 .$$
Give an orthonormal basis of $V^\perp$.

Exercise 4. Prove that the convolution $f * g$ between $C^1_0(\mathbb{R})$-functions, defined as $(f*g)(x) = \int f(x-y)g(y)\mathrm{d}y$, has the following properties:

(i) commutativity: $f * g = g * f$

(ii) associativity: $f * (g * h) = (f * g) * h$

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}(f * g) = (\frac{\mathrm{d}}{\mathrm{d}x} f) * g = f * (\frac{\mathrm{d}}{\mathrm{d}x} g)$.

Let now $\{h_n\}_{n=1}^\infty$ be a sequence of positive $C^\infty(\mathbb{R})$-functions with support in $[0,1]$ such that (a) $\forall n \int_{-\infty}^{\infty} h_n(x)\mathrm{d}x = 1$ and (b) $\forall \varepsilon > 0 \int_{|x|\geqslant\varepsilon} h_n(x)\mathrm{d}x \to 0$ as $n \to \infty$. (See also Problem 3 discussed in the Exercise/Tutorial session, where such a sequence is constructed explicitly.)

(iv) Prove that for any $f \in C_0^\infty(\mathbb{R})$ one has $f * h_n \to f$ uniformly as $n \to \infty$.