Spin dependent point interactions
Models of quantum measurement processes and quantum environments

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Outline

a) Point Interactions in one particle quantum mechanics
b) Spin-dependent point interactions
c) Models of interaction particle point-field
d) Solvable models of quantum environment

Physical motivations The atomic structure exhibits multiple length scales.

$$+$$

No natural minimal length is reasonably hypothesized

$$\Downarrow$$

Practical and theoretical interest in investigating the limit

$$\text{linear range} \longrightarrow 0.$$

$\mathcal{M}$ configuration space of a classical system
H_0 the s.a. operator in $L^2(\mathcal{M})$ generating the free dynamics.
Y a closed subset of $\mathcal{M}$
H in $L^2(\mathcal{M})$ is said to describe an interaction supported in Y if H and H_0 act in the same way on smooth functions with support in $\mathcal{M} \setminus Y$.

The case $\mathcal{M} = \mathbb{R}^d$, $\mathbf{Y} = \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_N}\}$ a finite set of points of $\mathbb{R}^d$

$$H_0 = -\Delta \qquad \text{with domain} \qquad D(H_0) = H^2(\mathbb{R}^d) \tag{2.1}$$

The operator

$$\breve{H}_Y = H_0 \restriction C_0^\infty(\mathbb{R}^3 \setminus \mathbf{Y}) \tag{2.2}$$

Any self-adjoint extension of $\breve{H}_{\mathbf{Y}}$ different from H_0 is the hamiltonian of a quantum particle interacting with "point scatterers" placed in $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_N}\}$.

For $\lambda > 0$, let G^λ be the fundamental solution of the Helmholtz equation

$$(-\Delta_{\mathbf{x}} + \lambda)\, G^\lambda(\mathbf{x} - \mathbf{y}) = \delta_{\mathbf{y}} \qquad \mathbf{y} \in \mathbb{R}^d$$

$$\widetilde{G}^\lambda(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + \lambda} \quad \sim \quad \text{indicates Fourier transform}$$

$$\begin{aligned}
G(\mathbf{x}) &= \frac{e^{-\sqrt{\lambda}|\mathbf{x}|}}{4\pi|\mathbf{x}|} && \in L^2(\mathbb{R}^3) \quad d = 3 \\
G(\mathbf{x}) &= \frac{\imath}{4}\mathcal{H}_0(\sqrt{\lambda}\mathbf{x}) && \in L^2(\mathbb{R}^2) \quad d = 2 \\
G(\mathbf{x}) &= \frac{e^{-\sqrt{\lambda}|\mathbf{x}|}}{2\sqrt{\lambda}} && \in H^1(\mathbb{R}) \quad d = 1
\end{aligned}$$

An easy but fundamental consequence is that

- In $d = 1$ the domain of the adjoint operator $\breve{H}^*_{\mathbf{Y}}$ contains the functions $G^\lambda(\cdot - \mathbf{y_i})$ and their derivatives.
- in $d = 2, 3$ the domain of the adjoint operator $\breve{H}^*_{\mathbf{Y}}$ contains the functions $G^\lambda(\cdot - \mathbf{y_i})$ (that are singular at the positions of the point scatterers).
- $G^\lambda \notin L^2(\mathbb{R}^d)$ for $d > 3 \implies$ the only self-adjoint extension of $\breve{H}_{\mathbf{Y}}$ is H_0 no point interaction Hamiltonians can be defined for $d > 3$.

All the self-adjoint restrictions of $\breve{H}^*_{\mathbf{Y}}$ are classified using one of the many powerful machineries now available.

For $d = 3$:

$\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)$ an N-ple of real numbers. The following relations define domain and action of operators in the family of self-adjoint extensions $H_{\boldsymbol{\beta},\mathbf{Y}}$ of $\tilde{H}_{\mathbf{Y}}$ on $L^2(\mathbb{R}^3)$

$$D(H_{\boldsymbol{\beta},\mathbf{Y}}) = \Bigg\{ u \in L^2(\mathbb{R}^3) \mid u = \phi^\lambda + \sum_{i=1}^{N} q_i G^\lambda(\cdot - \mathbf{y_i}),$$

