A Lace-Expansion Analysis of Random Spatial Models

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A lace-expansion analysis of random spatial models

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Chapter 1
Introduction

Since its invention in 1985 [31], the *lace expansion* has become a powerful tool for proving mean-field behavior in various spatial stochastic systems, such as the self-avoiding walk, percolation, oriented percolation, the contact process, lattice trees and -animals, and the Ising model. In this thesis we discuss a generalized lace expansion approach that holds for self-avoiding walk, percolation and the Ising model. Our analysis covers the classical nearest-neighbor model as well as various spread-out cases. Particular attention is given to those spread-out models where the underlying step distribution has infinite variance, so-called *long-range* models. Among various other results, we show that a sufficiently long range can reduce the upper critical dimension, above which the system shows mean-field behavior. Later on we discuss the behavior of critical percolation on a high-dimensional torus, and the scaling limit of long-range self-avoiding walk.

We study percolation, self-avoiding walk and the Ising model on the hypercubic lattice $\mathbb{Z}^d$. We consider $\mathbb{Z}^d$ as a complete graph, i.e., the graph with vertex set $\mathbb{Z}^d$ and corresponding edge set $\mathbb{Z}^d \times \mathbb{Z}^d$. We will refer to the edges as *bonds* and to the vertices as *sites*. We assign each (undirected) bond $\{x, y\}$ a weight $D(x-y)$, where D is a probability distribution to be specified in Section 3.1 below. If $D(x-y) = 0$, then we can omit the bond $\{x, y\}$. We will always assume that $D(0) = 0$, hence there are no self-loops to consider. The most prominent example for D is the nearest-neighbor case, where D is given by

$$D(x) = \frac{1}{2d} \mathbb{1}_{\{|x|=1\}}, \qquad x \in \mathbb{Z}^d.$$

However, we do not rule out the possibility that D has unbounded support. Throughout the thesis, we denote by $|\cdot|$ the Euclidean norm on $\mathbb{Z}^d$ and $\mathbb{1}_E$ represents the indicator function of the event E.

We start by introducing the models that we shall consider, i.e., self-avoiding walk, percolation and the Ising model.

1.1 Percolation

Percolation has been proposed as a model of porous media by Broadbent and Hammersley [28, 29], but evolved into a core model of contemporary probability. Despite its relatively simple definition, percolation shows a tremendously rich structure and offers a variety of inspiring and challenging problems with links to various other statistical mechanical models.

Consider the set of *bonds*, which are unordered pairs of lattice sites. We set each bond $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$ *occupied*, independently of all other bonds, with probability $zD(y-x)$ and *vacant* otherwise. Thus for the nearest-neighbor model, each nearest-neighbor bond is occupied with probability $z/(2d)$. The corresponding product measure is denoted by $\mathbb{P}_z$ with corresponding expectation $\mathbb{E}_z$. We require $z \in [0, (\sup_x D(x))^{-1}]$ to ensure that $0 \le zD(y-x) \le 1$.

Percolation studies the (random) subgraph of occupied bonds. We write $\{x \leftrightarrow y\}$ for the event that x and y are connected on the subgraph of occupied bonds, i.e., there exists a path of occupied bonds from x to y. When the event $\{x \leftrightarrow y\}$ occurs we call the vertices x and y *connected*. For $x \in \mathbb{Z}^d$, the set

$$\mathcal{C}(x) := \{y \in \mathbb{Z}^d \mid y \leftrightarrow x\} \tag{1.1.1}$$

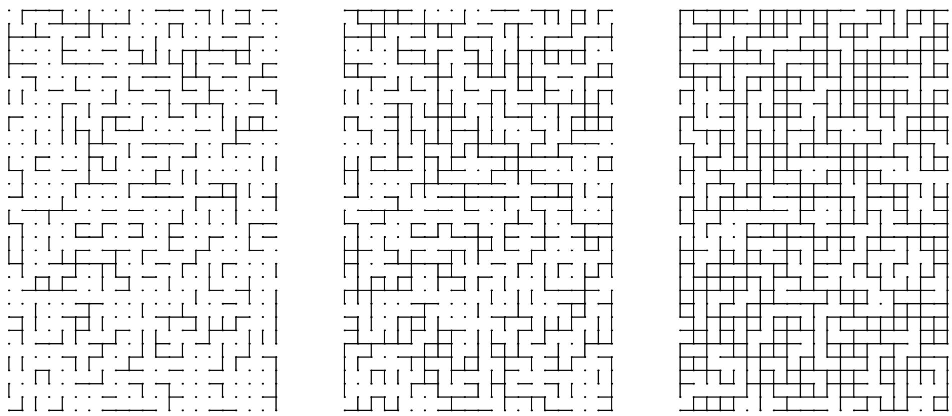


Figure 1.1: This figure shows realizations of nearest-neighbor percolation on (a finite subset of) the two-dimensional lattice. Nearest-neighbor bonds are occupied with probability $p = 1/3$ (left), $p = 1/2$ (middle) and $p = 2/3$ (right). Since $z = 2dp$, this corresponds to $z = 4/3$, $z = 2$ and $z = 8/3$, respectively. As Kesten [79] proves, $p = 1/2$ corresponds to the critical case.

of connected vertices is called the *cluster* of x. It is the size and geometry of these clusters that we are interested in. Due to the shift invariance of the model, we can restrict attention to the cluster at the origin $\mathcal{C} := \mathcal{C}(0)$.

One of the first results about percolation, due to Hammersley [29, 52, 53], shows that percolation observes a phase transition, see also [50, Section 1.4]. By means of a Peierls type argument, Hammersley shows that if z is sufficiently small (though positive), then $\mathcal{C}$ is $\mathbb{P}_z$-a.s. finite. On the other hand, if $d \geq 2$ and the nearest neighbors are in the support of D, then z can be taken so large that the probability that the size of the cluster $\mathcal{C}$ is infinite,

$$\theta(z) := \mathbb{P}_z(|\mathcal{C}| = \infty), \tag{1.1.2}$$

is strictly greater than zero. This has been extended to power-law spread-out percolation in one dimension by Newman and Schulman [90]. See Fig. 1.1 for a percolation realization in the two-dimensional nearest-neighbor model.

Since $z \mapsto \theta(z)$ is non-decreasing, there exists some critical value z_c where this probability turns positive,

$$z_c = \inf\{z \mid \theta(z) > 0\}. \tag{1.1.3}$$

There is also a second characterization of z_c as

$$z_c = \sup \{z \mid \mathbb{E}_z|\mathcal{C}| < \infty\}, \tag{1.1.4}$$

where

$$\mathbb{E}_z|\mathcal{C}| = \sum_{x\in\mathbb{Z}^d} \mathbb{P}_z(0 \leftrightarrow x) \tag{1.1.5}$$

is the expected number of vertices connected to the origin. Menshikov [89] as well as Aizenman and Barsky [3] proved equivalence of (1.1.3) and (1.1.4).

For a general account of percolation we refer to the monograph by Grimmett [50] and the proceedings article by Kesten [82]. The substantial progress on the rigorous understanding of two-dimensional percolation is summarized in Werner's Park City lecture notes [104].

The behavior at or nearby the critical value z_c is one of the major questions of statistical mechanics, and is the central question for the present thesis. We use the notion of *critical* *exponents* to describe this behavior. While the existence of these critical exponents is folklore, there is no general argument proving this. We write $f(z) \asymp g(z)$ if the ratio $f(z)/g(z)$ is bounded away from 0 and infinity, for some appropriate limit.

To this end, we consider the critical exponents γ_{P}, β_{P}, and δ_{P} defined by

$$\mathbb{E}_z|\mathcal{C}| \asymp (z_c - z)^{-\gamma_{\mathrm{P}}} \qquad \text{as } z \nearrow z_c, \tag{1.1.6}$$

$$\theta(z) \asymp (z - z_c)^{\beta_{\mathrm{P}}} \qquad \text{as } z \searrow z_c, \tag{1.1.7}$$

$$\mathbb{P}_{z_c}(|\mathcal{C}| \ge n) \asymp \frac{1}{n^{1/\delta_{\mathrm{P}}}} \qquad \text{as } n \to \infty. \tag{1.1.8}$$

The exponent γ_{P} describes the asymptotic behavior in the *sub*critical regime $\{z < z_c\}$, and characterizes the divergence of the expected cluster size when reaching the critical point. The exponent β_{P} describes the behavior in the *super*critical regime $\{z > z_c\}$, while the exponent δ_{P} bounds the upper tail of the cluster size *at* criticality. A number of other critical exponents has been considered, see [50, Section 9.1]. There are also certain relations between the various critical exponents, known as scaling and hyperscaling postulates.

It is evident that $\theta(z) = 0$ for $z < z_c$ by (1.1.2). We should remark that the map $z \mapsto \theta(z)$ is right-continuous, [50, Lemma 8.3]. If the critical exponent β_{P} exists and exceeds 0, then

$$\theta(z_c) = 0. \tag{1.1.9}$$

For example, in the nearest-neighbor case, (1.1.9) is known to hold for the two-dimensional lattice (as implied by the results of Harris [63] and Kesten [79]), and in high dimension (currently $d \ge 19$) it follows from the fact that $\beta_{\mathrm{P}} = 1$, see also Theorem 3.16 below. Though widely believed to hold, it is an open problem to show (1.1.9) in dimensions $3 \le d \le 18$.

1.2 Self-avoiding walk

Self-avoiding walk is a model at the intersection of probability, statistical mechanics, theoretical chemistry and combinatorics.

For every lattice site $x \in \mathbb{Z}^d$, we denote by

$$\mathcal{W}_n(x) = \{(w_0, \dots, w_n) \mid w_0 = 0,\ w_n = x,\ w_i \in \mathbb{Z}^d, 1 \le i \le n-1\} \tag{1.2.1}$$

the set of all n-step walks from the origin 0 to x. We call a walk $w \in \mathcal{W}_n(x)$ *self-avoiding* if $w_i \ne w_j$ for $i \ne j$ with $i, j \in \{0, \dots, n\}$. We define $c_0(x) = \delta_{0,x}$ and, for $n \ge 1$,

$$c_n(x) := \sum_{w \in \mathcal{W}_n(x)} \prod_{i=1}^{n} D(w_i - w_{i-1})\, \mathbb{1}_{\{w \text{ is self-avoiding}\}}. \tag{1.2.2}$$

where D is as in Section 3.1. For the nearest-neighbor case, $(2d)^n c_n(x)$ is equal to the *number* of n-step (nearest-neighbor) self-avoiding paths from 0 to x.

We write $c_n := \sum_{x \in \mathbb{Z}^d} c_n(x)$ and observe

$$c_{n+m} \le c_n\, c_m \tag{1.2.3}$$

by neglecting avoidance between the two parts of the walk. This shows that $\log c_n$ is a subadditive sequence in n, and hence the limit

$$\mu := \lim_{n\to\infty} c_n^{1/n} = \inf_n c_n^{1/n} \tag{1.2.4}$$

exists, cf. [88, Lemma 1.2.2]. The limit μ is known as the *connective constant.*

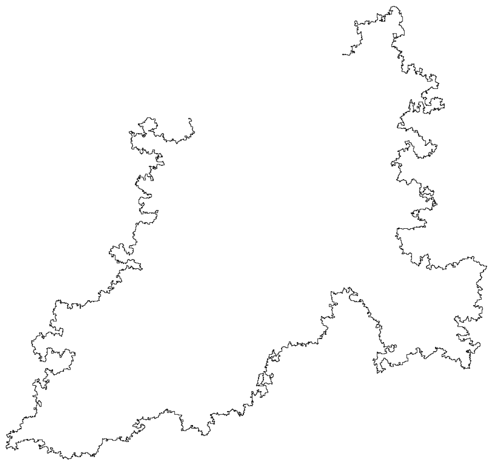


Figure 1.2: Realization of a 10 000 step self-avoiding walk for the two-dimensional nearest-neighbor model.[1]

The self-avoiding walk measure is the measure $\mathbb{Q}_n$ on the set of n-step paths $\mathcal{W}_n$ defined by

$$\mathbb{Q}_n(w) := \frac{1}{c_n} \prod_{i=1}^{n} D(w_i - w_{i-1}) \, \mathbb{1}_{\{w \text{ is self-avoiding}\}} \qquad (1.2.5)$$

A nearest-neighbor self-avoiding walk sampled from $\mathbb{Q}_n$ is depicted in Fig. 1.2.

Having said what a self-avoiding walk is, it might be worthwhile to remark what it is *not*. It is certainly not Markovian, because "history" plays a major role here. Even more, it is not even a stochastic process, because the sequence $(\mathbb{Q}_n)_{n\geq 0}$ does not form a consistent family of measures.

One of the motivations to study self-avoiding walk is the modeling of long chains of linear polymers in a good solvent. For example, polyethylene is a polymer that consists of many CH_2 groups (so-called monomers, in this case one carbon atom and two hydrogen atoms), that are lined up to form a long chain with eventually a CH_3 group at either end. The monomers are connected via chemical bonds that have a fixed distance, and also certain prescribed possible angles, which suggests using paths in a lattice to model the flexibility of the polymers. On the other hand, no two monomers can be at the same position in space, and this excluded volume constraint indicates that these paths should be self-avoiding. A very accessible discussion of modeling real polymers by self-avoiding walk paths can be found in Flory's 1974 Nobel lecture [42].

There are many related questions with no or very few rigorous answers. For example, polymers in a bad solvent also observe the excluded volume constraint, but on the other hand it is energetically favorable for the polymer to touch itself in order to minimize surface with the solvent. For this model there are two competing effects: the self-repellency resulting from the excluded volume constraint makes the walk more spatially extent, whereas the self-attraction makes the walk more compact. Suppose that the self-attraction is controlled by a parameter κ, then it is predicted that the repellency dominates if κ is sufficiently close to 0, and the end-to-end distance of the polymer scales like n^ν for some $\nu \geq 1/2$. As κ increases above a certain threshold κ_c, then the polymer collapses to a length scale of $n^{1/d}$. This phase transition point κ_c is known in physics as the *theta-point*, and it is believed

[1] I thank Vincent Beffara for Fig. 1.2, and also for his help with producing Fig. 1.1.

that *at* κ_c the two effects cancel out and the end-to-end distance is Gaussian for $d \geq 3$. The only rigorous result in that aspect is the existence of an uncollapsed phase for the case where D is given as in (3.1.5) with $h(x) = \mathrm{e}^{-|x|}$ and $d > 4$ due to Ueltschi [103].

For self-avoiding walk, we define the critical exponent $\hat{\gamma}_{\mathrm{S}}$ by

$$c_n \asymp \mu^n n^{\hat{\gamma}_{\mathrm{S}}-1} \qquad \text{as } n \to \infty. \tag{1.2.6}$$

See also below (1.4.9) for a related version of the exponent $\hat{\gamma}_{\mathrm{S}}$.

1.3 Ising model

The Ising model as a model of ferromagnetism was introduced by Ernst Ising in his 1924 thesis [73], but credits should be shared with his advisor Wilhelm Lenz who actually suggested the model to Ising. It is one of the fundamental models for disordered media, and attracts enormous interest in the physics and mathematics community.

For the Ising model we consider the space $\{-1,1\}^{\mathbb{Z}^d}$ of spin configurations on the hypercubic lattice, with a probability distribution thereon. For a formal definition, we consider a finite subset $\Lambda \subset \mathbb{Z}^d$, and for every spin configuration $\boldsymbol{\varphi} = \{\varphi_x \mid x \in \Lambda\} \in \{-1,1\}^\Lambda$ the energy given by the Hamiltonian

$$\mathcal{H}_\Lambda(\boldsymbol{\varphi}) = -\sum_{\{x,y\}\in\Lambda\times\Lambda} J(y-x)\,\varphi_x\,\varphi_y, \tag{1.3.1}$$

where J and D are related via the identity

$$D(x) = \frac{\tanh(zJ(x))}{\sum_{y\in\mathbb{Z}^d}\tanh(zJ(y))}, \tag{1.3.2}$$

and z is the inverse temperature. For example, in the nearest-neighbor case, $D = J$. The probability of a configuration $\boldsymbol{\varphi} \in \{-1,1\}^\Lambda$ is given by

$$\frac{\exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}{\sum_{\boldsymbol{\varphi}\in\{-1,1\}^\Lambda}\exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}. \tag{1.3.3}$$

For the Ising model, J is known as the *spin-spin coupling.* If $J \geq 0$ (which is equivalent to $D \geq 0$, and always satisfied in our setting) then the model is called *ferromagnetic*, hence alignment of spins is energetically preferable. The ferromagnetic Ising model is used to model cooperative phenomena in a rather general sense. For example, an application in Social sciences is presented by Contucci and Ghirlanda [37]. See Fig. 1.3 for realizations of the Ising model on a finite box.

As a general reference for the Ising model we refer to the monographs by Fernández, Fröhlich and Sokal [41], and Bovier [26].

It is most remarkable that percolation and the Ising model have a joint generalization known as the *random cluster model.* This relation was discovered and formulated by Fortuin and Kasteleyn [44, 45, 46], and also documented in Fortuin's doctoral thesis [43]. Their finding was motivated by Kasteleyn's observation that both percolation and the Ising model obey so-called "series/parallel laws", similar to those that Kirchhoff [83] observed for electrical networks. See [51] and references therein.

We will mainly study the spin correlation function,

$$\langle\varphi_0\,\varphi_x\rangle_\Lambda = \frac{\sum_{\boldsymbol{\varphi}\in\{-1,1\}^\Lambda}\varphi_0\,\varphi_x\exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}{\sum_{\boldsymbol{\varphi}\in\{-1,1\}^\Lambda}\exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}, \qquad x \in \mathbb{Z}^d, \tag{1.3.4}$$

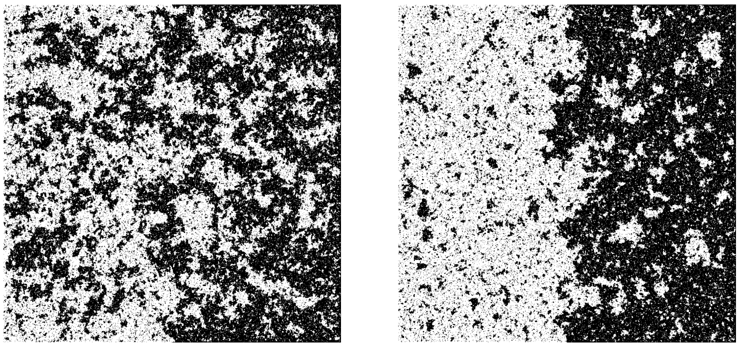


Figure 1.3: Realization of the two-dimensional nearest-neighbor Ising model on a finite box Λ. Here Λ is a box of size 1000×1000, and boundary conditions are chosen such that the left half of the box has negative boundary conditions (white), and the right half has positive boundary conditions (black). The J-term in (1.3.1) equals $J(y-x) = \mathbb{1}_{\{|y-x|=1\}}$, with $z = z_c = \log(1+\sqrt{2}) \approx 0.881374$ (left) and $z = 0.900000$ (right).[2]

in the *thermodynamic limit* when $\Lambda \nearrow \mathbb{Z}^d$. Here the limit is taken over any non-decreasing sequence of Λ's converging to $\mathbb{Z}^d$. This limit exists and is independent from the chosen sequence of Λ's due to Griffiths' second inequality [49]. The susceptibility $\chi(z)$ is then defined as

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \varphi_0 \, \varphi_x \rangle_\Lambda. \tag{1.3.5}$$

Also for the Ising model we require that the support of D contains the nearest neighbors of 0. This enables a Peierls' argument [91] showing that a (finite) critical threshold $z_c \in (0, \infty)$ exists, where the susceptibility $\chi(z)$ diverges as $z \nearrow z_c$. This is exemplified in [41, Sect. 2.1].

We further consider the *magnetization*

$$M(z,h) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\sum_{\boldsymbol{\varphi} \in \{-1,1\}^\Lambda} \varphi_0 \exp\{-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}) + h \sum_{y \in \Lambda} \varphi_y\}}{\sum_{\boldsymbol{\varphi} \in \{-1,1\}^\Lambda} \exp\{-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}) + h \sum_{y \in \Lambda} \varphi_y\}}, \tag{1.3.6}$$

and write $M(z, 0^+)$ for the limit $\lim_{h \searrow 0} M(z,h)$. The magnetization gives rise to another characterization of z_c, namely $z_c = \inf\{z \mid M(z,0^+) > 0\}$. As proved by Aizenman, Barsky and Fernández [4], these two versions of z_c are equivalent.

For the Ising model, we consider the critical exponents γ_I, β_I, δ_I defined by

$$\chi(z) \asymp (z_c - z)^{-\gamma_\mathrm{I}} \qquad \text{as } z \nearrow z_c, \tag{1.3.7}$$

$$M(z, 0+) \asymp (z - z_c)^{\beta_\mathrm{I}} \qquad \text{as } z \searrow z_c, \tag{1.3.8}$$

$$M(z_c, h) \asymp h^{1/\delta_\mathrm{I}} \qquad \text{as } h \searrow 0. \tag{1.3.9}$$

1.4 Two-point function and susceptibility

We study the critical behavior of percolation, self-avoiding walk and the Ising model in a unified way. For this, we need to introduce some notation. For percolation we define the

[2]Pictures by Vincent Beffara

function $G_z(x)$ for $x \in \mathbb{Z}^d$ by

$$G_z(x) = \mathbb{P}_z(0 \leftrightarrow x), \tag{1.4.1}$$

being the probability of the event that there is a path consisting of occupied bonds from 0 to x. For self-avoiding walk, we define $G_z(x)$ as the Green's function,

$$G_z(x) = \sum_{n=0}^{\infty} c_n(x)\, z^n, \tag{1.4.2}$$

whereas for the Ising model, we consider the spin correlation G_z as the *thermodynamic limit*

$$G_z(x) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \varphi_0\, \varphi_x \rangle_\Lambda = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\sum_{\boldsymbol{\varphi} \in \{-1,1\}^\Lambda} \varphi_0\, \varphi_x \exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}{\sum_{\boldsymbol{\varphi} \in \{-1,1\}^\Lambda} \exp(-z\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}. \tag{1.4.3}$$

We will refer to G_z as the *two-point function.* This is inspired by the fact that $G_z(x)$ describes features of the models depending on the two points 0 and x.

We further consider the Fourier transform of $G_z(x)$, $\hat{G}_z(k)$, and introduce the critical exponent η by

$$\hat{G}_{z_c}(k) \asymp \frac{1}{|k|^{(\alpha\wedge 2)-\eta}} \qquad \text{as } k \to 0, \tag{1.4.4}$$

with $\eta = \eta_{\mathrm{P}}$ for percolation, $\eta = \eta_{\mathrm{S}}$ for self-avoiding walk and $\eta = \eta_{\mathrm{I}}$ for the Ising model. The construction of $\hat{G}_{z_c}(k)$ is discussed in Section 3.4, and $\alpha \wedge 2$ is to be interpreted as 2 if the variance of D exists, and as $\alpha = \sup\{\kappa \mid \text{the } \kappa\text{th moment of } D \text{ exists}\}$ if the variance of D does not exist. We refer to Section 3.1 for a formal definition of α.

The *susceptibility* is defined as

$$\chi(z) := \sum_{x \in \mathbb{Z}^d} G_z(x). \tag{1.4.5}$$

For percolation, the susceptibility is equal to the expected cluster size $\chi(z) = \mathbb{E}_z|\mathcal{C}|$. For all our three models the *mean-field bound*

$$\chi'(z) \leq \text{const}\ \chi(z)^2 \tag{1.4.6}$$

holds for some positive constant. Eq. (1.4.6) is a consequence of the fact that the models are self-repellent. For example, for self-avoiding walk we use (1.2.3) to bound

$$\chi'(z) = \sum_{n=0}^{\infty} (n+1)\, c_{n+1} z^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{n+1} z^n \leq \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_m c_{n-m} c_1 z^m z^{n-m} = \chi(z)^2. \tag{1.4.7}$$

For percolation, the proof of (1.4.6) follows from (3.5.20) after taking the appropriate limit. For the Ising model, this mean-field bound is a consequence of the Lebowitz inequality [86].

For all three models we can express z_c, the critical value of z, as

$$z_c = \sup\,\{z \mid \chi(z) < \infty\}\,. \tag{1.4.8}$$

For self-avoiding walk, z_c is the convergence radius of the power series (1.4.2), and hence $z_c = \mu^{-1}$ by (1.2.4).

For self-avoiding walk, we shall consider the critical exponent γ_{S}, defined by

$$\chi(z) \asymp (z_c - z)^{-\gamma_{\mathrm{S}}} \qquad \text{as } z \nearrow z_c, \tag{1.4.9}$$

rather then the formerly introduced exponent $\hat{\gamma}_S$, see (1.2.6). As in the other two models, the exponent γ_S describes divergence of the susceptibility. The two exponents $\hat{\gamma}_S$ and γ_S are presumably the same. Indeed, if $\hat{\gamma}_S$ exists, then also γ_S exists and $\gamma_S = \hat{\gamma}_S$, cf. [88, Sect. 1.3]. For the reverse problem, we aim to conclude the large n behavior of the sequence c_n from the divergence of its generating function $\chi(z)$. This is known as a Tauberian problem, and does not follow a priori. We shall encounter more examples of Tauberian problems throughout the thesis. Theorem 3.16 contains a result about γ_S, and Corollary 5.4 provides a result about $\hat{\gamma}_S$.

1.5 Mean-field behavior

We have argued that all three models, percolation, self-avoiding walk and the Ising model, exhibit a phase transition at the (model-dependent) critical value z_c, and the critical behavior is characterized by means of critical exponents.

It is believed that critical exponents are *universal*, i.e., minor modifications of the model, like changes in the underlying graph, leave the general asymptotic behavior, as described by the critical exponents, unchanged (although they do change the specific value of z_c).

Mean-field behavior and upper critical dimension. We next describe the general picture on a somewhat non-rigorous level. The values of the critical exponents do depend on the dimension d. It is predicted that there is an *upper critical dimension* d_c, such that the critical exponents take the same value for all $d > d_c$. These values are the *mean-field* values of the critical exponents. On the other hand, for $d < d_c$ the values of the critical exponents are different from the mean-field value. At the critical dimension, i.e. $d = d_c$, typically the mean-field values apply, but there is a logarithmic correction present.[3]

The mean-field values for percolation are $\gamma_P = 1$, $\beta_P = 1$, $\delta_P = 2$ and $\eta_P = 0$, which coincide with the corresponding critical exponents obtained for percolation on an infinite regular tree, as obtained in [50, Section 10.1]. This fact can be interpreted in the sense that local interactions are qualitatively less important for $d > d_c$. It also supports the belief, that critical clusters above the upper critical dimensions have an "almost" tree-like structure, with only small loops. That is why the infinite regular tree is also called a *mean-field model* for percolation.

For self-avoiding walk, the mean-field values are the same values that are obtained for simple random walk, i.e., $\gamma_S = 1$ and $\eta_S = 0$. This suggests that for $d > d_c$ the self-avoidance constraint effects only the microscopic scale, but becomes invisible on a macroscopic level, and simple random walk is the mean-field model for self-avoiding walk. A stronger statement in that direction is contained in Chapter 5.

The mean-field values for the Ising model are obtained if in (1.3.1) one of the two factors, φ_x or φ_y, is replaced by its average. Then the model becomes explicitly solvable, and yields $\gamma_I = 1$, $\beta_I = 1/2$, $\delta_I = 3$ and $\eta_I = 0$. In fact, this calculation is also the origin of the term "mean-field".

In Chapters 2 and 3 we use the lace expansion to show that these critical exponents exist and take their mean-field values in sufficiently high dimensions for the nearest-neighbor version of D, or if d exceeds some critical dimension d_c and D is sufficiently "spread-out" (to be made precise in Chapter 3).

Spatial vs. non-spatial models. The Ising model on the lattice $\mathbb{Z}^d$ is a *spatial* model, because the underlying geometry of the lattice heavily influences the behavior of the model. It turns out that the mean-field values for the Ising model coincide with the critical exponents that are obtained for the Curie-Weiss model, which is the Ising model on the *complete*

[3] There is recent progress on the rigorous understanding of the renormalization group approach for self-avoiding walk for $d = d_c$ by Brydges and Slade, see also [30].

graph. The Curie-Weiss model is an example of a *non-spatial* model, because the geometry of the lattice disappears on the complete graph, where every vertex is a direct neighbor of every other vertex. That is, the behavior of the spatial model in high dimensions is the same as for the non-spatial model.

A similar effect for percolation is described in Chapter 4, where we consider percolation on a high-dimensional *torus* with V vertices. We show that the largest cluster for critical percolation on the torus has order $V^{2/3}$ vertices, and the very same asymptotic is observed for percolation on the complete graph. The latter is known as the Erdős-Rényi random graph model, whence the $V^{2/3}$-scaling is called *random graph asymptotic.* The random graph asymptotic of high-dimensional tori is another example of a model where the behavior of the high-dimensional spatial model (percolation on torus) coincides with that of the non-spatial model (percolation on the complete graph).

1.6 Overview

The present thesis is based on the following three research papers:

[64] M. Heydenreich. Long-range self-avoiding walk converges to α-stable processes. Preprint (2008).

[65] M. Heydenreich and R. van der Hofstad. Random graph asymptotics on high-dimensional tori. *Comm. Math. Phys.*, **270**(2):335–358, 2007.

[67] M. Heydenreich, R. van der Hofstad and A. Sakai. Mean-field behaviour of finite- and long-range Ising model, percolation and self-avoiding walk. *J. Statist. Phys.*, **132**(6):1001–1049, 2008.

The general setup of the thesis is as follows. In Chapter 2 we perform the lace expansion for percolation and self-avoiding walk, and derive diagrammatic bounds. These derivations are well-known and not presented in full detail; the focus is on the understanding of the diagrams, which are heavily used to derive the diagrammatic bounds. The lace expansion for the Ising model, which was obtained by Sakai [95], is much more involved and is therefore not presented here. Bounds on the lace expansion coefficients for the Ising model are derived in Appendix A, based on the diagrammatic bounds in [95].

In Chapter 3, which is based on [67], we use the lace expansion to prove the infrared bound in Theorem 3.7. As a consequence of this infrared bound, we derive the critical exponents above the upper critical dimension, see Theorem 3.16. Furthermore, several other lace expansion results are briefly discussed at the end of the chapter.

The comparison between percolation on the infinite lattice $\mathbb{Z}^d$ and the high-dimensional torus is the central topic of Chapter 4. This chapter is based on paper [65]. The main result of the chapter is Theorem 4.2, stating that for critical percolation on a high-dimensional torus with V vertices, the largest cluster has order $V^{2/3}$ vertices.

Finally, Chapter 5 deals with the scaling limit of long-range self-avoiding walk in high dimensions, [64]. We prove that (rescaled) long-range self-avoiding walk converges in distribution to Brownian motion, or to an α-stable Lévy-motion, depending on the step distribution D.

Chapter 2
Lace expansion

Consider the two-point function $G_z(x)$, defined in (1.4.1)–(1.4.3). For each of the three models, i.e., for percolation, self-avoiding walk, and the Ising model, the *lace expansion* obtains an expansion formula of the form

$$G_z(x) = \delta_{0,x} + \tau(z)\,(D * G_z)\,(x) + (G_z * \Phi_z)\,(x) + \Psi_z(x). \tag{2.0.1}$$

Here we write δ for the Kronecker delta function, and we let $\tau(z) = z$ for percolation and self-avoiding walk, and $\tau(z) = \sum_{y\in\mathbb{Z}^d} \tanh(zJ(y))$ for the Ising model (compare to (1.3.2)). The *lace-expansion coefficients* $\Phi_z(x)$ and $\Psi_z(x)$ depend on the particular model, but above their respective upper critical dimension they obey similar bounds. In the present chapter we derive a representation of the lace-expansion coefficients Φ_z and Ψ_z, and prove *diagrammatic bounds* on them. The name 'diagrammatic bounds' stems from the fact that Φ_z and Ψ_z can be represented using certain diagrams, and these diagrams are heavily used in obtaining the bounds.

Both the expansion and the diagrammatic bounds are well-known in the literature. In the present thesis we do the expansion briefly and only sketch the derivation of the diagrammatic bounds; full expansions and detailed derivations of the diagrammatic bounds are performed in [25] for percolation, in [68, 102] for self-avoiding walk, and in [95] for the Ising model.

2.1 Lace expansion for percolation

The lace expansion for percolation was first derived by Hara and Slade in [56], and is based on an inclusion-exclusion argument. The expansion itself holds quite generally for any connected graph, finite or infinite, even for non-regular graphs. Since we are using Fourier analysis later on, we nevertheless restrict our attention to the case where the graph has vertex set $\mathbb{Z}^d$, and edge set $\big\{\{x,y\} \mid x,y\in\mathbb{Z}^d, x\neq y\big\}$.

In our account, we follow the presentation in [25, Sect. 3 and 4] (the same derivation is contained in the monograph [102]), at some places even verbatim.

Throughout this section we fix a particular value $z \in [0, z_c)$, and further omit the z-dependence from the notation; e.g., we write $G(x)$ for $G_z(x)$, etc.

The expansion. Our aim is to derive the expansion formula

$$G(x) = \delta_{0,x} + z\,(D * G)\,(x) + z\,(\Pi_M * D * G)\,(x) + \Pi_M(x) + R_M(x) \tag{2.1.1}$$

for $M = 0, 1, 2, \ldots$. By $*$ we denote matrix multiplication, which on regular graphs reduces to convolution. The function $\Pi_M\colon \mathbb{Z}^d \to \mathbb{R}$ is the central quantity in the expansion, and $R_M(x)$ is a remainder term. Here M indicated the level to which the expansion is carried out, and the dependence of Π_M on M is given by

$$\Pi_M(x) = \sum_{N=0}^{M} (-1)^N \pi^{(N)}(x). \tag{2.1.2}$$

The alternating sign in (2.1.2) arises via inclusion-exclusion. When the expansion converges, one has

$$\lim_{M\to\infty} \sum_x |R_M(x)| = 0 \tag{2.1.3}$$

for each $x \in \mathbb{Z}^d$. We shall later fix M sufficiently large (so that (3.3.37) and (3.3.38) below are satisfied for $K = 4$). Then (2.1.1) is equivalent to (2.0.1) by letting $\tau(z) = z$, and

$$\Phi_z(x) = z(D * \Pi_M)(x), \qquad x \in \mathbb{Z}^d, \tag{2.1.4}$$

$$\Psi_z(x) = \Pi_M(x) + R_M(x), \qquad x \in \mathbb{Z}^d. \tag{2.1.5}$$

Notation and definitions. At some places we will write the two-point function G with two arguments, with the convention that $G(x, y) = G(y - x)$. We call two vertices x and y *doubly connected* (and write $x \Leftrightarrow y$), if $x = y$ or there are (at least) two bond-disjoint paths from x to y consisting of occupied bonds. Given a bond configuration, we call an (occupied or vacant) bond b *pivotal* for the event $\{x \leftrightarrow y\}$, if $\{x \leftrightarrow y\}$ occurs if b is made occupied, and $\{x \leftrightarrow y\}$ does not occur if b is made vacant. We denote by $\mathrm{Piv}(x, y)$ the set of bonds that are *occupied and pivotal* for the event $\{x \leftrightarrow y\}$. Although bonds are generally regarded as undirected, it is convenient to consider pivotal bonds as *directed* bonds, i.e., $(u, v) \in \mathrm{Piv}(x, y)$ if $x \leftrightarrow u$, $v \leftrightarrow y$, and x is not connected to y if the bond $\{u, v\}$ is made vacant.

For a set of vertices $A \subset \mathbb{Z}^d$ we define $\{x \leftrightarrow y \text{ off } A\}$ to be the event that x is connected to y *after* all bonds with endpoints in the set A are made vacant. We write $G^A(x, y) := \mathbb{P}(x \leftrightarrow y \text{ off } A)$. Also, we write $\{x \xleftrightarrow{A} y\} := \{x \leftrightarrow y\} \setminus \{x \leftrightarrow y \text{ off } A\}$ for the event that either every connected path from x to y has at least one bond with endpoint in A or $x = y \in A$. This leads to the identity

$$G^A(x, y) = G(x, y) - \Big(G(x, y) - G^A(x, y)\Big) = G(x, y) - \mathbb{P}(x \xleftrightarrow{A} y), \tag{2.1.6}$$

which plays a crucial role in the expansion later on.

Finally, we write $\tilde{\mathcal{C}}^{(u,v)}(x)$ for the *restricted cluster* of x which consists of all vertices connected to x *after* setting the bond (u, v) vacant.

Derivation of (2.1.1). We are now expanding $G(x) = \mathbb{P}(0 \leftrightarrow x)$. We start by introducing a schematic representation, see Figure 2.1. To begin the expansion, we define

$$\pi^{(0)}(x) := \mathbb{P}(0 \Leftrightarrow x) - \delta_{0,x} \tag{2.1.7}$$

and distinguish configurations with $0 \leftrightarrow x$ according to whether or not there is a double connection by writing

$$G(x) = \delta_{0,x} + \pi^{(0)}(x) + \mathbb{P}(0 \leftrightarrow x, 0 \not\Leftrightarrow x). \tag{2.1.8}$$

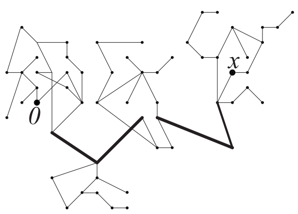


Figure 2.1: On the left there is a possible cluster containing the vertices 0 and x, with all 4 occupied pivotal bonds shown bold. On the right is a schematic representation of the configuration as a “string of sausages”.

If 0 is connected to x, but not doubly, then $\mathrm{Piv}(0,x)$ is nonempty. There is therefore a *unique* element $(u,v) \in \mathrm{Piv}(0,x)$ such that (u,v) is occupied and $0 \Leftrightarrow u$ (the "*first*" occupied and pivotal bond), and we can write

$$\mathbb{P}\big(0 \leftrightarrow x, 0 \not\Leftrightarrow x\big) = \sum_{(u,v)} \mathbb{P}\big(0 \Leftrightarrow u, (u,v) \in \mathrm{Piv}(0,x)\big). \tag{2.1.9}$$

The sum in (2.1.9) is over all directed bonds (u,v). Now comes the essential part of the expansion. Ideally, we would like to factor the probability on the right hand side of (2.1.9) as

$$\mathbb{P}(0 \Leftrightarrow u)\, \mathbb{P}((u,v) \text{ is occupied})\, \mathbb{P}(v \leftrightarrow x) = \big(\delta_{0,u} + \pi^{(0)}(u)\big)\, zD(v-u)\, G(x-v). \tag{2.1.10}$$

The expression (2.1.10) would lead to (2.1.1) with $\Pi_M = \pi^{(0)}$ and $R_M = 0$. However, (2.1.9) does not factor in this way, because (u,v) being pivotal implies that the cluster $\tilde{\mathcal{C}}^{(u,v)}(0)$ (which is the cluster containing 0 after setting the bond (u,v) vacant) is constrained not to intersect the cluster $\tilde{\mathcal{C}}^{(u,v)}(x)$. We are taking this constraint into account with the following lemma. To this end, we define the events $E'(v,y;A)$ and $E(x,u,v,y;A)$ by

$$\begin{aligned} E'(v,y;A) &:= \{v \xleftrightarrow{A} y\} \cap \big\{\nexists (u',v') \in \mathrm{Piv}(v,y) \text{ such that } v \xleftrightarrow{A} u'\big\}, && (2.1.11)\\ E(x,u,v,y;A) &:= E'(x,u;A) \cap \big\{(u,v) \in \mathrm{Piv}(x,y)\big\}, && (2.1.12) \end{aligned}$$

where $u,v,x,y \in \mathbb{Z}^d$, and $A \subset \mathbb{Z}^d$ is a nonempty set of vertices. These events are best understood graphically:

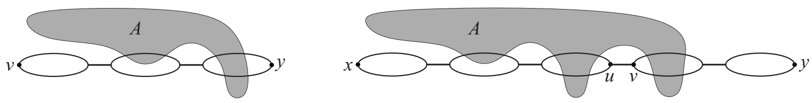


Figure 2.2: Pictorial representation of $E'(v,y;A)$ (left) and $E(x,u,v,y;A)$ (right).

