Effective dynamics for shrinking waveguides

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Mathematical challenges of zero-range Physics:
rigorous results and open problems

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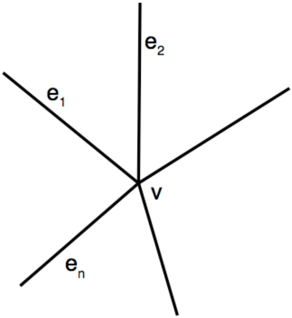
The star-graph



Hilbert space: $\mathbb{H}_{\mathcal{G}} := \bigoplus_{j=1}^{N} L^2((0,\infty), ds_j)$

$\psi \in \mathbb{H}_{\mathcal{G}}$ is a vector with N components

$\psi = (\psi_1, ..., \psi_N)$

$\psi_j \in L^2((0,\infty), ds_j)$

With scalar product

$\langle \psi, \phi \rangle = \sum_{j=1}^{N} (\psi_j, \phi_j)_{L^2((0,\infty))}$

Laplacian on $\mathbb{H}_\mathcal{G}$ (see also, e.g., Kostrykin and Schrader '99, Harmer '00)

Let Π be an orthogonal projection acting in $\mathbb{C}^N$

Let Θ be a selfadjoint operator in $\operatorname{Ran}(\Pi^\perp) \subseteq \mathbb{C}^N$, with $\Pi^\perp = 1 - \Pi$.

Define the operator $-\Delta_\mathcal{G}^{\Pi,\Theta}$:

$$-\Delta_\mathcal{G}^{\Pi,\Theta}\psi = (-\psi_1'', ..., -\psi_N'')$$

$$\begin{aligned} D(-\Delta_\mathcal{G}^{\Pi,\Theta}) = \big\{\psi = (\psi_1, ..., \psi_N) : \; & \psi_j \in H^2((0,\infty)) \\ & \Pi\psi(0) = 0 \\ & \Pi^\perp\psi'(0) + \Theta\Pi^\perp\psi(0) = 0\big\} \end{aligned}$$

with

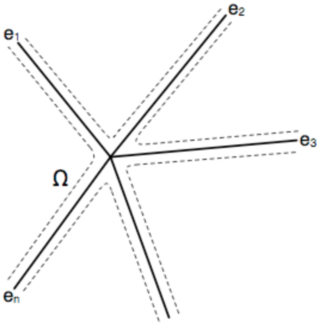
$$\psi(0) = (\psi_1(0), ..., \psi_N(0)) \quad \text{and} \qquad \psi'(0) = (\psi_1'(0), ..., \psi_N'(0))$$

Remark:

$$0 = \langle\psi, -\Delta_\mathcal{G}^{\Pi,\Theta}\phi\rangle - \langle -\Delta_\mathcal{G}^{\Pi,\Theta}\psi, \phi\rangle = (\psi(0), \phi'(0))_{\mathbb{C}^N} - (\psi'(0), \phi(0))_{\mathbb{C}^N}$$

Too many free parameters in the coupling conditions in the vertex!

The coupling should be understood in terms of free parameters models!!



$$\Pi\psi(0) = 0 \qquad \Pi^{\perp}\psi'(0) + \Theta\Pi^{\perp}\psi(0) = 0$$

Depending on a spectral property of the junction, we will obtain the following two operators on the graph:

- The decoupling (or Dirichlet) Laplacian $\mathbf{h}^{\mathbf{dec}}$: $\psi(0) = 0$, $\Pi = 1$
- The weighted Kirchhoff Laplacian $\mathbf{h}^{\beta}$: take $\beta_1, ..., \beta_N$ with $\beta_j \in \mathbb{C}$

$$\beta_j\psi_i(0) = \beta_i\psi_j(0)$$
$$\sum_{i=1}^{N} \bar{\beta}_i\psi_i'(0) = 0$$

$\Pi = 1 - \Lambda$ and $\Theta = 0$

$$(\Lambda)_{ij} = \frac{\beta_i\bar{\beta}_j}{\sum_{k=1}^{N}|\beta_k|^2}\,, \quad i,j = 1, ..., N\,,$$

When all the constants β_j are equal this defines the so called Kirchhoff (or standard) Laplacian and the boundary conditions read $\psi_1(0) = ... = \psi_N(0)$ and $\sum_{i=1}^{N}\psi_i'(0) = 0$.