$$\phi^\lambda \in H^2(\mathbb{R}^3),\; q_i \in \mathbb{C}, \phi^\lambda(\mathbf{y_i}) = \sum_{j=1}^{N} (\Gamma^\lambda_{\boldsymbol{\beta},\mathbf{Y}})_{ij} q_j \Bigg\}$$

$$(H_{\boldsymbol{\beta},\mathbf{Y}} + \lambda) u = (-\Delta + \lambda)\phi^\lambda$$

where

$$(\Gamma^\lambda_{\boldsymbol{\beta},\mathbf{Y}})_{ij} = \left(\beta_i + \frac{\sqrt{\lambda}}{4\pi}\right)\delta_{ij} - G^\lambda(\mathbf{y_i} - \mathbf{y_j})\big(1 - \delta_{ij}\big)$$

for $\lambda > 0$ large enough to make $\Gamma^\lambda_{\boldsymbol{\beta},\mathbf{Y}}$ invertible.

$u \in D(H_{\boldsymbol{\beta},\mathbf{Y}})$ consists of a "regular part" ϕ^λ plus the "potential" of the point charges q_i, where

$$q_i = 4\pi \lim_{\mathbf{x}\to\mathbf{y_i}} |\mathbf{x} - \mathbf{y_i}| u(\mathbf{x}) \tag{2.3}$$

Moreover

$$\lim_{\mathbf{x}\to\mathbf{y_i}} \left(u(\mathbf{x}) - \frac{q_i}{4\pi|\mathbf{x} - \mathbf{y_i}|} \right) = \beta_i \, q_i$$

a singular boundary condition satisfied by u at each point $\mathbf{y_i}$.
In the past

$$\frac{\partial}{\partial r_i}(r_i u)\Big|_{\mathbf{x}=\mathbf{y_i}} = 4\pi\beta_i \, (r_i u)\Big|_{\mathbf{x}=\mathbf{y_i}} , \qquad r_i = |\mathbf{x} - \mathbf{y_i}| \tag{2.4}$$

The parameters β_i give a measure of the strength of the interaction (for $\beta_i = \pm\infty$ one has the free laplacian)

Minimal set of parameters: the strength parameters β_i and the positions $\mathbf{y_i}$.

Properties of $H_{\boldsymbol{\beta},\mathbf{Y}}$

- The resolvent operator $(H_{\boldsymbol{\beta},\mathbf{Y}} - z)^{-1}, \quad z = -\lambda \quad$ for $\lambda > 0$ sufficiently large in order that $\Gamma^{\lambda}_{\boldsymbol{\beta},\mathbf{Y}}$ is invertible

$$(H_{\boldsymbol{\beta},\mathbf{Y}} + \lambda)^{-1}(\mathbf{x},\mathbf{x'}) = G^{\lambda}(\mathbf{x}-\mathbf{x'}) + \sum_{i,j=1}^{N} \left(\Gamma^{\lambda}_{\boldsymbol{\beta},\mathbf{Y}}\right)^{-1}_{ij} G^{\lambda}(\mathbf{x}-\mathbf{y}_i) G^{\lambda}(\mathbf{x'}-\mathbf{y}_j)$$

- Spectrum of $H_{\beta,\mathbf{Y}}$

$$\sigma_{ac}(H_{\beta,\mathbf{Y}}) = [0,\infty)\,, \qquad \sigma_p(H_{\beta,\mathbf{Y}}) \subset (-\infty,0)$$

- $H_{\beta,\mathbf{Y}}$ has at most N negative eigenvalues counting multiplicity.
- $E<0$ is an eigenvalue of $H_{\beta,\mathbf{Y}}$ if and only if

$$det\,\Gamma^{|E|}_{\beta,\mathbf{Y}} = 0$$

  and its multiplicity is the same as the multiplicity of the eigenvalue zero of the matrix $\Gamma^{|E|}_{\beta,\mathbf{Y}}$.
- The corresponding eigenfunctions of $H_{\beta,\mathbf{Y}}$ are explicitly computable in terms of the eigenvectors of $\Gamma^{|E|}_{\beta,\mathbf{Y}}$ corresponding to the eigenvalue zero.