Lemma 2.1 ([102, Lemma 10.1]). *For $z \geq 0$ such that there is a.s. no infinite cluster ($z < z_c$), $x,y,u,v \in \mathbb{Z}^d$, and $A \subset \mathbb{Z}^d$,*

$$\mathbb{P}\big(E(x,u,v,y;A)\big) = z\, D(v-u)\, \mathbb{E}\Big(\mathbb{1}_{E'(x,u;A)}\, G^{\tilde{\mathcal{C}}^{(u,v)}(0)}(v,x)\Big). \tag{2.1.13}$$

The expectation on the right hand side of (2.1.13) needs some explanation: the set $\tilde{\mathcal{C}}^{(u,v)}(0)$ in the superscript of $G^{\tilde{\mathcal{C}}^{(u,v)}(0)}(v,x)$ should be understood as *fixed* (i.e., deterministic) w.r.t. the restricted two-point function. On the other hand, it is a *random* set with respect to the expectation $\mathbb{E}$.

For a formal proof of Lemma 2.1 we refer to [102, Proof of Lemma 10.1], and appeal to Fig. 2.2 instead. We note that $E'(0,x;\mathbb{Z}^d) = \{0 \Leftrightarrow x\}$ and

$$E(0,v,u,x;\mathbb{Z}^d) = \{0 \Leftrightarrow u, (u,v) \in \mathrm{Piv}(0,x)\},$$

hence Lemma 2.1 with $A = \mathbb{Z}^d$ implies

$$\mathbb{P}\big(0 \Leftrightarrow u, (u,v) \in \mathrm{Piv}(0,x)\big) = zD(v-u)\, \mathbb{E}\Big(\mathbb{1}_{\{0 \Leftrightarrow u\}} G^{\tilde{\mathcal{C}}^{(u,v)}(0)}(v,x)\Big). \tag{2.1.14}$$

We now combine (2.1.8)–(2.1.9) with (2.1.14) and apply (2.1.6) with $A = \tilde{\mathcal{C}}^{(u,v)}(0)$ to obtain

$$\begin{aligned} G(x) &= \delta_{0,x} + \pi^{(0)}(x) + \sum_{(u,v)} zD(v-u)\, \mathbb{E}_0\Bigg(\mathbb{1}_{\{0 \Leftrightarrow u\}} \Big(G(x-v) - \mathbb{P}_1\Big(v \xleftrightarrow{\tilde{\mathcal{C}}_0^{(u,v)}(0)} x \Big)\Big)\Bigg) \\ &= \delta_{0,x} + \pi^{(0)}(x) + \sum_{(u,v)} \Big(\delta_{0,u} + \pi^{(0)}(u)\Big)\, zD(v-u)\, G(x-v) \\ &\quad - \sum_{(u,v)} zD(v-u)\, \mathbb{E}_0\Bigg(\mathbb{1}_{\{0 \Leftrightarrow u\}}\, \mathbb{P}_1\Big(v \xleftrightarrow{\tilde{\mathcal{C}}_0^{(u,v)}(0)} x \Big)\Bigg). \end{aligned} \tag{2.1.15}$$

Here we have introduced subscripts for $\tilde{\mathcal{C}}$ and the expectations to indicate to which expectation $\tilde{\mathcal{C}}$ belongs. With $R_0(x)$ equal to the last line of (2.1.15) (including the minus sign) this proves (2.1.1) for $M = 0$.

We are now proceeding with the expansion. For any (nonempty) set of vertices $A \subset \mathbb{Z}^d$ we consider configurations in which $v \xleftrightarrow{A} x$. An occupied pivotal bond (u', v') is called a *cutting bond* for the event $\{v \xleftrightarrow{A} x\}$ if $v \xleftrightarrow{A} u'$ and (u', v') is the *first* such pivotal bond. Equivalently, (u', v') is a *cutting bond* for $\{v \xleftrightarrow{A} x\}$ if $(u', v') \in \mathrm{Piv}(v, x)$ and $E'(v, u'; A)$ occurs. In any configuration, there is either 0 or 1 cutting bond. For example, in Figure 2.2 there is no cutting bond on the left, and (u, v) is a cutting bond for $\{x \xleftrightarrow{A} y\}$ on the right.

A partition of the event $\{v \xleftrightarrow{A} x\}$ according to the location of the cutting bond yields

$$\{v \xleftrightarrow{A} x\} = E'(v, x; A) \,\dot{\cup}\, \dot{\bigcup_{(u',v')}} E(v, u', v', x; A), \tag{2.1.16}$$

where the first term $E'(v, x; A)$ consists of configurations where there is *no* cutting bond. By using Lemma 2.1 in the first line, and identity (2.1.6) in the second,

$$\begin{aligned} \mathbb{P}\big(v \xleftrightarrow{A} x\big) &= \mathbb{P}\big(E'(v,x;A)\big) + \sum_{(u',v')} zD(u',v')\, \mathbb{E}\Big(\mathbb{1}_{E'(v,u';A)}\, G^{\tilde{\mathcal{C}}^{(u',v')}(0)}(v',x)\Big) \\ &= \mathbb{P}\big(E'(v,x;A)\big) + \sum_{(u',v')} zD(u',v')\, \mathbb{P}\big(E'(v,u';A)\big)\, G(v',x) \\ &\quad - \sum_{(u',v')} zD(u',v')\, \mathbb{E}_1\Bigg(\mathbb{1}_{E'(v,u';A)}\, \mathbb{P}_2\Big(v' \xleftrightarrow{\tilde{\mathcal{C}}_1^{(u',v')}(0)} x\Big)\Bigg). \end{aligned} \tag{2.1.17}$$

Recall that the subscripts for $\tilde{\mathcal{C}}$ and the expectations indicate to which expectation $\tilde{\mathcal{C}}$ belongs.

Defining

$$\pi^{(1)}(x) = \sum_{(u,v)} zD(v-u)\, \mathbb{E}_0\Big(\mathbb{1}_{\{0 \Leftrightarrow u\}}\, \mathbb{P}_1\big(E'(u, x; \tilde{\mathcal{C}}_0^{(u,v)}(0))\big)\Big), \tag{2.1.18}$$

we insert (2.1.6) (with $A = \tilde{\mathcal{C}}_0^{(u,v)}(0)$) into (2.1.17) and obtain

$$\begin{aligned} G(x) &= \delta_{0,x} + \pi^{(0)}(x) - \pi^{(1)}(x) + \sum_{(u,v)} \Big(\delta_{0,u} + \pi^{(0)}(u) - \pi^{(1)}(u)\Big)\, zD(v-u)\, G(v,x) \\ &\quad + \sum_{(u,v)} zD(v-u) \sum_{(u',v')} zD(v'-u') \\ &\qquad \times \mathbb{E}_0\left(\mathbb{1}_{\{0\Leftrightarrow u\}}\, \mathbb{E}_1\Big(\mathbb{1}_{E'(v,u';\tilde{\mathcal{C}}_0^{(u,v)}(0))}\, \mathbb{P}_2\Big(v' \xleftrightarrow{\tilde{\mathcal{C}}_1^{(u',v')}(v)} x\Big)\Big)\right). \end{aligned} \tag{2.1.19}$$

This proves (2.1.1) for $M = 1$ and $R_1(x)$ equal to the last two lines of (2.1.19).

A repeated use of (2.1.17) proves (2.1.1) recursively with

$$\begin{aligned} \pi^{(N)}(x) = &\sum_{(u_0,v_0)} zD(v_0-u_0) \cdots \sum_{(u_{N-1},v_{N-1})} zD(v_{N-1}-u_{N-1})\, \mathbb{E}_0\, \mathbb{1}_{\{0\Leftrightarrow u_0\}} \\ &\times \mathbb{E}_1\, \mathbb{1}_{E'(v_0,u_1;\tilde{\mathcal{C}}_0)} \cdots \mathbb{E}_{N-1}\, \mathbb{1}_{E'(v_{N-2},u_{N-1};\tilde{\mathcal{C}}_{N-2})}\, \mathbb{E}_N\, \mathbb{1}_{E'(v_{N-1},x;\tilde{\mathcal{C}}_{N-1})} \end{aligned} \tag{2.1.20}$$

and remainder term

$$\begin{aligned} R_M(x) = &(-1)^{M+1} \sum_{(u_0,v_0)} zD(v_0-u_0) \cdots \sum_{(u_M,v_M)} zD(v_M-u_M)\, \mathbb{E}_0\, \mathbb{1}_{\{0\Leftrightarrow u_0\}} \\ &\times \mathbb{E}_1\, \mathbb{1}_{E'(v_0,u_1;\tilde{\mathcal{C}}_0)} \cdots \mathbb{E}_{M-1}\, \mathbb{1}_{E'(v_{M-2},u_{M-1};\tilde{\mathcal{C}}_{M-2})} \\ &\times \mathbb{E}_M\left(\mathbb{1}_{E'(v_{M-1},u_M;\tilde{\mathcal{C}}_{M-1})}\, \mathbb{P}_{M+1}\big(v_M \xleftrightarrow{\tilde{\mathcal{C}}_M} x\big)\right) \end{aligned} \tag{2.1.21}$$

where we abbreviate $\tilde{\mathcal{C}}_j = \tilde{\mathcal{C}}_j^{(u_j,v_j)}(v_{j-1})$, with $v_{-1} = 0$.

Since $\mathbb{P}_{M+1}\big(v_M \xleftrightarrow{\tilde{\mathcal{C}}_M} x\big) \leq \mathbb{P}_{M+1}\big(v_M \leftrightarrow x\big) = G(v_M, x)$ it follows that

$$|R_M(x)| \leq \sum_{(u_M,v_M)} \pi^{(M)}(u_M)\, zD(v_M-u_M)\, G(v_M,x). \tag{2.1.22}$$

By expanding $\mathbb{P}(0 \leftrightarrow x)$, we have obtained exact expressions for $\Pi_M(x)$ and $R_M(x)$. Recalling (2.1.4)–(2.1.5), these identities imply exact expressions for $\Phi_z(x)$ and $\Psi_z(x)$ in (2.0.1). We proceed by deriving upper bounds on Π_M and R_M, known as diagrammatic bounds. These bounds are used later on to show that Φ_z and Ψ_z are actually small.

Diagrammatic bounds. We proceed by decomposing the lace-expansion coefficients $\pi^{(N)}$ into certain diagrams to be introduced now. Denote

$$\tilde{G}(x) := z(D * G)(x) \tag{2.1.23}$$

and define the diagrams

$$T := (G * G * G)(0), \qquad \tilde{T} := (\tilde{G} * G * G)(0); \tag{2.1.24}$$

$$B_k := \sum_{x\in\mathbb{Z}^d} [1-\cos(k\cdot x)]\, \tilde{G}(x)\, G(x), \qquad W_k := \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} [1-\cos(k\cdot x)]\, G(x)\, G(x+y); \tag{2.1.25}$$

and

$$H_k := \sup_{a_1,a_2} \sum_{s,t,u,v,w} [1-\cos(k\cdot(t-u))]\, G(0,u)\, G(u,t)\, \tilde{G}(t,v+a_2) \times G(u,w)\, G(t,w)\, G(w,s)\, \tilde{G}(a_1,s)\, G(s,v). \tag{2.1.26}$$

In (2.1.26) and the remainder of the section, all sums and suprema are taken over $\mathbb{Z}^d$ unless stated otherwise.

Proposition 2.2 (Diagrammatic bounds for percolation [25, Prop. 4.1])**.**

$$\sum_x \pi^{(0)}(x) \le \tilde{T}, \tag{2.1.27}$$

$$\sum_x [1-\cos(k\cdot x)]\, \pi^{(0)}(x) \le B_k, \tag{2.1.28}$$

and, for $N \ge 1$,

$$\sum_x \pi^{(N)}(x) \le T\left(2T\tilde{T}\right)^N, \tag{2.1.29}$$

$$\sum_x [1-\cos(k\cdot x)]\, \pi^{(N)}(x) \le (4N+3)\left[TW_k\left(2\tilde{T}+(1+z)NT\right)\left(2T\tilde{T}\right)^{N-1} + (N-1)\left(\tilde{T}^2 W_k + H_k\right) T^2 \left(2T\tilde{T}\right)^{N-2}\right]. \tag{2.1.30}$$

Furthermore, for $N=1$ *we can improve* (2.1.30) *to*

$$\sum_x [1-\cos(k\cdot x)]\, \pi^{(1)}(x) \le B_k + 31\, T\tilde{T}W_k. \tag{2.1.31}$$

Why do we need these bounds? As we shall see, (2.1.27)–(2.1.31) imply bounds on the Fourier transform of Π_M, introduced in (2.1.2). In particular, we get upper bounds on $|\hat{\Pi}_M(0)|$ and $|\hat{\Pi}_M(0)-\hat{\Pi}_M(k)|$. Together with (2.1.22), these bounds allow for sufficient control on Π_M and R_M in the expansion identity (2.1.1). The bounds on Π_M and R_M involve the two-point function G only, and we apply a clever 'consistency' argument (the bootstrap lemma in Section 3.4.3) to obtain bounds on G. This analysis is carried out in Sections 3.3 and 3.4 below, and Proposition 2.2 is the main ingredient for the proof of Proposition 3.2 in the percolation case.

We shall prove Proposition 2.2 only for the cases $N=0$ and $N=1$. A full proof uses induction over N, and can be found in [25, Prop. 4.1][1].

BK-inequality. The main tool for the decomposition of the diagrams is the BK-inequality, which is due to van den Berg and Kesten [17]. In order to state the inequality we need to introduce the notion of *increasing events* and *disjoint occurrence.* We call E an *increasing* event, if the occurrence of E on a given configuration $\omega \in \{0,1\}^{\mathbb{Z}^d\times\mathbb{Z}^d}$ implies that E also occurs on ω' with $\omega' \ge \omega$ (pointwise comparison). Casually speaking, making vacant bonds occupied is "harmless" for increasing events. In our setting we will mainly consider events of the form that some points are connected via paths of occupied bonds, e.g. $\{0 \leftrightarrow x\}$, and these are clearly increasing events. By $E_1 \circ E_2$ we denote the *disjoint occurrence* of the

[1]Note that $p\Omega$ in the notation of [25] corresponds to z in our notation.

increasing events E_1 and E_2, and an edge configuration $\omega \in \{0,1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ belongs to $E_1 \circ E_2$ if the following holds: The set of edges can be partitioned into two sets, K and K^c, such that E_1 occurs if all edges in K^c are made vacant, and E_2 occurs if all edges in K are made vacant.

Proposition 2.3 (BK-inequality [17]). *For increasing events E_1 and E_2,*

$$\mathbb{P}(E_1 \circ E_2) \le \mathbb{P}(E_1)\, \mathbb{P}(E_2).$$

For example, the event that there is a double connection between 0 and x can be written as $\{0 \Leftrightarrow x\} = \{0 \leftrightarrow x\} \circ \{0 \leftrightarrow x\}$, and hence the BK-inequality yields

$$\mathbb{P}(0 \Leftrightarrow x) = \mathbb{P}\big(\{0 \leftrightarrow x\} \circ \{0 \leftrightarrow x\}\big) \le \mathbb{P}(0 \leftrightarrow x)^2. \tag{2.1.32}$$

Proof of (2.1.27) *and* (2.1.28). For $x \neq 0$ we have that

$$\{0 \leftrightarrow x\} = \bigcup_{y \in \mathbb{Z}^d \setminus \{0\}} \big(\{\text{bond } \{0,y\} \text{ is occupied}\} \circ \{y \leftrightarrow x\}\big). \tag{2.1.33}$$

Consequently, the BK-inequality implies

$$G(x) \le \delta_{0,x} + \sum_{y \in \mathbb{Z}^d} zD(y)\, G(x-y) = \delta_{0,x} + \tilde{G}(x). \tag{2.1.34}$$

Recalling the definition of $\pi^{(0)}(x)$ in (2.1.7), we use (2.1.32), then (2.1.34), and finally $G(x) \le (G * G)(x)$ to obtain

$$\sum_x \pi^{(0)}(x) = \sum_{x \neq 0} \mathbb{P}(0 \Leftrightarrow x) \le \sum_{x \neq 0} G(x)^2 \le \sum_x G(x)\, \tilde{G}(x) \le (G * G * \tilde{G})(0) = \tilde{T} \tag{2.1.35}$$

and

$$\sum_x [1 - \cos(k \cdot x)]\, \pi^{(0)}(x) \le \sum_x [1 - \cos(k \cdot x)]\, \tilde{G}(x)\, G(x) = B_k. \tag{2.1.36}$$

□

Proof of (2.1.29) *for* $N = 1$. We rewrite (2.1.18) with Fubini's theorem as

$$\pi^{(1)}(x) = \sum_{(u,v)} zD(v-u)\, \mathbb{P}\Big(\{0 \Leftrightarrow u\}_0 \cap E'\big(u, x; \tilde{\mathcal{C}}_0^{(u,v)}(0)\big)_1\Big). \tag{2.1.37}$$

Here, $\mathbb{P}$ denotes the product measure on two copies of $\mathbb{Z}^d$, that are coupled via the restricted cluster $\tilde{\mathcal{C}}_0^{(u,v)}(0)$. Events with subscript 0 or 1 indicate connections on either of the two copies. Eq. (2.1.37) is best understood graphically, see Figure 2.3.

It is evident from Figure 2.3 that on the product space,

$$\begin{aligned} &\{0 \Leftrightarrow u\}_0 \cap E'\big(u, x; \tilde{\mathcal{C}}_0^{(u,v)}(0)\big)_1 \\ &\quad \subset \bigcup_{w_1, w_2, w_3} \big\{\{0 \leftrightarrow u\} \circ \{0 \leftrightarrow w_1\} \circ \{w_1 \leftrightarrow u\} \circ \{w_1 \leftrightarrow w_2\} \circ \{w_2 \leftrightarrow w_3\}\big\}_0 \\ &\qquad\qquad \cap \big\{\{v \leftrightarrow w_3\} \circ \{w_2 \leftrightarrow x\} \circ \{x \leftrightarrow w_3\}\big\}_1 \end{aligned} \tag{2.1.38}$$

so that the BK-inequality allows for the upper bound

$$\begin{aligned} \pi^{(1)}(x) \le \sum_{(u,v)} zD(v-u) \sum_{w_1, w_2, w_3} &G(0,u)\, G(0,w_1)\, G(w_1,u)\, G(w_1,w_2)\, G(w_2,w_3) \\ &\times G(v,w_3)\, G(w_2,x)\, G(x,w_3). \end{aligned} \tag{2.1.39}$$

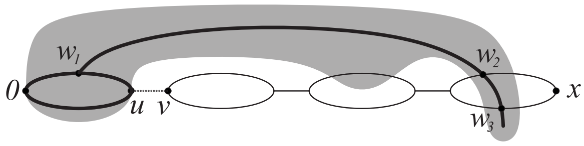
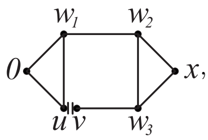
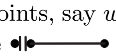


Figure 2.3: Schematic representation of $\pi^{(1)}(x)$. The two graphs are coupled via the restricted cluster $\tilde{\mathcal{C}}_0^{(u,v)}(0)$, which is shown gray. Connections within graph 0 are printed bold, whereas connections within graph 1 have thin lines.

The sum over directed bonds (u, v) can be replaced by the double sum over all $u, v \in \mathbb{Z}^d$, because since $D(0) = 0$ there is no contribution from terms with $u = v$. A graphical representation for Eq. (2.1.39) is the following:

$$\pi^{(1)}(x) \leq \sum_{\substack{u,v \\ w_1,w_2,w_3}} \qquad (2.1.40)$$



where a line between two points, say w_1 and w_2, represents a two-point function $G(w_1, w_2)$, and the double-dashed line  represents $\sum_v zD(v-u)\, G(v, w_3) = \tilde{G}(u, w_3)$.

We have upper bounded the rather complicated diagram describing $\pi^{(1)}(x)$ into a structure consisting of only two-point functions G, and we achieved this by repeated use of the BK-inequality. This procedure is called *decomposition of the diagrams*, and it can be applied similarly to higher order terms $\pi^{(N)}(x)$ (cf. [25, (4.1)–(4.11)]).

The last step now is to identify the triangle diagrams that are hidden in (2.1.39). We remark that

$$(G * G * G)(x) \leq (G * G * G)(0) = T, \quad (G * G * \tilde{G})(x) \leq (G * G * G)(0) = \tilde{T}. \qquad (2.1.41)$$

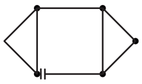
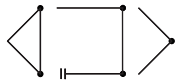
This can be seen via translating the quantities on the left side of the inequality sign into Fourier space and using $\hat{G}(k) \geq 0$ for all k by [9, Lemma 3.3]. For example, for the first inequality in (2.1.41) we bound

$$(G{*}G{*}G)(x) = \int_{[-\pi,\pi)^d} \mathrm{e}^{ik\cdot x}\, \hat{G}(k)^3 \frac{\mathrm{d}k}{(2\pi)^d} \leq \int_{[-\pi,\pi)^d} \underbrace{|\,\mathrm{e}^{ik\cdot x}\,|}_{=1} \underbrace{|\hat{G}(k)|^3}_{=\hat{G}(k)^3} \frac{\mathrm{d}k}{(2\pi)^d} = (G{*}G{*}G)(0).$$

We use (2.1.41) to bound

$$\begin{aligned}\sum_x \pi^{(1)}(x) &\leq \underbrace{\sum_{u,w_1} G(0,u)\, G(0,w_1)\, G(w_1,u)}_{=T} \underbrace{\left(\sup_u \sum_{w_2,w_3} G(w_1,w_2)\, G(w_2,w_3)\, \tilde{G}(u,w_3)\right)}_{\leq \tilde{T}} \\ &\quad \times \underbrace{\left(\sup_{w_3} \sum_x G(w_2,x)\, G(x,w_3)\right)}_{\leq T \text{ using } G(x,w_3) \leq (G*G)(x,w_3)} \\ &\leq T^2 \tilde{T}.\end{aligned} \qquad (2.1.42)$$

Again, the bound in (2.1.42) has a graphical interpretation as

$$\sum_x \pi^{(1)}(x) =$$  $$\le$$  , (2.1.43)

where the three objects on the right hand side correspond to the three factors in (2.1.42), and vertices marked with a black dot are summed over. This finishes the proof of (2.1.29) for $N = 1$. □

Proof of (2.1.31). We denote by Θ the right hand side of (2.1.39) without the sum,

$$\begin{aligned}\Theta(s,u,w_1,w_2,w_3,x) = {} & G(s,u)\,G(u,w_1)\,G(w_1,s)\\ & \times G(w_1,w_2)\,G(w_2,w_3)\,\tilde G(w_3,u)\,G(w_2,x)\,G(x,w_3),\end{aligned} \tag{2.1.44}$$

so that (2.1.39) can be rewritten as

$$\sum_x \pi^{(1)}(x) \le \sum_{u,w_1,w_2,w_3,x} \Theta(0,u,w_1,w_2,w_3,x).$$

Insertion of the identity

$$1 = \mathbb{1}_{\{0=u=w_1\}}\,\mathbb{1}_{\{w_2=w_3=x\}} + \mathbb{1}_{\{0=u=w_1\}}\big(1-\mathbb{1}_{\{w_2=w_3=x\}}\big) + (1-\mathbb{1}_{\{0=u=w_1\}}). \tag{2.1.45}$$

into the right hand side yields

$$\sum_x [1-\cos(k\cdot x)]\pi^{(1)}(x) \le (I) + (II) + (III), \tag{2.1.46}$$

where

$$(I) = \sum_x [1-\cos(k\cdot x)]\,G(x)\,\tilde G(x), \tag{2.1.47}$$

$$(II) = \sum_{w_2,w_3,x} [1-\cos(k\cdot x)]\,(1-\mathbb{1}_{\{w_2=w_3=x\}})\,\Theta(0,0,0,w_2,w_3,x), \tag{2.1.48}$$

$$(III) = \sum_{\substack{u,x,\\ w_1,w_2,w_3}} [1-\cos(k\cdot x)]\,(1-\mathbb{1}_{\{0=u=w_1\}})\,\Theta(0,u,w_1,w_2,w_3,x). \tag{2.1.49}$$

Clearly, $(I) = B_k$, cf. (2.1.25). Hence it suffices to show that $(II)+(III) \le 31\,T\tilde T W_k$.

To this end, we need the following trigonometric bound. For real numbers $t_1,\dots,t_n$ one can show that

$$1-\cos\left(\sum_{j=1}^n t_j\right) \le (2n+1)\sum_{j=1}^n [1-\cos t_j]\,, \tag{2.1.50}$$

cf. [25, (4.51)]. We apply (2.1.50) to (II) with $t_1 = k\cdot w_3$ and $t_2 = k\cdot(x-w_3)$ to obtain

$$\begin{aligned}&\sum_{\substack{w_2,w_3,x\\ |\{w_2,w_3,x\}|\ge 2}} [1-\cos(k\cdot x)]\,G(0,w_2)\,G(w_2,w_3)\,\tilde G(w_3,0)\,G(w_2,x)\,G(x,w_3)\\ &\le \sum_{w_2,w_3,x} 5\,[1-\cos(k\cdot w_3)]\,\big(1-\mathbb{1}_{\{w_2=w_3=x\}}\big)G(0,w_2)\,G(w_2,w_3)\,\tilde G(w_3,0)\,G(w_2,x)\,G(x,w_3)\\ &\quad+\sum_{w_2,w_3,x} 5[1-\cos(k\cdot(x-w_3))]\,\big(1-\mathbb{1}_{\{w_2=w_3=x\}}\big)\\ &\qquad\times G(0,w_2)\,G(w_2,w_3)\,\tilde G(w_3,0)\,G(w_2,x)\,G(x,w_3)\end{aligned} \tag{2.1.51}$$

For the first summand on the right hand side of (2.1.51) we replace w_2 and x by $w_2 + w_3$ and $x + w_3$, and use translation invariance of the model to obtain the equivalent expression

$$\sum_{w_2,w_3,x} 5\,[1-\cos(k\cdot w_3)]\,\big(1-\mathbb{1}_{\{w_2=w_3=x\}}\big)\;\tilde G(w_3,0)\,G(w_2,w_3)\,G(0,w_2)\,G(w_2,x)\,G(x,w_3). \tag{2.1.52}$$

This is further bound above by

$$\underbrace{5\sum_{w_2,x}\big(1-\mathbb{1}_{\{w_2=w_3=x\}}\big)\,G(0,w_2)\;G(w_2,x)\,G(x,0)}_{\le T}\underbrace{\sum_{w_3}[1-\cos(k\cdot w_3)]\,\tilde G(w_3,0)\,G(w_2,w_3)}_{\le W_k} \tag{2.1.53}$$

Generally we have that

$$0 \le T - 1 \le \tilde T, \tag{2.1.54}$$

where the lower bound comes from the contribution $x = y = 0$ in $T = \sum_{x,y} G(0,x)\,G(x,y)$ $G(y,0)$, and the upper bound arises from that fact that if at least one of the two, x or y, is nonzero, then there is one of the two-point functions with a nonzero contribution, and the upper bound follows with (2.1.34). Therefore, (2.1.53) is smaller or equal to $5\,T\tilde T W_k$. For the second summand on the right hand side of (2.1.51), we bound from above by

$$\underbrace{5\sum_{w_2,w_3} G(0,w_2)\,G(w_2,w_3)\,\tilde G(w_3,0)}_{=\tilde T}\underbrace{\sum_{x}[1-\cos(k\cdot(x-w_3))]\,\hat G(w_3,x)\,G(w_2,w_3)}_{\le W_k}. \tag{2.1.55}$$

A combination of (2.1.51)–(2.1.55) shows

$$(II) \le 2\cdot 5\,T\tilde T W_k.$$

For the bound on (III) it is convenient to use translation invariance again yielding

$$\begin{aligned}&\sum_{\substack{u,x,\\ w_1,w_2,w_3}}[1-\cos(k\cdot x)]\,\big(1-\mathbb{1}_{\{0=u=w_1\}}\big)\,\Theta(0,u,w_1,w_2,w_3,x)\\ &=\sum_{\substack{s,x,\\ w_1,w_2,w_3}}[1-\cos(k\cdot(x-s))]\,\big(1-\mathbb{1}_{\{0=u=w_1\}}\big)\,\Theta(s,0,w_1,w_2,w_3,x).\end{aligned} \tag{2.1.56}$$

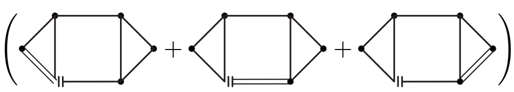
We use again (2.1.50), this time with $J = 3$ and $t_1 = s$, $t_2 = w_3$, and $t_3 = x - w_3$. This obtains $(III) \le 7\big[(IIIa) + (IIIb) + (IIIc)\big]$, where

$$(IIIa) = \sum_{\substack{s,x,\\ w_1,w_2,w_3}}[1-\cos(k\cdot s)]\,\Theta(s,0,w_1,w_2,w_3,x) \tag{2.1.57}$$

$$(IIIb) = \sum_{\substack{s,x,\\ w_1,w_2,w_3}}[1-\cos(k\cdot w_3)]\,\big(1-\mathbb{1}_{\{0=u=w_1\}}\big)\,\Theta(s,0,w_1,w_2,w_3,x) \tag{2.1.58}$$

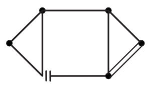
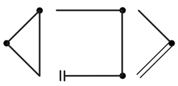
$$(IIIc) = \sum_{\substack{s,x,\\ w_1,w_2,w_3}}[1-\cos(k\cdot(x-w_3))]\,\Theta(s,0,w_1,w_2,w_3,x). \tag{2.1.59}$$

In terms of the diagrammatical representation of (2.1.43), this bound may be depicted as

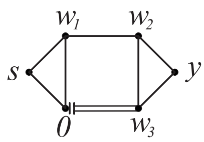
$(III) \le 7$  , (2.1.60) where a double line between two points, say v_1 and v_2, denotes a factor $[1 - \cos(k(v_2 - v_1))]\, G(v_1, v_2)$. For the third term we bound straightforwardly

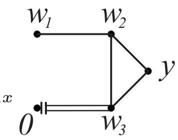
$$\begin{aligned}(IIIc) &\le \sum_{s,w_1} G(0,s)\, G(s,w_1)\, G(w_1,0) \sum_{w_2,w_3} G(w_1,w_2)\, G(w_2,w_3)\, \tilde{G}(w_3,0) \\ &\quad \times \sum_x G(w_2,x)\, G(x,w_3)\, [1 - \cos(k\cdot(x - w_3))] \\ &\le T\tilde{T}W_k,\end{aligned} \tag{2.1.61}$$

or "translated" into diagrams,

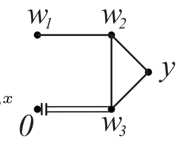
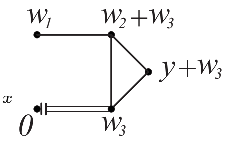
$(IIIc) \quad = \quad$  $\quad \le \quad$  $\quad = \quad T\tilde{T}W_k. \qquad (2.1.62)$

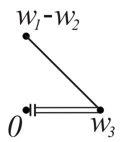
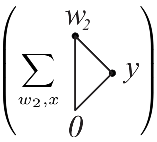
The term $(IIIa)$ obeys the same bound, as can be seen after another index shift (similar to (2.1.56)). For the bound on $(IIIb)$ we use translation invariance to shift a line in the diagram, and this works as follows:

$(IIIb) \quad = \quad \sum_{s,x,w_1,w_2,w_3} \big(1 - \mathbb{1}_{\{0=s=w_1\}}\big)$ 

$\le \quad \tilde{T} \quad \sup_{w_1} \sum_{w_2,w_3,x}$  $\qquad (2.1.63)$

by (2.1.54), whereas translation invariance yields

$\sup_{w_1} \sum_{w_2,w_3,x}$  $\quad = \quad \sup_{w_1} \sum_{w_2,w_3,x}$ 

$\le \quad \left(\sup_{w_1,w_2} \sum_{w_3}\right.$  $\left.\right) \quad \left(\sum_{w_2,x}\right.$  $\left.\right) = W_k\, T. \qquad (2.1.64)$

This shows $(IIIb) \le T\tilde{T}W_k$, and together with (2.1.60) and (2.1.62), $(III) \le 7 \cdot 3\, T\tilde{T}W_k$. We summarize that

$$\sum_x [1 - \cos(k\cdot x)]\, \pi^{(1)}(x) \le (I) + (II) + (III) \le B_k + 10\, T\tilde{T}W_k + 21\, T\tilde{T}W_k,$$

as desired. □

The inequality (3.3.42) is obtained by similar methods, [25, Prop. 4.1].

2.2 Lace expansion for self-avoiding walk

The lace expansion for self-avoiding walk was first derived by Brydges and Spencer [31]. They provide an algebraic expansion using graphs. A special class of graphs that play an important role here, the laces, gave the lace expansion its name. An alternative approach is based on an inclusion-exclusion argument, and was first derived by Slade [99]. We refer the reader to [68, Sect. 2.2.1] or [102, Sect. 3] for a full derivation of the expansion and the diagrammatic bounds. Here we shall only present a sketch of the argument, based on the presentation in [68].

The lace expansion for self-avoiding walk obtains an identity of the form

$$c_{n+1}(x) = (D * c_n)(x) + \sum_{m=2}^{n+1} (\pi_m * c_{n+1-m})(x) \tag{2.2.1}$$

for certain quantities $\pi_m(x)\colon \mathbb{Z}^d \to \mathbb{R}$, $m \geq 2$. We multiply (2.2.1) by z^{n+1} and sum over $n \geq 0$. By letting $\Phi_z(x) = \sum_{m=2}^{\infty} \pi_m(x) z^m$ and recalling $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$ this yields

$$G_z(x) = \delta_{0,x} + z(D * G_z)(x) + (G_z * \Phi_z)(x), \tag{2.2.2}$$

which is equivalent to 2.0.1 with $\tau(z) = z$ and $\Psi_z(x) = 0$.

The expansion. To derive (2.2.1), we define $R^{(1)}_{n+1}(x)$ by

$$c_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} D(y)\, c_n(x-y) - R^{(1)}_{n+1}(x). \tag{2.2.3}$$

The term $R^{(1)}_{n+1}(x)$ is the contribution of walks that contribute to the first term on the right-hand side of (2.2.3), but not on the left-hand side. Therefore, this contribution is due to paths that have at least one self-intersection. Since the first term on the right-hand side of (2.2.3) can alternatively be seen as the contribution from concatenations of a step from 0 to some y and a self-avoiding walk from y to x, this self-intersection must be at the origin. The inclusion-exclusion derivation of the lace expansion studies the correction term $R^{(1)}_{n+1}(x)$ in more detail by using inclusion-exclusion on the avoidance properties of the paths involved.

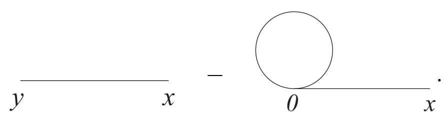
Let $\mathcal{P}^{(1)}_{n+1}(x)$ be the set of paths $\omega \in \mathcal{W}_{n+1}(x)$ which contribute to $R^{(1)}_{n+1}(x)$, i.e., the walks ω for which there exists an $l \in \{2, \ldots, n+1\}$ (depending on ω) with $\omega(l) = 0$ and $\omega(i) \neq \omega(j)$ for all $i \neq j$ with $\{i,j\} \neq \{0,l\}$. For the special case $x = 0$, $\mathcal{P}^{(1)}_{n+1}(0)$ is the set of $(n+1)$-step self-avoiding polygons. For $x \neq 0$, $\mathcal{P}^{(1)}_{n+1}(x)$ is the set of self-avoiding polygons followed by a self-avoiding walk from 0 to x, with the total length being $n+1$ and with the walk and polygon mutually avoiding. Then, by definition,

$$R^{(1)}_{n+1}(x) = \sum_{\omega \in \mathcal{P}^{(1)}_{n+1}(x)} W(\omega), \tag{2.2.4}$$

where $W(\omega) := \prod_{i=1}^{|\omega|} D(\omega_i - \omega_{i-1})$ denotes the weight of the path ω.

Diagrammatically the right-hand side of (2.2.3) can be represented by

$$\sum_{y \in \mathbb{Z}^d} D(y) \cdot$$



(2.2.5)

The line in the first term indicates an n-step walk from y to x which is unconstrained, apart from the fact that it should be self-avoiding.

We proceed by again applying the inclusion-exclusion relation to $R^{(1)}_{n+1}(x)$. Indeed, we ignore the mutual avoidance constraint of the polygon and self-avoiding walk that together form $\omega \in \mathcal{P}^{(1)}_{n+1}(x)$, and then make up for the overcounted paths by excluding the walks where the polygon and the self-avoiding walk do intersect. For $y \in \mathbb{Z}^d$, let

$$\pi^{(1)}_m(y) = \delta_{0,y} \sum_{\omega \in \mathcal{P}^{(1)}_m(0)} W(\omega) = \delta_{0,y}\,(D * c_{m-1})(0), \tag{2.2.6}$$

and define $R^{(2)}_{n+1}(x)$ by

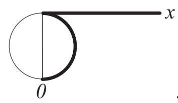
$$R^{(1)}_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} \sum_{m=2}^{n+1} \pi^{(1)}_m(y)\, c_{n+1-m}(x-y) - R^{(2)}_{n+1}(x). \tag{2.2.7}$$

The next step is to investigate $R^{(2)}_{n+1}(x)$, which involves walks consisting of a self-avoiding polygon and a self-avoiding walk from 0 to x, of total length $n+1$, where the self-avoiding polygon and the self-avoiding walk have an intersection point additional to their intersection at the origin. Let $\mathcal{P}^{(2)}_{n+1}(x)$ be the subset of walks of $\mathcal{W}_{n+1}(x)$ satisfying these requirements. Then we clearly have

$$R^{(2)}_{n+1}(x) = \sum_{\omega \in \mathcal{P}^{(2)}_{n+1}(x)} W(\omega). \tag{2.2.8}$$

Diagrammatically, we can represent (2.2.7) as follows:

$$R^{(1)}_{n+1}(x) \quad = \quad \sum_{m=2}^{n+1} (\pi^{(1)}_m * c_{n+1-m})(x) \quad - \tag{2.2.9}$$



The two thick lines are mutually avoiding, so that they together form a self-avoiding walk. The walk and polygon may intersect more than once, and we focus on the *first* intersection point.