For any $z \in \mathbb{C}\backslash\mathbb{R}$ denote by $\mathbf{r}^{\mathbf{dec}}(z)$ and $\mathbf{r}^{\beta}(z)$ the resolvents of $\mathbf{h}^{\mathbf{dec}}$ and $\mathbf{h}^{\beta}$

$$\mathbf{r}^{\mathbf{dec}}(z) = (\mathbf{h}^{\mathbf{dec}} - z)^{-1} \qquad \mathbf{r}^{\beta}(z) = (\mathbf{h}^{\beta} - z)^{-1}$$

For any vector $f \in \mathbb{H}_{\mathcal{G}}$

$$\mathbf{r}^{\mathbf{dec}}(z)(f_1, ..., f_N) = (r_0(z)f_1, ..., r_0(z)f_N)\,,$$

with

$$\big(r_0(z)f\big)(s) := \int_0^\infty \left(-\frac{e^{i\sqrt{z}|s-s'|}}{2i\sqrt{z}} + \frac{e^{i\sqrt{z}s}e^{i\sqrt{z}s'}}{2i\sqrt{z}} \right) f(s')ds'$$

Then

$$\mathbf{r}^{\beta}(z)(f_1, ..., f_N) = (r_0(z)f_1 + q_1 e^{i\sqrt{z}\,\cdot}, ..., r_0(z)f_N + q_N e^{i\sqrt{z}\,\cdot})\,,$$

with

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \frac{i\Lambda}{\sqrt{z}} \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} \;\; ; \quad p_j = \big(r_0(z)f_j\big)'(0)$$

Note that

$$\left(-\frac{d^2}{ds^2} - z\right) r_0(z)f = f \qquad \left(-\frac{d^2}{ds^2} - z\right) q_j e^{i\sqrt{z}s} = 0$$

1d approximating problem for $N = 2$ (see also Golovaty and Hryniv '10, Exner and Manko '13):

Consider the Hamiltonian $\mathbf{h}_\varepsilon : L^2((0,\infty)) \oplus L^2((0,\infty)) \oplus L^2((-1,1), \varepsilon ds)$

$$\mathbf{h}_\varepsilon \psi = \left(-\psi_1'', -\psi_2'', \frac{1}{\varepsilon^2} \left(-\psi_v'' + V\psi_v \right) \right)$$

$\psi \in D(\mathbf{h}_\varepsilon)$ satisfies the coupling conditions

$$\psi_1(0) = \psi_v(-1) \qquad \psi_2(0) = \psi_v(1)$$

$$\psi_1'(0) = -\frac{1}{\varepsilon}\psi_v'(-1) \qquad \psi_2'(0) = \frac{1}{\varepsilon}\psi_v'(1)$$

For any

$$f = (f_1, f_2, 0) \qquad f_j \in L^2((0,\infty))$$

we look for

$$\psi_\varepsilon = (\mathbf{h}_\varepsilon - z)^{-1} f$$

We denote by h_v the "auxiliary" Hamiltonian in $L^2((-1,1))$

$$D(h_v) := \{\psi_v \in H^2((-1,1)) |\, \psi_v'(\pm 1) = 0\}$$

$$h_v := -\frac{d^2}{ds^2} + V$$

For $n = 1, 2, 3, ...$ we denote by ϕ_n the (real, orthonormal) eigenfunctions of h_v and by λ_n the corresponding eigenvalues arranged in increasing order

