SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS WITH (MANY) DELTA SHELLS

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1. Introduction

   Reduction to study of ∞-many δ-interactions on $\mathbb{R}_+$

2. Properties of such systems of δ-interactions

3. Back to δ-shells

4. Further remarks

1. Introduction

Interest in discussing

$$" -\Delta + \sum_{j=1}^{\infty} \alpha_j \, \delta_{y_j} "$$

$y_j \in Y$, $j \in \mathbb{N}$ "delta centers" $Y \subset \mathbb{R}^d$

$\alpha_j \in \alpha (\subset \mathbb{R})$ "charges"

with delta inter. at ∞-many points goes back to

R. Kronig, L. Penney 1931

($Y = \mathbb{Z}$, α_j indep. of j)

General description as self-adjoint operators in

Grossmann, Høegh-Krohn, Mebkhout '80 ...

Høegh-Krohn, Gesztesy, Holden, A. '88 (book)

'04 (2nd ed., App. P. Exner)

works in $d = 1, 2, 3$

strong resolv. limit from

case $|Y| < \infty$ ...

Case of crystals particularly well studied ..

Also interest in studying

$$H_{R,\alpha} = \text{"} -\Delta + \sum_{j=1}^{\infty} \alpha_j \, \delta(|x| - r_j) \text{"}$$

$R = \{r_j\}_{j \in \mathbb{N}}$, $r_j > 0$, $r_j \uparrow +\infty$

"delta shell model"

Green, Moszkowski '65
nuclear physics

Blinder '78 " +
solid state physics

P. Lloyd '65

Rubio, Garcia-Moliner '67
solid state physics
(band structure in periodic case ...)

Math. rigorous:

Antoine, Gesztesy, Shabani
'87

Shabani et al '88...

Exner Fraas 2007

(radially periodic case)

My talk based on

Aleksey Kostenko
Mark Malamud
Hagen Neidhardt, + A.
2013

Basic idea: connect to point interactions on $\mathbb{R}_+$, via PARTIAL WAVES ANALYSIS let $U : L^2(\mathbb{R}_+, r^{d-1}) \to L^2(\mathbb{R}_+)$

(d = space dimension, $d = 2, 3$)

$$(Uf)(r) := r^{\frac{d-1}{2}} f(r)$$

unitary

$$L^2(\mathbb{R}^d) = L^2(\mathbb{R}_+, r^{d-1} dr) \otimes L^2(S^{d-1})$$

$$= \bigoplus_{l=0}^{\infty} U^{-1} L^2(\mathbb{R}_+) \otimes \mathcal{H}_l$$

$\mathcal{H}_l := \{$eigenspace to l-th EV $\varkappa_l = -l(l+d-2)$ of Δ_{LB} on $L^2(S^{d-1})\}$

$$H_{R,\alpha} = \bigoplus_{l=0}^{\infty} U^{-1} h_{R,\alpha}^{(l)} U \otimes \varkappa_l \mathbb{1}_{\mathcal{H}_l}$$

to be defined

$$h_{R,\alpha}^{(l)} = \text{"} -\frac{d^2}{dr^2} + \frac{l(d)}{r^2} + \sum_{j=1}^{\infty} \alpha_j \delta_{r_j}(r) \text{"}$$

$$\ell(d) := \frac{(d-1)(d-3)}{4} - \varkappa_\ell$$

$h^{(\ell)}_{R,\alpha}$ "Bessel-type operator" in $L^2(\mathbb{R}_+)$

$$h^{(\ell)}_{R,\alpha} = \text{"} \overset{\circ}{h}{}^{(\ell)}_{R,\alpha} + \sum_{j=1}^{\infty} \alpha_j \, \delta_{r_j}(r) \text{"}$$

Rigorously: $h^{(\ell)}_{R,\alpha} = \overline{\overset{\circ}{h}{}^{(\ell)}_{R,\alpha}}$,

with

$$\overset{\circ}{h}{}^{(\ell)}_{R,\alpha} = -\frac{d^2}{dr^2} + \frac{\ell(d)}{r^2}$$

$$D(\overset{\circ}{h}{}^{(\ell)}_{R,\alpha}) = \{ f \in C^\infty(\mathbb{R}_+ \setminus R),$$

$\overset{\circ}{h}{}^{(\ell)}_{R,\alpha} f \in L^2(\mathbb{R}_+)$, $f(r_j+) = f(r_j-)$,

$f'(r_j+) - f'(r_j-) = \alpha_j f(r_j)$,

$$\lim_{r \downarrow 0} r^{\ell(d)} \left((\ell(d)+1) f(r) - r f'(r) \right) = 0$$

if $\ell(d) \in [-\frac{1}{2}, \frac{1}{2})$ $\}$

Let us consider first $d = 3$, ~~[illegible]~~

(~~as in Andrade~~ $\cancel{|Y| < \infty}$ $|Y| = 1$

(finitely many similar ...)