We then again perform inclusion-exclusion, neglecting the avoidance between the portions of the self-avoiding walk before and after this first intersection, and again subtracting a correction term. Due to the fact that we look at the *first* intersection point of the self-avoiding walk and the self-avoiding polygon, the three self-avoiding walks in the Θ-shaped diagram are also *mutually avoiding* each other. We define $R^{(3)}_{n+1}(x)$ by

$$R^{(2)}_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} \sum_{m=2}^{n+1} \pi^{(2)}_m(y)\, c_{n+1-m}(x-y) - R^{(3)}_{n+1}(x), \tag{2.2.10}$$

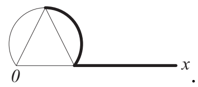
where $\pi^{(2)}_m(y)$ is defined by

$$\pi^{(2)}_m(x) = \sum_{\substack{m_1, m_2, m_3 \geq 1 \\ m_1 + m_2 + m_3 = m}} \prod_{j=1}^{3} \sum_{\omega_j \in \mathcal{W}_{m_j}(x)} W(\omega_i)\, I(\omega_1, \omega_2, \omega_3), \tag{2.2.11}$$

and $I(\omega_1, \omega_2, \omega_3)$ is equal to 1 if the ω_i are all self-avoiding and mutually avoiding each other (apart from their common start- and endpoint), and otherwise equals 0. We do not write down an explicit formula for $R^{(3)}_{n+1}(x)$, as this already gets quite involved.

This inclusion-exclusion step can be diagrammatically represented as

$$R^{(2)}_{n+1}(x) \quad = \quad \sum_{m=2}^{n+1} (\pi^{(2)}_m * c_{n+1-m})(x) \quad - \tag{2.2.12}$$



The process of using inclusion-exclusion is continued indefinitely and leads to

$$c_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} D(y)\, c_n(x-y) + \sum_{y\in\mathbb{Z}^d}\sum_{m=2}^{n+1} \pi_m(y)\, c_{n+1-m}(x-y), \tag{2.2.13}$$

where

$$\pi_m(y) = \sum_{N=1}^{\infty} (-1)^N \pi^{(N)}_m(y). \tag{2.2.14}$$

Explicit expressions for the $\pi^{(N)}$ for $N \geq 3$ are given e.g. in [68, (2.2.32)].

Diagrammatic bounds. As in (2.1.23), we write

$$\tilde{G}_z(x) = z(D * G_z)(x) \tag{2.2.15}$$

and define

$$\tilde{B}(z) := \sup_x (\tilde{G}_z * G_z)(x) \tag{2.2.16}$$

and

$$\tilde{H}_k(z) := \sup_x\, [1-\cos(k\cdot x)]\, \tilde{G}_z(x). \tag{2.2.17}$$

We further write

$$\Pi^{(N)}_z(x) := \sum_{m=2}^{\infty} \pi^{(N)}_m(x)\, z^m \tag{2.2.18}$$

so that

$$\Phi_z(x) = \sum_{N=1}^{\infty} (-1)^N\, \Pi^{(N)}_z(x). \tag{2.2.19}$$

Proposition 2.4 (Diagrammatic bounds for self-avoiding walk)**.** *For $z \geq 0$ and $N \geq 2$,*

$$\sum_x \Pi^{(1)}_z(x) \leq \|\tilde{G}_z\|_\infty, \qquad \sum_x \Pi^{(N)}_z(x) \leq \|\tilde{G}_z\|_\infty\, \tilde{B}(z)^{N-1}, \tag{2.2.20}$$

and

$$\sum_x [1-\cos(k\cdot x)]\, \Pi^{(1)}_z(x) = 0, \quad \sum_x [1-\cos(k\cdot x)]\, \Pi^{(N)}_z(x) \leq \frac{N}{2}\,(N+1)\, \tilde{H}_k(z)\, \tilde{B}(z)^{N-1}. \tag{2.2.21}$$

Inserting the bounds of the proposition into (2.2.19) yields

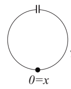
$$\sum_x |\Phi_z(x)| \quad \leq \quad \|\tilde{G}_z\|_\infty \sum_{N=1}^{\infty} \tilde{B}(z)^{N-1}, \tag{2.2.22}$$

$$\sum_x [1-\cos(k\cdot x)]\, |\Phi_z(x)| \quad \leq \quad \tilde{H}_k(z) \sum_{N=2}^{\infty} \frac{N}{2}\,(N+1)\; \tilde{B}(z)^{N-1}. \tag{2.2.23}$$

We provide a proof of the proposition for $N = 1, 2, 3$, and refer to the monograph by Slade [102, Chapter 4] for the bounds on higher order contributions.

Proof of Proposition 2.4 for $N = 1, 2, 3$. For $N = 1$ we multiply (2.2.6) by z^m and sum over $m \geq 2$ to obtain

$$\Pi_z^{(1)}(x) = \sum_{m=2}^{\infty} \pi_m^{(1)}(x)\, z^m \leq \delta_{0,x}\, \tilde{G}_z(x) = \qquad (2.2.24)$$



and hence

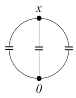
$$\sum_x \Pi_z^{(1)}(x) = \tilde{G}_z(0) \leq \|\tilde{G}_z\|_\infty, \quad \sum_x [1 - \cos(k \cdot x)]\, \Pi_z^{(1)}(x) = [1 - \cos(k \cdot 0)]\, \tilde{G}(0) = 0. \qquad (2.2.25)$$

For $N = 2$ we recall (2.2.11). Since $G_z(0) = 1$ and, for $x \neq 0$ and $n \geq 1$, $c_n(x) \leq (D * c_{n-1})(x)$ by allowing that the $(n-1)$-step walk on the right hand side returns to the origin. Consequently,

$$G_z(x) \leq \delta_{0,x} + z\,(D * G_z)(x). \qquad (2.2.26)$$

We obtain an upper bound on $\pi_m^{(2)}(x)$ if the indicator $I(\omega_1, \omega_2, \omega_3)$ is replaced by the (weaker) requirement that ω_1, ω_2, and ω_3 are self-avoiding (but not necessarily mutually avoiding). Whence,

$$\Pi_z^{(2)}(x) \leq \quad = (G_z(x) - \delta_{0,x})^3, \qquad (2.2.27)$$



where the lines in the diagram indicate three two-point functions that cannot shrink to a single point (because the three walks in (2.2.11) have length at least 1). Consequently, by (2.2.26),

$$\sum_x \Pi_z^{(2)}(x) \leq \sum_x \tilde{G}_z(x)^2\, G_z(x) \leq \|\tilde{G}_z(x)\|_\infty \sum_x \tilde{G}_z(x)\, G_z(x) = \|\tilde{G}_z(x)\|_\infty\, \tilde{B}(z) \qquad (2.2.28)$$

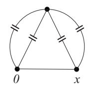
and

$$\sum_x [1 - \cos(k \cdot x)]\, \Pi_z^{(2)}(x) \leq \sum_x [1 - \cos(k \cdot x)]\, \tilde{G}_z(x)^2\, G_z(x) \leq \tilde{H}_k(z)\, \tilde{B}(z), \qquad (2.2.29)$$

as desired.

For $N = 3$, there is no explicit expression given here, but it is suggested in (2.2.12) (and shown in [102, p. 21]) that $\pi_z^{(3)}(x)$ can be expressed by

$$\pi_z^{(3)}(x) = \quad . \qquad (2.2.30)$$



The unlabeled vertex on the top of the diagram is summed over, and lines with a double dash indicate walks of length ≥ 1. There are mutual avoidances between some of these walks, and we obtain an upper bound by neglecting them. By multiplying z^m and summing over $m \geq 2$, and using (2.2.26),

$$\Pi_z^{(3)}(x) \leq \sum_y G_z(x)\, (G_z(y) - \delta_{0,y})^2\, (G_z(y - x) - \delta_{x,y})^2 \leq G_z(x) \Big((\tilde{G}_z\, G_z) * (\tilde{G}_z)^2 \Big)(x). \qquad (2.2.31)$$

We bound this expression from above by

$$\sum_x \Pi_z^{(3)}(x) \leq \underbrace{\sum_y \tilde{G}_z(y)\, G_z(y)}_{\leq \tilde{B}(z)} \underbrace{\sum_x G_z(x) \tilde{G}_z(y - x)}_{\leq \tilde{B}(z)} \underbrace{\tilde{G}_z(y - x)}_{\leq \|\tilde{G}\|_\infty}. \qquad (2.2.32)$$

and similarly $\sum_x [1 - \cos(k \cdot x)]\, \Pi_z^{(3)}(x) \leq H_k(z)\, B(z)^2$. □

2.3 Lace expansion for the Ising model

The lace expansion for the Ising model has been established recently by Sakai [95]. It is similar in spirit to a high-temperature expansion. The starting point is to rewrite the two-point function (spin-spin correlation) using the random-current representation. This gives rise to a representation involving bonds, in that showing some similarities to a percolation configuration. The lace expansion is then performed using ideas from the lace expansion for percolation, however, it is considerably more involved.

For the Ising model on a finite graph Λ, Sakai in [95, Prop. 1.1] proved the expansion formula

$$G_z^\Lambda(x) = \delta_{0,x} + \tau(z)\left(D * G_z^\Lambda\right)(x) + \tau(z)\left(D * \Pi_M^\Lambda * G_z^\Lambda\right)(x) + \Pi_M^\Lambda(x) + R_M^\Lambda(x), \quad (2.3.1)$$

where the z-dependence of Π_M^Λ and R_M^Λ is omitted from the notation. Note that R_M^Λ here corresponds to $(-1)^{M+1} R_{p;\Lambda}^{(M+1)}$ in [95]. Here M refers to the level of the expansion, and G_z^Λ denotes the finite-volume two-point function. This is equivalent to (2.0.1) if we let

$$\Phi_z^\Lambda(x) = \tau(z)(D * \Pi_M^\Lambda)(x), \qquad x \in \mathbb{Z}^d, \quad (2.3.2)$$

$$\Psi_z^\Lambda(x) = \Pi_M^\Lambda(x) + R_M^\Lambda(x), \qquad x \in \mathbb{Z}^d, \quad (2.3.3)$$

then choose M so large that (3.3.54) and (3.3.55) below are satisfied for a certain K, say $K = 4$, and subsequently taking the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$. Note that, if comparing (2.3.1) to [95, (1.11)], we explicitly extract the $\delta_{0,x}$-term from the Π-term in [95], i.e., $\Pi_{p;\Lambda}^{(M)}(x)$ in [95] corresponds to $\Pi_M^\Lambda(x) + \delta_{0,x}$ in this thesis.

We have chosen not to demonstrate the expansion for the Ising model, nor the derivation of the diagrammatic bounds. Instead we refer to [95]. Since already the statement of the bounding diagrams is quite complicated, we defer the statement of the bounds to the appendix. After the statement of the diagrammatic bounds in Proposition A.2 we also show how they imply bounds on the lace expansion coefficients Φ and Ψ, needed for Assumption 3.2 below.

Chapter 3
Convergence of the lace expansion and the infrared bound

In Chapter 2 we have performed the lace expansion and derived diagrammatic bounds. The aim of this chapter is to use the results of Chapter 2 to prove an infrared bound for the two-point function. This infrared bound is formulated in Theorem 3.7, and this is the main result of the chapter. As an application of the infrared bound we derive critical exponents in Section 3.5. In fact, the lace expansion has been used to prove various other results concerning the critical behavior of these models, like the identification of scaling limits and the correlation length in the critical case. A short overview of (some of) these results is given in Section 3.6.

In order to study the various models in a unified way, we set up a generalized framework. Our strategy is as follows: We make two assumptions in terms of the general framework, Assumptions 3.1 and 3.2. Subsequently we prove the infrared bound in terms of the general framework. Assumption 3.1 is actually an assumption on the step distribution D. We prove in Section 3.2 that it is satisfied for the nearest-neighbor model if d is sufficiently large, and for so-called spread-out models (to be defined in Section 3.1) if $d > d_c$ and the spread-out parameter L is sufficiently large. Assumption 3.2 concerns bounds on the lace expansion coefficients that were introduced in the previous chapter. The proof that Assumption 3.2 is satisfied by percolation, self-avoiding walk and the Ising model uses specific model-dependent properties, and is therefore performed separately for each of the three models in Section 3.3. Nevertheless, Assumption 3.2 provides a bound that holds simultaneously for all three models, and this is what we use in Section 3.4 to prove the infrared bound.

Fourier analysis is an important tool in this chapter. Unless specified otherwise, k will always denote an element from the Fourier dual of the discrete lattice, which is the torus $[-\pi,\pi)^d$. The Fourier transform of a summable function $f\colon \mathbb{Z}^d \to \mathbb{C}$ is defined as $\hat f(k) = \sum_{x\in\mathbb{Z}^d} f(x)\, \mathrm{e}^{ik\cdot x}$, and its inverse Fourier transform is given by

$$f(x) = \int_{k\in[-\pi,\pi)^d} \mathrm{e}^{-ik\cdot x}\; \hat f(k)\, \frac{\mathrm{d}k}{(2\pi)^d}.$$

We benefit from the fact that the Fourier transform of a convolution is the product of the two Fourier transforms: $\widehat{f*g} = \hat f\, \hat g$. In accordance with the usual notation of their elements, we will refer to $\mathbb{Z}^d$ as x-space and to the Fourier dual $[-\pi,\pi)^d$ as k-space.

We use the following Landau symbols throughout the thesis. For functions f,g, we write $f(x) = O\big(g(x)\big)$ for a limit $x\to a$ if $|f(x)/g(x)|$ is bounded as $x\to a$. By $f(x) = o\big(g(x)\big)$ we denote the stronger version $|f(x)/g(x)| \to 0$. For example, $O(1)$ represents a uniformly bounded term, and $o(1)$ denotes a vanishing term.

3.1 The step distribution D: 3 versions

Let D denote a probability distribution on $\mathbb{Z}^d$ that is symmetric under reflections in coordinate hyperplanes and rotations by $\pi/2$. We refer to D as a *step* distribution, having in mind a random walker taking independent steps distributed according to D. Without loss of generality we henceforth assume that there is no mass at the origin, i.e. $D(0) = 0$.

We consider three different versions of D. While we explicitly state our main results for these versions, they actually hold more generally under a random walk condition formulated in Assumption 3.1 below. The first version is the *nearest-neighbor model*, where D is the uniform distribution on the nearest neighbors, i.e.,

$$D(x) = \frac{1}{2d} \mathbb{1}_{\{|x|=1\}}, \qquad x \in \mathbb{Z}^d. \tag{3.1.1}$$

This nearest-neighbor version of D corresponds to the classical model for the study of self-avoiding walk, percolation, and the Ising model, see e.g. [41, 50, 88].

We further consider two versions of *spread-out* models. They involve some spread-out parameter L, which is typically chosen large. In order to stress the L-dependence of D we will write D_L in the definitions, but later omit the subscript. In the *finite-variance spread-out* model we require D_L to satisfy the following conditions[1]:

(D1) There is an $\varepsilon > 0$ such that

$$\sum_{x \in \mathbb{Z}^d} |x|^{2+\varepsilon} D_L(x) < \infty.$$

(D2) There is a constant C such that, for all $L \geq 1$,

$$\|D_L\|_\infty \leq C L^{-d}.$$

(D3) There exist constants c_1, $c_2 > 0$ such that

$$1 - \hat{D}_L(k) \geq c_1 L^2 |k|^2 \qquad \text{if } \|k\|_\infty \leq L^{-1}, \tag{3.1.2}$$

$$1 - \hat{D}_L(k) > c_2 \qquad \text{if } \|k\|_\infty \geq L^{-1}, \tag{3.1.3}$$

$$1 - \hat{D}_L(k) < 2 - c_2, \qquad k \in [-\pi, \pi)^d. \tag{3.1.4}$$

Example. Let h be a non-negative bounded function on $\mathbb{R}^d$ which is almost everywhere continuous, and symmetric under the lattice symmetries of reflection in coordinate hyperplanes and rotations by ninety degrees. Assume that there is an integrable function H on $\mathbb{R}^d$ with $H(te)$ non-increasing in $t \geq 0$ for every unit vector $e \in \mathbb{R}^d$, such that $h(x) \leq H(x)$ for all $x \in \mathbb{R}^d$. Assume further that the $(2+\varepsilon)$-th moment of h exists for some $\varepsilon > 0$. The monotonicity and integrability hypotheses on H imply that $\sum_x h(x/L) < \infty$ for all L, with $x/L = (x_1/L, \ldots, x_d/L)$. Then

$$D_L(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)}, \qquad x \in \mathbb{Z}^d, \tag{3.1.5}$$

obeys the conditions (D1)–(D3), whenever L is large enough (cf. [71, Appendix A]). For $h(x) = \mathbb{1}_{\{0 < \|x\|_\infty \leq 1\}}$ we obtain the *uniform spread-out model* with

$$D_L(x) = \frac{1}{(2L+1)^d - 1} \mathbb{1}_{\{0 < \|x\|_\infty \leq L\}}, \qquad x \in \mathbb{Z}^d. \tag{3.1.6}$$

In the *spread-out power-law model* we replace assumptions (D1) and (D3) by the condition that there exists an $\alpha > 0$ such that

(D1′) all $\varepsilon > 0$ satisfy

$$\sum_{x \in \mathbb{Z}^d} |x|^{\alpha - \varepsilon} D_L(x) < \infty;$$

[1] These conditions coincide with Assumption D in [71].

(D3′) there exist constants $c_1, c_2 > 0$ such that

$$1 - \hat{D}_L(k) \geq c_1 L^\alpha |k|^\alpha \qquad \text{if } \|k\|_\infty \leq L^{-1}, \tag{3.1.7}$$

$$1 - \hat{D}_L(k) > c_2 \qquad \text{if } \|k\|_\infty \geq L^{-1}, \tag{3.1.8}$$

$$1 - \hat{D}_L(k) < 2 - c_2, \qquad k \in [-\pi, \pi)^d. \tag{3.1.9}$$

The condition (D2)=(D2′) remains unchanged.

As an example, let D_L be of the form (3.1.5), but instead of the existence of the $(2+\varepsilon)$-th moment of h, require h to decay as $|x|^{-d-\alpha}$ as $|x| \to \infty$. In particular, there exist positive constants c_h and l_h such that

$$h(x) \geq c_h |x|^{-d-\alpha}, \qquad \text{whenever } |x| \geq l_h. \tag{3.1.10}$$

In this setting, the κ^{th} moment $\sum_{x\in\mathbb{Z}^d} |x|^\kappa D_L(x)$ does not exist if $\kappa \geq \alpha$, but exists and equals $O(L^\alpha)$ if $\kappa < \alpha$. Take e.g.

$$h(x) = (|x| \vee 1)^{-d-\alpha}, \tag{3.1.11}$$

so that D_L has the form

$$D_L(x) = \frac{(|x/L| \vee 1)^{-d-\alpha}}{\sum_{y\in\mathbb{Z}^d} (|y/L| \vee 1)^{-d-\alpha}}, \qquad x \in \mathbb{Z}^d. \tag{3.1.12}$$

Chen and Sakai [33, Prop. 1.1] showed that, analogously to the finite-variance spread-out model, the spread-out power-law model (3.1.12) satisfies conditions (D1′)–(D3′).

Note that the spread-out power-law model with parameter $\alpha > 2$ satisfies the finite variance condition (D1), and hence is covered in the finite variance case. For simplicity we further write $\alpha \wedge 2$ indicating the minimum of α and 2 in the spread-out power-law case, and 2 in the nearest-neighbor case or in the finite-variance spread-out case.

For the finite-variance spread-out model and the spread-out power-law model we require that the support of D contains the nearest neighbors of 0, see the discussion below (1.4.3).

3.2 The random walk condition

We start this section with preliminary considerations about random walks. Given a step distribution D, we consider the *random walk two-point function* or *Green's function* of the random walk defined by

$$C_z(x) = \sum_{n=0}^{\infty} D^{*n}(x)\, z^n, \tag{3.2.1}$$

where D^{*n} is the n-fold convolution of D and $D^{*0}(x)\, z^0 = \delta_{0,x}$. By conditioning on the first step we obtain

$$C_z(x) = \delta_{0,x} + z\,(D * C_z)\,(x). \tag{3.2.2}$$

Taking the Fourier transform and solving for $\hat{C}_z(k)$ yields

$$\hat{C}_z(k) = \frac{1}{1 - z\hat{D}(k)}, \qquad z < 1. \tag{3.2.3}$$

We define the *model parameter* s to be 2 for self-avoiding walk or the Ising model, and 3 for percolation. This choice is motivated by the fact that the lace expansion coefficients for percolation are bounded by the triangle diagram $T(z) = G_z^{*3}(0)$ and derivations thereof, whereas the lace expansion coefficients for self-avoiding walk and the Ising model are bounded by quantities related to the bubble diagram $B(z) = G_z^{*2}(0)$. We make an assumption on the step distribution D.

Assumption 3.1 (Random walk s-condition)**.** *There exists $\beta > 0$ such that*

$$\sup_{x\in\mathbb{Z}^d} D(x) \leq \beta \tag{3.2.4}$$

and

$$\int_{[-\pi,\pi)^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^s}\,\frac{\mathrm{d}k}{(2\pi)^d} \leq \beta. \tag{3.2.5}$$

We will later need that Assumption 3.1 holds for β sufficiently small, and the specific amount of smallness required in (3.2.4)–(3.2.5) will be specified in the proofs in Section 3.4. Be aware that the *critical exponents* β_{P} and β_{I} have no relation with the β introduced here.

For $s = 2$ we call (3.2.5) the random walk bubble condition. This is inspired by the fact that its x-space analogue reads

$$(D * C_1 * D * C_1)(0) \leq \beta. \tag{3.2.6}$$

In other words, we have an (ordinary) random walk from 0 to x of at least one step, and a second walk from x to 0 and subsequently sum over x. Correspondingly, for $s = 3$, we obtain the x-space representation

$$(C_1 * D * C_1 * D * C_1)(0) \leq \beta, \tag{3.2.7}$$

and refer to (3.2.5) as the random walk triangle condition. See the graphical representation in Figure 3.1.

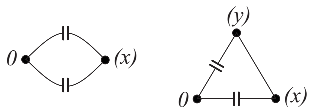


Figure 3.1: Graphical representation of the random walk bubble diagram in (3.2.6) and the random walk triangle diagram in (3.2.7). A line between two points, say x and y, represents the two-point function $C_1(y-x)$, a line with a double dash in the middle requires at least one step, e.g. a line between 0 and x represents $(D * C_1)(x)$. Vertices labeled in brackets are summed over $\mathbb{Z}^d$.

Proposition 3.1. *Assumption 3.1 is satisfied for arbitrarily small β if d is chosen sufficiently large in the nearest-neighbor model (at least $d > 4s$) or $d > d_c = s(\alpha \wedge 2)$ and L is sufficiently large in the spread-out models. More specifically, the assumption holds with $\beta = O(d^{-1})$ in the nearest-neighbor case, and $\beta = O(L^{-d})$ in the spread-out cases.*

A proof of Proposition 3.1 is contained in [25, Sect. 2.2.2], where finite tori are considered. Restriction to the infinite lattice gives rise to a noteworthy simplification, which we shall present in the following.

Proof of Proposition 3.1 for the nearest-neighbor model. We follow [25, Sect. 2.2.2]. Since $\|D\|_\infty = (2d)^{-1}$, the bound (3.2.4) is satisfied for d sufficiently large, and it remains to prove (3.2.5).

By the symmetry of D we have

$$\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x)\,\cos(k\cdot x) = \frac{1}{d}\sum_{j=1}^{d}\cos(k_j), \qquad k = (k_1,\dots,k_d) \in [-\pi,\pi)^d. \tag{3.2.8}$$

Since $1-\cos t \geq 2\pi^{-2}t^2$ for $|t| \leq \pi$, this implies the infrared bound

$$1-\hat{D}(k) \geq \frac{2}{\pi^2}\frac{|k|^2}{d}. \tag{3.2.9}$$

The Cauchy-Schwarz inequality[2] yields

$$\begin{aligned}&\int_{[-\pi,\pi)^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^s}\frac{\mathrm{d}k}{(2\pi)^d} \\ &\leq \left(\int_{[-\pi,\pi)^d} \hat{D}(k)^4\,\frac{\mathrm{d}k}{(2\pi)^d}\right)^{1/2}\left(\int_{[-\pi,\pi)^d} \frac{1}{[1-\hat{D}(k)]^{2s}}\frac{\mathrm{d}k}{(2\pi)^d}\right)^{1/2}\end{aligned} \tag{3.2.10}$$

First we show that the first term on the right hand side of (3.2.10) is small if d is large. Note that $\int_{[-\pi,\pi)^d} \hat{D}(k)^4\ (2\pi)^{-d}\,\mathrm{d}k = D^{*4}(0)$ is the probability that a nearest-neighbor random walk returns to its starting point after the fourth step. This is bounded from above by $c(2d)^{-2}$ with c being a well-chosen constant, because the first two steps must be compensated by the last two. Finally, the square root yields the upper bound $O(d^{-1})$.

It remains to show that the second term on the right of (3.2.10) is bounded uniformly in d. The infrared bound (3.2.9) gives

$$\int_{[-\pi,\pi)^d} \frac{1}{[1-\hat{D}(k)]^{2s}}\frac{\mathrm{d}k}{(2\pi)^d} \leq \frac{\pi^{4s}}{2^{2s}}\int_{[-\pi,\pi)^d} \frac{d^{\,2s}}{|k|^{4s}}\frac{\mathrm{d}k}{(2\pi)^d}. \tag{3.2.11}$$

The right hand side of (3.2.11) is finite if $d > 4s$. For $A > 0$ and $m > 0$,

$$\frac{1}{A^m} = \frac{1}{\Gamma(m)}\int_0^\infty t^{m-1}\,\mathrm{e}^{-tA}\,\mathrm{d}t. \tag{3.2.12}$$

Applying this with $A = |k|^2/d$ and $m = 2s$ yields

$$\frac{1}{\Gamma(2s)}\frac{\pi^{4s}}{2^{2s}}\int_0^\infty t^{2s-1}\left(\int_{-\pi}^{\pi}\left(\mathrm{e}^{-t\theta^2}\right)^{1/d}\frac{\mathrm{d}\theta}{2\pi}\right)^d \mathrm{d}t \tag{3.2.13}$$

as an upper bound for (3.2.11). This is non-increasing in d, because $\|f\|_p \leq \|f\|_q$ for $0 < p \leq q \leq \infty$ on a probability space by Lyapunov's inequality. □

Proof of Proposition 3.1 for the spread-out models. We again follow [25, Sect. 2.2.2]. Obviously (3.2.4) is implied by condition (D2)/(D2′) for sufficiently large L, hence it remains to prove (3.2.5).

The power-law spread-out model with $\alpha > 2$ satisfies the finite variance condition (D1) with $\varepsilon < \alpha - 2$. Note further that (D3) and (D3′) agree when the exponent in the first inequality is taken $\alpha \wedge 2$.

We separately consider the regions $\|k\|_\infty \leq L^{-1}$ and $\|k\|_\infty > L^{-1}$. By (3.1.2), (3.1.7) and the bound $\hat{D}(k)^2 \leq 1$, the corresponding contributions to the integral are

$$\int_{k:\|k\|_\infty \leq L^{-1}} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^s}\frac{\mathrm{d}k}{(2\pi)^d} \leq \frac{1}{c_1^s L^{(\alpha\wedge 2)s}}\int_{k:\|k\|_\infty \leq L^{-1}} \frac{1}{|k|^{(\alpha\wedge 2)s}}\frac{\mathrm{d}k}{(2\pi)^d} \leq C_{d,c_1}L^{-d} \tag{3.2.14}$$

[2] The Hölder inequality gives better bounds here. In particular, it requires $d > 2s$ only, cf. (2.19) in [25].

if $d > (\alpha \wedge 2)s$, where C_{d,c_1} is a constant depending (only) on d and c_1, and by (3.1.3), (3.1.8),

$$\int_{k:\|k\|_\infty > L^{-1}} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^s} \frac{\mathrm{d}k}{(2\pi)^d} \le c_2{}^{-s} \int_{k:\|k\|_\infty > L^{-1}} \hat{D}(k)^2 \frac{\mathrm{d}k}{(2\pi)^d} \le O\big(L^{-d}\big). \tag{3.2.15}$$

In the last step we used assumption (D2) / (D2′) to see that

$$\begin{aligned} \int_{k\in[-\pi,\pi)^d} \hat{D}(k)^2 \frac{\mathrm{d}k}{(2\pi)^d} &= (D*D)(0) = \sum_{y\in\mathbb{Z}^d} D(y)^2 \\ &\le \sum_{y\in\mathbb{Z}^d} D(y)\, \|D\|_\infty = \|D\|_\infty \le O\big(L^{-d}\big). \end{aligned} \tag{3.2.16}$$

□

3.3 Bounds on the lace expansion coefficients

We recall that the function $\tau\colon z \mapsto \tau(z)$ is given by $\tau(z) = z$ for self-avoiding walk and percolation, and

$$\tau(z) = \sum_{y\in\mathbb{Z}^d} \tanh(zJ(y)) \tag{3.3.1}$$

for the Ising model, cf. (2.0.1).

Assuming the existence of $\hat{\Phi}_z(k)$ and $\hat{\Psi}_z(k)$, we apply Fourier transformation to the identity in (2.0.1) to obtain

$$\hat{G}_z(k) = \frac{1+\hat{\Psi}_z(k)}{1-\tau(z)\hat{D}(k)-\hat{\Phi}_z(k)}, \qquad z < z_c. \tag{3.3.2}$$

We will see that, for $z = 0$, $\hat{\Psi}_0(k) \equiv 0$ and $\hat{\Phi}_0(k) \equiv 0$ for all three models. Eq. (3.3.2) suggests that moreover $\hat{G}_z(k) \approx \big(1-\tau(z)\hat{D}(k)\big)^{-1}$ if $\hat{\Psi}_z(k)$ and $\hat{\Phi}_z(k)$ are sufficiently small, and this is what we aim to prove now.

We introduce the quantity

$$\lambda_z := 1 - \frac{1}{\hat{G}_z(0)} = 1 - \frac{1}{\chi(z)} \in [0,1]. \tag{3.3.3}$$

Then λ_z satisfies the equality

$$\hat{G}_z(0) = \hat{C}_{\lambda_z}(0). \tag{3.3.4}$$

The idea of the proof of the infrared bound is motivated by the intuition that $\hat{G}_z(k)$ and $\hat{C}_{\lambda_z}(k)$ are comparable in size and, moreover, the discretized second derivative

$$\Delta_k \hat{G}_z(l) := \hat{G}_z(l-k) + \hat{G}_z(l+k) - 2\hat{G}(l) \tag{3.3.5}$$

is bounded by

$$U_{\lambda_z}(k,l) := \frac{200}{\hat{C}_{\lambda_z}(k)} \Big\{ \hat{C}_{\lambda_z}(l-k)\hat{C}_{\lambda_z}(l) + \hat{C}_{\lambda_z}(l)\hat{C}_{\lambda_z}(l+k) + \hat{C}_{\lambda_z}(l-k)\hat{C}_{\lambda_z}(l+k) \Big\}. \tag{3.3.6}$$

More precisely, we will show that the function $f\colon [0, z_c) \to \mathbb{R}$, defined by

$$f := f_1 \vee f_2 \vee f_3 \tag{3.3.7}$$

with

$$f_1(z) := \tau(z), \qquad f_2(z) := \sup_{k \in [-\pi,\pi)^d} \frac{\hat{G}_z(k)}{\hat{C}_{\lambda_z}(k)}, \tag{3.3.8}$$

and

$$f_3(z) := \sup_{k,l \in [-\pi,\pi)^d} \frac{|\Delta_k \hat{G}_z(l)|}{U_{\lambda_z}(k,l)}, \tag{3.3.9}$$

is small, given that β in Assumption 3.1 is sufficiently small. To make this rigorous, we need the following assumption:

Assumption 3.2 (Bounds on the lace expansion coefficients)**.** *If Assumption 3.1 holds and, for some $K > 0$, the inequality $f(z) \le K$ holds uniformly for $z \in (0, z_c)$, then there exists a constant $c_K > 0$ (independent of β) such that, for all $k \in [-\pi,\pi)^d$,*

$$\left|\hat{\Psi}_z(k)\right| \le c_K \beta, \qquad \left|\hat{\Phi}_z(k)\right| \le c_K \beta \tag{3.3.10}$$

and

$$\sum_x [1-\cos(k\cdot x)]\,|\Psi_z(x)| \le c_K\beta\,\hat{C}_{\lambda_z}(k)^{-1}, \quad \sum_x [1-\cos(k\cdot x)]\,|\Phi_z(x)| \le \tau(z)\, c_K\beta\,\hat{C}_{\lambda_z}(k)^{-1} \tag{3.3.11}$$

where Φ_z and Ψ_z refer to the model-dependent coefficients in the expansion formula (2.0.1).

Indeed, the assumption is satisfied by our three models:

Proposition 3.2. *If Assumption 3.1 holds for a value of β that is sufficiently small, then also Assumption 3.2 holds for self-avoiding walk, percolation and the Ising model.*

The relevant bounds have been proven by Slade [102] for self-avoiding walk, by Borgs et al. [25] for percolation (on finite graphs), and by Sakai [95] for the Ising model. We demonstrate the proof of Proposition 3.2 subject to the diagrammatic bounds in Chapter 2.

3.3.1 Preliminaries on the proof of Proposition 3.2

Before we start proving the proposition separately for each of the three models, we formulate (and prove) some general bounds that hold simultaneously for all three models.

We occasionally make use of the bound

$$0 \le 1 - \hat{D}(k) \le 2\hat{C}_{\lambda_z}(k)^{-1}, \qquad k \in [-\pi,\pi)^d, \tag{3.3.12}$$

which itself is a consequence of

$$0 \le \hat{C}_{\lambda_z}(k)\,[1 - \hat{D}(k)] = 1 + \frac{\lambda_z - 1}{1 - \lambda_z \hat{D}(k)} \hat{D}(k) \le 2. \tag{3.3.13}$$

Lemma 3.3. *Assume that for a model we have $f(z) \le K$ for some $K > 1$, uniformly for $z \in (0, z_c)$. Assume further that Assumption 3.1 holds for a certain value of $s = 2, 3, \ldots$, and*

$$G_z(x) \le \delta_{0,x} + \tilde{G}_z(x) = \delta_{0,x} + \tau(z)\,(D * G_z)(x), \tag{3.3.14}$$

as well as $G_z(0) = 1$. *Then the following bounds hold:*

$$G_z^{*s}(x) \leq \delta_{0,x} + K^2 s^2 (2K)^s \beta \tag{3.3.15}$$

$$\sup_x (\tilde{G}_z * G_z^{*(s-1)})(x) \leq K^2 s (2K)^s \beta. \tag{3.3.16}$$

We note that (3.3.14) is satisfied by percolation and self-avoiding walk with $\tau(z) = z$ (cf. (2.1.34) and (2.2.26)), and for the Ising model with $\tau(z)$ given by (3.3.1).

Proof. We first prove (3.3.16). For $x \in \mathbb{Z}^d$,

$$\tilde{G}_z * G_z^{*s}(x) = \tau(z) \sum_{w_1,\ldots,w_s} D(w_1)\, G_z(w_2 - w_1) \ldots G_z(w_s - w_{s-1})\, G_z(x - w_s). \tag{3.3.17}$$

We split the sum in (3.3.17) into two summands, one term corresponding to $w_1 = \cdots = w_s = x$, and the second term for the remaining part $(w_2 - w_1, \ldots, w_s - w_{s-1}, x - w_s) \neq (0, \ldots, 0)$. This yields

$$\tilde{G}_z * G_z^{*s}(x) \leq \tau(z)\, D(x) + \tau(z)^2 s \left(D^{*2} * G_z^{*s}\right)(x) \tag{3.3.18}$$

by (3.3.14), where the factor s comes from the fact that there are s possibilities for a nonzero displacement. We now bound $\tau(z) \leq K$ by using our assumption $f_1(z) \leq K$, bound further $D(x) \leq \beta$ by (3.2.4) and

$$\begin{aligned}
(D^{*2} * G_z^{*s})(x) &= \int_{[-\pi,\pi)^d} \mathrm{e}^{-ik\cdot x}\, \hat{D}(k)^2\, \hat{G}_z(k)^s \frac{\mathrm{d}k}{(2\pi)^d} \\
&\leq K^s \int_{[-\pi,\pi)^d} \mathrm{e}^{-ik\cdot x}\, \hat{D}(k)^2\, \hat{C}_{\lambda_z}(k)^s \frac{\mathrm{d}k}{(2\pi)^d} \\
&\leq (2K)^s \int_{[-\pi,\pi)^d} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^s} \frac{\mathrm{d}k}{(2\pi)^d} \\
&\leq (2K)^s \beta,
\end{aligned} \tag{3.3.19}$$

by Parseval's identity in the first line, $f_2(z) \leq K$ in the second line, (3.3.12) in the third line, and (3.2.5) in the last line. This proves (3.3.16).

For (3.3.15), we proceed similarly. The sum in

$$G_z^{*s}(x) = \sum_{w_1,\ldots,w_{s-1}} G_z(w_1)\, G_z(w_2 - w_1) \ldots G_z(w_s - w_{s-1})\, G_z(x - w_s) \tag{3.3.20}$$

contributes 1 if $(w_1, w_2 - w_1, \ldots, w_s - w_{s-1}, x - w_s) = (0, \ldots, 0)$. Otherwise, there is a nonzero displacement for at least one of the two-point functions, such that $G_z \leq \tilde{G}_z$ for this. Hence,

$$G_z^{*s}(x) \leq \delta_{0,x} + s \left(\sup_x \tilde{G}_z * G_z^{*(s-1)}(x) \right), \tag{3.3.21}$$

which proves (3.3.15). □

Lemma 3.4. *Under the assumptions of Lemma 3.3,*

$$\sum_x [1 - \cos(k\cdot x)]\, \tilde{G}_z(x)\, G_z^{*(s-2)}(y - x) \leq \tau(z)\, \hat{C}_{\lambda_z}(k)^{-1} \begin{cases} O(\beta) & \text{if } y = 0; \\ O(1) & \text{if } y \neq 0. \end{cases} \tag{3.3.22}$$

The constants in the O-terms depend only on K and s, and we use the convention that $G_z^{*0}(x) = \delta_{0,x}$.