For any $z \in \mathbb{C} \backslash \mathbb{R}$ we denote by $r_v(z)$ the resolvent of h_v,

$$r_v(z; s, s') := (h_v - z)^{-1}(s, s') = \sum_n \frac{\phi_n(s)\phi_n(s')}{\lambda_n - z}$$

Proposition

For any

$$f = (f_1, f_2, 0) \qquad f_j \in L^2((0,\infty))$$

One has that $\psi_\varepsilon = (\mathbf{h}_\varepsilon - z)^{-1} f$ is given by

$$\psi_{1,\varepsilon}(s) = \big(r_0(z) f_1\big)(s) + q_{1,\varepsilon} e^{i\sqrt{z}s}$$
$$\psi_{2,\varepsilon}(s) = \big(r_0(z) f_2\big)(s) + q_{2,\varepsilon} e^{i\sqrt{z}s}$$
$$\psi_{v,\varepsilon}(s) = \varepsilon\big[\xi_{1,\varepsilon} r_v(\varepsilon^2 z; s, -1) + \xi_{2,\varepsilon} r_v(\varepsilon^2 z; s, 1)\big]$$

With the constants $q_{1,\varepsilon}$, $q_{2,\varepsilon}$, $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ fixed by the relations

$$\xi_{1,\varepsilon} = (p_1 + i\sqrt{z} q_{1,\varepsilon}) \; ; \quad p_1 = \big(r_0(z) f_1\big)'(0)$$
$$\xi_{2,\varepsilon} = (p_2 + i\sqrt{z} q_{2,\varepsilon}) \; ; \quad p_2 = \big(r_0(z) f_2\big)'(0)$$

and

$$\begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \begin{pmatrix} \varepsilon r_v(\varepsilon^2 z; -1, -1) & \varepsilon r_v(\varepsilon^2 z; -1, 1) \\ \varepsilon r_v(\varepsilon^2 z; 1, -1) & \varepsilon r_v(\varepsilon^2 z; 1, 1) \end{pmatrix} \begin{pmatrix} \xi_{1,\varepsilon} \\ \xi_{2,\varepsilon} \end{pmatrix}$$

We distinguish two cases:

Case 1. Zero is not an eigenvalue of the Hamiltonian h_v.
Then

$$r_v(\varepsilon^2 z; \pm1, \pm1) = \sum_n \frac{\phi_n(\pm1)\phi_n(\pm1)}{\lambda_n - \varepsilon^2 z} = \mathcal{O}(1)$$

Case 2. Zero is an eigenvalue of the Hamiltonian h_v. Denote by ϕ^* the eigenfunction corresponding to the eigenvalue zero. One has $\phi^{*\prime}(\pm1) = 0$, moreover the constants

$$\beta_1 := \phi^*(-1)\,; \quad \beta_2 := \phi^*(1)\,.$$

are real and we assume that they are not both equal to zero.
Then

$$r_v(\varepsilon^2 z; -1, -1) = \sum_n \frac{\phi_n(-1)\phi_n(-1)}{\lambda_n - \varepsilon^2 z} = -\frac{\beta_1\beta_1}{\varepsilon^2 z} + \mathcal{O}(1)$$

$$r_v(\varepsilon^2 z; 1, 1) = \sum_n \frac{\phi_n(1)\phi_n(1)}{\lambda_n - \varepsilon^2 z} = -\frac{\beta_2\beta_2}{\varepsilon^2 z} + \mathcal{O}(1)$$

$$r_v(\varepsilon^2 z; -1, 1) = \sum_n \frac{\phi_n(-1)\phi_n(1)}{\lambda_n - \varepsilon^2 z} = -\frac{\beta_1\beta_2}{\varepsilon^2 z} + \mathcal{O}(1)$$

$$r_v(\varepsilon^2 z; 1, -1) = \sum_n \frac{\phi_n(1)\phi_n(-1)}{\lambda_n - \varepsilon^2 z} = -\frac{\beta_2\beta_1}{\varepsilon^2 z} + \mathcal{O}(1)$$