Explicitly for $N = 1$

- $$\sigma_p(H_{\beta,\mathbf{y}}) = \emptyset \quad \text{for} \quad \beta \geq 0\,, \qquad \sigma_p(H_{\beta,\mathbf{y}}) = \{-(4\pi\beta)^2\} \quad \text{for} \quad \beta < 0$$

  the normalized eigenfunction for $\beta < 0$ associated to the eigenvalue is

  $$\zeta_{\beta,\mathbf{y}}(\mathbf{x}) = \sqrt{2|\beta|}\,\frac{e^{-4\pi|\beta||\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$$

- generalized eigenfunction associated to the absolutely continuous spectrum

  $$\phi^{\pm}_{\beta,\mathbf{y}}(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\left(e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{i\mathbf{k}\cdot\mathbf{y}}}{4\pi\beta \pm i|\mathbf{k}|}\,\frac{e^{\mp i|\mathbf{k}||\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)$$

Spin-dependent point interactions

Phenomenology showed that neutron-nuclei scattering was strongly dependent on neutron and/or nuclei spin.

To take this dependence into account in point interactions models, it is enough to consider **multi-channel** point interaction hamiltonians.
To each possible **scattering channel** (defined by a particular spin state of neutron and nucleus) it was associated a dynamical strength parameter β of the zero range interaction.

Spin-dependent point interactions

Consider an array of N 1/2-spins placed in $\{y_1, \dots, y_N\}$ $\quad y_i \in \mathbb{R}^d$.
The state space of the j-th spin is $\mathbb{C}^2$.
Notation:

- $\hat{\sigma}_j^{(1)}$ is the first Pauli matrix relative to the j-th spin $j = 1, \dots, N$
- χ_{σ_j} is the normalized eigenvector of $\hat{\sigma}_j^{(1)}$ with eigenvalue ± 1

$$\hat{\sigma}_j^{(1)} \chi_{\sigma_j} = \sigma_j \chi_{\sigma_j} \qquad \sigma_j = \pm 1\,;\ \|\chi_{\sigma_j}\|_{\mathbb{C}^2} = 1\,;\ j = 1, \dots, N\,.$$

The Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{S}_N\,,$$

where

$$\mathbb{S}_N = \overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{N}$$

Spin-dependent point interactions

$\mathcal{X}_{\underline{\sigma}} = \chi_{\sigma_1} \otimes \cdots \otimes \chi_{\sigma_N}$, where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$.

$$\Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi \in \mathcal{H} \quad \psi_{\underline{\sigma}} \in L^2(\mathbb{R}^d) \quad \forall \underline{\sigma}$$

The scalar product in $\mathcal{H}$

$$\langle \Psi, \Phi \rangle = \sum_{\underline{\sigma}} (\psi_{\underline{\sigma}}, \phi_{\underline{\sigma}})_{L^2} \qquad \Psi, \Phi \in \mathcal{H}\,.$$

The free hamiltonian in $\mathcal{H}$

$$D(H) = H^2(\mathbb{R}^d) \otimes \mathbb{S}_N$$

$$H = -\frac{\hbar^2}{2m} \Delta \otimes \mathbb{I}_{\mathbb{S}_N} + \sum_{j=1}^{N} \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{S}_j \qquad \alpha_j \in \mathbb{R}$$

$$\mathbf{S}_j = \overbrace{\mathbb{I}_{\mathbb{C}^2} \otimes \cdots \otimes \hat{\sigma}_j^{(1)} \otimes \cdots \otimes \mathbb{I}_{\mathbb{C}^2}}^{N} \qquad j = 1, \ldots, N$$

is self-adjoint and generates a free and independent dynamics of a point particle in $\mathbb{R}^d$ and N spins.

$$H\Psi = \sum_{\underline{\sigma}} \big(-\Delta + \underline{\alpha}\,\underline{\sigma} \big) \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi \in \mathcal{H}$$

where $\underline{\alpha} \equiv (\alpha_1, \ldots, \alpha_N)$ and $\underline{\alpha}\,\underline{\sigma} = \sum_{j=1}^{N} \alpha_j \sigma_j$.
Consider the (non self-adjoint) symmetric operator H_0 on $\mathcal{H}$

$$D(H_0) = C_0^{\infty}(\mathbb{R}^d \setminus Y) \otimes \mathbb{S}_N$$

$$H_0 = -\Delta \otimes \mathbb{I}_{\mathbb{S}_N} + \sum_{j=1}^{N} \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{S}_j \qquad \alpha_j \in \mathbb{R}$$

All the self-adjoint extensions of H_0 are characterizable and each one of them can be taken as generator of the dynamics of the quantum system: particle + spins.