~~Δ~~ Condition only for $l(d) = -\varkappa_l =$

$$h_{R,\alpha}^{(l)} = \overset{\circ}{h}{}_{R,\alpha}^{(l)}$$

$= l(l+1) \in [-\frac{1}{2}, \frac{1}{2})$

$l = 0$ and then

$$\lim_{r \downarrow 0} [f(r) - r f'(r)] = 0$$

yields

$f(0_+) = 0$

$h^{(\ell)}_{R,\alpha}$ 1d model with ∞-many deltas

Study of $H_{R,\alpha}$ now reduced to study of $h^{(\ell)}_{R,\alpha}$

2. STUDY OF $h^{(\ell)}_{R,\alpha}$ (Bessel-type operators with point interactions)

No complication to look at this as special case of

$$\text{"}\boxed{-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}} + \sum_j \alpha_j \delta_{r_j}(r) \boxed{+ q(r)}\text{"}$$

ℓ as parameter $\geq -\frac{1}{2}$

q "potential"

Set $\boxed{} := \overset{\circ}{h}{}^{(\ell)}_{R,\alpha,q}$

with $D(\overset{\circ}{h}{}^{(\ell)}_{R,\alpha,q}) = \{ f \in W^{2,1}_c(\mathbb{R}_+ \setminus R) |$

$$f(r_j+) = f(r_j-)\,,\ f'(r_j+) - f'(r_j-) = \alpha_j\, f(r_j)\,,$$

$$\mathring{h}^{(\ell)}_{R,\alpha,q} f \in L^2(\mathbb{R}_+) \text{ and if } \ell \in [-\tfrac{1}{2}, \tfrac{1}{2})\,,\ \lim \left[r^{\ell} \left((\ell+1) f(r) - r f'(r) \right] = 0 \right\}$$

$\mathring{h}^{(\ell)}_{R,\alpha,q}$ symm., closure $\exists$, called $h^{(\ell)}_{R,\alpha,q}$ by definition

By results of Kostenko + Malamud: study of $h^{(\ell)}_{R,\alpha,q}$ reduced to study of associated Jacobi matrix (quite explicit!) $B_{R,\alpha,q}$ (acting in ℓ^2, with elements depending on model parameters α_j, y_j, q)

Theor: Let $q \in l^\infty(\mathbb{R}_+)$

(Kostenko, Malamud, A 2013) Assume $\sup_k d_k < \infty$

$d_k := r_k - r_{k-1}$

1) $h^{(l)}_{R,\alpha,q}$ self-adjoint resp. lower semibounded (≥ 0) iff

Jacobi $B_{R,\alpha,0}$ self-adjoint resp l.s.b. ($\geq 0, l \geq 0$)

(thus: if certain explicit conditions on α_j, γ_j satisfied)

1)' $h^{(l)}_{R,\alpha,q}$ l.s.b. $\Rightarrow$ $h^{(l)}_{R,\alpha,q}$ s.a.

2) defect indices equal, ≤ 1 (and expressed by those of $B_{R,\alpha,0}$)

3) if $h^{(l)}_{R,\alpha,q}$ self-adjoint then:

$\sigma(h^{(l)}_{R,\alpha,q})$ discrete

$$\Leftrightarrow \begin{cases} d_k \underset{k\to\infty}{\to} 0 \\ \sigma(B_{R,\alpha,q}) \text{ discrete} \end{cases}$$

4) $\sigma_-(h^{(l)}_{R,\alpha,0})$ discrete $\Leftrightarrow$ $\sigma_-(B_{R,\alpha,0})$ discrete

$\sigma_-(h^{(l)}_{R,\alpha,0})$ finite $\Leftarrow$ $\sigma_-(B_{R,\alpha,0})$ finite

$$\kappa_-\left(h^{(\ell)}_{R,\alpha,0}\right) \le \kappa_-\left(B_{R,\alpha,0}\right)$$

\# EVs ≤ 0

Theor (Kostenko, Malamud, A. 2013)

1) Sufficient for $h^{(\ell)}_{R,\alpha,q}$ l. s. b.

are

$$\begin{cases} \text{1a)} \quad \sup_{r>0} \int_r^{r+1} |q_-(t)|\,dt < \infty \qquad q_- = \tfrac{1}{2}[q(t) - |q(t)|] \\ \text{1b)} \quad \sup_{r>0} \sum_{r_k \in [r, r+1]} |\alpha_k^-| < \infty \end{cases}$$