Theorem

Case 1.

$$\begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \mathcal{O}(\varepsilon)$$

Case 2.

$$\begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \frac{i\Lambda}{\sqrt{z}} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \mathcal{O}(\varepsilon)$$

where we denoted by Λ the projection

$$\Lambda = \frac{1}{\beta_1^2 + \beta_2^2} \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & \beta_2^2 \end{pmatrix}$$

Corollary

Let $\psi_\varepsilon = (\mathbf{h}_\varepsilon - z)^{-1} f$. Then

Case 1.

$$\|(\psi_{1,\varepsilon}, \psi_{2,\varepsilon}) - \mathbf{r}^{\mathbf{dec}}(z)(f_1, f_2)\|_{\mathbb{H}_\mathcal{G}} \to 0$$

Case 2.

$$\|(\psi_{1,\varepsilon}, \psi_{2,\varepsilon}) - \mathbf{r}^{\beta}(z)(f_1, f_2)\|_{\mathbb{H}_\mathcal{G}} \to 0$$

We consider two straight tubes (the edges) of width δ connected by a smooth junction (the vertex) of uniform width δ and length of order ε

- The "edges": $E_{1,\delta}$ and $E_{2,\delta}$ to which is associated the Hilbert space

$$L^2((0,\infty)\times(0,1), dsdu)$$

$s \in (0,\infty)$ and $u \in (0,1)$

- The "vertex": $V_{\delta,\varepsilon}$

Take a curve $C : (-1,1) \to \mathbb{R}^2$

$$C(s) := (\gamma_1(s), \gamma_2(s)) \qquad {\gamma_1'}^2(s) + {\gamma_2'}^2(s) = 1$$

$$V_{\delta/\varepsilon} = \Big\{(x,y) \in \mathbb{R}^2 \text{ s.t. } (x,y) = (\gamma_1(s), \gamma_2(s)) + u\delta/\varepsilon \hat{n}(s)$$
$$\text{with } s \in (-1,1), u \in (0,1)\Big\}$$

Then $V_{\delta,\varepsilon} = \varepsilon V_{\delta/\varepsilon}$. In this way $\theta_\varepsilon = \theta$

Moreover the associated Hilbert space is

$$L^2((-1,1) \times (0,1), \varepsilon \rho_{\delta/\varepsilon} ds du)$$

The function $\rho_{\delta/\varepsilon} = \rho_{\delta/\varepsilon}(s,u)$ is defined by

$$\rho_{\delta/\varepsilon}(s,u) = 1 + u\delta/\varepsilon \gamma(s) \,,$$

γ is the signed curvature

$$\gamma(s) := \gamma_2'(s)\gamma_1''(s) - \gamma_1'(s)\gamma_2''(s)$$

The waveguide is obtained by identifying the boundary of $E_{1,\delta}$ corresponding to $s = 0$ with the boundary of $V_{\delta,\varepsilon}$ corresponding to $s = -1$ and the boundary of $E_{2,\delta}$ corresponding to $s = 0$ with the boundary of $V_{\delta,\varepsilon}$ corresponding to $s = 1$.

We denote by $\mathbb{H}_{\delta,\varepsilon}$ the complex Hilbert space

$$\mathbb{H}_{\delta,\varepsilon} := L^2((0,\infty)\times(0,1))\oplus L^2((0,\infty)\times(0,1))\oplus L^2((-1,1)\times(0,1),\varepsilon\rho_{\delta/\varepsilon}dsdu)$$