Spin-dependent point interactions

A family of Hamiltonians giving rise to a dynamics with particle-spins interaction in $d = 3$ is the following:
With

$$G^z(x) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|}\,; \qquad z \in \mathbb{C}\backslash\mathbb{R}^+\,, \quad \Im\sqrt{z} > 0\,.$$

and

$$\begin{aligned}
&(\Gamma_\gamma(z))_{j\underline{\sigma},j'\underline{\sigma}'} = 0 && j \neq j'\,;\underline{\sigma} \neq \underline{\sigma}' \\
&(\Gamma_\gamma(z))_{j\underline{\sigma},j'\underline{\sigma}} = -G^{z-\underline{\sigma}\,\underline{\alpha}}(y_j - y_{j'}) && j \neq j' \\
&(\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}'} = 0 && \sigma_k \neq \sigma'_k \text{ for } k \neq j \\
&(\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}'} = \sigma'_j\, i\rho && \sigma'_j \neq \sigma_j \text{ and } \sigma_k = \sigma'_k \ \text{ for } k \neq j \\
&(\Gamma_\gamma(z))_{j\underline{\sigma},j\underline{\sigma}} = \frac{\sqrt{z - \underline{\sigma}\,\underline{\alpha}}}{4\pi i} + \beta
\end{aligned}$$

The operators H_γ, in the one parameter family defined as follows

$$D(H_\gamma) := \Big\{ \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \chi_{\underline{\sigma}} \in \mathcal{H} :$$

$$\Psi = \Psi^z + \sum_{j\underline{\sigma}, j'\underline{\sigma}'} (\Gamma_\gamma(z))^{-1}_{j\underline{\sigma}, j'\underline{\sigma}'} \psi^z_{\underline{\sigma}'}(y_{j'}) G^{z - \underline{\sigma}\,\underline{\alpha}}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}};$$

$$\Psi^z = \sum_{\underline{\sigma}} \psi^z_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in D(H)\,; z \in \rho(H_\gamma) \Big\}\,.$$

with action on its domain given by

$$(H_\gamma - z)\Psi = (H - z)\Psi^z\,; \qquad z \in \rho(H_\gamma)\,.$$

They are self-adjoint Hamiltonians describing a particle exchanging energy with the spin array.[1]

[1] C. Cacciapuoti, R. Carlone, R.F. *Spin-dependent point potentials in one and three dimensions* J. Phys. A: Math. Theor. **40** 249 (2007)

Spin-dependent point interactions

Boundary conditions satisfied by functions in the domain

$$A_{j\underline{\sigma},j'\underline{\sigma}'}q_{j'\underline{\sigma}'} = f_{j\underline{\sigma}}$$

where

$$q_{j\underline{\sigma}} = \lim_{|\mathbf{x}-\mathbf{y_j}|\to 0} 4\pi\,|\mathbf{x}-\mathbf{y_j}|\psi_{\underline{\sigma}}(\mathbf{x}) \quad f_{j\underline{\sigma}} = \lim_{|\mathbf{x}-\mathbf{y_j}|\to 0}\left[\psi_{\underline{\sigma}}(\mathbf{x}) - \frac{q_{j\underline{\sigma}}}{4\pi\,|\mathbf{x}-\mathbf{y_j}|}\right],$$

$$\begin{aligned}
A_{j\underline{\sigma},j'\underline{\sigma}'} &= 0 \qquad \forall j \neq j' \\
A_{j\underline{\sigma},j\underline{\sigma}'} &= 0 \qquad \text{if for some } \; k \neq j,\ \sigma_k \neq \sigma'_k \\
A_{j\underline{\sigma},j\underline{\sigma}'} &= a_{j\sigma_j,j\sigma'_j}\,; \quad \text{otherwise} \\
a_{j\sigma_j,j\sigma'_j} &= \beta\,\delta_{\sigma_j,\sigma'_j} + \sigma_j i\rho(1-\delta_{\sigma_j,\sigma'_j}) \quad \text{with } \beta,\rho \in \mathbb{R}
\end{aligned}$$