Proof. We use (2.1.50) for $n = 2$ yielding

$$[1-\cos(k\cdot x)]\,\tilde G_z(x) = \tau(z)\sum_w [1-\cos(k\cdot x)]\,D(w)\,G_z(x-w)$$
$$\le 5\tau(z)\sum_w \Big([1-\cos(k\cdot w)]\,D(w)\,G_z(x-w) + [1-\cos(k\cdot(x-w))]\,D(w)\,G_z(x-w)\Big). \tag{3.3.23}$$

We abbreviate $G_{z,k}(x) = \cos(k\cdot x)\,G(x)$ and use (3.3.23) to obtain

$$\sum_x [1-\cos(k\cdot x)]\,\tilde G_z(x)\,G_z^{*(s-2)}(y-x) \;\le\; 5\tau(z)\sum_{w\neq 0}[1-\cos(k\cdot w)]\,D(w)\,G_z^{*(s-1)}(y-w)$$
$$+\,5\tau(z)\,\big(D*(G_z-G_{z,k})*G_z^{*(s-2)}\big)(y). \tag{3.3.24}$$

We first consider the first summand on the right hand side of (3.3.24). There is no contribution to the sum for $w = 0$, because $1-\hat D(0) = 0$. Furthermore,

$$\sum_w [1-\cos(k\cdot w)]\,D(w) = 1-\hat D(k) \le 2\hat C_{\lambda_z}(k)^{-1} \tag{3.3.25}$$

by (3.3.12), whereas

$$\sup_{w\neq 0} G_z^{*(s-1)}(y-w) \le \delta_{w,y} + O(\beta) \tag{3.3.26}$$

by (3.3.15). Therefore,

$$\sum_{w\neq 0}[1-\cos(k\cdot w)]\,D(w)\,G_z^{*(s-1)}(y-w) \le \hat C_{\lambda_z}(k)^{-1}\begin{cases} O(\beta) & \text{if } y = 0,\\ O(1) & \text{if } y\neq 0,\end{cases} \tag{3.3.27}$$

We now consider the second summand on the right hand side of (3.3.24). We note that the Fourier transform of $G_z - G_{z,k}(x)$ is $\hat G_z(l) - 1/2\,(\hat G_z(l-k)+\hat G_z(l+k))$, and this is equal to $-1/2\,\Delta_k \hat G_z(l)$. Whence by Parseval's identity,

$$\big(D*(G_z-G_{z,k})*G_z^{*(s-2)}\big)(y) = \int_{[-\pi,\pi)^d} \mathrm{e}^{-il\cdot y}\,\hat D(l)\left(-\frac12\Delta_k\,\hat G_z(l)\right)\hat G_z(l)^{s-2}\,\frac{\mathrm{d}l}{(2\pi)^d}. \tag{3.3.28}$$

Our bounds $f_2 \le K$ and $f_3 \le K$ imply that (3.3.28) is bounded above by

$$\frac{100\,K^{s-1}}{\hat C_{\lambda_z}(k)}\int_{[-\pi,\pi)^d}\hat D(l)\Big(\hat C_{\lambda_z}(l-k)\,\hat C_{\lambda_z}(l) + \hat C_{\lambda_z}(l)\,\hat C_{\lambda_z}(l+k)$$
$$+\,\hat C_{\lambda_z}(l-k)\,\hat C_{\lambda_z}(l+k)\Big)\,\hat C_{\lambda_z}(l)^{s-2}\,\frac{\mathrm{d}l}{(2\pi)^d}. \tag{3.3.29}$$

Denoting $C_{\lambda_z,k}(x) := \cos(k\cdot x)\,C_{\lambda_z}(x)$, we observe that $|C_{\lambda_z,k}(x)| \le C_{\lambda_z}(x)$ and

$$\hat C_{\lambda_z,k}(l) = \frac12\left(\hat C_{\lambda_z}(l-k)+\hat C_{\lambda_z}(l+k)\right). \tag{3.3.30}$$

Since $\hat C_{\lambda_z}(l-k)\,\hat C_{\lambda_z}(l) + \hat C_{\lambda_z}(l)\,\hat C_{\lambda_z}(l+k) = \hat C_{\lambda_z}(l)\,\hat C_{\lambda_z,k}(l)$ and

$$\hat C_{\lambda_z}(l-k)\,\hat C_{\lambda_z}(l+k) \;=\; \frac14\Big[\hat C_{\lambda_z}(l-k)+\hat C_{\lambda_z}(l+k)\Big]^2 - \frac14\Big[\hat C_{\lambda_z}(l-k)-\hat C_{\lambda_z}(l+k)\Big]^2$$
$$\le\; \frac14\Big[\hat C_{\lambda_z}(l-k)+\hat C_{\lambda_z}(l+k)\Big]^2 = \hat C_{\lambda_z,k}(l)^2, \tag{3.3.31}$$

we can use again Parseval's identity to bound (3.3.29) further from above by

$$\begin{aligned}100\,K^{s-1}\hat{C}_{\lambda_z}(k)^{-1}\Big((D*C_{\lambda_z,k}*C_{\lambda_z}^{*(s-1)})(0)+(D*C_{\lambda_z,k}*C_{\lambda_z,k}*C_{\lambda_z}^{*(s-2)})(0)\Big)\\ \leq 100\,K^{s-1}\hat{C}_{\lambda_z}(k)^{-1}2\,(D*C_{\lambda_z}^{*(s)})(0)\\ \leq 200\,K^{s-1}\hat{C}_{\lambda_z}(k)^{-1}(D*C_1^{*(s)})(0).\end{aligned} \tag{3.3.32}$$

Applying (3.2.2) iteratively and using $C_1(x)\leq(C_1*C_1)(x)$ yields

$$\begin{aligned}(D*C_1^{*s})(x)&=(D^{*2}*C_1^{*s})(x)+(D^{*2}*C_1^{*(s-1)})(x)+\cdots+(D^{*2}*C_1)(x)+D(x)\\ &\leq D(x)+s\,(D^{*2}*C_1^{*s})(x).\end{aligned} \tag{3.3.33}$$

Hence $(D*C_1^{*(s)})(0)\leq s\beta$ by using $D(0)=0$ and (3.2.5). A combination of this inequality with (3.3.28)–(3.3.32) shows

$$\big(D*(G_z-G_{z,k})*G_z^{*(s-2)}\big)(y)\leq\hat{C}_{\lambda_z}(k)^{-1}\,O(\beta) \tag{3.3.34}$$

where the bounding constant depends only on s and K. Together with (3.3.24) and (3.3.27), this proves the claim. □

3.3.2 Proof of Proposition 3.2 for percolation

We begin by stating certain bounds on the lace expansion coefficients Π_M and R_M, introduced in Section 2.1. Thereafter we derive the bounds on Φ_z and Ψ_z, needed for Proposition 3.2.

Proposition 3.5 (Bounds on the lace expansion coefficients for percolation [25])**.** *Fix $z\in(0,z_c)$. If $f(z)$ of (3.3.7) obeys $f(z)\leq K$, then there are positive constants c_K' and $\beta_0=\beta_0(K)$, such that the following holds: If Assumption 3.1 holds for some $\beta\leq\beta_0$, then for all $M=0,1,2,\dots,$*

$$\sum_x|\Pi_M(x)|\leq c_K'\beta, \tag{3.3.35}$$

$$\sum_x[1-\cos(k\cdot x)]\,|\Pi_M(x)|\leq c_K'\beta\hat{C}_{\lambda_z}(k)^{-1}, \tag{3.3.36}$$

and for M sufficiently large (depending on K and z),

$$\sum_x|R_M(x)|\leq\beta, \tag{3.3.37}$$

$$\sum_x[1-\cos(k\cdot x)]\,|R_M(x)|\leq\beta\hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.38}$$

For the proof we make use of the framework developed in Section 2.1, and in particular Proposition 2.2 is of crucial importance.

Proof of Proposition 3.5. Lemma 3.3 and Lemma 3.4, both for $s=3$, imply that there exists a constant $\check{c}_K$ such that

$$T(z)\leq 1+\check{c}_K\,\beta,\qquad\tilde{T}(z)\leq\check{c}_K\,\beta, \tag{3.3.39}$$

$$B_k(z) \leq z\, \check{c}_K\, \beta\, \hat{C}_{\lambda_z}(k)^{-1}, \qquad W_k(z) \leq z\, \check{c}_K\, \hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.40}$$

The bound

$$H_k(z) \leq \check{c}_K\, \hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.41}$$

is not proven here, we refer to [25, Lemma 5.7] instead. Together with Proposition 2.2, these bounds prove that there exists $\bar{c}_K$ (depending on K only) such that

$$\sum_x \pi^{(N)}(x) \quad \leq \quad (\bar{c}_K \beta)^{N \vee 1}, \tag{3.3.42}$$

$$\sum_x [1-\cos(k\cdot x)]\, \pi^{(N)}(x) \quad \leq \quad (\bar{c}_K \beta)^{(N-1)\vee 1}\, \hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.43}$$

Recalling (2.1.2), the bounds (3.3.35)–(3.3.36) follow from summing (3.3.42)–(3.3.43) over $N \geq 0$ assuming β is small enough that $\bar{c}_K \beta \leq 1/2$.

For the bounds on the remainder R_M in (3.3.37)–(3.3.38), we recall from (2.1.22) that

$$|R_M(x)| \leq K \sum_{u,v} \pi^{(M)}(u)\, D(v-u)\, G_z(x-v). \tag{3.3.44}$$

Consequently, $\sum_x |R_M(x)| \leq \hat{\pi}^{(M)}(0)\, \chi(z)$, and this can be made arbitrarily small by taking M large enough, proving (3.3.37). For (3.3.38) we use once more (2.1.50) (this time with $n=3$) to obtain

$$\begin{aligned}\sum_x [1-\cos(k\cdot x)]\, |R_M(x)| \quad \leq \quad & 7K\,[1-\hat{D}(k)]\, \hat{\pi}^{(M)}(0)\, \chi(z) \\ & + 7K\left(\hat{\pi}^{(M)}(0) - \hat{\pi}^{(M)}(k)\right)\chi(z) \\ & + 7K\, \hat{\pi}^{(M)}(0)\left(\hat{G}_z(0) - \hat{G}_z(k)\right). \end{aligned}\tag{3.3.45}$$

The first summand can be made smaller than $\beta/3\, \hat{C}_{\lambda_z}(k)^{-1}$ by using (3.3.12) and (3.3.35). Similarly the second summand by (3.3.36). Finally, for the third summand we apply the bound $f_3 \leq K$ on $\hat{G}_z(0) - \hat{G}_z(k) = -1/2\, \Delta_k \hat{G}_z(0)$ and use calculations similar to (3.3.28)–(3.3.32) showing $\hat{G}_z(0) - \hat{G}_z(k) \leq \hat{C}_{\lambda_z}(k)^{-1} O(1)$. Again, we take M large and appeal to (3.3.35) showing that also the third line is bounded above by $\beta/3\, \hat{C}_{\lambda_z}(k)^{-1}$. This proves (3.3.38). □

We now use Proposition 3.5 to prove Proposition 3.2 for percolation:

Proof of Proposition 3.2 for percolation. Recall (2.1.4)–(2.1.5). The bounds on $\Psi_z(x)$ in (3.3.10)–(3.3.11) follow directly from Proposition 3.5 if M is chosen so large that (3.3.37)–(3.3.38) is satisfied.

For the bounds on $\Phi_z(x) = z(D * \Pi_M)(x)$ we use the estimate

$$[1-\cos(t_1+t_2)] \leq 5\left([1-\cos t_1] + [1-\cos t_2]\right), \qquad t_1, t_2 \in \mathbb{R}, \tag{3.3.46}$$

(see (2.1.50)) to obtain

$$\begin{aligned}\sum_x [1-\cos(k\cdot x)]\, |\Phi_z(x)| \quad \leq \quad & 5 \sum_x z \sum_y \big([1-\cos(k\cdot y)] \\ & \qquad + [1-\cos(k\cdot(x-y))]\big) D(y)\, |\Pi_M(x-y)| \\ \leq \quad & 5z \sum_x [1-\hat{D}(k)]\, |\Pi_M(x-y)| \\ & \qquad + 5z \sum_x [1-\cos(k\cdot(x-y))]\, |\Pi_M(x-y)| \\ \leq \quad & 5z\left(2c'_K \beta \hat{C}_{\lambda_z}(k)^{-1} + c'_K \beta \hat{C}_{\lambda_z}(k)^{-1}\right). \end{aligned}\tag{3.3.47}$$

by (3.3.35)–(3.3.36) and (3.3.12). □

3.3.3 Proof of Proposition 3.2 for self-avoiding walk

Proof of Proposition 3.2 for self-avoiding walk. Since $\Psi(x) \equiv 0$ for self-avoiding walk, there is nothing to prove for $\hat{\Psi}_z(k)$. By (2.2.22)–(2.2.23) it is sufficient to prove that there is a constant $\check{c}_K$ depending on K only, such that

$$\|\tilde{G}\|_\infty \le \check{c}_K \beta, \qquad \tilde{B}(z) = \sup_x (\tilde{G} * G)(x) \le \check{c}_K \beta, \tag{3.3.48}$$

and

$$\tilde{H}_k(z) \le z\, \check{c}_K\, \hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.49}$$

Then, taking β small enough such that $\check{c}_K\beta \le 1/2$, the geometric series in (2.2.22)–(2.2.23) are summable, and $c_K = 2\check{c}_K$ is sufficient.

The required bounds on $\tilde{B}(z)$ and $\tilde{H}_k(z)$ follow directly from Lemmas 3.3 and 3.4 with $s = 2$ and $\tau(z) = z$.

For the bound on $\|\tilde{G}\|_\infty$ we proceed similarly as in the proof of Lemma 3.3: By (2.2.26),

$$\tilde{G}_z(x) = z\,(D * G_z)(x) \le z\, D(x) + z^2\, (D^{*2} * G_z)(x). \tag{3.3.50}$$

For the first summand we bound $z\,D(x) \le K\beta$ by the $f_1(z) = z \le K$ and (3.2.4). For the second summand on the right hand side of (3.3.50) we estimate

$$\begin{aligned} z^2\,(D^{*2} * G_z)(x) &= z^2 \int_{[-\pi,\pi)^d} \mathrm{e}^{-ik\cdot x}\, \hat{D}(k)^2\, \hat{G}_z(k) \frac{\mathrm{d}k}{(2\pi)^d} \\ &\le K^3 \int_{[-\pi,\pi)^d} \hat{D}(k)^2\, \hat{C}_{\lambda_z}(k) \frac{\mathrm{d}k}{(2\pi)^d} \le 4K^3 \int_{[-\pi,\pi)^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2} \frac{\mathrm{d}k}{(2\pi)^d} \le 4K^3\beta, \end{aligned} \tag{3.3.51}$$

where the second line uses the bounds $f_1 \le K$ and $f_2 \le K$ in the first inequality, (3.3.12) and $[1-\hat{D}(k)] \le 2$ in the second inequality, and Assumption 3.1 in the final inequality. Thus $\|\tilde{G}\|_\infty \le 5K^3\beta$, and this finishes the proof of Proposition 3.2 for self-avoiding walk. □

3.3.4 Proof of Proposition 3.2 for the Ising model

For Π_M^Λ and R_M^Λ we have the following bounds:

Proposition 3.6 (Diagrammatic estimates for the Ising model from [95])**.** *Fix $z \in (0, z_c)$. If $f(z)$ of (3.3.7) obeys $f(z) \le K$, then there are positive constants c'_K and $\beta_0 = \beta_0(K)$, such that the following holds: If Assumption 3.1 holds for some $\beta \le \beta_0$, then for all $M = 0, 1, 2, \dots,$*

$$\sum_x |\Pi_M^\Lambda(x)| \le c'_K \beta, \tag{3.3.52}$$

$$\sum_x [1 - \cos(k\cdot x)]\, |\Pi_M^\Lambda(x)| \le c'_K \beta \hat{C}_{\lambda_z}(k)^{-1}, \tag{3.3.53}$$

and for M sufficiently large (depending on K and z),

$$\sum_x |R_M^\Lambda(x)| \le \beta, \tag{3.3.54}$$

$$\sum_x [1 - \cos(k\cdot x)]\, |R_M^\Lambda(x)| \le \beta \hat{C}_{\lambda_z}(k)^{-1}. \tag{3.3.55}$$

These bounds hold uniformly in Λ.

A similar statement was proved by Sakai [95, Proposition 3.2]. However, since the hypothesis of Proposition 3.6 is different from that in [95], it is not so obvious how Proposition 3.6 follows from the results in [95]. In Appendix A we explain how the statement in [95, Prop. 3.2] can be modified to obtain the desired bounds (3.3.52)–(3.3.55).

Given Proposition 3.6, we prove Proposition 3.2 for the Ising model as in the percolation case, now using Proposition 3.6 instead of Proposition 3.5. We refrain from repeating the argument.

3.4 Infrared bound

Our main result is the following infrared behavior:

Theorem 3.7 (Infrared bound)**.** *Fix $s = 2$ for self-avoiding walk and the Ising model, and $s = 3$ for percolation. Let d sufficiently large in the nearest-neighbor case (at least $d > 4s$), or $d > 2s$ and L sufficiently large in the finite-variance spread-out case, or $d > s(\alpha \wedge 2)$ and L sufficiently large in the spread-out power-law case. Then*

$$\hat{G}_z(k) = \frac{1 + O(\beta)}{\chi(z)^{-1} + \tau(z)[1 - \hat{D}(k)]} \tag{3.4.1}$$

uniformly for $z \in [0, z_c)$ and $k \in [-\pi, \pi)^d$.

The terminology ‘infrared bound’ originates from the fact that (3.4.1) particularly identifies the asymptotics of $\hat{G}_z(k)$ for small k, which presumably (via a Tauberian theorem) determines the behavior of $G_z(x)$ for large x. This is reminiscent of the relation between wave length and frequency, where low frequency (infrared light) corresponds to long wave lengths.

The infrared bound is well-known in several cases. Hara and Slade proved the infrared bound for the nearest-neighbor case and the finite-variance spread-out case, for self-avoiding walk [58, 57] (see also [88, Theorem 6.1.6]) as well as for percolation [56]. Fröhlich, Simon and Spencer [47] proved the upper bound in (3.4.1) for the Ising model under the *reflection positivity* assumption, which holds e.g. for the nearest-neighbor case. We discuss reflection positivity in more detail in Section 3.4.1.

For the critical case (i.e., $z = z_c$) we have

$$1 \le \tau(z_c) \le 1 + O(\beta), \tag{3.4.2}$$

where the lower bound is a consequence of $G_z(x) \le \delta_{0,x} + \tau(z)\,(D * G_z)(x)$ (see also (3.5.2) below), and the upper bound emerges from (3.3.8) and the fact that $f(z) \le 1 + O(\beta)$, see (3.4.5) below. The function $G_{z_c}(x) = \lim_{z \nearrow z_c} G_z(x)$ is not in $\ell^1(\mathbb{Z}^d)$, hence the Fourier transform does not exist. However, the diagrammatic bounds of the lace expansion coefficients in Proposition 3.2 and the dominated convergence theorem guarantee the absolute convergence of the various sums involved defining $\hat{\Psi}_z(k)$ and $\hat{\Phi}_z(k)$, which shows that the critical quantities $\hat{\Psi}_{z_c}(k)$ and $\hat{\Phi}_{z_c}(k)$ are well-defined. This justifies the introduction of $\hat{G}_{z_c}(k)$ as a solution to (3.3.2) with $z = z_c$. Note that we do not assume any continuity of $z \mapsto \hat{\Psi}_z(k)$ and $z \mapsto \hat{\Phi}_z(k)$ to do this. Nevertheless, we can extend (3.4.1) to the critical case $z = z_c$, and further use (3.4.2) to obtain

$$\hat{G}_{z_c}(k) = \frac{1 + O(\beta)}{1 - \hat{D}(k)}. \tag{3.4.3}$$

An issue of interest is the (left-) continuity of $\hat{G}_z(k)$ at $z = z_c$. In particular, the identity

$$G_{z_c}(x) = \int_{[-\pi,\pi)^d} \mathrm{e}^{-ik\cdot x}\, \hat{G}_{z_c}(k) \frac{\mathrm{d}k}{(2\pi)^d}, \qquad x \in \mathbb{Z}^d, \tag{3.4.4}$$

would follow from the the fact that $\hat{\Psi}_z(k)$ and $\hat{\Phi}_z(k)$ are left-continuous at $z = z_c$, as explained by Hara [55, Appendix A]. The left-continuity of $\hat{\Psi}_z(k)$ and $\hat{\Phi}_z(k)$ at $z = z_c$ indeed holds for self-avoiding walk (by Abel's Theorem) and for percolation (by [55, Lemma A.1]), but a proof for the Ising model is not known.

We shall prove the following generalized version of Theorem 3.7. By Proposition 3.1, Theorem 3.8 below immediately implies Theorem 3.7.

Theorem 3.8 (Infrared bound, generalized form). *Fix $s = 2$ for self-avoiding walk and the Ising model, and $s = 3$ for percolation. If Assumption 3.1 is satisfied for β sufficiently small, then (3.4.1) holds uniformly for $z \in [0, z_c)$ and $k \in [-\pi, \pi)^d$.*

3.4.1 Discussion of related literature

There is numerous work on the application of the lace expansion, see the lecture notes by Slade [102] and references therein. We give more references below at places where we use lace expansion methodology and need particular results. We now briefly summarize the results known for *long-range* systems.

For **percolation**, Hara and Slade [56] proved the infrared bound for the finite-variance spread-out case when D has exponential tails. The study of long-range percolation with power law spread-out bonds started in the 1980's by considering the one-dimensional case [10, 90, 97]. These papers study the case where occupation probabilities are given by (3.1.12) with $\alpha \in (0, 1]$ and prove criteria for the existence of an infinite cluster. For example, Aizenman–Newman [10] show that if $D(x)\,|x|^2 \to 1$ as $|x| \to \infty$ in one dimension, and $D(1)$ is sufficiently large, then there exists a critical infinite cluster and hence the percolation probability $z \mapsto \theta(z)$ is *discontinuous* at z_c. This is compatible with our results, which imply that there is no infinite cluster at criticality for $d > 3\alpha$ and large L (and here $\alpha = 1$). Berger [18] uses a renormalization argument to show that in dimension $d = 1, 2$ for all $z \geq 0$ the infinite cluster (if it exists) is transient if $0 < \alpha < d$ and recurrent if $\alpha \geq d$. He further concludes that in the d-dimensional case ($d \geq 1$) there is no infinite cluster at criticality if $0 < \alpha < d$. The question whether there exists an infinite critical cluster for $d \geq 2$ and $\alpha \geq d$ [18, Question 6.4] is answered negatively by the present thesis for $d > 6$ and L sufficiently large.

In recent work, Chen and Sakai [33, 34] study **oriented percolation** in the spread-out power-law case. Using similar methods, they prove that the two-point function in oriented percolation obeys an infrared bound if $d > 2(\alpha \wedge 2)$, which implies mean-field behavior of the model.

Long-range **self-avoiding walk** has rarely been studied. Yang and Klein [105] showed that weakly self-avoiding walk in dimension $d \geq 3$ jumping m lattice sites *along the coordinate axes* with probability proportional to $1/m^2$ converges to a Cauchy process (as for ordinary random walk with such step distribution). A similar result for strictly self-avoiding walk has been obtained by Cheng [35].

A long-range **Ising model** in one dimension has been studied by Aizenman, Chayes, Chayes, and Newman [5]. Similar to the percolation result in [10], they prove that in the one-dimensional case where $D(x)\,|x|^2 \to 1$ as $|x| \to \infty$, the spontaneous magnetization $M(z, 0+)$ has a discontinuity at the critical point z_c.

The infrared bound for the Ising model was proved in [47] for $d > \alpha \wedge 2$ for a class of models obeying the *reflection positivity* (RP) property. The class of models satisfying (RP) includes the nearest-neighbor model (where $D(x) = (2d)^{-1}\mathbb{1}_{\{|x|=1\}}$), exponential decaying potentials (where $D(x) \propto \exp\{-\mu\|x\|_1\}$ for $\mu > 0$), power-law decaying interactions (where $D(x) \propto |x|^{-s}$ for $s > 0$), and combinations thereof. For a definition of (RP) and a discussion of the above mentioned models, we refer to [20]. Nevertheless, (RP) fails in most cases for small perturbations of these models, although it is believed that the asymptotics still hold. Moreover, (RP) only implies the upper bound in (3.4.1), in that implying that the critical exponent η (when it exists) is nonnegative. Our approach using the lace expansion does *not* require reflection positivity, it is more universal in the choice of D, and also gives a matching lower bound in (3.4.1), yielding $\eta = 0$. On the other hand, our approach requires that the dimension d or the spread-out parameter L are sufficiently large, a limitation that one may not expect to reflect the physics. The literature for the long-range Ising model in higher dimensions based on (RP) arguments is summarized by Aizenman and Fernández [7], who also identify $2(\alpha \wedge 2)$ as upper critical dimension.[3]

3.4.2 Proof of the infrared bound

The proof of Theorem 3.8 will follow from the following proposition:

Proposition 3.9. *Suppose we are given a model with some model-dependent constant $s \in \{2, 3, \dots\}$, and a two-point function G_z of the form (2.0.1), where the step distribution D satisfies Assumption 3.1 (for some sufficiently small $\beta > 0$), and Φ_z and Ψ_z satisfy Assumption 3.2. Assume further that $\chi'(z) \le O\big(\chi(z)^2\big)$, $z \in [0, z_c)$. Then*

$$f(z) \le 1 + O(\beta) \tag{3.4.5}$$

uniformly for $z < z_c$.

The assumption $\chi'(z) \le O\big(\chi(z)^2\big)$ in Proposition 3.9 is only needed to prove continuity of f. However, it does hold for our models, cf. (1.4.6).

Proof of Theorem 3.8 (and Theorem 3.7) subject to Proposition 3.9. Propositions 3.1 and 3.2 validate Assumptions 3.1 and 3.2. With these assumptions, the prerequisites of Proposition 3.9 are satisfied and (3.4.5) holds provided β sufficiently small. The latter can be achieved by taking d or L large enough, by Proposition 3.1. Then we again use Assumption 3.2 to obtain (3.3.10)–(3.3.11),

It remains to show that (3.3.10)–(3.3.11) imply (3.4.1). To this end, let

$$m_z = 1 - \tau(z) - \hat{\Phi}_z(0). \tag{3.4.6}$$

Then

$$\hat{G}_z(k) = \frac{1 + \hat{\Psi}_z(k)}{1 - \tau(z)\hat{D}(k) - \hat{\Phi}_z(k)} = \frac{1 + \hat{\Psi}_z(k)}{m_z + \tau(z)[1 - \hat{D}(k)] + [\hat{\Phi}_z(0) - \hat{\Phi}_z(k)]}. \tag{3.4.7}$$

By the first inequality in (3.3.10) and the second in (3.3.11),

$$\hat{G}_z(k) = \frac{1 + O(\beta)}{m_z + \tau(z)\,[1 - \hat{D}(k)] + \tau(z)\,O(\beta)\,\hat{C}_{\lambda_z}(k)^{-1}}. \tag{3.4.8}$$

Evaluating (3.4.7) for $k = 0$ yields

$$\chi(z) = \hat{G}_z(0) = \frac{1 + \hat{\Psi}_z(0)}{m_z}, \tag{3.4.9}$$

and the first inequality in (3.3.10) implies

$$m_z = (1 + O(\beta))\,\chi(z)^{-1}. \tag{3.4.10}$$

Furthermore, by (3.2.3) and (3.3.3),

$$\hat{C}_{\lambda_z}(k)^{-1} = 1 - \lambda_z \hat{D}(k) = 1 - \hat{D}(k) + \chi(z)^{-1}\hat{D}(k). \tag{3.4.11}$$

[3] The value of δ in [7, (1.2)] should be 3.

A combination of (3.4.8), (3.4.10), (3.4.11) in conjunction with $|\hat{D}(k)| \leq 1$ and $\tau(z) \leq O(1)$ proves

$$\hat{G}_z(k) = \frac{1 + O(\beta)}{(1 + O(\beta))\,\chi(z)^{-1} + \tau(z)\,(1 + O(\beta))\,[1 - \hat{D}(k)]}, \tag{3.4.12}$$

which implies (3.4.1). □

3.4.3 The bootstrap argument

The remainder of the section is devoted to the proof of Proposition 3.9, which is based on the following lemma:

Lemma 3.10 (The bootstrap / forbidden region argument)**.** *Let f be a continuous function on the interval $[0, z_c)$, and assume that $f(0) \leq 3$. Suppose for each $z \in (0, z_c)$ that if $f(z) \leq 4$, then in fact $f(z) \leq 3$. Then $f(z) \leq 3$ for all $z \in [0, z_c)$.*

Proof. This is a straightforward application of the intermediate value theorem for continuous functions, see also [102, Lemma 5.9]. □

The bootstrap argument in Lemma 3.10 is often used in lace expansion, see e.g. [88, Section 6.2]. An alternative approach that involves an induction argument has been applied in [71], see also the lecture notes by van der Hofstad [68].

We shall now prove that the function f defined in (3.3.7) obeys the prerequisites of Lemma 3.10. We therefore have to show that $f(0) \leq 3$, that f is continuous on $[0, z_c)$, and that $f(z) \leq 4$ implies $f(z) \leq 3$ for $z \in (0, z_c)$. The latter is referred to as the *improvement of the bounds*.

Let us first check that $f(0) \leq 3$. Clearly, $f_1(0) = 0$. Note that $\hat{\Psi}_0(k) \equiv 0$ and $\hat{\Phi}_0(k) \equiv 0$. This leads to $\hat{G}_0(k) \equiv 1$ and $\lambda_0 = 0$, hence $f_2(0) = 1$ and $f_3(0) = 0$.

Next we want to prove continuity of f. To this end, we need the following lemma:

Lemma 3.11 (Continuity of equicontinuous functions)**.** *Let $(f_\alpha)_{\alpha \in A}$ be an equicontinuous family of functions on an interval $[t_1, t_2]$, i.e., for every given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f_\alpha(s) - f_\alpha(t)| < \varepsilon$ whenever $|s - t| < \delta$, uniformly in $\alpha \in A$. Furthermore, suppose that $\sup_{\alpha \in A} f_\alpha(t) < \infty$ for each $t \in [t_1, t_2]$. Then $t \mapsto \sup_{\alpha \in A} f_\alpha(t)$ is continuous on $[t_1, t_2]$.*

A proof of this standard result can be found e.g. in [102, Lemma 5.12].

Lemma 3.12 (Continuity)**.** *Assume that, for $z \in (0, z_c)$, $\chi'(z) \leq c\chi(z)^2$ for some constant c. Then, the function f defined in (3.3.7) is continuous on $(0, z_c)$.*

Proof. It is sufficient to show that f_1, f_2 and f_3 are continuous. The continuity of f_1 is obvious. We show that f_2 and f_3 are continuous on the closed interval $[0, z_c - \varepsilon]$ for any $\varepsilon > 0$ by taking derivatives with respect to z and bound it uniformly in k on $[0, z_c - \varepsilon]$.

We do f_2 first. To this end, we consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{\hat{G}_z(k)}{\hat{C}_{\lambda_z}(k)} = \frac{1}{\hat{C}_{\lambda_z}(k)^2} \left[\hat{C}_{\lambda_z}(k) \frac{\mathrm{d}\hat{G}_z(k)}{\mathrm{d}z} - \hat{G}_z(k) \left.\frac{\mathrm{d}\hat{C}_\lambda(k)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_z} \frac{\mathrm{d}\lambda_z}{\mathrm{d}z} \right]. \tag{3.4.13}$$

We proceed by showing that each of the terms on the right hand side is uniformly bounded in k and $z \in [0, z_c - \varepsilon]$, and hence the derivative is bounded. First we recall the definition of λ_z in (3.3.3) to see that

$$\frac{1}{2} \leq \frac{1}{1 - \lambda_z \hat{D}(k)} = \hat{C}_{\lambda_z}(k) \leq \hat{C}_{\lambda_z}(0) = \chi(z). \tag{3.4.14}$$

Furthermore, $\chi(z) \le \chi(z_c-\varepsilon)$, and the latter is finite by the definition of z_c in (1.4.8). For every $k \in [-\pi,\pi)^d$, the two-point function is bounded from above by

$$|\hat{G}_z(k)| \le |\hat{G}_z(0)| = \chi(z) \le \chi(z_c-\varepsilon), \tag{3.4.15}$$

For the derivative of the two-point function, we bound

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\hat{G}_z(k)\right| = \left|\sum_x \mathrm{e}^{ik\cdot x}\,\frac{\mathrm{d}}{\mathrm{d}z}G_z(x)\right| \le \sum_x \frac{\mathrm{d}}{\mathrm{d}z}G_z(x) = \frac{\mathrm{d}}{\mathrm{d}z}\sum_x G_z(x) = \chi'(z), \tag{3.4.16}$$

where the exchange in the order of sum and derivative is validated by the fact that both $\sum_x \mathrm{e}^{ik\cdot x}\, G_z(x)$ and $\sum_x G_z(x)$ are uniformly convergent series of functions. By the assumed mean-field bound $\chi'(z) \le c\chi(z)^2$, (3.4.16) is bounded above by $c\chi(z_c-\varepsilon)^2$.

Moreover, we obtain from (3.2.3) that $|\mathrm{d}\hat{C}_\lambda(k)/\mathrm{d}\lambda| \le \hat{C}_\lambda(k)^2$, and, for $\lambda=\lambda_z$, this is in turn bounded by $\chi(z_c-\varepsilon)^2$, cf. (3.4.14). Finally, $|\mathrm{d}\lambda_z/\mathrm{d}z| = \chi'(z)/\chi(z)^2 \le c$ by (3.3.3) and our assumption.

We treat f_3 in exactly the same way as f_2, and omit the details here. □

3.4.4 Improvement of the bounds

The following lemma covers the remaining prerequisite of Lemma 3.10 and thus proves the final ingredient needed for the proof of Proposition 3.9.

Lemma 3.13 (Improvement of the bounds)**.** *There exists a constant $c>0$ such that if the assumptions of Proposition 3.9 are satisfied for some sufficiently small β and if $f(z) \le 4$, then $f(z) \le 1 + c\beta$ for all $z \in (0,z_c)$. In particular, if β is small enough, then $f(z) \le 3$.*

The following lemma will help us for the improvement of the bound on f_3.

Lemma 3.14 (Slade [102])**.** *Suppose that $a(x) = a(-x)$ for all $x \in \mathbb{Z}^d$, and let*

$$\hat{A}(k) = \frac{1}{1-\hat{a}(k)}. \tag{3.4.17}$$

Then, for all $k,l \in [-\pi,\pi)^d$,

$$\begin{aligned} \left|\Delta_k \hat{A}(l)\right| \;\le\; & \Big(\hat{A}(l-k)+\hat{A}(l+k)\Big)\,\hat{A}(l)\,\Big(\widehat{|a|}(0)-\widehat{|a|}(k)\Big) \\ & + 8\hat{A}(l-k)\,\hat{A}(l)\,\hat{A}(l+k)\,\Big(\widehat{|a|}(0)-\widehat{|a|}(l)\Big)\,\Big(\widehat{|a|}(0)-\widehat{|a|}(k)\Big). \end{aligned} \tag{3.4.18}$$

By $\widehat{|a|}$ we denote the Fourier transform of the absolute value of a. The reader shall be warned that $\widehat{|a|} \ne |\hat{a}|$. The proof of Lemma 3.14 uses several bounds on trigonometric quantities, and can be found in [102, Lemma 5.7].

Proof of Lemma 3.13. Fix $z \in (0,z_c)$ arbitrarily and assume $f(z) \le 4$. Our general strategy will be to show that f_i for $i=1,2,3$ is smaller then $(1+\text{const}\,\beta)$ and thus, by taking β small, $f(z) \le 3$.

The bound on f_1 is easy. First note that $\lambda_z = 1-\chi(z)^{-1} \le 1$. Using (3.3.3) along with (3.4.6)–(3.4.9) and Proposition 3.2 (with $K=4$) we obtain

$$\begin{aligned} f_1(z) &= \lambda_z\Big(1+\hat{\Psi}_z(0)\Big) - \hat{\Phi}_z(0) - \hat{\Psi}_z(0) \\ &\le \lambda_z\Big(1+|\hat{\Psi}_z(0)|\Big) + |\hat{\Phi}_z(0)| + |\hat{\Psi}_z(0)| \\ &\le 1 + 3\,c_4\beta. \end{aligned} \tag{3.4.19}$$

The bound on f_2 is slightly more involved. We write $\hat{G}_z = \hat{N}/\hat{F}$, with

$$\hat{N}(k) = \frac{1+\hat{\Psi}_z(k)}{1+\hat{\Psi}_z(0)}, \qquad \hat{F}(k) = \frac{1-\tau(z)\hat{D}(k)-\hat{\Phi}_z(k)}{1+\hat{\Psi}_z(0)}. \tag{3.4.20}$$

Recall from (3.2.3) that $\hat{C}_{\lambda_z}(k) = [1-\lambda_z\hat{D}(k)]^{-1}$ and, by (3.3.2) and (3.3.3),

$$\lambda_z = 1 - \frac{1-\tau(z)-\hat{\Phi}_z(0)}{1+\hat{\Psi}_z(0)}. \tag{3.4.21}$$

This yields

$$\frac{\hat{G}_z(k)}{\hat{C}_{\lambda_z}(k)} = \hat{N}(k) + \hat{G}_z(k)\left[1-\lambda_z\hat{D}(k)-\hat{F}(k)\right], \tag{3.4.22}$$

where

$$1-\lambda_z\hat{D}(k)-\hat{F}(k) = \frac{[1-\hat{D}(k)]\hat{\Psi}_z(0) + [\hat{\Phi}_z(k)-\hat{\Phi}_z(0)]\hat{D}(k) + [1-\hat{D}(k)]\hat{\Phi}_z(k)}{1+\hat{\Psi}_z(0)}.$$

By taking $c_4\beta \le 1/2$, we obtain the bound

$$\frac{1+\ell c_4\beta}{1-c_4\beta} \le 1 + (2\ell+2)\,c_4\beta, \qquad \ell = 0,1,2,\ldots, \tag{3.4.23}$$

which we use frequently below. For example, together with Assumption 3.2, it enables us to bound

$$\left|\hat{N}(k)\right| = \left|\frac{1+\hat{\Psi}_z(k)}{1+\hat{\Psi}_z(0)}\right| \le \frac{1+|\hat{\Psi}_z(k)|}{1-|\hat{\Psi}_z(0)|} \le 1 + 4\,c_4\beta.$$

Together with (3.3.12) we obtain in the same fashion that

$$\begin{aligned}\left|1-\lambda_z\hat{D}(k)-\hat{F}(k)\right| &\le \frac{[1-\hat{D}(k)]\,|\hat{\Psi}_z(0)| + |\hat{\Phi}_z(k)-\hat{\Phi}_z(0)| + [1-\hat{D}(k)]\,|\hat{\Phi}_z(k)|}{1-|\hat{\Psi}_z(0)|} \\ &\le \frac{2c_4\beta[1-\hat{D}(k)] + c_4\beta\hat{C}_{\lambda_z}(k)^{-1}}{1-c_4\beta} \le 12\,c_4\beta\,\hat{C}_{\lambda_z}(k)^{-1}\end{aligned}$$

By our assumption that $\hat{G}_z(k) \le 4\hat{C}_{\lambda_z}(k)$ (which follows from $f(z) \le 4$) and the above inequalities, we can bound (3.4.22) from above by

$$\left|\frac{\hat{G}_z(k)}{\hat{C}_{\lambda_z}(k)}\right| \le 1 + 4\,c_4\beta + 4\cdot 12\,c_4\beta\left|\hat{C}_{\lambda_z}(k)\,\hat{C}_{\lambda_z}(k)^{-1}\right| = 1 + 52\,c_4\beta. \tag{3.4.24}$$

for every $k \in [-\pi,\pi)^d$. This proves the bound on f_2.