In $\mathbb{H}_{\delta,\varepsilon}$ we define the quadratic form $Q_{\delta,\varepsilon}$

$$\begin{aligned} Q_{\delta,\varepsilon}[\Phi,\Psi] := \sum_{k=1,2}\int_0^\infty\int_0^1 \left[\overline{\frac{\partial\Phi_k}{\partial s}}\frac{\partial\Psi_k}{\partial s} + \frac{1}{\delta^2}\overline{\frac{\partial\Phi_k}{\partial u}}\frac{\partial\Psi_k}{\partial u}\right]dsdu \\ + \int_{-1}^1\int_0^1\left[\frac{1}{\varepsilon^2\rho^2_{\delta/\varepsilon}}\overline{\frac{\partial\Phi_v}{\partial s}}\frac{\partial\Psi_v}{\partial s} + \frac{1}{\delta^2}\overline{\frac{\partial\Phi_v}{\partial u}}\frac{\partial\Psi_v}{\partial u}\right]\varepsilon\rho_{\delta/\varepsilon}dsdu\,. \end{aligned}$$

Let $\mathring{C}^\infty$ be the set

$$\begin{aligned} \mathring{C}^\infty := \{\Psi = (\Psi_1,\Psi_2,\Psi_v) |\, \Psi_1,\Psi_2 \in C_0^\infty(\overline{E}),\ \Psi_v\in C^\infty(V)\,; \\ \Psi\big|_{u=0,1} = 0\,; \\ \left[\partial_s^k\Psi_1\right](0,u) = \left[(-\varepsilon)^{-k}\partial_s^k\Psi_v\right](-1,u) \\ \left[\partial_s^k\Psi_2\right](0,u) = \left[\varepsilon^{-k}\partial_s^k\Psi_v\right](1,u),\ \forall k\in\mathbb{N}_0\}\,, \end{aligned}$$

We denote by $-\Delta^D_{\delta,\varepsilon}$ the unique selfadjoint operator in $\mathbb{H}_{\delta,\varepsilon}$ associated to the quadratic form $Q_{\delta,\varepsilon}$

The Operator $-\Delta^D_{\delta,\varepsilon}$ is unitarily equivalent to

$$H_{\delta,\varepsilon} : L^2((0,\infty)\times(0,1)) \oplus L^2((0,\infty)\times(0,1)) \oplus L^2((-1,1)\times(0,1), \varepsilon ds du)$$

defined by

$$H_{\delta,\varepsilon}(\psi_1,\psi_2,\psi_v) = \left(\left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\delta^2}\frac{\partial^2}{\partial u^2}\right]\psi_1, \left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\delta^2}\frac{\partial^2}{\partial u^2}\right]\psi_2, \frac{1}{\varepsilon^2}L_{\delta/\varepsilon}\psi_v\right),$$

where $L_{\delta/\varepsilon}$ has the "expansion"

$$L_{\delta/\varepsilon} \simeq -\frac{\partial^2}{\partial s^2} - \frac{\gamma^2(s)}{4} - \frac{1}{(\delta/\varepsilon)^2}\frac{\partial^2}{\partial u^2} + \mathcal{O}(\delta/\varepsilon)$$

Spectrum

$$\sigma_{cont}(H_{\delta,\varepsilon}) = \left[\frac{\pi^2}{\delta^2}, \infty\right)$$

Denote by $\chi(u)$ the "transverse" eigenfunction

$$\chi(u) := \sqrt{2}\sin(\pi u)$$

We note that

$$\chi(0) = \chi(1) = 0 \quad \text{and} \qquad -\frac{1}{\delta^2}\frac{d^2}{du^2}\chi = \frac{\pi^2}{\delta^2}\chi$$

For any vector $\Xi = (f_1\chi, f_2\chi, 0)$ with $f_1, f_2 \in L^2((0,\infty))$ we look for an approximate solution of the resolvent equation

$$\Phi = \left(H_{\delta,\varepsilon} - \frac{\pi^2}{\delta^2} - z\right)^{-1}\Xi$$