In this case, for $\ell \ge -\frac{1}{2}$:

$$\kappa_-\left(h^{(\ell)}_{R,\alpha,q}\right) < \frac{1}{2\ell+1}\left(\int_0^\infty r|q_-(r)|\,dr + \sum_k |\alpha_k^-|\, r_k\right)$$

2) Let 1a), 1b) be satisfied: then

$\sigma\left(h^{(\ell)}_{R,\alpha,q}\right)$ discrete $\Leftrightarrow$ $\forall \varepsilon > 0$:

$$\int_r^{r+\varepsilon} q(t)\,dt + \sum_{\substack{k \\ r_k \in (r, r+\varepsilon)}} \alpha_k \xrightarrow[r \to +\infty]{} +\infty$$

(Molchanov type estimate)

If moreover

$$\lim_{r \to +0} \int_r^{r+1} |q(t)|\,dt = 0$$

$$\lim_{r \to +\infty} \sum_{k:\, r_k \in [r, r+1]} |\alpha_k| = 0$$

then

$$\sigma_c\left(h^{(\ell)}_{R,\alpha,q}\right) = \mathbb{R}_+ .$$

Rem for $|Y| < \infty$ many more detailed results ...

- Applications to δ-shells model

Th. 1) $H_{R,\alpha}$ s. a. $\Leftrightarrow$ $B_{R,\alpha}$ s. a
l. s. b. l. s. b.

2) If $n_\pm(B_{R,\alpha}) = 1$ then

$$n_\pm(H_{R,\alpha}) = +\infty$$

3) i) If $\sum_k d_k^2 = \infty$ then

$$H_{R,\alpha} \text{ s. a.}$$

ii) If $\sum_k d_k^2 < \infty$, $d_{k-1} d_{k+1} \geq d_k^2 \ \forall k$ and

$$\sum_k d_{k+1} \left| \alpha_k + \frac{1}{d_k} + \frac{1}{d_{k+1}} \right| < \infty$$

then $H_{R,\alpha}$ only symm.

with $n_\pm(H_{R,\alpha}) = +\infty$

Rem. Complements (~ 1):

a) if $H_{R,\alpha}$ l.s.b. then $H_{R,\alpha}$ s.a.

b) $\sup\limits_{r>0} \sum\limits_{r_k\in[r,r+1]} |\alpha_k^-| < \infty$

$\Rightarrow H_{R,\alpha}$ l.s.b.

($\Leftarrow$ $H_{R,\alpha}$ l.s.b., $\alpha_k = \alpha_k^- \ \forall k$)

[x] ($\underset{\text{Exner, Fraas}}{\Rightarrow}$ $\sigma_{ess}(H_{R,\alpha}) = [\inf \sigma_{ess}(h^{(0)}_{R,\alpha}), +\infty)$)

Moreover criterium for discreteness:

c) under b) above if $d_k \xrightarrow[k\to\infty]{} 0$ then:

$\sigma(H_{R,\alpha})$ discrete $\Leftrightarrow$ $\forall \varepsilon > 0$

$$\int_r^{r+\varepsilon} q(t)\,dt + \sum \alpha_k \xrightarrow[r\to+\infty]{} +\infty$$

d) criterium for essential spectrum:
if we have 1 b) in Theor. p. 10
and $\lim\limits_{r\to+\infty} \sum\limits_{r_k\in[r,r+1]} |\alpha_k| = 0$
then

$$\sigma_{ess}(H_{R,\alpha}) = \sigma_{ess}(h^{(\ell)}_{R,\alpha}) = \mathbb{R}_+$$

$\exists$ also results on # boundstates:

e.g.
$$\kappa_-(H_{R,\alpha}) = \sum_{\ell \in \mathbb{N}_0} \dim(\mathcal{H}_\ell)\,\underbrace{\kappa_-(h^{(\ell)}_{R,\alpha})}$$

$\exists$ bounds!

Rem.

For $|Y| < \infty$ many other detailed results, e.g.

- $H_{\alpha,R}$ finitely many boundstates
- approximations in norm resolvent limit by local short-range potentials V, where $\alpha_j \delta_{r_j}$ replaced by
$$\frac{\lambda_j(\varepsilon)}{\varepsilon^2} V\left(\frac{\cdot - r_j}{\varepsilon}\right), \quad \varepsilon > 0$$
$$\alpha_j = \lambda_j'(0) \int_0^\infty V(r)\,dr$$
....
- also models with δ-shells + δ_0 + Coulomb

Antoine, Gesztesy, Høegh-Krohn, Streit,

A '84 ...

- δ'-shell models Shabani et al. '92...