It remains to show the bound on f_3. In the following, we write K for a positive constant, whose value may change from line to line. Furthermore, we write

$$\hat{G}_z(k) = \frac{\hat{b}(k)}{1-\hat{a}(k)}, \qquad \text{where } \hat{b}(k) = 1+\hat{\Psi}_z(k), \;\; \hat{a}(k) = \tau(z)\hat{D}(k) + \hat{\Phi}_z(k). \tag{3.4.25}$$

A straightforward calculation (see also [33, (4.18)]) shows that

$$\Delta_k \hat{G}_z(l) = \frac{\Delta_k \hat{b}(l)}{1-\hat{a}(l)} + \sum_{\sigma\in\{1,-1\}} \frac{\big(\hat{a}(l+\sigma k)-\hat{a}(l)\big)\,\big(\hat{b}(l+\sigma k)-\hat{b}(l)\big)}{(1-\hat{a}(l))\,(1-\hat{a}(l+\sigma k))} + \hat{b}(l)\,\Delta_k\Big[\frac{1}{1-\hat{a}(l)}\Big]. \tag{3.4.26}$$

We now bound all three summands in (3.4.26), and start with the first one:

$$\left|\frac{\Delta_k \hat{b}(l)}{1-\hat{a}(l)}\right| = \left|\frac{\Delta_k \hat{b}(l)}{\hat{b}(l)}\right| \left|\hat{G}_z(l)\right| = \left|\frac{\Delta_k \hat{\Psi}_z(l)}{1+\hat{\Psi}_z(l)}\right| \left|\hat{G}_z(l)\right| \le \left|\Delta_k \hat{\Psi}_z(l)\right| 2(1+K\beta)\,\hat{C}_{\lambda_z}(l), \tag{3.4.27}$$

where the last bound uses (3.3.11) to bound the denominator, and (3.4.24). A basic calculation shows that any function $g\colon \mathbb{Z}^d \to \mathbb{R}$ with $g(x)=g(-x)$ satisfies

$$|\Delta_k \hat{g}(l)| \le \sum_x [1-\cos(k\cdot x)]\,|g(x)|\,, \tag{3.4.28}$$

cf. [25, (5.32)]. We apply this bound with $g(x)=\Psi_z(x)$, combine it with (3.4.27) and (3.3.11), and use $\hat{C}_{\lambda_z}(l\pm k)\ge 1/2$ and the definition of $U_{\lambda_z}(l,k)$ in (3.3.6) to obtain

$$\left|\frac{\Delta_k \hat{b}(l)}{1-\hat{a}(l)}\right| \le K\beta\,\hat{C}_{\lambda_z}(k)^{-1}\hat{C}_{\lambda_z}(l) \le O(\beta)\,U_{\lambda_z}(l,k). \tag{3.4.29}$$

The second term in (3.4.26) is bounded as follows. First, since

$$|\,\mathrm{e}^{il\cdot x}(\mathrm{e}^{i(\pm k\cdot x)}-1)| \le |\sin(k\cdot x)| + 1-\cos(k\cdot x), \tag{3.4.30}$$

we obtain

$$\big|\hat{b}(l\pm k)-\hat{b}(l)\big| = \big|\hat{\Psi}_z(l\pm k)-\hat{\Psi}_z(l)\big| \le \sum_x |\sin(k\cdot x)|\,\big|\Psi_z(x)\big| + \sum_x [1-\cos(k\cdot x)]\,\big|\Psi_z(x)\big|. \tag{3.4.31}$$

The second term on the right hand side of (3.4.31) is bounded by $O(\beta)\,\hat{C}_{\lambda_z}(k)^{-1}$; on the first term we apply the Cauchy-Schwarz inequality and (3.3.10)–(3.3.11):

$$\begin{aligned}\sum_x |\sin(k\cdot x)|\,\big|\Psi_z(x)\big| &\le \Big(\sum_{x\neq 0} |\Psi_z(x)|\Big)^{1/2}\Big(\sum_{x\neq 0}\sin(k\cdot x)^2\,|\Psi_z(x)|\Big)^{1/2}\\ &\le O(\beta)^{1/2}\Big(\sum_{x\neq 0}[1-\cos(k\cdot x)]\,|\Psi_z(x)|\Big)^{1/2}\\ &\le O(\beta)\,\hat{C}_{\lambda_z}(k)^{-1/2}. \end{aligned}\tag{3.4.32}$$

Furthermore,

$$\hat{a}(l\pm k)-\hat{a}(l) = \tau(z)\Big(\hat{D}(l\pm k)-\hat{D}(l)\Big) + \Big(\hat{\Phi}_z(l\pm k)-\hat{\Phi}_z(l)\Big). \tag{3.4.33}$$

In a similar fashion as (3.4.31)–(3.4.32), we bound $\Big|\hat{\Phi}_z(l\pm k)-\hat{\Phi}_z(l)\Big| \le O(\beta)\,\hat{C}_{\lambda_z}(k)^{-1/2}$ and

$$\begin{aligned}\Big|\hat{D}(l\pm k)-\hat{D}(l)\Big| &\le \Big(\sum_x D(x)\Big)^{1/2}\Big(\sum_x [1-\cos(k\cdot x)]\,D(x)\Big)^{1/2}\\ &\quad + \sum_x [1-\cos(k\cdot x)]\,D(x) \qquad (3.4.34)\\ &= 1\cdot[1-\hat{D}(k)]^{1/2} + [1-\hat{D}(k)]\\ &\le 2\hat{C}_{\lambda_z}(k)^{-1/2} + 2\hat{C}_{\lambda_z}(k)^{-1} \le O(1)\,\hat{C}_{\lambda_z}(k)^{-1/2}, \qquad (3.4.35)\end{aligned}$$

where the last line uses (3.3.12). The combination of (3.4.31)–(3.4.35) and (3.4.19) yields

$$\big(\hat a(l\pm k)-\hat a(l)\big)\,\big(\hat b(l\pm k)-\hat b(l)\big)\le O(\beta)\,\hat C_{\lambda_z}(k)^{-1}. \tag{3.4.36}$$

On the other hand, by (3.4.24)–(3.4.25),

$$\frac{1}{1-\hat a(l+\sigma k)}=\frac{1}{\hat b(l+\sigma k)}\hat G(l+\sigma k)\le (1+O(\beta))\,\hat C_{\lambda_z}(l+\sigma k),\quad \sigma\in\{-1,0,1\}. \tag{3.4.37}$$

Combining (3.4.36) and (3.4.37) yields

$$\frac{\big(\hat a(l\pm k)-\hat a(l)\big)\,\big(\hat b(l\pm k)-\hat b(l)\big)}{(1-\hat a(l))\,(1-\hat a(l\pm k))}\le O(\beta)\,\hat C_{\lambda_z}(k)^{-1}\,\hat C_{\lambda_z}(l)\,\hat C_{\lambda_z}(l\pm k)\le O(\beta)\,U_{\lambda_z}(l,k). \tag{3.4.38}$$

For the third term in (3.4.26) we argue that $|\hat b(l)|=1+|\hat\Psi_z(l)|\le 1+c_4\beta$ by our assumption on $\hat\Psi_z$. In order to apply Lemma 3.14 to bound $\Delta_k(1-\hat a(l))^{-1}$, we estimate

$$\hat A(l):=\frac{1}{1-\hat a(l)}=\frac{1}{\hat b(l)}\,\hat G_z(l)\le(1+2c_4\beta)\,(1+51c_4\beta)\,\hat C_{\lambda_z}(l)\le(1+K\beta)\hat C_{\lambda_z}(l) \tag{3.4.39}$$

by Assumption 3.2 and (3.4.24), and

$$\begin{aligned}\widehat{|a|}(0)-\widehat{|a|}(k) &= \sum_x[1-\cos(k\cdot x)]\,\big|\tau(z)D(x)+\Phi_z(x)\big|\\ &\le \tau(z)[1-\hat D(k)]+\sum_x[1-\cos(k\cdot x)]\,\big|\Phi_z(x)\big|\\ &\le (2(1+c_4\beta)+c_4\beta)\,\hat C_{\lambda_z}(k)^{-1}\le 5\,\hat C_{\lambda_z}(k)^{-1},\end{aligned}$$

where the last line uses again (3.3.12) and, as usual, requires a certain smallness of β (here we need $c_4\beta\le 1$). Plugging these estimates into (3.4.18) yields

$$\begin{aligned}\Delta_k\frac{1}{1-\hat a(l)}\le&\frac{(1+K\beta)^3\cdot 8\cdot 5^2}{\hat C_{\lambda_z}(k)}\\ &\Big\{\hat C_{\lambda_z}(l-k)\hat C_{\lambda_z}(l)+\hat C_{\lambda_z}(l)\hat C_{\lambda_z}(l+k)+\hat C_{\lambda_z}(l-k)\hat C_{\lambda_z}(l+k)\Big\},\end{aligned} \tag{3.4.40}$$

so that finally

$$\frac{|\Delta_k\hat G_z(l)|}{U_{\lambda_z}(k,l)}\le(1+K\beta), \tag{3.4.41}$$

as required. In conclusion $f_3(z)\le 1+K\beta$, and thus we obtain the improved bound $f(z)\le 1+O(\beta)$. □

Proof of Proposition 3.9. Note first that f is continuous on $(0,z_c)$ by Lemma 3.12 and the assumed mean-field bound $\chi'(z)\le\mathrm{const}\,\chi(z)^2$. Whence the prerequisites of Lemma 3.10 are satisfied by Lemma 3.13 and the fact that $f(0)=1$. Therefore, $f(z)\le 3$ for all $z<z_c$. Moreover, Lemma 3.13 shows that, if $f\le 4$, then in fact $f\le 1+O(\beta)$. Hence $f(z)\le 1+O(\beta)$, uniformly for $z<z_c$. □

3.5 Critical exponents

By discarding the term $\chi(z)^{-1}$ in (3.4.1), we obtain from Theorem 3.7 that (under the assumptions formulated there)

$$\hat{G}_z(k) \leq \frac{1+O(\beta)}{\tau(z)[1-\hat{D}(k)]} \tag{3.5.1}$$

uniformly for $z < z_c$.

Note that the bound

$$G_z(x) - \delta_{0,x} \leq \tau(z)\,(D * G_z)\,(x). \tag{3.5.2}$$

holds in all our three models: for percolation see (2.1.34), for self-avoiding walk see (2.2.26), and for the Ising model we use [95, (4.2)] in the infinite-volume limit, see also (A.8) in the appendix. Thus for $s = 2$,

$$B(z) = \sum_x G_z(x)^2 \leq 1 + \sum_x \tau(z)^2\,(D * G_z)\,(x)^2 \leq 1 + \tau(z)^2 \int_{[-\pi,\pi)^d} \hat{D}(k)^2\,\hat{G}_z(k)^2 \frac{\mathrm{d}k}{(2\pi)^d}. \tag{3.5.3}$$

A combination of (3.5.1) and (3.5.3) gives rise to

$$B(z) \leq 1 + O(1) \int_{[-\pi,\pi)^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2} \frac{\mathrm{d}k}{(2\pi)^d} \leq 1 + O(\beta), \tag{3.5.4}$$

where we use that the integrated term is $O(\beta)$ by Assumption 3.1 and Proposition 3.1 below. A similar calculation gives the corresponding result for $s = 3$. More specifically,

$$T(z) = \sum_{x,y} G_z(0,x)\,G_z(x,y)\,G_z(y,0) \leq 1 + O(\beta) \qquad \text{when } s = 3. \tag{3.5.5}$$

The bounds (3.5.3)–(3.5.5) hold uniformly for $z < z_c$ under the assumptions in Theorem 3.7. Note that in (3.5.5) we write $G_z(x,y) = G_z(x-y)$. We call $B(z)$ the *bubble diagram* and $T(z)$ the *triangle diagram*.

The two-point function $G_z(x)$ seen as a function of z (for fixed x) is continuous on $[0, z_c]$. For self-avoiding walk this fact follows from monotone convergence, and for percolation it is a consequence of Aizenman, Kesten and Newman [8]. A general argument that holds for all our three models is the following: the quantity $G_z(x)$ can be realized as an increasing limit (finite volume approximation) of a function which is continuous and non-decreasing in z, hence $G_z(x)$ is left-continuous (cf. [55, Appendix A]). It follows that (3.5.3)–(3.5.5) even hold at criticality, i.e. when $z = z_c$. In particular, this implies the *bubble condition* (i.e., $B(z_c) < \infty$) or the *triangle condition* (i.e., $T(z_c) < \infty$) for $s = 2$ or 3, respectively. We formulate this fact as a corollary:

Corollary 3.15 (Bubble/Triangle condition)**.** *Under the assumptions in Theorem 3.7, $B(z_c) \leq 1 + O(\beta)$ for $s = 2$ (self-avoiding walk and Ising model), and $T(z_c) \leq 1 + O(\beta)$ for $s = 3$ (percolation).*

The bubble/triangle condition is important since it implies *mean-field* behavior of the model, which is formulated in the next theorem. We recall from (3.4.3) that the infrared bound in (3.4.1) extends to the critical case $z = z_c$ as

$$\hat{G}_{z_c}(k) = \frac{1+O(\beta)}{1-\hat{D}(k)}. \tag{3.5.6}$$

We now use Theorem 3.7 to establish the existence of the formerly introduced critical exponents.

Theorem 3.16 (Critical exponents)**.**

(i) ***Percolation.*** *Consider the percolation model ($s = 3$). Under the assumptions in Theorem 3.7, the critical exponents $\gamma_{\mathrm{P}} = 1$, $\beta_{\mathrm{P}} = 1$ and $\delta_{\mathrm{P}} = 2$ for percolation exist.*

(ii) ***Self-avoiding walk.*** *Consider the self-avoiding walk model ($s = 2$). Under the assumptions in Theorem 3.7, the critical exponent $\gamma_{\mathrm{S}} = 1$ for the self-avoiding walk exists.*

(iii) ***Ising model.*** *Consider the Ising model ($s = 2$). Under the assumptions in Theorem 3.7, the critical exponents $\gamma_{\mathrm{I}} = 1$, $\beta_{\mathrm{I}} = 1/2$ and $\delta_{\mathrm{I}} = 3$ for the Ising model exist.*

(iv) *For all three models, under the assumptions in Theorem 3.7 and if $1-\hat{D}(k) \asymp |k|^{\alpha\wedge 2}$, then*

$$\hat{G}_{z_c}(k) \asymp \frac{1}{|k|^{\alpha\wedge 2}} \qquad \text{as } k \to 0, \tag{3.5.7}$$

i.e., the critical exponents $\eta_{\mathrm{S}} = \eta_{\mathrm{P}} = \eta_{\mathrm{I}} = 0$ exist.

The derivation of the critical exponents from the bubble-/triangle condition (Corollary 3.15) is well-known in the literature. However, the mode of convergence required for the existence of the critical exponents varies, and some derivations are stated only for finite range models. We therefore add a more detailed discussion of the literature here.

For self-avoiding walk, the existence (and the value) of the critical exponent γ_{S} is based on the inequality

$$\frac{z_c}{z_c - z} \leq \chi(z) \leq B(z_c)\left(\frac{z_c}{z_c - z} + 1\right). \tag{3.5.8}$$

Thus the bubble condition (3.5.3) is sufficient to prove that γ_{S} exists and $\gamma_{\mathrm{S}} = 1$. The inequality (3.5.8) is derived from a differential inequality in [102, Theorem 2.3], which was proved there for uniform spread-out models. The derivation still holds for infinite-range spread-out models due to the multiplicative structure of the *weights* of the self-avoiding walks in (1.2.2). A version of (3.5.8) appeared earlier in [27, (5.30)–(5.33)].

The derivation of the exponents $\gamma_{\mathrm{P}} = 1$, $\beta_{\mathrm{P}} = 1$ and $\delta_{\mathrm{P}} = 2$ from the triangle condition is due to Aizenman–Newman [9] and Barsky–Aizenman [14]. While $\beta_{\mathrm{P}} = 1$ has been proven in [14] in the fully general setting of partially oriented percolation models, for $\gamma_{\mathrm{P}} = 1$ only the upper bound is known in full generality [9]. The corresponding lower bound is stated in [9] only for the nearest-neighbor model. In Section 3.5.1 below we show how the arguments in [9] can be adjusted for an infinite range model. Also $\delta_{\mathrm{P}} = 2$ is known in full generality [14], but only for a slightly different version of δ_{P}. We state the different version of δ_{P} and prove equivalence of the two versions in Section 3.5.2.

For the Ising model, it has been proven by Aizenman [1, Proposition 7.1] that the bubble condition implies $\gamma_{\mathrm{I}} = 1$ as long as $|J| = \sum_x J(x) < \infty$ (which is equivalent to $\sum_x D(x) < \infty$). Under the same condition, Aizenman and Fernández [6] proved the existence and mean-field values of the critical exponents β_{I} and δ_{I}.

The statement in *(iv)* is an immediate consequence of (3.5.6). The lower bound in $1 - \hat{D}(k) \asymp |k|^{\alpha\wedge 2}$ follows from (D3)/(D3'). The upper bound indeed holds for a number of examples, and in particular if D is chosen as in the nearest-neighbor model (3.1.1), the finite-variance spread-out model (3.1.6) or the spread-out power-law model (3.1.12) with $\alpha \neq 2$, cf. [33, 71]. However, if D is chosen as in (3.1.12) with $\alpha = 2$, then $1 - \hat{D}(k) \asymp (L|k|)^2 \log(\pi/(L|k|))$, cf. [33, Prop. 1.1].

Given (3.5.7) it is folklore that

$$G_{z_c}(x) \asymp |x|^{-d+(\alpha\wedge 2)} \tag{3.5.9}$$

holds in the general setting considered here. Partial results towards (3.5.9) have been obtained. Indeed, Hara, van der Hofstad and Slade [62] proved (3.5.9) in the finite-range spread-out setting for self-avoiding walk and percolation, Hara [55] proved it in the nearest-neighbor setting, and Sakai [95] proved it for the Ising model in finite-range spread-out and nearest-neighbor setting.

3.5.1 Derivation of $\gamma_{\mathrm{P}} = 1$ for percolation

Aizenman and Newman [9] prove that the triangle condition $T(z_c) < \infty$ implies that the critical exponent γ_{P} for percolation exists, and satisfies $\gamma_{\mathrm{P}} = 1$. That is to say, they show $\chi(z) \asymp (z_c - z)^{-1}$ as $z \nearrow z_c$. The lower bound $\gamma_{\mathrm{P}} \geq 1$ in [9, Prop. 3.1] holds for any homogeneous bond percolation model. On the other hand, the upper bound $\gamma_{\mathrm{P}} \leq 1$ is stated in [9, Prop. 3.1] for the nearest-neighbor model only. The aim of this section is to show how the derivation in [9] can be extended to long range systems.

The argument requires a finite volume and range approximation in order to apply Russo's formula. We denote by

$$\mathbb{T}_r := [-r, r]^d \cap \mathbb{Z}^d$$

a cube of sidelength $2r+1$. In order to achieve translation invariance, we equip the cube with periodic boundary conditions, that is, $\mathbb{T}_r$ is a torus. In [9] free boundary conditions were used. We write $G^{(R)}_{z,\mathbb{T}_r}(x,y)$ for the probability that the points x and y are connected on the torus using only bonds $\{u,v\}$ of length $|u-v| \leq R$. For $r > R$ (which we always assume), this is equivalent to removing all bonds from $\mathbb{T}_r$ with length larger than R. Define accordingly the *restricted expected cluster size* by

$$\chi^{(R)}_{\mathbb{T}_r}(z) := \sum_{x \in \mathbb{T}_r} G^{(R)}_{z,\mathbb{T}_r}(0,x), \tag{3.5.10}$$

and the *restricted triangle diagram* by

$$\nabla^{(R)}_{\mathbb{T}_r}(z) := \sum_{\substack{v,s,t \in \mathbb{T}_r \\ |v| \leq R}} D(v)\, G^{(R)}_{z,\mathbb{T}_r}(v,s)\, G^{(R)}_{z,\mathbb{T}_r}(s,t)\, G^{(R)}_{z,\mathbb{T}_r}(t,0). \tag{3.5.11}$$

We proceed as follows. We fix $\varepsilon > 0$ small, and first show that for $z < z_c - \varepsilon$,

$$\left(1 - \nabla^{(R)}_{\mathbb{T}_r}(z_c - \varepsilon) - e_R\right)(z_c - z - \varepsilon) \leq \frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z)} - \frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z_c - \varepsilon)} \leq (z_c - z - \varepsilon) \tag{3.5.12}$$

holds uniformly in r and R, where $e_R = o(1)$ as $R \to \infty$. We argue that indeed, for $z < z_c - \varepsilon$,

$$\lim_{R \to \infty} \lim_{r \to \infty} \chi^{(R)}_{\mathbb{T}_r}(z) = \chi(z), \tag{3.5.13}$$

and, for every $R > 0$,

$$\nabla^{(R)}_{\mathbb{T}_r}(z_c - \varepsilon) \leq \nabla(z_c - \varepsilon) + o(1) \qquad \text{as } r \nearrow \infty, \tag{3.5.14}$$

where $\nabla(z) = (D * G_z * G_z * G_z)(0)$. Note that $\nabla(z)$ differs from $T(z)$ by the extra displacement D. Then, taking $r \to \infty$ followed by $R \to \infty$, we obtain for every $\varepsilon > 0$,

$$(1 - \nabla(z_c - \varepsilon))\,(z_c - z - \varepsilon) \leq \frac{1}{\chi(z)} - \frac{1}{\chi(z_c - \varepsilon)} \leq z_c - z - \varepsilon. \tag{3.5.15}$$

The limit $\varepsilon \searrow 0$ then yields

$$(1 - \nabla(z_c))\,(z_c - z) \leq \frac{1}{\chi(z)} \leq z_c - z. \tag{3.5.16}$$

since $\chi(z_c - \varepsilon)^{-1} \searrow 0$ as $\varepsilon \searrow 0$.

It follows from the infrared bound (3.4.1) and Assumption 3.1, together with the Cauchy-Schwarz inequality, that $\nabla(z_c) \leq O(\beta^{1/2})$. Thus (3.5.16) implies $\gamma_{\mathrm{P}} = 1$ if β in Theorem 3.7 is sufficiently small, which suffices for our needs. It is possible to extend the argument to any finite triangle diagram (rather than small triangle diagrams only) by using ultraviolet regularization, as done in [9, Lemma 6.3].

We start by proving (3.5.12), and recall the notion of pivotal bonds from Section 2.1. A crucial tool in the proof is Russo's formula [50, Theorem 2.25], stating that

$$\frac{\mathrm{d}}{\mathrm{d}z} G^{(R)}_{z,\mathbb{T}_r}(x,y) = \sum_{\substack{(u,v)\in\mathbb{T}_r\times\mathbb{T}_r \\ |u-v|\leq R}} D(v-u)\, \mathbb{P}^{(R)}_{z,\mathbb{T}_r}((u,v) \text{ is pivotal for } x \leftrightarrow y), \qquad x,y\in\mathbb{T}_r. \tag{3.5.17}$$

The factor $D(v-u)$ arises from the chain rule and the fact that the bond (u,v) is occupied with probability $zD(v-u)$. Since

$$\{(u,v) \text{ is pivotal for } x \leftrightarrow y\} \subset \big\{\{x \leftrightarrow u\} \circ \{v \leftrightarrow y\}\big\} \cup \big\{\{x \leftrightarrow v\} \circ \{u \leftrightarrow y\}\big\}, \tag{3.5.18}$$

(3.5.17) and the BK-inequality (Proposition 2.3) imply

$$\frac{\mathrm{d}}{\mathrm{d}z} G^{(R)}_{z,\mathbb{T}_r}(x,y) \leq \sum_{u,v\in\mathbb{T}_r} D(v-u)\, \mathbb{P}^{(R)}_{z,\mathbb{T}_r}(x \leftrightarrow u)\, \mathbb{P}^{(R)}_{z,\mathbb{T}_r}(v \leftrightarrow y). \tag{3.5.19}$$

Summing over y yields the upper bound

$$\frac{\mathrm{d}}{\mathrm{d}z}\chi^{(R)}_{\mathbb{T}_r}(z) \;\leq\; \Big(\sum_{u\in\mathbb{T}_r} G^{(R)}_{z,\mathbb{T}_r}(x,u)\Big)\Big(\sum_{v\in\mathbb{T}_r} D(v-u)\Big)\Big(\sum_{y\in\mathbb{T}_r} G^{(R)}_{z,\mathbb{T}_r}(v,y)\Big) \;\leq\; \chi^{(R)}_{\mathbb{T}_r}(z)^2 \tag{3.5.20}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[-\frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z)}\right] \leq 1. \tag{3.5.21}$$

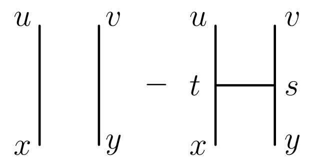
Integration over the interval $(z, z_c - \varepsilon)$ yields

$$\frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z)} - \frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z_c - \varepsilon)} \leq z_c - z - \varepsilon. \tag{3.5.22}$$

For the lower bound in (3.5.12) we use arguments as in [102, Section 9.4] to obtain

$$\begin{aligned} &\mathbb{P}^{(R)}_{z,\mathbb{T}_r}((u,v) \text{ is pivotal for } x \leftrightarrow y) \\ &\geq G^{(R)}_{z,\mathbb{T}_r}(x,u)\, G^{(R)}_{z,\mathbb{T}_r}(v,y) - \sum_{s,t\in\mathbb{T}_r} G^{(R)}_{z,\mathbb{T}_r}(x,t)\, G^{(R)}_{z,\mathbb{T}_r}(t,s)\, G^{(R)}_{z,\mathbb{T}_r}(t,u)\, G^{(R)}_{z,\mathbb{T}_r}(s,v)\, G^{(R)}_{z,\mathbb{T}_r}(s,y). \end{aligned} \tag{3.5.23}$$

=



(The contribution to the second line in (3.5.23) with u and v interchanged is hidden there, but is incorporated in the next line when we sum over both, u and v.) With Russo's formula (3.5.17),

$$\frac{\mathrm{d}}{\mathrm{d}z}\chi^{(R)}_{\mathbb{T}_r}(z) \geq \chi^{(R)}_{\mathbb{T}_r}(z)^2 \sum_{|v|\leq R} D(v) - \chi^{(R)}_{\mathbb{T}_r}(z)^2 \sum_{\substack{v,s,t\in\mathbb{T}_r\\ |v|\leq R}} D(v)\, G^{(R)}_{z,\mathbb{T}_r}(v,s)\, G^{(R)}_{z,\mathbb{T}_r}(s,t)\, G^{(R)}_{z,\mathbb{T}_r}(t,0). \tag{3.5.24}$$

Since $\sum_{v\in\mathbb{Z}^d} D(v) = 1$, the quantity $e_R := \sum_{|v|>R} D(v)$ is $o(1)$ as $R\to\infty$. Recalling the definition of $\nabla^{(R)}_{\mathbb{T}_r}(z)$ in (3.5.11) we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[-\frac{1}{\chi^{(R)}_{\mathbb{T}_r}(z)}\right] \geq (1-e_R) - \nabla^{(R)}_{\mathbb{T}_r}(z) \geq 1 - \nabla^{(R)}_{\mathbb{T}_r}(z_c-\varepsilon) - e_R \tag{3.5.25}$$

for $z < z_c - \varepsilon$, and an integrated version of this proves (3.5.12).

We now consider (3.5.13) and fix $z < z_c - \varepsilon$. We write $\mathbb{E}^{(R)}_{z,\mathbb{T}_r}|\mathcal{C}|$ for the expected cluster size under the measure $\mathbb{P}^{(R)}_{z,\mathbb{T}_r}$, i.e., $\mathbb{E}^{(R)}_{z,\mathbb{T}_r}|\mathcal{C}| = \chi^{(R)}_{\mathbb{T}_r}(z)$. We further denote by $\partial_R\mathbb{T}_r := \mathbb{T}_{r+R}\setminus\mathbb{T}_r$ the boundary of $\mathbb{T}_r$ of thickness R. Hence,

$$\mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}| = \mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}|\mathbb{1}_{\{0\not\leftrightarrow\partial_R\mathbb{T}_r\}} + \mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}|\mathbb{1}_{\{0\leftrightarrow\partial_R\mathbb{T}_r\}}. \tag{3.5.26}$$

In the first summand, $\mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}$ can be replaced by $\mathbb{E}^{(R)}_z$ (the expected cluster size on the infinite lattice, where bonds are restricted to have length $\leq R$), because the indicator guarantees $\mathcal{C}\subset\mathbb{T}_r$. This leads to

$$\mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}| = \mathbb{E}^{(R)}_z|\mathcal{C}| - \mathbb{E}^{(R)}_z|\mathcal{C}|\mathbb{1}_{\{0\leftrightarrow\partial_R\mathbb{T}_r\}} + \mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}|\mathbb{1}_{\{0\leftrightarrow\partial_R\mathbb{T}_r\}}. \tag{3.5.27}$$

By the tree graph bound [9] and the monotonicity of $\mathbb{E}^{(R)}_z|\mathcal{C}|$ in R,

$$\mathbb{E}^{(R)}_z|\mathcal{C}|^2 \leq \left(\mathbb{E}^{(R)}_z|\mathcal{C}|\right)^3 \leq \chi(z)^3, \tag{3.5.28}$$

and hence the Cauchy-Schwarz inequality yields

$$\mathbb{E}^{(R)}_z|\mathcal{C}|\mathbb{1}_{\{0\leftrightarrow\partial_R\mathbb{T}_r\}} \leq \chi(z)^{3/2}\,\mathbb{P}_z(0\leftrightarrow\partial_R\mathbb{T}_r)^{1/2}. \tag{3.5.29}$$

For $z < z_c - \varepsilon$, the first factor on the right is finite, and the latter vanishes as $r\to\infty$. For the last summand in (3.5.27), we bound as follows:

$$\mathbb{E}^{(R)}_{z,\mathbb{T}_{r+R}}|\mathcal{C}|\mathbb{1}_{\{0\leftrightarrow\partial_R\mathbb{T}_r\}} \leq (2(r+R)+1)^d\,\mathbb{P}^{(R)}_{z,\mathbb{T}_{r+R}}(0\leftrightarrow\partial_R\mathbb{T}_r), \tag{3.5.30}$$

but, for $r > R$,

$$\mathbb{P}^{(R)}_{z,\mathbb{T}_{r+R}}(0\leftrightarrow\partial_R\mathbb{T}_r) \leq \mathbb{P}_{z,\mathbb{T}_{r+R}}(|\mathcal{C}|\geq r/R) \leq \mathbb{P}_z(|\mathcal{C}|\geq r/R) \leq \exp\left\{-\frac{r}{2R\chi(z)^2}\right\}, \tag{3.5.31}$$

where in the first bound we use the fact that occupied bonds have length $\leq R$ in the restricted model, the second bound utilizes the fact that clusters on the torus are a.s. smaller than clusters in the infinite lattice (Proposition 4.1), and the third bound uses [9, Prop. 5.1]. The expression on the right hand side of (3.5.31) decays exponentially as r increases, hence the right hand side of (3.5.30) vanishes and (3.5.13) is established once we have shown that $\mathbb{E}^{(R)}_z|\mathcal{C}| \to \mathbb{E}_z|\mathcal{C}|$ as $R\to\infty$.

This is done as follows. We write $G_z^{(R)}$ and $\chi^{(R)}$ for the model on the infinite lattice where bonds are restricted to have length $\leq R$. Then obviously $\chi(z) \geq \chi^{(R)}(z)$. Furthermore,

$$G_z(x) - G_z^{(R)}(x) = \mathbb{P}_z\left(0 \leftrightarrow x, \exists \text{ pivotal bond } (u,v) \text{ for } \{0 \leftrightarrow x\} \text{ with } |u-v| > R\right),$$

hence, using the BK-inequality,

$$\chi(z) - \chi^{(R)}(z) \leq \chi(z)^2 \left(z \sum_{v:|v|>R} D(v) \right).$$

Again, this vanishes as $R \to \infty$, because $z < z_c - \varepsilon$ and $\sum_v D(v) = 1$.

It remains to prove (3.5.14). We use again the coupling of Proposition 4.1 to write

$$\mathbb{P}^{(R)}_{z_c-\varepsilon,\mathbb{T}_{r+R}}(0 \leftrightarrow x) \leq \mathbb{P}_{z_c-\varepsilon}(0 \leftrightarrow x) + \mathbb{P}_{z_c-\varepsilon}(0 \leftrightarrow \partial_R \mathbb{T}_r). \tag{3.5.32}$$

Since the contribution from terms involving $\mathbb{P}_{z_c-\varepsilon}(0 \leftrightarrow \partial_R \mathbb{T}_r)$ is again exponentially small in r (cf. (3.5.31)), we readily obtain (3.5.14).

3.5.2 Derivation of $\delta_{\mathrm{P}} = 2$ for percolation

Barsky and Aizenman [14] showed that the triangle condition implies $\beta_{\mathrm{P}} = 1$ and $\delta_{\mathrm{P}} = 2$, where they used the mean-field bounds $\beta_{\mathrm{P}} \leq 1$ and $\delta_{\mathrm{P}} \geq 2$ due to [32] and [3], respectively. It should be noted that in these references a different version of δ_{P} is considered, namely $\hat{\delta}_{\mathrm{P}}$ given by

$$M(z_c, h) := \sum_{k=1}^{\infty} [1 - \mathrm{e}^{-kh}]\, \mathbb{P}_{z_c}(|\mathcal{C}| = k) \asymp h^{1/\hat{\delta}_{\mathrm{P}}} \qquad \text{as } h \to \infty. \tag{3.5.33}$$

The quantity M is known as *magnetization*. If we consider the critical exponents in terms of slowly varying functions only (and not our stronger version $\asymp$), then the equivalence of δ_{P} and $\hat{\delta}_{\mathrm{P}}$ can be seen directly via a Tauberian Theorem (e.g. [40, Theorem XIII.5.2]).

Our version of δ_{P} can be derived from (3.5.33), as we show now for the mean-field value $\delta_{\mathrm{P}} = 2$. In particular, we show that

$$c/\sqrt{n} \leq M(z_c, 1/n) \leq C/\sqrt{n}, \qquad 0 < c \leq C < \infty, \tag{3.5.34}$$

implies $\tilde{c}/\sqrt{n} \leq \mathbb{P}_{z_c}(|\mathcal{C}| \geq n) \leq \tilde{C}/\sqrt{n}$ for certain constants $\tilde{c}, \tilde{C} \in (0, \infty)$.

For an upper bound on $\mathbb{P}_{z_c}(|\mathcal{C}| \geq n)$ we bound

$$\begin{aligned} \mathbb{P}_{z_c}(|\mathcal{C}| \geq n) &= \sum_{k=n}^{\infty} \mathbb{P}_{z_c}(|\mathcal{C}| = k) \leq \sum_{k=n}^{\infty} \frac{1 - \mathrm{e}^{-k/n}}{1 - \mathrm{e}^{-1}}\, \mathbb{P}_{z_c}(|\mathcal{C}| = k) \\ &\leq \left[1 - \mathrm{e}^{-1}\right]^{-1} \sum_{k=1}^{\infty} \left[1 - \mathrm{e}^{-k/n}\right] \mathbb{P}_{z_c}(|\mathcal{C}| = k) \\ &= \left[1 - \mathrm{e}^{-1}\right]^{-1} M(p_c, 1/n), \end{aligned} \tag{3.5.35}$$

and hence $\mathbb{P}_{z_c}(|\mathcal{C}| \geq n) \leq \tilde{C}/\sqrt{n}$ for $\tilde{C} = [1 - \mathrm{e}^{-1}]^{-1} C$.

The lower bound is more involved. For every $\varepsilon > 0$ we obtain

$$\begin{aligned} \mathbb{P}_{z_c}(|\mathcal{C}| \geq n) &\geq \sum_{k=n}^{\infty} \left[1 - \mathrm{e}^{-\varepsilon k/n}\right] \mathbb{P}_{z_c}(|\mathcal{C}| = k) \\ &= M(p_c, \varepsilon/n) - \sum_{k=1}^{n-1} \left[1 - \mathrm{e}^{-\varepsilon k/n}\right] \mathbb{P}_{z_c}(|\mathcal{C}| = k). \end{aligned}$$

We exploit $1 - \mathrm{e}^{-x} \leq x$ to bound further

$$\sum_{k=1}^{n-1} \left[1 - \mathrm{e}^{-\varepsilon k/n}\right] \mathbb{P}_{z_c}(|\mathcal{C}| = k) \leq \frac{\varepsilon}{n} \sum_{k=1}^{n-1} k \, \mathbb{P}_{z_c}(|\mathcal{C}| = k).$$

Note

$$\sum_{k=1}^{n-1} k \, \mathbb{P}_{z_c}(|\mathcal{C}| = k) = \sum_{k=1}^{n-1} \sum_{l=1}^{k} \mathbb{P}_{z_c}(|\mathcal{C}| = k) = \sum_{l=1}^{n-1} \sum_{k=l}^{n-1} \mathbb{P}_{z_c}(|\mathcal{C}| = k) \leq \sum_{l=1}^{n-1} \mathbb{P}_{z_c}(|\mathcal{C}| \geq l),$$

whence

$$\mathbb{P}_{z_c}(|\mathcal{C}| \geq n) \geq M(p_c, \varepsilon/n) - \frac{\varepsilon}{n} \sum_{k=1}^{n-1} \mathbb{P}_{z_c}(|\mathcal{C}| \geq k).$$

We apply (3.5.35) and compare with (3.5.34) to obtain

$$\mathbb{P}_{z_c}(|\mathcal{C}| \geq n) \geq \frac{c\sqrt{\varepsilon}}{\sqrt{n}} - \frac{\varepsilon}{n} \underbrace{\sum_{k=1}^{n-1} \frac{C}{[1 - \mathrm{e}^{-1}]\sqrt{k}}}_{\leq 2C[1-\mathrm{e}^{-1}]^{-1}\sqrt{n}}. \tag{3.5.36}$$

This proves that $\mathbb{P}_{z_c}(|\mathcal{C}| \geq n) \geq \tilde{c}/\sqrt{n}$ with $\tilde{c} = c\sqrt{\varepsilon} - 2\varepsilon C[1 - \mathrm{e}^{-1}]^{-1}$, and $\tilde{c} > 0$ as long as ε is small enough. With a modification in (3.5.36), the argument can be extended to the case $\delta_{\mathrm{P}} \neq 2$, but we refrain from giving this argument.