The resolvent of H_γ $R_\gamma(z) = (H_\gamma - z)^{-1}$, is the finite rank perturbation of $R_0(z)$ given by

$$R_\gamma^{\underline{\sigma}}(z) = R^{\underline{\sigma}}(z) + \sum_{j,j'\underline{\sigma}'}\left((\Gamma_\gamma(z))^{-1}\right)_{j\underline{\sigma},j'\underline{\sigma}'}$$

$$|G^{\bar{z}-\underline{\alpha}\cdot\underline{\sigma}'}(\cdot - y_j)\otimes\mathcal{X}_{\underline{\sigma}'}\rangle\langle G^{z-\underline{\alpha}\cdot\underline{\sigma}'}(\cdot - y_j)\otimes\mathcal{X}_{\underline{\sigma}'}| \quad z \in \rho(H_\gamma)$$

The explicit form of the generalized eigenfunctions for high particle energy is

$$\Phi^{\underline{\sigma}}_{\underline{\gamma}}(\lambda,\omega) = \frac{(\lambda-\underline{\alpha}\,\underline{\sigma})^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}}\left[e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega\,\cdot}\otimes\mathcal{X}_{\underline{\sigma}}+\right.$$

$$\left.+\sum_{j',\underline{\sigma}',j}(\Gamma_{\gamma}(\lambda))^{-1}_{j'\underline{\sigma}',j\underline{\sigma}}e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}}\omega y_j}\frac{e^{-i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}'}|\cdot-y_{j'}|}}{4\pi|\cdot-y_{j'}|}\otimes\mathcal{X}_{\underline{\sigma}'}\right];$$

$\lambda \geq \underline{\sigma}\,\underline{\alpha}\,,\ \lambda \geq \underline{\sigma}'\,\underline{\alpha}\,,$

The generalization to higher spin values is straightforward.
In particular the quantum system made of a particle interacting with a point "atom" having a finite number of energy states $j\alpha$ for $j=1,2,\dots,M$ can show a complex spectral structure.

Some particular features in the definition of H^A are noteworthy:
- The Hamiltonians one obtains for $\rho = 0$ do not show any term indicating interaction between particle and spins. They correspond to point potential Hamiltonians for the particle, together with free evolution of the spins. Among the self-adjoint extensions of H there are Hamiltonians where β is taken spin-dependent ($\beta(\underline{\sigma})$). The latter were the Hamiltonians used to analyze neutron scattering by (fixed) nuclei.
- ρ is the coupling constant of the particle-spin interaction. If ρ is different from zero, the particle, in addition to the zero-range interaction with the points, can exchange energy with the spins.
- The spectrum of H^A can have a very rich structure. In particular, several eigenstates embedded in the continuum when $\rho = 0$ turn into resonances when $\rho \neq 0$ as a consequence of the interaction particle-spin.

Modelling the interaction of a particle with a zero-dimensional field: the particle interacts via point forces with a quantum system with a finite number of eigenstates.

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Let $H_{\beta,0}$ the multi-channel point interaction hamiltonian with all the strength parameters equal to β and $\rho = 0$. The spectrum of $H_{\beta,0}$ is obtained gluing together the spectra in each channel

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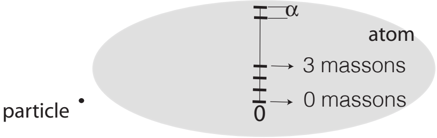

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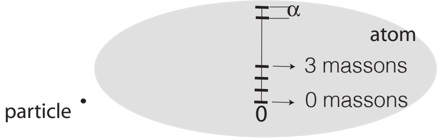

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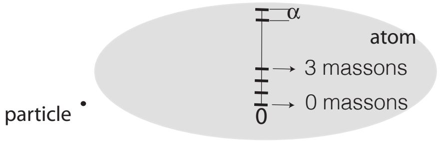

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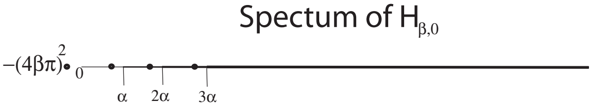