Definition

For any vector $\Xi = (f_1\chi, f_2\chi, 0)$ with $f_1, f_2 \in L^2((0,\infty))$ we denote by Ψ the vector $\Psi_\varepsilon = (\Psi_{1,\varepsilon}, \Psi_{2,\varepsilon}, \Psi_{v,\varepsilon})$, with

$$\begin{aligned}
\Psi_{1,\varepsilon}(s,u) &= \psi_{1,\varepsilon}(s)\chi(u) \\
\Psi_{2,\varepsilon}(s,u) &= \psi_{2,\varepsilon}(s)\chi(u) \\
\Psi_{v,\varepsilon}(s,u) &= \psi_{v,\varepsilon}(s)\chi(u)\,,
\end{aligned}$$

Where $\psi_\varepsilon = (\psi_{1,\varepsilon}, \psi_{2,\varepsilon}, \psi_{v,\varepsilon})$ was defined before.

The "auxiliary" Hamiltonian is

$$h_v := -\frac{d^2}{ds^2} - \frac{\gamma^2}{4}$$

$$D(h_v) := \{\psi_v \in H^2((-1,1)) \,|\, \psi_v'(\pm 1) = 0\}$$

Theorem

For any $\Xi = (f_1\chi, f_2\chi, 0)$ *with* $f_1, f_2 \in L^2((0,\infty))$, $\Psi_\varepsilon \in D(H_{\delta,\varepsilon})$, *moreover for all* $z \in \mathbb{C}\backslash\mathbb{R}$ *the following estimates hold true:*

Case 1.

$$\left\|\Psi_\varepsilon - \left(H_{\delta,\varepsilon} - \frac{\pi^2}{\delta^2} - z\right)^{-1}\Xi\right\|_{\mathbb{H}_\varepsilon} \leqslant c\frac{\delta}{\varepsilon^{3/2}}\|\Xi\|_{\mathbb{H}_\varepsilon}\,;$$

Case 2.

$$\left\|\Psi_\varepsilon - \left(H_{\delta,\varepsilon} - \frac{\pi^2}{\delta^2} - z\right)^{-1}\Xi\right\|_{\mathbb{H}_\varepsilon} \leqslant c\left(\frac{\delta}{\varepsilon^{3/2}} + \frac{\delta}{\varepsilon^{5/2}}\right)\|\Xi\|_{\mathbb{H}_\varepsilon}$$

where c *is a constant which does not depend on* ε, f_1, f_2.

Theorem

Take $\delta = \delta(\varepsilon) \ll \varepsilon$. *Define the function*

$$\Phi_\varepsilon = (\Phi_{1,\varepsilon}, \Phi_{2,\varepsilon}, \Phi_{v,\varepsilon}) = \left(H_{\delta,\varepsilon} - \frac{\pi^2}{\delta^2} - z \right)^{-1} (f_1\chi, f_2\chi, 0)$$

Then the projection of Φ_ε *on the edges of the graph*

$$\phi_{1,\varepsilon}(s) = (\chi, \Phi_{1,\varepsilon}(s,\cdot))_{L^2((-1,1))} \qquad \phi_{2,\varepsilon}(s) = (\chi, \Phi_{2,\varepsilon}(s,\cdot))_{L^2((-1,1))}$$

Then, as $\varepsilon \to 0$,

Case 1.

$$\|(\phi_{1,\varepsilon}, \phi_{2,\varepsilon}) - \mathbf{r}^{\mathbf{dec}}(z)(f_1, f_2)\|_{\mathbb{H}_{\mathcal{G}}} \to 0$$

Case 2.

$$\|(\phi_{1,\varepsilon}, \phi_{2,\varepsilon}) - \mathbf{r}^{\beta}(z)(f_1, f_2)\|_{\mathbb{H}_{\mathcal{G}}} \to 0$$