3.6 Related lace expansion results

We now have seen how the lace expansion can be used to derive an infrared bound, and we discussed various consequences of it. In fact, the lace expansion is used in much wider context to obtain a variety of results on the high-dimensional behaviour of statistical mechanical models. Some of these results shall be briefly reviewed here.

Asymptotics for the critical two-point function $G_z(x)$. In Theorem 3.7 we have identified the asymptotics of the Fourier transform of the critical two-point function $\hat{G}_{z_c}(k)$ as $|k| \to 0$. A natural question arising is the asymptotics of the (x-space) critical two-point function $G_{z_c}(x)$ as $|x| \to \infty$. Obtaining the x-space behavior from the k-space result is a notoriously difficult problem, and such results (which are known as Tauberian theorems) typically involve corrections by slowly varying functions. The following theorem provides strong bounds on the decay of $G_{z_c}(x)$, and is obtained by doing the analysis of the lace expansion in x-space directly (in contrast to our approach in Section 3.4). The theorem is proven for finite-range systems only.

Theorem 3.17 (Asymptotics for the critical two-point function)**.** *Consider percolation, self-avoiding walk or the Ising model with D given by* (3.1.1) *(nearest-neighbor case) with d sufficiently large, or D given by* (3.1.5) *(finite-variance spread-out model) with $d > 3s$ and L sufficiently large with the additional requirement that the function h is supported in $[-1, 1]^d$. Then there exists a constant A such that*

$$G_{z_c}(x) = \frac{A}{(|x| + 1)^{d-2}} \left(1 + o(1)\right). \tag{3.6.1}$$

The power law bound (3.6.1) is sometimes interpreted as the statement that the critical exponent η exists and is equal to 0. In fact, this is a stronger version than (3.5.7). In Chapter 4 we shall use the following weaker form of (3.6.1): there exist constants $c_\tau, C_\tau > 0$ such that

$$\frac{c_\tau}{(|x|+1)^{d-2}} \leq G_{z_c}(x) \leq \frac{C_\tau}{(|x|+1)^{d-2}}. \tag{3.6.2}$$

For percolation and self-avoiding walk, Takashi Hara [55] gives a proof of Theorem 3.17 for the nearest-neighbor case, and Hara, van der Hofstad and Slade [62] prove it for the uniform spread-out case. For the Ising model, both cases are proved by Sakai [95]. In fact, the results proven in [55, 62, 95] are even stronger in the sense that they include rather precise expressions for the constant A and good bounds on the error term.

Correlation length and exponential decay of the subcritical two-point function for percolation. The next theorem provides an upper bound on the two-point function for percolation in terms of the correlation length.

Theorem 3.18 (Correlation length and exponential decay of the subcritical two-point function)**.** *Consider percolation in the nearest-neighbor setting (with d being sufficiently large), or with the uniform spread out model* (3.1.6) *with $d > 6$ and L sufficiently large. For any $z < z_c$,*

$$G_z(x) \leq e^{-\|x\|_\infty/\xi(z)}, \tag{3.6.3}$$

where the correlation length $\xi(z)$ is defined by

$$\xi(z)^{-1} = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z\big((0,\dots,0) \longleftrightarrow (n,0,\dots,0)\big). \tag{3.6.4}$$

Furthermore, there exist constants $c_\xi, C_\xi > 0$ such that the correlation length satisfies

$$c_\xi\,(z_c - z)^{-1/2} \leq \xi(z) \leq C_\xi\,(z_c - z)^{-1/2}. \tag{3.6.5}$$

The exponential bound (3.6.3) is well-known in percolation theory, see e.g. Grimmett [50, Prop. 6.47]. Hara [54] proves the bound (3.6.5).

There is a rather loose interpretation of the term ‘correlation length’ in mathematical physics as ‘natural length scale’ of the process in the following sense: it is the minimal length scale on which percolation with parameter z differs qualitatively from percolation with parameter z_c.

Asymptotic of the critical threshold. The particular value of z_c is known only in certain special cases, for example on the two-dimensional nearest-neighbor lattice $z_c = 2d\,\frac{1}{2} = 2$ for percolation [63, 79], and $z_c = \log(1+\sqrt{2})$ for the Ising model. It is assumed that in higher dimensions z_c cannot be expressed in closed form, but the following expansion is proved using lace expansion methodology.

Theorem 3.19. *For the nearest-neighbor version of percolation and the self-avoiding walk, the critical threshold $z_c = z_c(d)$ obeys the following asymptotic expansion:*

(i) for percolation,

$$z_c = 1 + \frac{1}{2d} + \frac{7}{2(2d)^2} + O\big((2d)^{-3}\big); \tag{3.6.6}$$

(ii) for self-avoiding walk,

$$z_c^{-1} = \mu = 1 - \frac{1}{2d} - \frac{1}{(2d)^2} - \frac{3}{(2d)^3} - \frac{16}{(2d)^4} - \frac{102}{(2d)^5} - \frac{729}{(2d)^6} - \frac{5533}{(2d)^7} - \frac{42229}{(2d)^8} - \frac{288761}{(2d)^9} - \frac{1026328}{(2d)^{10}} + \frac{21070667}{(2d)^{11}} + \frac{780280468}{(2d)^{12}} + O\left(\frac{1}{(2d)^{13}}\right). \tag{3.6.7}$$

The asymptotic formula (3.6.6) has been proven by Hara and Slade [59] (and reproved, using simpler methods, by van der Hofstad and Slade [72]). Clisby, Liang and Slade [36] showed (3.6.7). See also [102, Sections 2.1 and 11.2] for a bibliography of partial results towards (3.6.6)–(3.6.7). It seems likely that these asymptotic expansions have radius of convergence zero, but there is no proof of this.

Incipient infinite cluster and scaling limit of high-dimensional percolation. Consider nearest-neighbor percolation. Then it is believed (and proved for $d = 2$ or $d \geq 19$) that $\theta(z_c) = 0$. On the other hand, $\chi(z_c) = \infty$ by (1.1.5). This means that critical percolation clusters are a.s. finite, but the expected size is infinite, and there exist clusters at all length scales. Moreover, if z is increased a tiny little bit, then immediately an infinite cluster emerges. These observations motivated the construction of an *incipient infinite cluster* (IIC), which can be thought of as a critical cluster that is artificially made infinite by considering an appropriate limit.

The first such construction of an IIC is due to Kesten [81]. Indeed, Kesten investigates the local configuration close to the origin in the two-dimensional lattice $\mathbb{Z}^2$, and offers two alternatives:

(i) To condition the origin to be connected to infinity at $z > z_c$ and take the limit $z \searrow z_c$.

(ii) To condition the critical cluster of the origin to be connected to the boundary of the box $\{-n, \ldots, 0, \ldots, n\}^2$ and take the limit $n \to \infty$.

Kesten proves that both limits exist and are equal. This limit is Kesten's incipient infinite cluster. Kesten was motivated to describe this IIC in order to study random walk on large critical clusters [80], for which physicists have performed simulations showing subdiffusive behavior.

Járai [76, 77] extended these results, and proved that several other natural conditioning and limiting schemes give the same limit. In one of these constructions, Járai takes a uniform point in the largest critical cluster on a box $\{0, \ldots, r-1\}^d$, shifts it to the origin and takes the limit $r \to \infty$. In [70], the lace expansion was used to extend the proof of existence of the IIC to high-dimensional percolation. The proof in [70] follows the proof in [69], where the IIC was constructed for spread-out oriented percolation above 4 spatial dimensions. Also Aizenman [2] introduces an IIC version suitable in high dimensions, this construction is discussed at length in Chapter 4.

Hara and Slade [60, 61] study the geometry of large critical clusters on the spatially rescaled nearest-neighbor lattice in sufficiently high dimension. Consider the quantity

$$\tau(x, n) := \mathbb{P}_{z_c}(0 \leftrightarrow x, |\mathcal{C}(0)| = n). \tag{3.6.8}$$

The authors show that the Fourier transform of (3.6.8), rescaled in the spatial variable by $n^{-1/4}$, converges to the two-point function of integrated super-Brownian excursion (ISE) scaled by $n^{-1/2}$. Integrated super-Brownian excursion is a random mass distribution on $\mathbb{R}^d$ introduced by Aldous [12] as a continuous-time branching process with branching at all time scales. It is known to be the scaling limit of critical branching random walk conditioned on the total mass being equal to 1, [11, 12]. Furthermore, ISE is known to be the scaling limit of high-dimensional lattice trees, [38].

The results by Hara and Slade [60, 61], which also include convergence of the corresponding 3-point function, provide solid support for the conjecture that the scaling limit of large critical clusters is integrated super-Brownian excursion. Furthermore, the results indicate that critical percolation clusters have an almost tree-like structure, where loops do not contribute in the scaling limit.

Exact enumerations of self-avoiding walks. Consider nearest-neighbor self-avoiding walk, and denote by $N_n := (2d)^n\, c_n$ the *number* of n-step nearest-neighbor self-avoiding walks on $\mathbb{Z}^d$. So far we reported only about asymptotic results obtained by lace expansion methodology. In a new development, Clisby, Liang and Slade [36] used the lace expansion to compute the *exact* value of N_n for *fixed* n. Among various other quantities they compute N_n for $n = 1, 2, \dots 24$ in all dimensions up to 12, and also the number of self-avoiding polygons in these dimensions up to length 24. This is a computationally hard problem, and in dimensions $d > 2$ little progress had been made before. The authors use the so-called two-step method to calculate the number of lace graphs, and then use the lace expansion to calculate N_n.

Chapter 4
Random graph asymptotics on high-dimensional tori

In this chapter we consider a somewhat different problem, namely the comparison of percolation on two different graphs: the infinite lattice $\mathbb{Z}^d$ and the (finite) d-dimensional torus, both in sufficiently high dimension. The basic question to be addressed is the following: Let p_c be the critical percolation threshold on $\mathbb{Z}^d$. What is then the size of the largest connected cluster for percolation on the torus with this parameter p_c? The main results in this chapter are Theorems 4.2 and 4.4. The key to the proof of these theorems is a coupling argument to be developed in Section 4.2. We do not use the lace expansion here, but our proofs rely on a variety of results that have been proven using lace expansion. In contrast to the previous chapters we consider here only the nearest-neighbor model (3.1.1) and the uniform spread-out model (3.1.6). In line with most literature on these models we will slightly change notation and write p for the probability that a certain bond will be occupied (e.g. for the nearest-neighbor model, p corresponds to $z/(2d)$ in the setup of Chapter 3).

4.1 Percolation on a torus

We consider Bernoulli bond percolation on the graph $\mathbb{G}$, where $\mathbb{G}$ is either the hypercubic lattice $\mathbb{Z}^d$, or the finite torus $\mathbb{T}_{r,d} = \{-\lfloor r/2 \rfloor, \dots, \lceil r/2 \rceil - 1\}^d$.

For $\mathbb{G} = \mathbb{Z}^d$, we consider two sets of bonds. In the *nearest-neighbor model* (3.1.1), two vertices x and y are linked by a bond whenever $|x-y| = 1$, whereas in the *(uniform) spread-out model* (3.1.6), they are linked whenever $0 < \|x-y\| \leq L$. Throughout the chapter we adopt the convention that $\|\cdot\|$ denotes the supremum norm. The degree of the graph, which we denote by Ω, is thus $\Omega = 2d$ in the nearest-neighbor case and $\Omega = (2L+1)^d - 1$ in the spread-out case. Bonds are occupied independently with probability p, and the two-point function is now denoted by

$$\tau_{\mathbb{Z},p}(x) := \mathbb{P}_{\mathbb{Z},p}(0 \longleftrightarrow x), \qquad x \in \mathbb{Z}^d. \tag{4.1.1}$$

All quantities related to $\mathbb{Z}^d$-percolation get a subscript $\mathbb{Z}$, e.g. we write $\mathcal{C}_{\mathbb{Z}}(x)$ for the cluster of x and $\chi_{\mathbb{Z}}(p)$ for the expected cluster size.

For $\mathbb{G} = \mathbb{T}_{r,d}$, we also consider two related settings:

(i) The nearest-neighbor torus: an edge joins vertices that differ by 1 (modulo r) in exactly one component. For d fixed and r large, this is a periodic approximation to $\mathbb{Z}^d$. Here $\Omega = 2d$ for $r \geq 3$. We study the limit in which $r \to \infty$ with $d > 6$ fixed, but large.

(ii) The spread-out torus: an edge joins vertices $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ if $0 < \max_{i=1,\dots,d} |x_i - y_i|_r \leq L$ (with $|\cdot|_r$ the metric on $\mathbb{Z}_r$). We study the limit $r \to \infty$, with $d > 6$ fixed and L large (depending on d) and fixed. This gives a periodic approximation to range-L percolation on $\mathbb{Z}^d$. Here $\Omega = (2L+1)^d - 1$ provided that $r \geq 2L+1$, which we will always assume.

We consider bond percolation on these tori with bond occupation probability p and write $\mathbb{P}_{\mathbb{T},p}$ and $\mathbb{E}_{\mathbb{T},p}$ for the product measure and corresponding expectation, respectively. We use the notation $\tau_{\mathbb{T},p}(\cdot)$, $\chi_{\mathbb{T}}(p)$ and $\mathcal{C}_{\mathbb{T}}(\cdot)$ analogously to the corresponding $\mathbb{Z}^d$-quantities.

In this chapter, we will investigate the size of the maximal cluster on $\mathbb{T}_{r,d}$, i.e.,

$$|\mathcal{C}_{\max}| := \max_{x\in\mathbb{T}_{r,d}} |\mathcal{C}_{\mathbb{T}}(x)|, \tag{4.1.2}$$

at the critical percolation threshold $p_c(\mathbb{Z}^d)$. An alternative definition for the critical percolation threshold on the torus, denoted by $p_c(\mathbb{T}_{r,d})$, was given in [24, (1.7)] as the solution to

$$\chi_{\mathbb{T}}(p_c(\mathbb{T}_{r,d})) = \lambda V^{1/3}, \tag{4.1.3}$$

where λ is a sufficiently small constant, and $V = |\mathbb{T}_{r,d}| = r^d$ denotes the volume of the torus. The definition of $p_c(\mathbb{T}_{r,d})$ in (4.1.3) is an *internal* definition only, due to the fact that [25] deals with rather general tori, for which an external definition (such as $p_c(\mathbb{Z}^d)$) does not always exist. On the other hand, the internal definition in (4.1.3) assumes *a priori* mean-field behaviour, and is therefore unsuitable outside this setting[1]. On the high-dimensional torus $\mathbb{T}_{r,d}$, we therefore have *two* sensible critical values, the externally defined $p_c(\mathbb{Z}^d)$, and the internally defined $p_c(\mathbb{T}_{r,d})$ in (4.1.3). One of the goals of this chapter is to investigate how close these two critical values are.

The most prominent example of percolation on a finite graph is the *random graph*, which is obtained by applying percolation to the *complete* graph. This has been first studied by Erdős and Rényi in 1960 [39]. They showed that, when p is scaled as $(1+\varepsilon)V^{-1}$, there is a phase transition at $\varepsilon = 0$. For $\varepsilon < 0$, the size of the largest cluster is proportional to $\log V$, whereas for $\varepsilon > 0$, it is proportional to V. For $\varepsilon = 0$, the size of the largest cluster divided by $V^{2/3}$ weakly converges to some (non-trivial) limiting random variable, while the expected cluster size is, as in (4.1.3), proportional to $V^{1/3}$. This follows from results by Aldous [13], see also [21, 74, 75, 87]. We will refer to the $V^{2/3}$-scaling as *random graph asymptotics*.

In this chapter, we study the size of the largest cluster on the torus for $p = p_c(\mathbb{Z}^d)$. It has been shown by Borgs, Chayes, van der Hofstad, Slade and Spencer [24, 25] that, if

$$p_c(\mathbb{Z}^d) = p_c(\mathbb{T}_{r,d}) + O(V^{-1/3}), \tag{4.1.4}$$

then, with probability at least $1 - O(\omega^{-1})$, $|\mathcal{C}_{\max}|$ is in between $\omega^{-1}V^{2/3}$ and $\omega V^{2/3}$ as $V \to \infty$, for $\omega \geq 1$ sufficiently large.

Aizenman [2] conjectured that this random graph asymptotics holds for the maximal critical cluster in dimension $d > 6$, as we explain in more detail below. Also in [25] it was conjectured that (4.1.4) holds. By means of a coupling argument, we prove that a slightly weaker statement than (4.1.4) (with a logarithmic correction in the lower bound, see (4.3.3) below) indeed holds for d sufficiently large in the nearest-neighbor model, or $d > 6$ and L sufficiently large in the spread-out model. Furthermore, we give a criterion which we believe to hold, and which implies (4.1.4) without logarithmic corrections.

Note that all our results assume that d is large in the nearest-neighbor model or $d > 6$ and L large in the spread-out model. That is, we require the torus to be in some sense high-dimensional. We do believe that the results hold for all $d > 6$ and $L \geq 1$, however, the proof relies on various lace expansion results, which require that the degree Ω is large. On the other hand, we do not expect these asymptotics to be true for $d < 6$.

Aizenman [2] studied a similar question, but now for percolation on a box of width r under *bulk* boundary conditions, where clusters are defined to be the intersection of the box $\{-\lfloor r/2\rfloor, \ldots, \lceil r/2\rceil - 1\}^d$ with clusters in the infinite lattice (and thus clusters need not to be connected within the box). He assumed $\tau_{\mathbb{Z},p_c}(x) \asymp \|x\|^{-(d-2)}$, which was established in [55, 62] for sufficiently large dimension, see Theorem 3.17. Aizenman showed that, under

[1] It is believed that this issue could be solved by considering $\lambda V^{1/(1+\delta_{\mathrm{P}})}$ as the right hand side of (4.1.3), where δ_{P} is the critical exponent introduced in (1.1.8).

this condition on the two-point function, the size of the largest connected component under bulk boundary conditions is, with high probability, bounded from above by a constant times $r^4 \log r$, and bounded from below by $\varepsilon_r r^4$ for any sequence $\varepsilon_r \to 0$ as $r \to \infty$. Furthermore, he conjectures that the r^4-scaling for the size of the largest cluster holds for dimension $d > 6$ also under *free* boundary conditions (where no connections outside the box are allowed), but changes to $V^{2/3} = r^{2d/3} \gg r^4$ under *periodic* boundary conditions. This indicates the importance of boundary conditions at criticality in high dimensions. We will further elaborate on the role of boundary conditions in Section 4.4.

4.2 A coupling result for clusters on the torus and on $\mathbb{Z}^d$

In this section, we prove that the cluster size for percolation on the torus is stochastically smaller than the one on $\mathbb{Z}^d$ by a coupling argument. We fix p, and omit the subscript p from the notation. We use subscripts $\mathbb{Z}$ and $\mathbb{T}$ to denote objects on $\mathbb{Z}^d$ and $\mathbb{T}_{r,d}$, respectively.

The goal of this section is to give a coupling of the $\mathbb{T}_{r,d}$-cluster and the $\mathbb{Z}^d$-cluster of the origin. This will be achieved by constructing these two clusters simultaneously from a percolation configuration on $\mathbb{Z}^d$, as we explain in more detail now.

The basic idea is that, on any graph, it is well-known that the law of a cluster $\mathcal{C}(0)$ can be described by subsequently exploring the bonds one can reach from 0. We will first describe this exploration of a cluster in some detail, before giving the coupling, which is described by a more elaborate way of exploring the percolation clusters on the torus and on $\mathbb{Z}^d$ simultaneously from a percolation configuration on $\mathbb{Z}^d$. The exploration process is defined in terms of *colors* of the bonds. Initially, all bonds are uncolored, which means that they have not yet been explored. During the exploration process we will color the bonds black if they are found to be occupied, and white if they are found to be vacant. Furthermore, we distinguish between active and inactive vertices. Initially, only the origin 0 is active, and all other vertices are inactive.

We now explore the bonds in the graph according to the following scheme. We order the vertices in an arbitrary way. Let v be the smallest active vertex. Now we explore (and color) all uncolored bonds that have an endpoint in v, i.e., we make the bond black with probability p and white with probability $1-p$, independently of all other bonds. In case we have assigned the black color, we set the vertex at the other end of the bond active (unless it was already active and none of its neighboring edges are now uncolored, in which case we make it inactive). In particular, the active vertices are those vertices that are part of a black bond, as well as an uncolored bond. Finally, after all bonds starting at v have been explored, we set v inactive. We repeat doing so until there are no more active vertices. In the latter case, the exploration process is completed, i.e., there are no more black bonds that share a common endpoint with an uncolored bond. The cluster $\mathcal{C}(0)$ is equal to the set of vertices that are part of the black bonds. When the graph is finite, then this procedure always stops. When the graph is infinite, then the exploration process continues forever precisely when $|\mathcal{C}(0)| = \infty$. This completes the exploration of a single cluster on a general graph.

The exploration of a single cluster will be extended to explore the cluster on the torus $\mathcal{C}_{\mathbb{T}}(0)$ and the cluster on the infinite lattice $\mathcal{C}_{\mathbb{Z}}(0)$ simultaneously from a percolation configuration on $\mathbb{Z}^d$. For $\mathcal{C}_{\mathbb{Z}}(0)$, the result of the exploration will be identical to the exploration of a single cluster described above. The related cluster $\mathcal{C}_{\mathbb{T}}(0)$ is a subset of all vertices that are r-equivalent to vertices part of a black bond. The main result in this section is Proposition 4.1, whose proof gives the details of this simultaneous construction of the two clusters.

To state the result, we need some notation. For $x, y \in \mathbb{T}_{r,d}$, we write $x \overset{\mathbb{T}}{\leftrightarrow} y$ when x is connected to y in the percolation configuration on the torus, while, for $x, y \in \mathbb{Z}^d$, we write $x \overset{\mathbb{Z}}{\leftrightarrow} y$ when x is connected to y in the percolation configuration on $\mathbb{Z}^d$. We write $x \overset{r}{\sim} y$

when $x \pmod r = y \pmod r$, and call x and y *r-equivalent* when $x \stackrel{r}{\sim} y$. Also, we call two distinct bonds $\{x_1, y_1\}$ and $\{x_2, y_2\}$ *r-equivalent* if there exists an element $z \in \mathbb{Z}^d$ such that $\{x_1, y_1\} = \{x_2 + rz, y_2 + rz\}$. We sometimes abbreviate r-equivalent to equivalent. For a directed bond $b = (x, y)$, we write $\underline{b} = x$ and $\overline{b} = y$, and for two bonds $b_1 = (x_1, y_1)$ and $b_2 = (x_2, y_2)$, we write $b_1 \stackrel{r}{\sim} b_2$ when $(x_1, y_1) = (x_2 + rz, y_2 + rz)$ for some $z \in \mathbb{Z}^d$. We recall from Section 2.1 that $A \circ B$ denotes the event that the increasing events A and B occur on disjoint sets of bonds.

Proposition 4.1 (The coupling). *Consider nearest-neighbor percolation for $r \geq 3$ or spread-out percolation for $r \geq 2L+1$, in any dimension. There exists a probability law $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$ on the joint space of $\mathbb{Z}^d$- and $\mathbb{T}_{r,d}$-percolation such that, for all events E,*

$$\mathbb{P}_{\mathbb{Z},\mathbb{T}}(\mathcal{C}_{\mathbb{T}}(0) \in E) = \mathbb{P}_{\mathbb{T}}(\mathcal{C}_{\mathbb{T}}(0) \in E), \qquad \mathbb{P}_{\mathbb{Z},\mathbb{T}}(\mathcal{C}_{\mathbb{Z}}(0) \in E) = \mathbb{P}_{\mathbb{Z}}(\mathcal{C}_{\mathbb{Z}}(0) \in E), \tag{4.2.1}$$

and $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$-almost surely, for all $x \in \mathbb{T}_{r,d}$,

$$\{0 \stackrel{\mathbb{T}}{\leftrightarrow} x\} \subseteq \bigcup_{y \in \mathbb{Z}^d :\, y \stackrel{r}{\sim} x} \{0 \stackrel{\mathbb{Z}}{\leftrightarrow} y\}. \tag{4.2.2}$$

In particular, $|\mathcal{C}_{\mathbb{T}}(0)| \leq |\mathcal{C}_{\mathbb{Z}}(0)|$. Moreover, for $x \stackrel{r}{\sim} y$, and $\mathbb{P}_{\mathbb{Z},\mathbb{T}}$-almost surely,

$$\begin{aligned} &\{0 \stackrel{\mathbb{Z}}{\leftrightarrow} y\} \cap \{0 \stackrel{\mathbb{T}}{\leftrightarrow} x\}^c \\ &\subseteq \bigcup_{b_1 \neq b_2 :\, b_1 \stackrel{r}{\sim} b_2} \bigcup_{z \in \mathbb{Z}^d} (0 \stackrel{\mathbb{Z}}{\leftrightarrow} z) \circ (z \stackrel{\mathbb{Z}}{\leftrightarrow} \underline{b}_1) \circ (z \stackrel{\mathbb{Z}}{\leftrightarrow} \underline{b}_2) \circ (b_2 \text{ is } \mathbb{Z}-occ.) \circ (\overline{b}_2 \stackrel{\mathbb{Z}}{\leftrightarrow} y). \end{aligned} \tag{4.2.3}$$

Equation (4.2.2) will be used to conclude that the expected cluster size on $\mathbb{T}_{r,d}$ is bounded from above by the one on $\mathbb{Z}^d$. In order to prove our main results, we use (4.2.3) to prove a related lower bound on the expected cluster size on $\mathbb{T}_{r,d}$ in terms of the one on $\mathbb{Z}^d$. See Sections 4.3.3 and 4.3.4 for details. The inequality on the cluster sizes of $\mathbb{T}_{r,d}$- and $\mathbb{Z}^d$-percolation also follows from the coupling used by Benjamini and Schramm [16, Theorem 1]. However, (4.2.3) does not follow immediately from their work.

Proof of Proposition 4.1. The exploration of a single cluster, as described above, will be generalized to construct $\mathcal{C}_{\mathbb{T}}(0)$ and $\mathcal{C}_{\mathbb{Z}}(0)$ simultaneously from a percolation configuration on $\mathbb{Z}^d$. The difference between percolation on the torus and on $\mathbb{Z}^d$ can be summarised by saying that, on the torus, r-equivalent bonds have the *same* occupation status, while on $\mathbb{Z}^d$, equivalent and distinct bonds have an *independent* occupation status. For the exploration of the torus $\mathcal{C}_{\mathbb{T}}(0)$, we have to make sure that we explore equivalent bonds at most *once*. We therefore introduce a third color, gray, indicating that the bond itself has not been explored yet, but one of its equivalent bonds has. Therefore, at each step of the exploration process, we have 4 different types of bonds on $\mathbb{Z}^d$:

- uncolored bonds, which have not been explored yet;
- black bonds, which have been explored and found to be occupied;
- white bonds, which have been explored and found to be vacant;
- gray bonds, of which an equivalent bond has been explored.

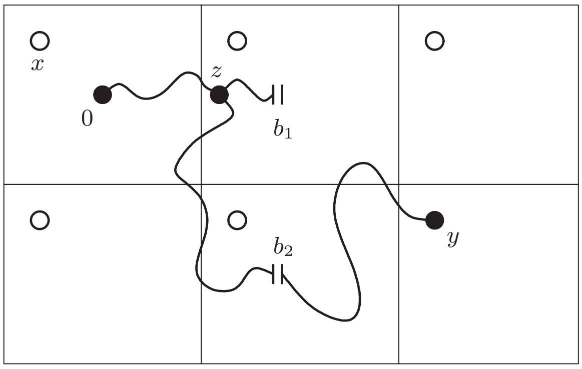


Figure 4.1: Illustration of the right hand side in (4.2.3). The bond b_1 has been explored first and is found to be $\mathbb{Z}$-vacant and $\mathbb{T}$-vacant. The bond b_2 is r-equivalent and thus has been explored in the $\mathbb{Z}$ exploration only, it is $\mathbb{T}$-vacant (determined by b_1 during the $\mathbb{T}$-exploration), but $\mathbb{Z}$-occupied.

As in the exploration of a single cluster, we number the vertices of $\mathbb{Z}^d$ in an arbitrary way, and start with all bonds uncolored and only the origin active. Then we repeat choosing the smallest active bond, and explore all uncolored bonds containing it. However, after exploring a bond (and coloring it black or white), we color all bonds that are r-equivalent to it gray. Again, this exploration is completed when there are no more active vertices. This is equivalent to the fact that there are no more black (and therefore occupied) bonds sharing a common endpoint with an uncolored bond.

The exploration process of $\mathcal{C}_{\mathbb{T}}(0)$ must be completed at some point, since the number of bonds within $\mathcal{C}_{\mathbb{T}}(0)$ is finite and vertices turn active only if a bond containing it is explored and is found to be occupied. We call the result of this exploration process the $\mathbb{T}$-*exploration.* The cluster $\mathcal{C}_{\mathbb{T}}(0)$ consists of all vertices in $\mathbb{T}_{r,d}$ that are contained in a bond that is r-equivalent to a black bond. However, we have embedded the cluster $\mathcal{C}_{\mathbb{T}}(0)$ into $\mathbb{Z}^d$, which will be useful when we also wish to describe the related cluster $\mathcal{C}_{\mathbb{Z}}(0)$.

For $\mathcal{C}_{\mathbb{Z}}(0)$, the exploration of the cluster is similar, but there are no gray bonds. We start with the final configuration of the $\mathbb{T}$-exploration, and set all vertices that are a common endpoint of a black bond and a gray bond active. Then we make all gray bonds uncolored again. From this setting we apply the coloring scheme that colors the uncolored bonds that contain an active vertex. The coloring scheme is the one for the exploration cluster on $\mathbb{Z}^d$, where no gray bonds are created. Again, we perform this exploration until there are no more active vertices, i.e., no more black bonds attached to uncolored bonds. The result is called the $\mathbb{Z}$-*exploration.* In particular, the black bonds in the $\mathbb{T}$-exploration are a subset of the black bonds in the $\mathbb{Z}$-exploration, which proves that $\{0 \overset{\mathbb{T}}{\leftrightarrow} x\} \subseteq \bigcup_{y \overset{r}{\sim} x} \{0 \overset{\mathbb{Z}}{\leftrightarrow} y\}$, and hence $|\mathcal{C}_{\mathbb{T}}(0)| \leq |\mathcal{C}_{\mathbb{Z}}(0)|$.[2]

We now show (4.2.3). When $\{0 \overset{\mathbb{Z}}{\leftrightarrow} y\}$ occurs, then picture all $\mathbb{Z}$-occupied paths from 0 to y in mind. Since $x \overset{r}{\sim} y$ and $\{0 \overset{\mathbb{T}}{\leftrightarrow} x\}^c$ occurs, each of these paths $\mathbb{Z}$-connecting 0 and y should contain a bond which is $\mathbb{T}$-vacant. Fix such a (self-avoiding) path $\omega \colon 0 \longleftrightarrow y$ that is $\mathbb{Z}$-occupied, and denote by b_2 the first bond that is $\mathbb{T}$-vacant, but $\mathbb{Z}$-occupied, so

[2] To obtain a coupling for the full percolation configurations on $\mathbb{T}_{r,d}$ and $\mathbb{Z}^d$, we can finally let all bonds that have not been explored be independently occupied with probability p, both in $\mathbb{T}_{r,d}$ and in $\mathbb{Z}^d$, independently of each other. However, we do not rely on the coupling of the percolation configuration, but only on the coupling of the clusters $\mathcal{C}_{\mathbb{T}}(0)$ and $\mathcal{C}_{\mathbb{Z}}(0)$.

that $(0 \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_2) \circ (b_2 \text{ is } \mathbb{Z}-\text{occ.}) \circ (\overline{b}_2 \overset{\mathbb{Z}}{\leftrightarrow} y)$. Due to our coupling, this implies that there exists a previously explored bond b_1 that is r-equivalent to b_2, which is ($\mathbb{T}$- and $\mathbb{Z}$-) vacant. This, in turn, implies that there exists a vertex z that is visited by ω such that $(z \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_1)$ without using any of the bonds in ω. Therefore, the event $(0 \overset{\mathbb{Z}}{\leftrightarrow} z) \circ (z \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_1) \circ (z \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_2) \circ (b_2 \text{ is } \mathbb{Z}-\text{occ.}) \circ (\overline{b}_2 \overset{\mathbb{Z}}{\leftrightarrow} y)$ occurs, see Figure 4.1. □

4.3 Random graph asymptotics

4.3.1 Results

Our first result gives asymptotic bounds on the size of the largest cluster.

Theorem 4.2. *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then there exist constants $b_1, b_2, C > 0$, such that for all $\omega_1 \geq C$ and $\omega_2 \geq 1$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega_1^{-1}V^{2/3}(\log V)^{-4/3} \leq |\mathcal{C}_{\max}| \leq \omega_2 V^{2/3}\Big) \geq 1 - \frac{b_1}{\omega_1^{3/2}(\log V)^2} - \frac{b_2}{\omega_2} \quad \text{as } r \to \infty. \tag{4.3.1}$$

The constant b_1 can be chosen as $288 \cdot 120^3$, and b_2 equal to b_6 in [24, Theorem 1.3].

One might compare this result with the work by Borgs, Chayes, van der Hofstad, Slade and Spencer [24, 25]. These authors prove the following:

Theorem 4.3 (Scaling window [24, 25])**.** *Assume the conditions in Theorem 4.2. Let $\lambda > 0$ and $\Lambda < \infty$. Then there is a finite positive constant b (depending on λ and Λ) such that, for $p = p_c(\mathbb{T}_{r,d}) + \Omega^{-1}\epsilon$ with $|\epsilon| \leq \Lambda V^{-1/3}$ and $\omega \geq 1$,*

$$\mathbb{P}_{\mathbb{T},p}\Big(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega}. \tag{4.3.2}$$

It should be noted that Theorem 4.3 is proven in [24] subject to a certain *triangle condition on the torus.* Using the lace expansion, this triangle condition was established in [25, Proposition 1.2 and Theorem 1.3] for $d > 6$ and sufficient spread-out or d sufficiently large for nearest-neighbor percolation.

To prove Theorem 4.2, we will use the coupling argument in Proposition 4.1 relating $\chi_{\mathbb{T}}(p)$ and $\chi_{\mathbb{Z}}(p)$ to show that there exists a constant $\Lambda \geq 0$ such that, when $r \to \infty$,

$$p_c(\mathbb{T}_{r,d}) - \frac{\Lambda}{\Omega}V^{-1/3}(\log V)^{2/3} \leq p_c(\mathbb{Z}^d) \leq p_c(\mathbb{T}_{r,d}) + \frac{\Lambda}{\Omega}V^{-1/3}. \tag{4.3.3}$$

Combining (4.3.3) with Theorem 4.3 (and with Theorem 4.6 stated below) yields (4.3.1). Inequality (4.3.1) implies that $|\mathcal{C}_{\max}|V^{-2/3}$ is a tight random variable, but it does not rule out that $|\mathcal{C}_{\max}|V^{-2/3} \to 0$ as $V \to \infty$.

Our method is crucially based on the results in [24, 25], but we also rely on various other mean-field results for percolation described in Section 3.6. The proof of each of these results relies on the *lace expansion*, but the lace expansion will not be used in this chapter.

Note that in [24, 25], $p_c(\mathbb{T}_{r,d})$ was defined as in (4.1.3). The results in [24, 25], however, do not establish rigorously that the exponent 1/3 in (4.1.3) is the only correct choice. Indeed, [24, 25] suggest that a smaller exponent would also do, since the supercritical results proved there are not sufficiently sharp. Theorem 4.2 shows that, at least in terms of the power of V, the scaling of $|\mathcal{C}_{\max}|$ at $p_c(\mathbb{Z}^d)$ and at $p_c(\mathbb{T}_{r,d})$ is identical, thus establishing that on $\mathbb{Z}^d$ the choice (4.1.3) *is* appropriate.

Unfortunately, the lower bound in Theorem 4.2 does not quite meet the upper bound. Under a condition on the percolation two-point function, we can prove the matching lower bound. To state this result, we introduce the quantity

$$\widetilde{\chi}_{\mathbb{Z}}(p,r) := \sup_y \sum_{z \overset{r}{\sim} y,\, \|z\| \geq \frac{r}{2}} \tau_{\mathbb{Z},p}(z). \tag{4.3.4}$$

Theorem 4.4. *Under the assumptions in Theorem 4.2, suppose that there exists a $K > 0$ such that, for $p = p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$ and some $C_K > 0$, the bound*

$$\widetilde{\chi}_{\mathbb{Z}}(p,r) \leq C_K V^{-2/3} \tag{4.3.5}$$

holds. Then, for all $\omega \geq 1$ and b equal to b_6 in [24, Theorem 1.3],

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\left(\frac{1}{\omega}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\right) \geq 1 - \frac{b}{\omega}. \tag{4.3.6}$$

Inequality (4.3.6) implies that $|\mathcal{C}_{\max}|V^{-2/3}$ is tight, and that each possible weak limit along any subsequence is non-zero. The result in (4.3.6) combined with the results in [24] would indicate that the scaling of $|\mathcal{C}_{\max}|$ at $p_c(\mathbb{T}_{r,d})$ and at $p_c(\mathbb{Z}^d)$ agree, thus showing that there is no significant difference between the internally and externally defined critical values.

Analogously to Theorem 4.2, we will show that when (4.3.5) holds, there exists a constant $\Lambda \geq 0$ such that, when $r \to \infty$,

$$p_c(\mathbb{T}_{r,d}) - \frac{\Lambda}{\Omega}V^{-1/3} \leq p_c(\mathbb{Z}^d) \leq p_c(\mathbb{T}_{r,d}) + \frac{\Lambda}{\Omega}V^{-1/3}, \tag{4.3.7}$$

and deduce (4.3.6) using Theorem 4.3.