Let $H_{\beta,\rho}$ the multi-channel point interaction hamiltonian with all the strength parameters equal to β and $\rho \neq 0$.
Similarly to what happens in the case of an electron in an hydrogen atom when the interaction with the quantum electromagnetic field is taken into account, all the eigenstates embedded in the continuous spectrum turn into resonances whereas the ground state moves slightly but still remaining on the real axis.
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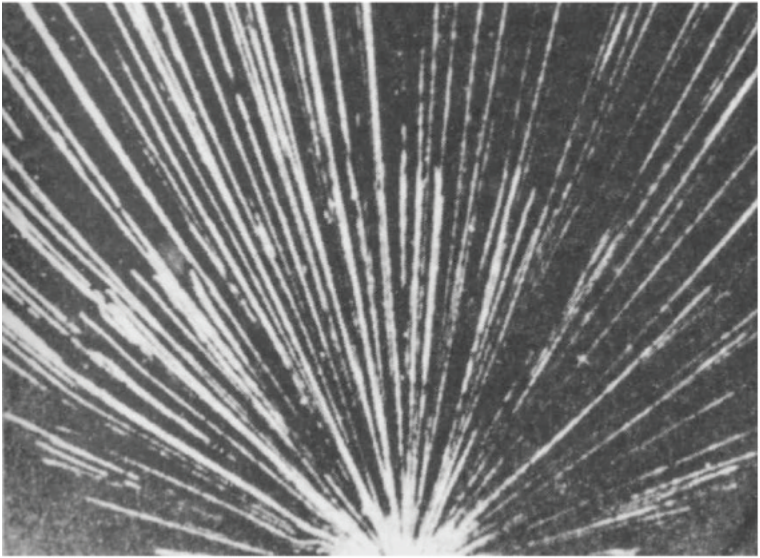
To test the scheme of a measurement process is necessary to analyze realistic models of quantum dynamics, describing the evolution of a microscopic and a macroscopic system in interaction. The goal is to prove that *"the objectification of a microevent is realized through different pointer states"* K. Hepp (1972).
A paradigmatic example is the cloud chamber:

- an extremely energetic alpha-particle (the microscopic system)
- evolves in an environment of atoms (the macro-system).

Ionization induces local formation of drops in a super-saturated vapor:

- each possible direction of the alpha-particle momentum (the micro event)
- is realized as a straight line of drops (the pointer state)

Solvable models of quantum environments



Modelling the cloud chamber with spin dependent point interactions.
Consider spins localized on a sphere of radius L centered in the origin. Two arrays, of $N/2$ spins each, are distributed around the opposite ends of a sphere diameter. Each group fills a region whose linear dimensions are much smaller than the sphere radius.
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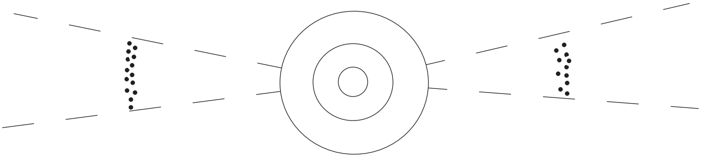

Consider an initial state Ψ_0 of the entire system corresponding to a configuration of spins $\underline{\sigma}_0$ with all spins down and a spherical wave (of high average energy) for the particle

$$\psi_0(x) = \psi_0(|x|) = \frac{1}{\pi^{3/4}\sqrt{\gamma}\left(1 - e^{-P_0^2\gamma^2}\right)^{1/2}} \frac{e^{-\frac{|x|^2}{2\gamma^2}}}{|x|} \sin\left(P_0|x|\right)$$

and prove that the long time behavior of the environment state is the incoherent sum of states where spins in a small cone around a particular direction flipped to the up state

The probability amplitude of an asymptotic transition to a state with a final configuration of spins $\underline{\sigma}_f$ is obtained analyzing first the initial state in terms of the generalized eigenfunctions

$$\left\langle \varphi^{\underline{\sigma}_f}_{\beta,\rho}(\lambda,\boldsymbol{\omega}),\Psi^0\right\rangle = \frac{(\lambda-\underline{\alpha}\,\underline{\sigma}_f)^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}}\sum_{j',j}(\Gamma_\gamma(\lambda))^{-1}_{j\underline{\sigma}_f,j'\underline{\sigma}_0}e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}_f}\omega y_j}$$
$$\left(\frac{e^{-i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}_0}|\cdot-y_{j'}|}}{4\pi|\cdot-y_{j'}|}\,,\,\psi_0\right)$$

where the scalar product is explicitly computable

The second step is to investigate the asymptotic free evolution of the modified initial state obtained from Möller's operator