Define the auxiliary Hamiltonian as the operator $-\Delta^{DN}_{\delta/\varepsilon}$ in the junction of width δ/ε

$$L^2((-1,1)\times(0,1), \rho_{\delta/\varepsilon} ds du)$$

where functions in $D(-\Delta^{DN}_{\delta/\varepsilon})$ satisfy the mixed boundary conditions

$$\Psi\Big|_{u=0,1} = 0\,; \quad \frac{\partial}{\partial s}\Psi\Big|_{s=\pm 1} = 0$$

The *Case 2* is associated to the existence of $\Phi^*_{\delta/\varepsilon}$ s.t.

$$\Big(-\Delta^{DN}_{\delta/\varepsilon} - \frac{\pi^2}{(\delta/\varepsilon)^2}\Big)\Phi^*_{\delta/\varepsilon} = \lambda_{\delta/\varepsilon}\Phi^*_{\delta/\varepsilon}$$

with

$$\lambda_{\delta/\varepsilon} = \mathcal{O}(\delta/\varepsilon)$$

and

$$\Phi^*_{\delta/\varepsilon}\Big|_{s=-1} \to \beta_1\chi \qquad \Phi^*_{\delta/\varepsilon}\Big|_{s=1} \to \beta_2\chi$$

Then study the limit as $\varepsilon \to 0$ of

$$\Big(-\Delta^{DN}_{\delta/\varepsilon} - \frac{\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z\Big)^{-1}\Big|_{s,s'=\pm 1}$$

Remark:

Neumann conditions on the external boundary of the waveguide

[Freidlin and Wentzel ’93, Raugel ’95, Kosugi ’00, Saitō ’01, Kuchment and Zeng ’01, Rubinstein and Schatzman ’01, Exner and Post ’05 ’07 ’09 ’12, Post ’06, Bonciocat ’08, Kuroda ’11]

No rescaling of the energy is needed, $\sigma(-\Delta^N_{\delta,\varepsilon}) = [0, \infty)$

The limit operator on the graph is characterized by Kirchhoff conditions in the vertex. In general there is no decoupling case (*Case 1*)

The auxiliary Hamiltonian $-\Delta^{NN}_{\delta/\varepsilon}$ has a zero eigenvalue (also for $\delta = \varepsilon$). The corresponding eigenfuction is the constant function. This gives $\beta_1 = \beta_2$.

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Consider the Hamiltonian $\mathbf{h}_\varepsilon : L^2((0,\infty)) \oplus L^2((0,\infty)) \oplus L^2((-1,1), \varepsilon ds)$

$$\mathbf{h}_\varepsilon \psi = \left(-\psi_1'', -\psi_2'', \frac{1}{\varepsilon^2} \left(-\psi_v'' + (1 + \lambda_1 \varepsilon) V \psi_v \right) \right)$$

$\psi \in D(\mathbf{h}_\varepsilon)$ satisfies the coupling conditions

$$\psi_1(0) = \psi_v(-1) \qquad \psi_2(0) = \psi_v(1)$$

$$\psi_1'(0) = -\frac{1}{\varepsilon} \psi_v'(-1) \qquad \psi_2'(0) = \frac{1}{\varepsilon} \psi_v'(1)$$

For any

$$f = (f_1, f_2, 0) \qquad f_j \in L^2((0,\infty))$$

we look for

$$\psi_\varepsilon = (\mathbf{h}_\varepsilon - z)^{-1} f$$

Then in *Case 2* the limit is $\mathbf{h}^{\beta,\alpha}$ defined by the boundary conditions

$$\beta_2 \psi_1(0) = \beta_1 \psi_2(0) \qquad \forall i \neq j$$

$$\bar{\beta}_1 \psi_1'(0) + \bar{\beta}_2 \psi_2'(0) = \alpha \left(\frac{f_1(0)}{2\beta_1} + \frac{f_2(0)}{2\beta_2} \right)$$

with

$$\alpha = \lambda_1 \int_{-1}^{1} V |\phi_v^*|^2 ds$$