We strongly believe that (4.3.5) holds, and we conclude this subsection by stating sufficient conditions for it. Indeed, (4.3.5) follows when the two-point function is sufficiently smooth. For example, the condition

$$\max_{\|z\| \geq \frac{r}{2},\, x \in \mathbb{T}_{r,d}} \frac{\tau_{\mathbb{Z},p}(z)}{\tau_{\mathbb{Z},p}(z+x)} \leq C \tag{4.3.8}$$

for some positive constant C (independent of r), where we consider $\mathbb{T}_{r,d} = \{-\lfloor r/2 \rfloor, \ldots, \lceil r/2 \rceil - 1\}^d$ as a subset of $\mathbb{Z}^d$, implies that

$$\tau_{\mathbb{Z},p}(z) \leq \frac{C}{V} \sum_{x \in \mathbb{T}_{r,d}} \tau_{\mathbb{Z},p}(z+x) \qquad \text{for } \|z\| \geq \frac{r}{2}. \tag{4.3.9}$$

Note that, for every $y \in \mathbb{Z}^d$, we have that $\sum_{z \overset{r}{\sim} y} \sum_{x \in \mathbb{T}_{r,d}} f(z+x) = \sum_{x \in \mathbb{Z}^d} f(x)$ for all functions $f\colon \mathbb{Z}^d \mapsto \mathbb{R}$. Consequently,

$$\begin{aligned} \widetilde{\chi}_{\mathbb{Z}}(p,r) &= \sup_y \sum_{z \overset{r}{\sim} y,\, \|z\| \geq \frac{r}{2}} \tau_{\mathbb{Z},p}(z) \leq \frac{C}{V} \sup_y \sum_{z \overset{r}{\sim} y,\, \|z\| \geq \frac{r}{2}} \sum_{x \in \mathbb{T}_{r,d}} \tau_{\mathbb{Z},p}(z+x) \\ &\leq \frac{C}{V} \sum_{x \in \mathbb{Z}^d} \tau_{\mathbb{Z},p}(x) = \frac{C}{V} \chi_{\mathbb{Z}}(p). \end{aligned} \tag{4.3.10}$$

Thus, for $p = p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$, we obtain by the fact that $\gamma_{\mathrm{P}} = 1$ (see Theorem 3.16, or (4.3.13) below) that $\widetilde{\chi}_{\mathbb{Z}}(p,r)$ is bounded from above by a constant multiple of $V^{-2/3}$.

Another approach to (4.3.5) is to split the sum over z in (4.3.4). Note that, for $p = p_c(\mathbb{Z}^d) - K_1\Omega^{-1}V^{-1/3}$, the sum due to $\|z\| \leq K_1V^{1/6}$ can be bounded, for any $K_1 > 0$, as

$$\sup_y \sum_{z \overset{r}{\sim} y,\, \frac{r}{2} \leq \|z\| \leq K_1V^{1/6}} \tau_{\mathbb{Z},p}(z) \leq C \sup_y \sum_{z \overset{r}{\sim} y,\, \frac{r}{2} \leq \|z\| \leq K_1V^{1/6}} (\|z\|+1)^{-(d-2)} \leq CK_1^2V^{-2/3}. \tag{4.3.11}$$

Therefore, we are left to give a bound on the contribution from $\|z\| \geq K_1V^{1/6}$. The restriction $\|z\| \geq K_1V^{1/6}$ is equivalent to $\|z\| \geq C_{K,K_1}\xi(p)$, where $\xi(p)$ denotes the correlation length. Indeed, $\xi(p)$ is comparable in size to $(p_c(\mathbb{Z}^d)-p)^{1/2}$ (see Theorem 3.18), and the constant C_{K,K_1} can be made arbitrarily large by taking K_1 large. However, in order to bound this contribution we miss sufficient bounds on $\tau_{\mathbb{Z},p}(z)$ for $\|z\| \geq C_{K,K_1}\xi(p)$.

4.3.2 Discussion of related literature

In this section we discuss the relation between Theorems 4.2 and 4.4 and the literature.

Borgs, Chayes, Kesten and Spencer [22, 23] consider the largest cluster in a finite box of width r under *free* boundary conditions, i.e., clusters are connected only within the box. They show that, for $p = p_c(\mathbb{Z}^d)$, the largest critical cluster scales like $V^{\delta_{\mathrm{P}}/(1+\delta_{\mathrm{P}})}$ (where δ_{P} is defined in (1.1.8)) under some conditions related to the so-called scaling and hyperscaling postulates. The hyperscaling postulates are proven in dimension $d = 2$, and are widely believed to hold up to the upper critical dimension 6. For the mean-field value $\delta_{\mathrm{P}} = 2$ we would obtain the $V^{2/3}$ asymptotics. However, in [22, 23], it was assumed that crossing probabilities of a cube of dimensions $(r, 3r, \ldots, 3r)$ remain uniformly bounded away from 1 as $r \to \infty$. In high dimensions, Aizenman [2] proves that any cube $\{0, \ldots, r\}^d$ has crossings with high probability, so that the results in [22, 23] do not apply. Also, in high dimensions, the hyperscaling relations are not valid. More specifically, one hyperscaling relation is that

$$2 - \eta = d\,\frac{\delta - 1}{\delta + 1}. \tag{4.3.12}$$

Under the conditions of Theorem 4.2, we have $\eta = 0$ and $\delta = 2$ (cf. Theorem 3.16), so that this hyperscaling relation fails for $d > 6$.

Theorems 4.2 and 4.4 study the scaling of the largest critical percolation cluster on the high-dimensional torus. These results indicate that the scaling limit of the largest critical cluster should be described by $|\mathcal{C}_{\max}|V^{-2/3}$. We conjecture that, at $p = p_c(\mathbb{Z}^d)$, the random variables $|\mathcal{C}_{\max}|V^{-2/3}$ converge as $r \to \infty$ to some (non-trivial) limiting distribution. It would be of interest to investigate whether, if the rescaled largest cluster converges, the limit law is identical to the limit of $|\mathcal{C}_{\max}|n^{-2/3}$ for the largest cluster of the random graph on n vertices, as identified by Aldous [13]. The convergence of $|\mathcal{C}_{\max}|V^{-2/3}$ would describe part of the *incipient infinite cluster* (IIC) for percolation on the torus, as described by Aizenman [2]. Aizenman's IIC is closely related to the scaling limit of percolation on large cubes, see [2, Section 5]. Mind also the warning at the bottom of [2, p. 553].

Recall our discussion of IIC in Section 3.6. We conjecture that, as $r \to \infty$, the law of local configurations around a uniform point in $\mathcal{C}_{\max}$ at criticality converges to the IIC as constructed in [70]. This result would give a natural link between the scaling limit of critical percolation on a large box in [2] and Kesten's notion of the IIC in [81].

According to the conjecture in [2], the size of the largest connected component $|\mathcal{C}_{\max}|$ on the cube $\{0, \ldots, r-1\}^d$ under *free* bondary conditions scales like r^4 (as under bulk boundary conditions), in contrast to the $V^{2/3}$-scaling under periodic boundary conditions. Such qualitatively different behavior between free and periodic boundary conditions has also been observed when studying loop-erased random walks and uniform spanning trees on a finite box in high dimensions, as we will explain now.

Choose two uniform points x and y from the d-dimensional box of side length r, with $d > 4$. We are interested in the graph distance between these two points on a uniform spanning tree. Pemantle [92] showed that this graph distance has the same distribution as the length of a loop-erased random walk starting in x and stopped when reaching y. Loop-erased random walk above 4 dimensions converges to Brownian motion (cf. [85, Section 7.7]), that is, it scales diffusively. This suggests that, under *free* boundary conditions, the graph distance between x and y scales like r^2. On the other hand, the combined results of Benjamini and Kozma [15] and Peres and Revelle [93] show that the distance between x and y on a uniformly chosen spanning tree on the *torus* $\mathbb{T}_{r,d}$ is of the order $V^{1/2} = r^{d/2} > r^2$ for $d > 4$. Schweinsberg [98] identifies the logarithmic correction for the scaling on the 4-dimensional torus.

4.3.3 The upper bound on the maximal critical cluster

In Theorem 3.16 we have proven that the critical exponent γ_{P} for percolation exists and is equal to 1 whenever d is sufficiently large in the nearest-neighbor model, or $d > 6$ and L sufficiently large in the spread-out model. That is, under the conditions of Theorem 4.2, there exists a constant $C_\chi > 1$ such that[3]

$$\frac{1}{\Omega\left(p_c(\mathbb{Z}^d) - p\right)} \leq \chi_{\mathbb{Z}}(p) \leq \frac{C_\chi}{\Omega\left(p_c(\mathbb{Z}^d) - p\right)} \qquad \text{as } p \nearrow p_c(\mathbb{Z}^d). \tag{4.3.13}$$

We will use (4.3.13) at various places throughout the proof of Theorems 4.2 and 4.4.

The following corollary establishes the upper bound on $p_c(\mathbb{Z}^d)$ and the upper bound on $|\mathcal{C}_{\max}|$ in Theorem 4.2. For the proof, we first use Proposition 4.1 to obtain that $\chi_{\mathbb{T}}(p) \leq \chi_{\mathbb{Z}}(p)$. Then we use (4.3.13) to turn this into relations between $p_c(\mathbb{T}_{r,d})$ and $p_c(\mathbb{Z}^d)$. Finally, using Theorem 4.3 we obtain a bound on $|\mathcal{C}_{\max}|$. We now present the details of the proof.

Corollary 4.5. *Under the conditions of Theorem 4.2 there exists a constant $\Lambda \geq 0$ such that, when $r \to \infty$,*

$$p_c(\mathbb{Z}^d) \leq p_c(\mathbb{T}_{r,d}) + \frac{\Lambda}{\Omega} V^{-1/3}. \tag{4.3.14}$$

Consequently, for b as in Theorem 4.3 and all $\omega \geq 1$,

$$\mathbb{P}_{\mathbb{T}, p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega}. \tag{4.3.15}$$

Proof. By Proposition 4.1,

$$\chi_{\mathbb{T}}(p_c(\mathbb{T}_{r,d})) \leq \chi_{\mathbb{Z}}(p_c(\mathbb{T}_{r,d})). \tag{4.3.16}$$

When $p_c(\mathbb{T}_{r,d}) \geq p_c(\mathbb{Z}^d)$, then (4.3.14) holds with $\Lambda = 0$, so we will next assume that $p_c(\mathbb{T}_{r,d}) < p_c(\mathbb{Z}^d)$. Using (4.1.3), (4.3.16) and (4.3.13), we obtain that

$$\lambda V^{1/3} \leq \frac{C_\chi}{\Omega\left(p_c(\mathbb{Z}^d) - p_c(\mathbb{T}_{r,d})\right)}, \tag{4.3.17}$$

so that

$$p_c(\mathbb{Z}^d) \leq p_c(\mathbb{T}_{r,d}) + \frac{C_\chi}{\lambda\Omega} V^{-1/3}, \tag{4.3.18}$$

which is (4.3.14) with $\Lambda = \lambda^{-1} C_\chi$.

[3]Strictly speaking, Theorem 3.16 implies only that $\chi_{\mathbb{Z}}(p)\,\Omega\left(p_c(\mathbb{Z}^d) - p\right)$ is bounded away from 0 and ∞. The fact that 1 is a lower bound follows from (3.5.16).

The bound (4.3.15) follows from the fact that, with $p = p_c(\mathbb{T}_{r,d}) + \Lambda\Omega^{-1}V^{-1/3} \geq p_c(\mathbb{Z}^d)$ by (4.3.14),

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq \mathbb{P}_{\mathbb{T},p}\Big(|\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega} \tag{4.3.19}$$

for some constant $b > 0$ depending on λ and Λ, and all $\omega \geq 1$. We have used Theorem 4.3 in the last bound. □

4.3.4 The lower bound on the maximal critical cluster

In this section, we will bound $\chi_{\mathbb{T}}(p) - \chi_{\mathbb{Z}}(p)$ from below. First we use the results and framework of Section 4.2 to prove such a lower bound in terms of $\widetilde{\chi}_{\mathbb{Z}}(p, r)$. Subsequently, assuming (4.3.5), we use Theorem 4.3, Corollary 4.5 and (4.3.2) to prove Theorem 4.4.

Some more work is required if we do not assume (4.3.5). We first deduce a bound on $\widetilde{\chi}_{\mathbb{Z}}(p, r)$ using the bounds in Theorems 3.17 and 3.18. Then we use this bound together with Lemma 4.7 to show Theorem 4.2 with the same ingredients as for the proof of Theorem 4.4.

Also the proof of the lower bound makes essential use of results by Borgs, Chayes, van der Hofstad, Slade and Spencer [24, 25] for percolation on $\mathbb{T}_{r,d}$. It turns out that for Theorem 4.2 it is not sufficient to use Theorem 4.3, since (4.3.3) does not match the assumptions of Theorem 4.3. Instead, we use the following theorem providing bounds for the subcritical phase.

Theorem 4.6 (Subcritical phase [24, 25])**.** *Under the conditions in Theorem 4.2, for λ sufficiently small and any $q \geq 0$,*

$$\Big(\lambda^{-1}V^{-1/3} + q\Big)^{-1} \leq \chi_{\mathbb{T}}\big(p_c(\mathbb{T}_{r,d}) - \Omega^{-1}q\big) \leq \Big(\lambda^{-1}V^{-1/3} + q/2\Big)^{-1}. \tag{4.3.20}$$

Also, for $p = p_c(\mathbb{T}_{r,d}) - \Omega^{-1}q$ and $\omega \geq 1$,

$$\mathbb{P}_{\mathbb{T},p}\left(|\mathcal{C}_{\max}| \geq \frac{\chi_{\mathbb{T}}^2(p)}{3600\,\omega}\right) \geq \left(1 + \frac{36\,\chi_{\mathbb{T}}^3(p)}{\omega\, V}\right)^{-1}. \tag{4.3.21}$$

Instead of the upper bound in (4.3.20), we will mainly use the cruder bound

$$\chi_{\mathbb{T}}\big(p_c(\mathbb{T}_{r,d}) - \Omega^{-1}q\big) \leq \frac{2}{q}. \tag{4.3.22}$$

Theorem 4.6 is implied by combining [24, Theorem 1.2] with [25, Proposition 1.2 and Theorem 1.3].

A lower bound on $\chi_{\mathbb{T}}(p)$ in terms of $\chi_{\mathbb{Z}}(p)$.

Lemma 4.7. *For all $p \in [0, 1]$ and $r \geq 3$ in the nearest-neighbor model or $r \geq 2L + 1$ in the spread-out model,*

$$\chi_{\mathbb{T}}(p) \geq \chi_{\mathbb{Z}}(p)\big(1 - \chi_{\mathbb{Z}}(p)\,\widetilde{\chi}_{\mathbb{Z}}(p, r) - p\,\Omega^2\chi_{\mathbb{Z}}(p)^2\widetilde{\chi}_{\mathbb{Z}}(p, r)\big). \tag{4.3.23}$$

Proof. The bound (4.3.23) will be achieved by comparing the two-point functions on the torus and on $\mathbb{Z}^d$. We will write $\mathbb{P} = \mathbb{P}_{\mathbb{Z},\mathbb{T}}$ and omit the percolation parameter p from the notation. Using (4.2.2), we write

$$\tau_{\mathbb{T}}(x) = \mathbb{P}\Big(\bigcup_{y\in\mathbb{Z}^d : y \overset{r}{\sim} x} \{0 \overset{\mathbb{Z}}{\leftrightarrow} y\}\Big) - \mathbb{P}\Big(\bigcup_{y\in\mathbb{Z}^d : y \overset{r}{\sim} x} \{0 \overset{\mathbb{Z}}{\leftrightarrow} y\} \cap \{0 \overset{\mathbb{T}}{\leftrightarrow} x\}^c\Big). \tag{4.3.24}$$

We further bound, using inclusion-exclusion,

$$\mathbb{P}\Big(\bigcup_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}\{0\overset{\mathbb{Z}}{\leftrightarrow}y\}\Big)\geq\sum_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}\mathbb{P}\big(0\overset{\mathbb{Z}}{\leftrightarrow}y\big)-\frac{1}{2}\sum_{y_1\neq y_2\in\mathbb{Z}^d: y_1,y_2\overset{r}{\sim}x}\mathbb{P}\big(0\overset{\mathbb{Z}}{\leftrightarrow}y_1,y_2\big),\tag{4.3.25}$$

so that

$$\tau_{\mathbb{T}}(x)\geq\sum_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}\tau_{\mathbb{Z}}(y)-\frac{1}{2}\sum_{y_1\neq y_2\in\mathbb{Z}^d: y_1,y_2\overset{r}{\sim}x}\mathbb{P}\big(0\overset{\mathbb{Z}}{\leftrightarrow}y_1,y_2\big)-\mathbb{P}\Big(\bigcup_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}\{0\overset{\mathbb{Z}}{\leftrightarrow}y\}\cap\{0\overset{\mathbb{T}}{\leftrightarrow}x\}^c\Big).\tag{4.3.26}$$

Summation over $x\in\mathbb{T}_{r,d}$ and using $\sum_{x\in\mathbb{T}_{r,d}}\sum_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}=\sum_{y\in\mathbb{Z}^d}$ yields that

$$\chi_{\mathbb{T}}(p)\geq\chi_{\mathbb{Z}}(p)-\chi_{\mathbb{T},1}(p)-\chi_{\mathbb{T},2}(p),\tag{4.3.27}$$

where

$$\chi_{\mathbb{T},1}(p)=\frac{1}{2}\sum_{y_1\neq y_2\in\mathbb{Z}^d: y_1\overset{r}{\sim}y_2}\mathbb{P}\big(0\overset{\mathbb{Z}}{\leftrightarrow}y_1,y_2\big),\tag{4.3.28}$$

$$\chi_{\mathbb{T},2}(p)=\sum_{x\in\mathbb{T}_{r,d}}\mathbb{P}\Big(\bigcup_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}\{0\overset{\mathbb{Z}}{\leftrightarrow}y\}\cap\{0\overset{\mathbb{T}}{\leftrightarrow}x\}^c\Big).\tag{4.3.29}$$

Here we use that the sum over x and over y_1 and y_2 such that $y_1,y_2\overset{r}{\sim}x$ is the same as the sum over y_1 and y_2 such that $y_1\overset{r}{\sim}y_2$. We are left to bound $\chi_{\mathbb{T},1}(p)$ and $\chi_{\mathbb{T},2}(p)$. We start by bounding $\chi_{\mathbb{T},1}(p)$. Using the tree-graph inequality [9],

$$\mathbb{P}\big(0\overset{\mathbb{Z}}{\leftrightarrow}x,y\big)\leq\sum_{z\in\mathbb{Z}^d}\tau_{\mathbb{Z}}(z)\,\tau_{\mathbb{Z}}(x-z)\,\tau_{\mathbb{Z}}(y-z),\tag{4.3.30}$$

we obtain

$$\begin{aligned}\chi_{\mathbb{T},1}(p)&\leq\frac{1}{2}\sum_{z}\sum_{y_1\neq y_2: y_1\overset{r}{\sim}y_2}\tau_{\mathbb{Z}}(z)\,\tau_{\mathbb{Z}}(y_1-z)\,\tau_{\mathbb{Z}}(y_2-z)\\&=\frac{1}{2}\chi_{\mathbb{Z}}(p)\sum_{y_1'\neq y_2': y_1'\overset{r}{\sim}y_2'}\tau_{\mathbb{Z}}(y_1')\,\tau_{\mathbb{Z}}(y_2').\end{aligned}\tag{4.3.31}$$

Here, and in the remainder of the proof, all sums over vertices will be over $\mathbb{Z}^d$ unless written explicitly otherwise.

Since $y_1'\neq y_2'$ and $y_1'\overset{r}{\sim}y_2'$, we must have that $\|y_1'\|\geq\frac{r}{2}$ or $\|y_2'\|\geq\frac{r}{2}$. By symmetry, these give the same contributions, so that,

$$\chi_{\mathbb{T},1}(p)\leq\chi_{\mathbb{Z}}(p)\sum_{y_1'}\tau_{\mathbb{Z}}(y_1')\sum_{y_2': y_1'\overset{r}{\sim}y_2',\|y_2'\|\geq\frac{r}{2}}\tau_{\mathbb{Z}}(y_2')\leq\chi_{\mathbb{Z}}(p)^2\,\widetilde{\chi}_{\mathbb{Z}}(p,r),\tag{4.3.32}$$

where we recall (4.3.4). We are left to prove that $\chi_{\mathbb{T},2}(p)\leq p\,\Omega^2\chi_{\mathbb{Z}}^3(p)\,\widetilde{\chi}_{\mathbb{Z}}(p,r)$.

We use (4.2.3) and note that the right hand side of (4.2.3) does not depend on x. Since $\sum_{x\in\mathbb{T}_{r,d}}\sum_{y\in\mathbb{Z}^d: y\overset{r}{\sim}x}=\sum_{y\in\mathbb{Z}^d}$, this brings us to

$$\chi_{\mathbb{T},2}(p)\leq\sum_{y\in\mathbb{Z}^d}\mathbb{P}\Big(\bigcup_{b_1\neq b_2: b_1\overset{r}{\sim}b_2}\bigcup_{z\in\mathbb{Z}^d}(0\overset{\mathbb{Z}}{\leftrightarrow}z)\circ(z\overset{\mathbb{Z}}{\leftrightarrow}\underline{b}_1)\circ(z\overset{\mathbb{Z}}{\leftrightarrow}\underline{b}_2)\circ(b_2\text{ is }\mathbb{Z}-\text{occ.})\circ(\overline{b}_2\overset{\mathbb{Z}}{\leftrightarrow}y)\Big).\tag{4.3.33}$$

Therefore, by the BK-inequality (Proposition 2.3),

$$\chi_{\mathbb{T},2}(p) \leq \sum_{y,z} \sum_{b_1 \neq b_2 : b_1 \overset{r}{\sim} b_2} \mathbb{P}\big((0 \overset{\mathbb{Z}}{\leftrightarrow} z) \circ (z \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_1) \circ (z \overset{\mathbb{Z}}{\leftrightarrow} \underline{b}_2) \circ (b_2 \text{ is } \mathbb{Z}-\text{occ.}) \circ (\overline{b}_2 \overset{\mathbb{Z}}{\leftrightarrow} y)\big)$$

$$\leq p \sum_{y,z} \sum_{b_1 \neq b_2 : b_1 \overset{r}{\sim} b_2} \tau_{\mathbb{Z}}(z)\, \tau_{\mathbb{Z}}(\underline{b}_1 - z)\, \tau_{\mathbb{Z}}(\underline{b}_2 - z)\, \tau_{\mathbb{Z}}(y - \overline{b}_2). \tag{4.3.34}$$

We can perform the sums over z, y to obtain, with $b_i' = b_i - z$,

$$\chi_{\mathbb{T},2}(p) \leq p\, \chi_{\mathbb{Z}}(p)^2 \sum_{b_1' \neq b_2' :\, b_1' \overset{r}{\sim} b_2'} \tau_{\mathbb{Z}}(\underline{b}_1')\, \tau_{\mathbb{Z}}(\underline{b}_2') = p\,\Omega^2 \chi_{\mathbb{Z}}(p)^2 \sum_{u \neq v :\, u \overset{r}{\sim} v} \tau_{\mathbb{Z}}(u)\, \tau_{\mathbb{Z}}(v), \tag{4.3.35}$$

where the factor Ω^2 arises from the number of choices for b_1' and b_2' for fixed $\underline{b}_1'$ and $\underline{b}_2'$. Therefore, by (4.3.32), we arrive at the bound

$$\chi_{\mathbb{T},2}(p) \leq p\,\Omega^2 \chi_{\mathbb{Z}}^3(p)\, \widetilde{\chi}_{\mathbb{Z}}(p, r). \tag{4.3.36}$$

The bounds (4.3.32) and (4.3.36) complete the proof of (4.3.23). □

Proof of the main results.

Proof of Theorem 4.4. We assume (4.3.5) and take $p = p_c(\mathbb{Z}^d) - K_1 \Omega^{-1} V^{-1/3}$ for K_1 sufficiently large. Choose V sufficiently large to ensure that $p > 0$. When $K_1 \geq K$, the bound (4.3.5) still holds. We obtain from Lemma 4.7 together with (4.3.13) and (4.3.5) that

$$\chi_{\mathbb{T}}(p) \geq K_1^{-1} V^{1/3} \left(1 - C_\chi C_K K_1^{-1} V^{-1/3} - p\,\Omega^2 C_\chi^2 C_K K_1^{-2}\right) \geq \widetilde{c}_{K_1} V^{1/3}, \tag{4.3.37}$$

where $\widetilde{c}_{K_1}$ is chosen appropriately. Let K_1 be so large that $p < p_c(\mathbb{T}_{r,d})$, which can be done by (4.3.14). Then, by (4.3.22),

$$\frac{2}{\Omega(p_c(\mathbb{T}_{r,d}) - p_c(\mathbb{Z}^d) + K_1 V^{-1/3})} \geq \chi_{\mathbb{T}}(p) \geq \widetilde{c}_{K_1} V^{1/3}, \tag{4.3.38}$$

so that

$$p_c(\mathbb{Z}^d) \geq p_c(\mathbb{T}_{r,d}) + \left(K_1 - \frac{2}{\widetilde{c}_{K_1} \Omega}\right) V^{-1/3}, \tag{4.3.39}$$

which is (4.3.7) with $\Lambda = \left(2\,\widetilde{c}_{K_1}^{-1} - \Omega K_1\right) \vee 0$. This, together with (4.3.14), permits using Theorem 4.3. By doing so, we obtain that, for p equal to the right hand side of (4.3.39),

$$\mathbb{P}_{\mathbb{T},\, p_c(\mathbb{Z}^d)}\left(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\right) \geq \mathbb{P}_{\mathbb{T},\, p}\left(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\right) \geq 1 - \frac{b}{\omega}. \tag{4.3.40}$$

This completes the proof of Theorem 4.4. □

Unfortunately, we cannot quite prove (4.3.5) so we will give cruder upper bounds on $\widetilde{\chi}_{\mathbb{Z}}(p, r)$. This is the content of the following lemma:

Lemma 4.8. *Under the conditions in Theorem 4.2, choose K sufficiently large, and let $R = K(\log V)(p_c(\mathbb{Z}^d) - p)^{-1/2}$. Then for all $p \leq p_c(\mathbb{Z}^d) - K\Omega^{-1} V^{-1/3}$,*

$$\widetilde{\chi}_{\mathbb{Z}}(p, r) \leq \frac{C_{\tilde{\chi}} R^2}{V}. \tag{4.3.41}$$

for some constant $C_{\tilde{\chi}} > 0$.

Note that it is here where the power of $\log V$ comes into play.

Proof. For $p \le p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$, we bound

$$\widetilde{\chi}_{\mathbb{Z}}(p,r) = \sup_y \sum_{z \overset{r}{\sim} y, \|z\| \ge \frac{r}{2}} \tau_{\mathbb{Z},p}(z) \le \sup_y \sum_{z \overset{r}{\sim} y, \|z\| \ge R} \tau_{\mathbb{Z},p}(z) + \sup_y \sum_{z \overset{r}{\sim} y, \frac{r}{2} \le \|z\| \le R} \tau_{\mathbb{Z},p}(z). \quad (4.3.42)$$

We start with the second contribution, for which we use (3.6.2). Since $\|z\| \ge \frac{r}{2}$, we have that

$$\tau_{\mathbb{Z},p}(z) \le \tau_{\mathbb{Z},p_c(\mathbb{Z}^d)}(z) \le \frac{C_\tau}{(|z|+1)^{d-2}} \le \frac{C_1}{V} \sum_{x \in \mathbb{T}_{r,d}} \frac{C_\tau}{(|z+x|+1)^{d-2}} \quad (4.3.43)$$

for constants $C_\tau, C_1 > 0$, where C_1 depends on the dimension d only. Therefore,

$$\begin{aligned} \sup_y \sum_{z \overset{r}{\sim} y : \frac{r}{2} \le \|z\| \le R} \tau_{\mathbb{Z},p}(z) &\le \frac{C_1}{V} \sup_y \sum_{x \in \mathbb{T}_{r,d}} \sum_{z \overset{r}{\sim} y : \|z\| \le R} \frac{C_\tau}{(|z+x|+1)^{d-2}} \\ &\le \frac{C_1}{V} \sum_{z : \|z\| \le 2R} \frac{C_\tau}{(|z|+1)^{d-2}} \le \frac{C_2 R^2}{V}, \end{aligned} \quad (4.3.44)$$

where the positive constant C_2 depends on d and L only. For the sum due to $\|z\| \ge R$, we use (3.6.3) and (3.6.5) in Theorem 3.18 to see that

$$\sup_y \sum_{z \overset{r}{\sim} y,\, \|z\| \ge R} \tau_{\mathbb{Z},p}(z) \le \sup_y \sum_{z \overset{r}{\sim} y,\, \|z\| \ge R} \exp\left\{-C_\xi^{-1} \|z\| \left(p_c(\mathbb{Z}^d) - p\right)^{1/2}\right\}. \quad (4.3.45)$$

Since $\|z\| \ge \frac{1}{d}(|z_1| + \cdots + |z_d|)$ for all $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$, (4.3.45) can be further bounded from above by

$$\begin{aligned} &\sum_{z : \|z\| \ge R} \exp\left\{-C_\xi^{-1} \|z\| \left(p_c(\mathbb{Z}^d) - p\right)^{1/2}\right\} \\ &\le \left(\sum_{z_1 : |z_1| \ge R} \exp\left\{-(dC_\xi)^{-1} |z_1| \left(p_c(\mathbb{Z}^d) - p\right)^{1/2}\right\} \right)^d \\ &\le C_3 \left(p_c(\mathbb{Z}^d) - p\right)^{-d/2} \exp\left\{-\frac{R}{C_\xi} \left(p_c(\mathbb{Z}^d) - p\right)^{1/2}\right\}, \end{aligned} \quad (4.3.46)$$

for some constant $C_3 > 0$. Since $R = K(\log V)(p_c(\mathbb{Z}^d) - p)^{-1/2}$, the exponential term can be bounded by V^{-K/C_ξ}. Furthermore, $\left(p_c(\mathbb{Z}^d) - p\right)^{-d/2} \le \left(K\Omega^{-1}\right)^{-d/2} V^{d/6}$ by our choice of p. Choose K so large that $K/C_\xi - d/6 > 1$. Then the upper bound is of the order $o(V^{-1})$. This, together with (4.3.44), proves the claim. □

We next use Lemma 4.8 to prove the lower bound in (4.3.3):

Lemma 4.9. *Under the conditions in Theorem 4.2, there exists a constant $\Lambda \ge 0$ such that*

$$p_c(\mathbb{Z}^d) \ge p_c(\mathbb{T}_{r,d}) - \frac{\Lambda}{\Omega} V^{-1/3} (\log V)^{2/3}. \quad (4.3.47)$$

Proof. By Lemma 4.8, for all $p \leq p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$,

$$\tilde{\chi}_{\mathbb{Z}}(p,r) \leq \frac{C_{\tilde{\chi}} K^2 (\log V)^2}{V\,(p_c(\mathbb{Z}^d) - p)}. \tag{4.3.48}$$

With (4.3.13), this can be further bounded as

$$\tilde{\chi}_{\mathbb{Z}}(p,r) \leq C_{\tilde{\chi}} K^2 \,\frac{(\log V)^2}{V}\,\Omega\,\chi_{\mathbb{Z}}(p). \tag{4.3.49}$$

Then, by Lemma 4.7 and $\chi_{\mathbb{Z}}(p) \geq 1$,

$$\chi_{\mathbb{T}}(p) \geq \chi_{\mathbb{Z}}(p)\left(1 - (1+p\Omega^2)\tilde{\chi}(p,r)\,\chi_{\mathbb{Z}}(p)^2\right) \tag{4.3.50}$$

if Ω and r are sufficiently large. Combining (4.3.49) and (4.3.50) yields

$$\chi_{\mathbb{T}}(p) \geq \chi_{\mathbb{Z}}(p)\left(1 - (1+p\Omega^2)\Omega\, C_{\tilde{\chi}} K^2\,\frac{(\log V)^2}{V}\,\chi_{\mathbb{Z}}(p)^3\right). \tag{4.3.51}$$

Let

$$\hat{p} := p_c(\mathbb{Z}^d) - \frac{\hat{C}}{\Omega} V^{-1/3}(\log V)^{2/3} \tag{4.3.52}$$

for some (sufficiently large) constant $\hat{C} > 0$. Depending on $\hat{C}$, we take V large to ensure that $\hat{p} > 0$. Then by (4.3.13),

$$\chi_{\mathbb{Z}}(\hat{p})^3 \leq \frac{C_\chi^3}{\Omega^3\,(p_c(\mathbb{Z}^d) - \hat{p})^3} = \left(\frac{C_\chi}{\hat{C}}\right)^3 V(\log V)^{-2}. \tag{4.3.53}$$

Substituting (4.3.53) into (4.3.51) for $p = \hat{p}$, using $\hat{p} \leq 1$ and the lower bound in (4.3.13) give

$$\chi_{\mathbb{T}}(\hat{p}) \geq \left(1 - \frac{(1+\Omega^2)\Omega\, C_{\tilde{\chi}} K^2 C_\chi^3}{\hat{C}^3}\right)\frac{1}{\hat{C}}\,\frac{V^{1/3}}{(\log V)^{2/3}}. \tag{4.3.54}$$

We make the $\hat{C}$ in (4.3.52) so large that

$$\hat{c} := \left(1 - \frac{(1+\Omega^2)\Omega\, C_{\tilde{\chi}} K^2 C_\chi^3}{\hat{C}^3}\right)\frac{1}{\hat{C}} > 0, \tag{4.3.55}$$

so that (4.3.54) simplifies to

$$\chi_{\mathbb{T}}(\hat{p}) \geq \hat{c}\,V^{1/3}(\log V)^{-2/3}. \tag{4.3.56}$$

The quantity

$$q := \Omega\,(p_c(\mathbb{T}_{r,d}) - \hat{p}) = \hat{C}V^{-1/3}(\log V)^{2/3} - \Omega\left(p_c(\mathbb{Z}^d) - p_c(\mathbb{T}_{r,d})\right), \tag{4.3.57}$$

is positive if V is large enough, by (4.3.14). Hence Theorem 4.6 is applicable, and (4.3.22) yields

$$\chi_{\mathbb{T}}(\hat{p}) = \chi_{\mathbb{T}}\left(p_c(\mathbb{T}_{r,d}) - q\,\Omega^{-1}\right) \leq \frac{2}{\Omega\,(p_c(\mathbb{T}_{r,d}) - \hat{p})}. \tag{4.3.58}$$

Merging (4.3.56) and (4.3.58), we arrive at

$$p_c(\mathbb{T}_{r,d}) - p_c(\mathbb{Z}^d) \leq \frac{1}{\Omega}\left[\frac{2}{\hat{c}} - \hat{C}\right] V^{-1/3}(\log V)^{2/3}, \tag{4.3.59}$$

which is (4.3.47) with $\Lambda = (2\hat{c}^{-1} - \hat{C}) \vee 0$. $\square$

Corollary 4.10. *Under the conditions in Theorem 4.2, there exists a constant $C > 0$ such that, for all $\omega_1 \geq C$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\left(|\mathcal{C}_{\max}| \geq \omega_1^{-1}(\log V)^{-4/3}\, V^{2/3}\right) \geq \left(1 + \frac{120^{3/2}\cdot 288}{\omega_1^{3/2}(\log V)^2}\right)^{-1}. \tag{4.3.60}$$

Proof. Take V so large that $\lambda^{-1} \leq \sqrt{\omega_1}\, 120^{-1}(\log V)^{2/3}$, and let

$$\hat{p} = p_c(\mathbb{T}_{r,d}) - \sqrt{\omega_1}\, 120^{-1}\Omega^{-1}V^{-1/3}(\log V)^{2/3}. \tag{4.3.61}$$

Then, by (4.3.20),

$$\chi_{\mathbb{T}}^2(\hat{p}) \geq \left(\frac{1}{\lambda^{-1}V^{-1/3} + \sqrt{\omega_1}\, 120^{-1}V^{-1/3}(\log V)^{2/3}}\right)^2 \geq 3600\, \omega_1^{-1}(\log V)^{-4/3}\, V^{2/3}. \tag{4.3.62}$$

This enables the bound

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\left(|\mathcal{C}_{\max}| \geq \omega_1^{-1}(\log V)^{-4/3}\, V^{2/3}\right) \geq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\left(|\mathcal{C}_{\max}| \geq \frac{\chi_{\mathbb{T}}^2(\hat{p})}{3600}\right). \tag{4.3.63}$$

By Lemma 4.9, $\hat{p} \leq p_c(\mathbb{Z}^d)$ for $\omega_1 \geq C$, and $C > 0$ large enough. Thus, we use (4.3.21) and (4.3.22) to bound (4.3.63) further from below by

$$\mathbb{P}_{\mathbb{T},\hat{p}}\left(|\mathcal{C}_{\max}| \geq \frac{\chi_{\mathbb{T}}^2(\hat{p})}{3600}\right) \geq \left(1 + \frac{36\,\chi_{\mathbb{T}}^3(\hat{p})}{V}\right)^{-1} \geq \left(1 + \frac{36\cdot 2^3\cdot 120^3}{\omega_1^{3/2}(\log V)^2}\right)^{-1}. \tag{4.3.64}$$

□

Combining our results from Sections 4.3.3 and 4.3.4, we finally prove Theorem 4.2:

Proof of Theorem 4.2. By Corollaries 4.5 and 4.10, for $\omega_1 \geq C$ for some sufficiently large C and $\omega_2 \geq 1$,

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\left(\omega_1^{-1}(\log V)^{-4/3}\, V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\right) \geq 1 - \left(\frac{\frac{120^3\cdot 288}{\omega_1^{3/2}(\log V)^2}}{1 + \frac{120^3\cdot 288}{\omega_1^{3/2}(\log V)^2}}\right) - \frac{b_2}{\omega_2}, \tag{4.3.65}$$

where the term in brackets on the right hand side vanishes for $V \to \infty$. Then b_1 in (4.3.1) can be taken as $120^3 \cdot 288$. This proves Theorem 4.2. □

4.4 The role of boundary conditions

In this section, we discuss the impact of boundary conditions on the geometry of the largest critical cluster, reflecting remarks made by Michael Aizenman in personal communication with Remco van der Hofstad (2005). The purpose of this discussion is to illustrate the consequences of bulk vs. periodic boundary conditions, apart from the previously considered $|\mathcal{C}_{\max}|$. In particular, the (conditional) probability of randomly chosen points to be connected behaves rather differently for the two different boundary conditions, as we will show now.

For the d-dimensional box $\{-\lfloor r/2\rfloor, \dots, \lceil r/2\rceil - 1\}^d$, we write $B_{r,d}$ if we consider it with bulk boundary conditions, and we write $\mathbb{T}_{r,d}$ if it is equipped with periodic boundary conditions. We fix $p = p_c(\mathbb{Z}^d)$ and further omit this subscript. Furthermore, we write C for a positive constant, whose value may change from line to line. Assume that the conditions in Theorem 4.2 are satisfied. Let X_1, X_2, X_3 and X_4 be 4 uniformly chosen vertices in $B_{r,d}$. Then Aizenman notices that, with bulk boundary conditions,

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \to 0, \tag{4.4.1}$$

as the width of the torus r tends to infinity. Indeed,

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) = \frac{\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3, X_4)}{\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4)}. \tag{4.4.2}$$

We can compute

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3, X_4) = V^{-4} \sum_{x \in B_{r,d}} \mathbb{E}_{\mathbb{Z}}[|C_{\mathbb{Z}}(x,r)|^3], \tag{4.4.3}$$

where $C_{\mathbb{Z}}(x,r)$ is the set of vertices $y \in B_{r,d}$ for which $x \stackrel{\mathbb{Z}}{\leftrightarrow} y$, and the right hand side will be bounded from above by the following lemma.

Lemma 4.11. *Under the conditions of Theorem 4.2, for $p = p_c(\mathbb{Z}^d)$ and $r \geq 3 \vee (2L+1)$,*

$$\mathbb{E}_{\mathbb{Z}}[|C_{\mathbb{Z}}(x,r)|^3] \leq Cr^{10}, \qquad \textit{for all } x \in B_{r,d}. \tag{4.4.4}$$

The proof will be given at the end of this section.