$$\left(\Psi^0_+\right)_{\underline{\sigma}_f}(\mathbf{x}) := \left(\Omega_+^{-1}\Psi^0\right)_{\underline{\sigma}_f}(\mathbf{x}) = \tag{2.5}$$
$$= \int_{\boldsymbol{\alpha}\cdot\underline{\sigma}_f}^{\infty} d\lambda \int_{S^2} d\boldsymbol{\omega}\, \phi_{\underline{\sigma}_f}(\mathbf{x},\lambda,\boldsymbol{\omega}) \left\langle \varphi^{\underline{\sigma}_f}_{\beta,\rho}(\lambda,\boldsymbol{\omega}), \Psi^0 \right\rangle$$

All the integrations are (almost) explicitly computable to give the final result expressed like a sum of waves of wavelength close to the one of the initial wave packet of the alpha particle.

The result

$$\begin{aligned}\left(\Psi^0_+\right)_{\underline{\sigma}_f}(\mathbf{x}) &= \sum_{j,j'} \Theta_{j\,j'}(P_0,\Delta E)\,\frac{4\,\pi}{\sqrt{P_0^2-\Delta E}}\,\frac{\sin\,\sqrt{P_0^2-\Delta E}\,|\mathbf{x}-\mathbf{y}_j|}{|\mathbf{x}-\mathbf{y}_j|}\\ &= \frac{1}{2^{1/2}\sqrt{\gamma}\;\pi^{7/4}}\sum_j\left[\sum_{j'}\left(\Gamma_{\beta,\rho}(P_0^2-N\alpha)\right)^{-1}_{j\,\underline{\sigma}_f,j'\,\underline{\sigma}_0}\right]\\ &\quad \frac{e^{-iP_0L}}{L}\,\frac{\sin\,\sqrt{P_0^2-\Delta E}\,|\mathbf{x}-\mathbf{y}_j|}{|\mathbf{x}-\mathbf{y}_j|}\end{aligned}$$

appears as the sum of spherical waves originating from each point scatterer with an initial phase $e^{i\sqrt{\lambda-\underline{\alpha}\,\underline{\sigma}_f}\omega y_j}$ depending on the point position

The final result reads:
The probability that the final state corresponds to any configuration of spins with a large number M of spins up is maximal if the M spins belong to the same group. Moreover, if the M spins all belong to one group, the probability that the momentum direction of the particle lies outside the cone connecting the origin to the group is negligible. [2]

[2] R.Figari,A. Teta *Quantum Dynamics of a Particle in a Tracking Chamber* Spinger 2014

The result is a consequence of two concurrent properties concerning the sum of spherical wave.
1) the Huyghens' principle: due to the assumptions on the linear size of the regions where points are placed and of the wavelength, the spherical waves interfere constructively only in the spatial regions reached by the "light rays" from the origin through the spins (in a regime of absence of diffraction).
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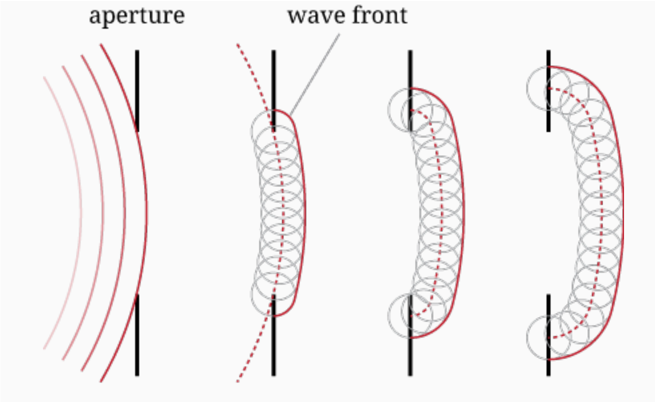

The amplitudes relative to transitions to a final spin configuration with a large number of flips in both regions occupied by the spins is smaller than those where the same number of flips appertains to only one group.

Open Problems

- Non linear spin dependent point interactions: transition parameters ρ depending on the value of the wave function on the scatterers
- more general initial conditions of the microscopic system and presence of external fields acting on the microscopic system
- environments modeled with self-interacting fields (e.g., spins ferromagnetically interacting among them), initially in a genuine meta-stable state. The non-linear self-interaction would enhance the response of the environment, which might show macroscopic modifications in finite time.