On the other hand,

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) = V^{-4} \sum_{x,y,u,v \in B_{r,d}} \mathbb{P}_{\mathbb{Z}}(x \stackrel{\mathbb{Z}}{\leftrightarrow} u,\ y \stackrel{\mathbb{Z}}{\leftrightarrow} v).$$

By the FKG-inequality, for all $x, y, u, v \in B_{r,d}$,

$$\mathbb{P}_{\mathbb{Z}}(x \stackrel{\mathbb{Z}}{\leftrightarrow} u, y \stackrel{\mathbb{Z}}{\leftrightarrow} v) \geq \mathbb{P}_{\mathbb{Z}}(x \stackrel{\mathbb{Z}}{\leftrightarrow} u)\, \mathbb{P}_{\mathbb{Z}}(y \stackrel{\mathbb{Z}}{\leftrightarrow} v), \tag{4.4.5}$$

so that

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \geq V^{-4} \left(\sum_{x,u \in B_{r,d}} \mathbb{P}_{\mathbb{Z}}(x \stackrel{\mathbb{Z}}{\leftrightarrow} u) \right)^2. \tag{4.4.6}$$

For fixed x, by (3.6.2),

$$\sum_{u \in B_{r,d}} \mathbb{P}_{\mathbb{Z}}(x \stackrel{\mathbb{Z}}{\leftrightarrow} u) \geq \sum_{u \in B_{r,d}} \frac{c_\tau}{(|x-u|+1)^{d-2}} \geq Cr^2. \tag{4.4.7}$$

Summing over x gives an extra factor V. We obtain that

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \geq CV^{-2}r^4. \tag{4.4.8}$$

Therefore, when $d > 6$,

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) &= \frac{\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3, X_4)}{\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4)} \\ &\leq \frac{V^{-3}r^{10}}{CV^{-2}r^4} = \frac{r^6}{CV} \to 0. \end{aligned} \tag{4.4.9}$$

All this changes when we consider the torus with periodic boundary conditions and we assume that

$$\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \asymp V^{1/3}. \tag{4.4.10}$$

Note that (4.4.10) is a consequence of (4.3.7), which follows from (4.3.5), and Theorem 4.6. In this case,

$$\mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_2, X_3, X_4) \geq \mathbb{P}_{\mathbb{T}}(X_1, X_2, X_3, X_4 \in \mathcal{C}_{\max}). \tag{4.4.11}$$

Thus, for $\omega \geq 1$ sufficiently large,

$$\begin{aligned} \mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_2, X_3, X_4) &\geq \mathbb{P}_{\mathbb{T}}\Big(X_1, X_2, X_3, X_4 \in \mathcal{C}_{\max}, |\mathcal{C}_{\max}| \geq \frac{1}{\omega} V^{2/3}\Big) \\ &\geq \omega^{-4} V^{-4/3} \mathbb{P}_{\mathbb{T}}\Big(|\mathcal{C}_{\max}| \geq \frac{1}{\omega} V^{2/3}\Big) \geq \frac{1}{2}\omega^{-4} V^{-4/3}. \end{aligned} \tag{4.4.12}$$

On the other hand, we have that

$$\mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \leq \mathbb{P}_{\mathbb{T}}\big((X_1 \longleftrightarrow X_2) \circ (X_3 \longleftrightarrow X_4)\big) + \mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_2, X_3, X_4), \tag{4.4.13}$$

and, by the BK-inequality,

$$\begin{aligned} \mathbb{P}_{\mathbb{T}}\big((X_1 \longleftrightarrow X_2) \circ (X_3 \longleftrightarrow X_4)\big) &\leq \mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_2)\, \mathbb{P}_{\mathbb{T}}(X_3 \longleftrightarrow X_4) \\ &= V^{-2} \chi_{\mathbb{T}}(p_c(\mathbb{Z}^d))^2 \leq C V^{-4/3}. \end{aligned} \tag{4.4.14}$$

Therefore, assuming (4.4.10), we obtain

$$\limsup_{V \to \infty} \mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) > 0. \tag{4.4.15}$$

The difference between (4.4.9) and (4.4.15) was conjectured by Aizenman. The obvious conclusion is that boundary conditions play a crucial role for high-dimensional percolation on finite cubes.

We do not know that (4.4.10) holds, so now we will investigate the changes using the results in Theorem 4.2 in the above discussion. Thus, we will use that, with high probability,

$$V^{2/3} (\log V)^{-4/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}, \tag{4.4.16}$$

and

$$\frac{1}{\omega} V^{1/3} (\log V)^{-2/3} \leq \chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \leq \omega V^{1/3}. \tag{4.4.17}$$

We will see that the conclusion weakens. Indeed, by (4.4.9),

$$\mathbb{P}_{\mathbb{Z}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \leq C r^6 V^{-1} = C r^{6-d} \to 0, \tag{4.4.18}$$

where the convergence is as an inverse power of r, while by an argument as in (4.4.12), now using (4.4.16),

$$\mathbb{P}_{\mathbb{T}}(X_1 \longleftrightarrow X_3 \mid X_1 \longleftrightarrow X_2, X_3 \longleftrightarrow X_4) \geq C (\log V)^{-\frac{16}{3}}, \tag{4.4.19}$$

which only converges to zero as a power of $\log r$. Therefore, the main conclusion that boundary conditions play an essential role is preserved.

We have argued that the largest critical cluster with bulk boundary conditions is much smaller than the one with periodic boundary conditions. We will now argue that, under the condition of Theorem 4.2, critical percolation clusters on the periodic torus $\mathbb{T}_{r,d}$ are similar to percolation clusters on a finite box with bulk boundary conditions, where the box has width $V^{1/6} = r^{d/6} \gg r$. Here we rely on the coupling in Proposition 4.1. In particular, when the origin 0 is $\mathbb{T}$-connected to a uniformly chosen point X, then, with high probability, there is no $\mathbb{Z}$-connection at distance $o(V^{1/6})$ from 0 to a point that is r-equivalent to X. In other words, an occupied path from 0 to X on the torus typically wraps around the torus several times.

This will be illustrated by the following calculation. Assume for the moment that $\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \asymp V^{1/3}$. This follows from our assumption (4.3.5). Choose the vertex X uniformly from the torus $\mathbb{T}_{r,d}$. Then, for any $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}_{\mathbb{Z},\mathbb{T}}\big(\exists y \in \mathbb{Z}^d \colon y \overset{r}{\sim} X, |y| \leq \varepsilon V^{1/6}, 0 \overset{\mathbb{Z}}{\leftrightarrow} y \mid 0 \overset{\mathbb{T}}{\leftrightarrow} X\big) \\ &\quad\leq \frac{\mathbb{P}_{\mathbb{Z},\mathbb{T}}\big(\exists y \in \mathbb{Z}^d \colon y \overset{r}{\sim} X, |y| \leq \varepsilon V^{1/6}, 0 \overset{\mathbb{Z}}{\leftrightarrow} y\big)}{\mathbb{P}_{\mathbb{Z},\mathbb{T}}\big(0 \overset{\mathbb{T}}{\leftrightarrow} X\big)}. \end{aligned} \tag{4.4.20}$$

For the denominator, we rewrite

$$\mathbb{P}_{\mathbb{Z},\mathbb{T}}\big(0 \overset{\mathbb{T}}{\leftrightarrow} X\big) = V^{-1}\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \geq CV^{-2/3}, \tag{4.4.21}$$

whereas the numerator in (4.4.20) is bounded from above by

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}_{\mathbb{Z},\mathbb{T}}\Big(y \overset{r}{\sim} X, |y| \leq \varepsilon V^{1/6}, 0 \overset{\mathbb{Z}}{\leftrightarrow} y\Big) \leq \frac{1}{V} \sum_{y \colon |y| \leq \varepsilon V^{1/6}} \frac{1}{(|y|+1)^{d-2}} \leq C\varepsilon^2 V^{-2/3}. \tag{4.4.22}$$

Thus,

$$\mathbb{P}_{\mathbb{Z},\mathbb{T}}\big(\exists y \in \mathbb{Z}^d \colon y \overset{r}{\sim} X, |y| \leq \varepsilon V^{1/6}, 0 \overset{\mathbb{Z}}{\leftrightarrow} y \mid 0 \overset{\mathbb{T}}{\leftrightarrow} X\big) \leq C\varepsilon^2. \tag{4.4.23}$$

We have seen that $|\mathcal{C}_{\max}| \asymp r^4$ under bulk boundary conditions, whereas $|\mathcal{C}_{\max}| \asymp V^{2/3}$ under periodic boundary conditions. Thus, the maximal critical percolation cluster on a high dimensional torus is of the order $|\mathcal{C}_{\max}| \asymp R^4$ with $R = V^{1/6} \gg r$, so that $\mathcal{C}_{\max}$ is 4-dimensional, but now in a box of width R. This suggests that percolation on a box $B_{r,d}$ with periodic boundary conditions is similar to percolation on the larger box $B_{R,d}$ under bulk boundary conditions, with $R = V^{1/6} \gg r$.

Without assuming (4.3.5), the lower bound on the denominator is only $CV^{-2/3}(\log V)^{2/3}$; thus we obtain the weaker bound

$$\mathbb{P}_{\mathbb{Z}}\Big(\exists y \in \mathbb{Z}^d \colon y \overset{r}{\sim} X, |y| \leq \varepsilon V^{1/6}(\log V)^{-1/3}, 0 \overset{\mathbb{Z}}{\leftrightarrow} y \mid 0 \overset{\mathbb{T}}{\leftrightarrow} X\Big) \leq \varepsilon^2. \tag{4.4.24}$$

The conclusion that occupied paths are long is preserved.

We conclude this section with the proof of Lemma 4.11.

Proof of Lemma 4.11. For all $x \in B_{r,d}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Z}}[|C_{\mathbb{Z}}(x,r)|^3] &= \sum_{s,t,u \in B_{r,d}} \mathbb{P}_{\mathbb{Z}}\left(x \overset{\mathbb{Z}}{\leftrightarrow} s \overset{\mathbb{Z}}{\leftrightarrow} t \overset{\mathbb{Z}}{\leftrightarrow} u\right) \\ &\leq 3 \sum_{\substack{s,t,u \in B_{r,d} \\ v,w \in \mathbb{Z}^d}} \mathbb{P}_{\mathbb{Z}}\left((x \overset{\mathbb{Z}}{\leftrightarrow} v) \circ (v \overset{\mathbb{Z}}{\leftrightarrow} s) \circ (v \overset{\mathbb{Z}}{\leftrightarrow} w) \circ (w \overset{\mathbb{Z}}{\leftrightarrow} t) \circ (w \overset{\mathbb{Z}}{\leftrightarrow} u)\right). \end{aligned} \tag{4.4.25}$$

Using the BK-inequality, this can be further bounded from above by

$$3\Big(\sup_{v\in\mathbb{Z}^d}\sum_{s\in B_{r,d}}\tau_{\mathbb{Z}}(s-v)\Big)^2\sum_{u\in B_{r,d}}\tau_{\mathbb{Z}}^{*3}(x-u), \tag{4.4.26}$$

where $\tau_{\mathbb{Z}}^{*3}$ denotes the threefold convolution of $\tau_{\mathbb{Z}}$. We begin to bound the expression in parenthesis. Fix $v\in\mathbb{Z}^d$. If the distance between v and the box $B_{r,d}$ is larger than r, then $\tau_{\mathbb{Z}}(s-v)\leq Cr^{-(d-2)}$ for all $s\in B_{r,d}$, by (3.6.2). Hence, in this case,

$$\sum_{s\in B_{r,d}}\tau_{\mathbb{Z}}(s-v)\leq Cr^2. \tag{4.4.27}$$

Otherwise, $\|s-v\|\leq 2r$ for all $s\in B_{r,d}$, and therefore, by (4.3.44),

$$\sum_{s\in B_{r,d}}\tau_{\mathbb{Z}}(s-v)\leq\sum_{z\in B_{2r,d}}\tau_{\mathbb{Z}}(z)\leq Cr^2. \tag{4.4.28}$$

Using [62, Prop. 1.7 (i)] for $d>6$, we see that, for all $z\in\mathbb{Z}^d$, the upper bound in (3.6.2) implies that

$$\tau_{\mathbb{Z}}^{*2}(z)\leq\frac{C}{(|z|+1)^{d-4}}, \tag{4.4.29}$$

which in turn implies, when $d>6$, so that $(d-2)+(d-4)>d$,

$$\tau_{\mathbb{Z}}^{*3}(z)=(\tau_{\mathbb{Z}}*\tau_{\mathbb{Z}}^{*2})(z)\leq\frac{C}{(|z|+1)^{d-6}}. \tag{4.4.30}$$

Thus we obtain, for $x\in B_{r,d}$,

$$\sum_{u\in B_{r,d}}\tau_{\mathbb{Z}}^{*3}(x-u)\leq\sum_{z\in B_{2r,d}}\tau_{\mathbb{Z}}^{*3}(z)\leq\sum_{z\in B_{2r,d}}\frac{C}{(|z|+1)^{d-6}}\leq Cr^6. \tag{4.4.31}$$

The combination of the bounds (4.4.25)–(4.4.31) yields the desired upper bound Cr^{10}. $\square$

Chapter 5
The scaling limit of long-range self-avoiding walk

In this chapter we consider the scaling limit of long-range self-avoiding walk in dimension $d > 2(\alpha \wedge 2)$. Under appropriate scaling we prove convergence to Brownian motion for $\alpha \geq 2$, and to α-stable Lévy motion for $\alpha < 2$. This complements results by Slade [100, 101], who proves convergence to Brownian motion for nearest-neighbor self-avoiding walk in high dimension.

5.1 Weak convergence of the end-to-end displacement

We take D as in Section 3.1, satisfying either (D1)–(D3) (finite-variance spread-out model) or (D1')–(D3') (power-law spread-out model). In this chapter we make the additional assumption that there is $\alpha > 0$ and a constant $v_\alpha > 0$ such that, as $|k| \to 0$,

$$1 - \hat{D}(k) \sim \begin{cases} v_\alpha |k|^{\alpha\wedge 2} & \text{if } \alpha \neq 2, \\ v_2 |k|^2 \log(1/|k|) & \text{if } \alpha = 2. \end{cases} \tag{5.1.1}$$

Our primary example of a D satisfying (5.1.1) is given in the following lemma:

Lemma 5.1 (Asymptotics of $\hat{D}(k)$)**.** *Let D be given as in* (3.1.5) *with*

$$h(x) = c|x|^{-d-\alpha}\,(1 + o(1)) \qquad \text{as } |x| \to \infty \tag{5.1.2}$$

for some constant c. For $\alpha \leq 2$ we assume further that $h(x)$ can be extended to a function on $\mathbb{R}^d$ that is rotation invariant. Then D satisfies (5.1.1) *with $v = O\big(L^{\alpha\wedge 2}\big)$.*

The proof is given in Section 5.1.1. We write

$$k_n := \begin{cases} k\,(v_\alpha n)^{-1/\alpha\wedge 2} & \text{if } \alpha \neq 2, \\ k\,(v_2 n \log \sqrt{n})^{-1/2} & \text{if } \alpha = 2, \end{cases} \tag{5.1.3}$$

so that

$$\lim_{n\to\infty} n\,[1 - \hat{D}(k_n)] = |k|^{\alpha\wedge 2}. \tag{5.1.4}$$

We consider self-avoiding walk with this step distribution D, and recall the basic definitions from Section 1.2.

Theorem 5.2 (Weak convergence of end-to-end displacement)**.** *Assume that D satisfies either (D1)–(D3) (finite-variance spread-out model) or (D1')–(D3') (power-law spread-out model), where the spread-out parameter L is sufficiently large. Assume further that also* (5.1.1) *holds. Then self-avoiding walk in dimension $d > d_c = 2(\alpha \wedge 2)$ satisfies*

$$\frac{\hat{c}_n(k_n)}{\hat{c}_n(0)} \to \exp\{-K_\alpha\,|k|^{\alpha\wedge 2}\} \qquad \text{as } n \to \infty, \tag{5.1.5}$$

where

$$K_\alpha = \Big(1+\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n\,\pi_n(x)\,{z_c}^{n-1}\Big)^{-1}\begin{cases}1, & \text{if } \alpha\le 2;\\ 1+(2d\,v_\alpha)^{-1}\displaystyle\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}|x|^2\,\pi_n(x)\,{z_c}^{n}, & \text{if } \alpha>2.\end{cases} \tag{5.1.6}$$

The quantities $\pi_n(x)$ appearing in the theorem are the lace expansion coefficients from Section 2.2. Under the conditions of Theorem 5.2, (5.1.44) and (5.1.72) below imply that K_α is a finite constant.

In Chapter 3 it is shown that long-range self-avoiding walk exhibits mean-field behaviour above dimension $d_c = 2(\alpha\wedge 2)$. More specifically, it is shown that under the conditions of Theorem 5.2, the Fourier transform of the critical two-point function satisfies $\hat G_{z_c}(k) = (1+O(\beta))/(1-\hat D(k))$, where $\beta = O(L^{-d})$ is an arbitrarily small quantity. Hence, on the level of Fourier transforms, the critical two-point functions of long-range self-avoiding walk and long-range *simple* random walk are very close. Indeed, this suggests that the two models behave similar for $d > d_c$, and we prove this belief in a rather strong form by showing that both objects have the same scaling limit.

Yang and Klein [105] prove a version of Theorem 5.2 for weakly self-avoiding walk jumping m lattice sites *along the coordinate axes* with probability proportional to $1/m^2$; and Cheng [35] extends their result to strictly self-avoiding walk with the same step distribution.

Chen and Sakai [34] prove an analogue of Theorem 5.2 for oriented percolation, and in fact our method of proving Theorem 5.2 is very much inspired by the method in [34]. The bounds on the diagrams are different for the two different models, but the general strategy works equally well with either model. In particular, the *spatial* fractional derivatives as in (5.1.49)–(5.1.51) are used for the first time in [34].

5.1.1 Asymptotics of the step function.

Before we start proving Theorem 5.2, we first prove Lemma 5.1.

Proof of Lemma 5.1. If the limit in (5.1.1) exists, then $0\le v_\alpha\le O(L^{\alpha\wedge 2})$ by [33, (1.7)]. We consider separately the cases $\alpha>2$ and $\alpha\le 2$.

Case $\alpha > 2$. Since $\cos(t) = 1 - t^2/2 + o(t^2)$ as $t\to 0$, we have

$$\begin{aligned}\hat D(k) &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{ik\cdot x}\,D(x) = \sum_{x\in\mathbb{Z}^d}\cos(k\cdot x)\,D(x)\\ &= \sum_{x\in\mathbb{Z}^d} D(x) - \sum_{x\in\mathbb{Z}^d}\left(\frac12\sum_{j=1}^d (k_j\,x_j)^2 + o\big(|k\cdot x|^2\big)\right)D(x)\\ &= 1-\frac12\sum_{x\in\mathbb{Z}^d}\left(\sum_{j=1}^d k_j^2\,x_j^2 + 2\sum_{1\le j\le n\le d} k_j\,k_n\,x_j\,x_n\right)D(x) + o\big(|k|^2\big).\end{aligned} \tag{5.1.7}$$

By reflection symmetry,

$$\sum_{x\in\mathbb{Z}^d}\sum_{1\le j\le n\le d} k_j\,k_n\,x_j\,x_n\,D(x) = 0.$$

Furthermore, as D is symmetric under rotations by ninety degree,

$$\sum_{x\in\mathbb{Z}^d} x_1^2\,D(x) = \sum_{x\in\mathbb{Z}^d} x_2^2\,D(x) = \cdots = \frac1d\sum_{x\in\mathbb{Z}^d}|x|^2\,D(x),$$

so that

$$\hat{D}(k) = 1 - \frac{|k|^2}{2d} \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) + o\big(|k|^2\big). \tag{5.1.8}$$

Setting $\sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = 2d\, v_\alpha$ proves the claim.

Case $\alpha \le 2$. The case $\alpha \le 2$ requires a more elaborate calculation. This part of the proof is adapted from Koralov and Sinai [84, Lemma 10.18], who consider the one-dimensional continuous case. We can write $D(x)$ as

$$D(x) = c\,\frac{1+g(x)}{|x|^{d+\alpha}}, \tag{5.1.9}$$

where c is a positive constant and g is a bounded function on $\mathbb{R}^d$ obeying $g(x) \to 0$ as $|x| \to 0$. By our assumption, g is rotation invariant for $|x| > M$. We might limit ourselves to the case $|k| \le 1/M$ and split the sum defining $\hat{D}(k)$ as

$$\hat{D}(k) = \sum_{|x|\le M} \mathrm{e}^{ik\cdot x}\, D(x) + \sum_{M<|x|\le 1/|k|} \mathrm{e}^{ik\cdot x}\, D(x) + \sum_{1/|k|<|x|} \mathrm{e}^{ik\cdot x}\, D(x). \tag{5.1.10}$$

Denote by S_1, S_2 and S_3 the three sums on the right hand side of (5.1.10). A calculation similar to (5.1.8) shows

$$S_1 = \sum_{|x|\le M} D(x) + O\big(|k|^2\big) = \sum_{|x|\le M} D(x) + \begin{cases} o\big(|k|^\alpha\big) & \text{if } \alpha < 2, \\ o\big(|k|^2 \log \frac{1}{|k|}\big) & \text{if } \alpha = 2. \end{cases} \tag{5.1.11}$$

For S_3 we substitue x by $y/|k|$ yielding

$$S_3 = |k|^{d+\alpha} \sum_{\substack{y\in|k|\mathbb{Z}^d \\ |y|>1}} c\,\frac{1+g(y/|k|)}{|y|^{d+\alpha}}\; \mathrm{e}^{ie_k\cdot y}, \tag{5.1.12}$$

where $e_k = k/|k|$ is the unit vector in direction k. By translation invariance of g and Riemann sum approximation we obtain

$$S_3 = |k|^\alpha \left(\int_{|y|\ge 1} c\,\frac{1+g(y/|k|)}{|y|^{d+\alpha}}\; \mathrm{e}^{iy_1}\, \mathrm{d}y + o(1) \right), \tag{5.1.13}$$

with y_1 being the first coordinate of the vector y. Finally, the dominated convergence theorem obtains

$$S_3 = |k|^\alpha c \int_{|y|\ge 1} \frac{\mathrm{e}^{iy_1}}{|y|^{d+\alpha}}\, \mathrm{d}y + o\big(|k|^\alpha\big), \tag{5.1.14}$$

where the integral contributes towards v_α.

Since D is symmetric, the sum defining S_2 can be split as

$$S_2 = \sum_{M<|x|\le 1/|k|} \Big(\mathrm{e}^{ik\cdot x} - 1 - ik\cdot x\Big)\, D(x) + \sum_{M<|x|} D(x) - \sum_{1/|k|<|x|} D(x). \tag{5.1.15}$$

Consider first the last sum. As before, we substitute x by $y/|k|$, use Riemann sum approximation and finally dominated convergence to obtain

$$\sum_{1/|k|<|x|} D(x) = |k|^{\alpha+d} \sum_{\substack{y\in|k|\mathbb{Z}^d \\ |y|>1}} c\,\frac{1+g(y/|k|)}{|y|^{d+\alpha}} = |k|^\alpha c \int_{|y|\ge 1} \frac{\mathrm{e}^{iy_1}}{|y|^{d+\alpha}}\, \mathrm{d}y + o\big(|k|^\alpha\big). \tag{5.1.16}$$

It remains to understand the first sum on the right hand side of (5.1.15). We treat this term with the same recipe as above yielding

$$\begin{aligned}&\sum_{M<|x|\le 1/|k|}\Big(\mathrm{e}^{ik\cdot x}-1-ik\cdot x\Big)\,D(x)\\&=|k|^{\alpha}c\int_{|k|\,M\le|y|\le 1}\frac{1+g(y/|k|)}{|y|^{d+\alpha}}\,\big(y_1^2+O\big(|y_1|^{2+\varepsilon}\big)\big)\;\mathrm{d}y+o\big(|k|^{\alpha}\big).\end{aligned}\tag{5.1.17}$$

For $\alpha<2$ the integral is uniformly bounded in k, and hence the dominated convergence theorem can be used one more time to obtain the desired asymptotics. However, if $\alpha=2$ then the dominating contribution towards (5.1.17) is

$$|k|^2\int_{|k|\,M\le|y|\le 1}\frac{y_1^2}{|y|^{d+\alpha}}\,\mathrm{d}y=\frac{|k|^2}{d}\int_{|k|\,M\le|y|\le 1}\frac{1}{|y|^d}\,\mathrm{d}y=\text{const}\,|k|^2\left(\log\frac{1}{|k|}+\log\frac{1}{M}\right).\tag{5.1.18}$$

Summarizing our calculations, we obtain

$$\hat{D}(k)=\sum_{x\in\mathbb{Z}^d}D(x)-v_\alpha|k|^\alpha+o\big(|k|^\alpha\big)=1-v_\alpha|k|^\alpha+o\big(|k|^\alpha\big)\tag{5.1.19}$$

for $\alpha<2$, and

$$\hat{D}(k)=1-v_\alpha|k|^2\log\frac{1}{|k|}+o\left(|k|^2\log\frac{1}{|k|}\right)\tag{5.1.20}$$

for $\alpha=2$, where v_α is composed of the various integrals arising during the proof. □

5.1.2 The scaling limit of the endpoint: Overview of proof

We recall the lace expansion for self-avoiding walk that was introduced in Section 2.2 (see also [68, Sect. 2.2.1] or [102, Sect. 3] for a detailed derivation). The lace expansion obtains an expansion of the form

$$c_{n+1}(x)=(D*c_n)(x)+\sum_{m=2}^{n+1}\left(\pi_m*c_{n+1-m}\right)(x)\tag{5.1.21}$$

for suitable coefficients $\pi_m(x)$, cf. (2.2.1). We multiply (5.1.21) by z^{n+1} and sum over $n\ge 0$. By letting

$$\Pi_z(x)=\sum_{m=2}^{\infty}\pi_m(x)z^m\tag{5.1.22}$$

for $z\le z_c$, and recalling $G_z(x)=\sum_{n=0}^\infty c_n(x)z^n$, this yields

$$G_z(x)=\delta_{0,x}+z(D*G_z)(x)+(G_z*\Pi_z)(x),\tag{5.1.23}$$

see also (2.2.2).

We proceed by proving Theorem 5.2 subject to certain bounds on the lace expansion coefficients $\pi_n(x)$ to be formulated below. A Fourier transformation of (5.1.23) yields

$$\hat{G}_z(k)=1+z\,\hat{D}(k)\,\hat{G}_z(k)+\hat{G}_z(k)\,\hat{\Pi}_z(k),\qquad k\in[-\pi,\pi)^d,\tag{5.1.24}$$

and this can be solved for $\hat{G}_z(k)$ as

$$\hat{G}_z(k)^{-1}=1-z\,\hat{D}(k)-\hat{\Pi}_z(k),\qquad k\in[-\pi,\pi)^d.\tag{5.1.25}$$

Since z_c is characterized by $\hat{G}_{z_c}(0)^{-1} = 0$, one has $\hat{\Pi}_{z_c}(0) = 1 - z_c$, and hence

$$\hat{G}_z(k)^{-1} = (z_c - z)\,\hat{D}(k) + \Big(\hat{\Pi}_{z_c}(k) - \hat{\Pi}_z(k)\Big) + z_c(1 - \hat{D}(k)) + \Big(\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\Big). \quad (5.1.26)$$

If we let

$$\begin{aligned} A(k) &:= \hat{D}(k) + \partial_z \hat{\Pi}_z(k)\big|_{z=z_c}, & (5.1.27)\\ B(k) &:= 1 - \hat{D}(k) + \frac{1}{z_c}\Big(\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\Big), & (5.1.28)\\ E_z(k) &:= \frac{\hat{\Pi}_{z_c}(k) - \hat{\Pi}_z(k)}{z_c - z} - \partial_z \hat{\Pi}_z(k)\big|_{z=z_c}, & (5.1.29) \end{aligned}$$

then

$$\begin{aligned} z_c\,\hat{G}_z(k) &= \frac{1}{[1 - z/z_c]\,(A(k) + E_z(k)) + B(k)} \\ &= \frac{1}{[1 - z/z_c]\,A(k) + B(k)} - \Theta_z(k), \end{aligned} \quad (5.1.30)$$

where

$$\Theta_z(k) = \frac{[1 - z/z_c]\,E_z(k)}{\big([1 - z/z_c]\,(A(k) + E_z(k)) + B(k)\big)\big([1 - z/z_c]\,A(k) + B(k)\big)}. \quad (5.1.31)$$

If $\hat{G}_z(k)^{-1}$ is understood as a function of z, then $A(k)$ denotes the linear contribution, $E_z(k)$ denotes the higher order contribution (which will turn out to be asymptotically negligible), and $B(k)$ denotes the constant term.

For the first term in (5.1.30) we write

$$\frac{1}{[1 - z/z_c]\,A(k) + B(k)} = \frac{1}{A(k) + B(k)} \sum_{n=0}^{\infty} \left(\frac{z}{z_c}\right)^n \left(\frac{A(k)}{A(k) + B(k)}\right)^n. \quad (5.1.32)$$

For $z < z_c$, we can write $\Theta_z(k)$ as a power series,

$$\Theta_z(k) = \sum_{n=0}^{\infty} \theta_n(k)\, z^n. \quad (5.1.33)$$

Since $\hat{G}_z(k) = \sum_{n=0}^{\infty} \hat{c}_n(k) z^n$ and $B(0) = 0$, we thus obtained

$$\hat{c}_n(k) = \frac{1}{z_c}\left(\frac{z_c^{-n}}{A(k) + B(k)}\left(\frac{A(k)}{A(k) + B(k)}\right)^n + \theta_n(k)\right), \quad \hat{c}_n(0) = \frac{1}{z_c}\left(\frac{z_c^{-n}}{A(0)} + \theta_n(0)\right). \quad (5.1.34)$$

In Section 5.1.4 we prove the following bound on the error term θ_n:

Lemma 5.3. *Under the conditions of Theorem 5.2, $|\theta_n(k)| \le O(z_c^{-n} n^{-\varepsilon})$ for all $\varepsilon \in \big(0, \big(\frac{d}{\alpha \wedge 2} - 2\big) \wedge 1\big)$ uniformly in $k \in [-\pi, \pi)^d$.*

Equation (5.1.34) and Lemma 5.3 imply the following corollary:

Corollary 5.4. *Under the conditions of Theorem 5.2,*

$$\hat{c}_n(0) = \Xi\, z_c^{-n} \left(1 + O(n^{-\varepsilon})\right), \quad (5.1.35)$$

where $\varepsilon \in \big(0, \big(d/(\alpha \wedge 2) - 2\big) \wedge 1\big)$ *and*

$$\Xi = [z_c\, A(0)]^{-1} = \left[z_c + \sum_{x \in \mathbb{Z}^d} \sum_{m=2}^{\infty} m\, \pi_m(x)\, z_c^m \right]^{-1} \in (0, \infty). \tag{5.1.36}$$

By (5.1.34) and Lemma 5.3, for $\varepsilon \in \big(0, \big(\frac{d}{\alpha \wedge 2} - 2\big) \wedge 1\big)$,

$$\begin{aligned}
\frac{\hat{c}_n(k_n)}{\hat{c}_n(0)} &= \big(1 + O(n^{-\varepsilon})\big) \frac{A(0)}{A(k_n) + B(k_n)} \left(\frac{A(k_n)}{A(k_n) + B(k_n)} \right)^n + O(n^{-\varepsilon}) \\
&= \big(1 + O(n^{-\varepsilon})\big) \frac{A(0)}{A(k_n) + B(k_n)} \\
&\quad \times \left(1 + \frac{-n(1 - \hat{D}(k_n))\, A(k_n)^{-1}\, B(k_n) [1 - \hat{D}(k_n)]^{-1}}{n} \right)^n + O(n^{-\varepsilon}).
\end{aligned} \tag{5.1.37}$$

As $n \to \infty$, we have that $n(1 - \hat{D}(k_n)) \to |k|^{\alpha \wedge 2}$ by (5.1.4),

$$A(k_n) \to A(0) = 1 + \sum_{x \in \mathbb{Z}^d} \sum_{m=2}^{\infty} m\, \pi_m(x)\, z_c^{m-1}.$$

The convergence

$$\lim_{n \to \infty} \frac{B(k_n)}{1 - \hat{D}(k_n)} = \begin{cases} 1, & \text{if } \alpha \le 2; \\ 1 + (2d\, v_\alpha)^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2\, \Pi_{z_c}(x), & \text{if } \alpha > 2. \end{cases} \tag{5.1.38}$$

follows directly from the following proposition:

Proposition 5.5. *Under the conditions of Theorem 5.2,*

$$\lim_{|k| \to 0} \frac{\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)}{1 - \hat{D}(k)} = \begin{cases} 0, & \textit{if } \alpha \le 2; \\ (2d\, v_\alpha)^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2\, \Pi_{z_c}(x), & \textit{if } \alpha > 2. \end{cases} \tag{5.1.39}$$

If a sequence h_n converges to a limit h, then $(1 + h_n/n)^n$ converges to e^h. The above estimates imply

$$\lim_{n \to \infty} -n(1 - \hat{D}(k_n))\, A(k_n)^{-1}\, B(k_n) [1 - \hat{D}(k_n)]^{-1} = -K_\alpha\, |k|^{\alpha \wedge 2}$$

and

$$\lim_{n \to \infty} \frac{A(0)}{A(k_n) + B(k_n)} = 1.$$

We thus have proved Theorem 5.2 subject to Lemma 5.3 and Proposition 5.5. We want to emphasize that the bounds on the lace expansion coefficients $\pi_n(x)$ enter the calculation only through (5.1.39) and the error bound in Lemma 5.3.

5.1.3 Bounding the lace expansion coefficients

In this section we prove an estimate on moments of the lace expansion coefficients $\pi_n(x)$. This estimate is used to prove Proposition 5.5. Let us begin by stating the moment estimate.

Lemma 5.6 (Finite moments of the lace expansion coefficients). *For $\alpha > 0$, $d > 2(\alpha \wedge 2)$ and L sufficiently large, we let*

$$\delta \begin{cases} \in \big(0\,,\,(\alpha \wedge 2) \wedge (2 - 2(\alpha \wedge 2))\big) & \text{if } \alpha \neq 2, \\ = 0 & \text{if } \alpha = 2. \end{cases} \tag{5.1.40}$$

Then, for any $z \le z_c$,

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^{\alpha \wedge 2 + \delta}\, |\pi_n(x)|\, z^n < \infty. \tag{5.1.41}$$

The fact that the $(\alpha \wedge 2 + \delta)$th moment of $\Pi_{z_c}(x)$ exists is the key to the proof of (5.1.39). Interestingly, there is a crossover between the phases $\alpha < 2$ and $\alpha > 2$, with $\alpha = 2$ playing a special role. A version of Lemma 5.6 in the setting of oriented percolation is contained in [34, Proposition 3.1].

Before we start with the proof of Lemma 5.6, we shall review some basic facts about structure and convergence of quantities related to $\pi_n(x)$ introduced in (5.1.21)–(5.1.22). For $n \ge 2$, $N \ge 1$, $x \in \mathbb{Z}^d$, there exist quantities $\pi_n^{(N)}(x) \ge 0$ such that

$$\pi_n(x) = \sum_{N=1}^{\infty} (-1)^N \pi_n^{(N)}(x). \tag{5.1.42}$$

A combination of Proposition 2.4 with (3.3.48) (recall $\beta = O(L^{-d})$) and Abel's limit theorem shows

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} \pi_n^{(N)}(x)\, z_c^n < O(L^{-d})^N, \tag{5.1.43}$$

where the constant in the O-term is uniform for all N. Consequently, (5.1.43) is summable in $N \ge 1$ provided that L is sufficiently large, and hence

$$\hat{\Pi}_{z_c}(k) \le \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} |\pi_n(x)|\, z_c^n < \infty. \tag{5.1.44}$$

Lemma 5.6 implies Proposition 5.5, as we will show now.

Proof of Proposition 5.5 subject to Lemma 5.6. We first prove the assertion for $\alpha \le 2$, and afterwards consider $\alpha > 2$.

For $\alpha \le 2$, we choose $\delta \ge 0$ as in (5.1.40), hence $\alpha + \delta \le 2$. Then we use $0 \le 1 - \cos(k \cdot x) \le O(|k \cdot x|^{\alpha+\delta})$ to estimate

$$\begin{aligned} \Big|\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\Big| &\le \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} [1 - \cos(k \cdot x)]\, |\pi_n(x)|\, z_c^n \\ &\le \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} O(|k \cdot x|^{\alpha+\delta})\, |\pi_n(x)|\, z_c^n \\ &\le O(1)\, |k|^{\alpha}\, |k|^{\delta} \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} |x|^{\alpha+\delta}\, |\pi_n(x)|\, z_c^n. \end{aligned} \tag{5.1.45}$$

We use (5.1.1) and Lemma 5.6 to bound further

$$\frac{|\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)|}{1 - \hat{D}(k)} = \begin{cases} O(|k|^{\delta}) & \text{if } \alpha < 2, \\ O(1/\log(1/|k|)) & \text{if } \alpha = 2, \end{cases} \tag{5.1.46}$$

which proves (5.1.39) for $\alpha \leq 2$.

For $\alpha > 2$, we fix $\delta \in (0, 2 \wedge (d-4))$. We apply the Taylor expansion

$$1 - \cos(k \cdot x) = \frac{1}{2}(k \cdot x)^2 + O(|k \cdot x|^{2+\delta}), \tag{5.1.47}$$

together with spatial symmetry of the model and Lemma 5.6 to obtain

$$\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k) = \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} [1 - \cos(k \cdot x)]\, \pi_n(x)\, z_c^n = \frac{|k|^2}{2d} \sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} |x|^2\, \pi_n(x)\, z_c^n + O(|k|^{2+\delta}). \tag{5.1.48}$$

Eq. (5.1.39) for $\alpha > 2$ now follows from (5.1.48) and (5.1.1). □

In the remainder of the section we prove Lemma 5.6. A key point in the proof is the use of a new form of (spatial) fractional derivative, first applied by Chen and Sakai [34] in the context of oriented percolation.

Proof of Lemma 5.6. For $t > 0$, $\zeta \in (0, 2)$, we let

$$K'_\zeta := \int_0^\infty \frac{1 - \cos(v)}{v^{1+\zeta}}\, \mathrm{d}v \in (0, \infty), \tag{5.1.49}$$

yielding

$$t^\zeta = \frac{1}{K'_\zeta} \int_0^\infty \frac{1 - \cos(ut)}{u^{1+\zeta}}\, \mathrm{d}u. \tag{5.1.50}$$

For $\alpha > 0$ and $d > 2(\alpha \wedge 2)$, we choose δ as in (5.1.40). For $x \in \mathbb{Z}^d$ we write $x = (x_1, \ldots, x_d)$. Then by reflection and rotation symmetry of $\pi_n(x)$,

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^{\alpha \wedge 2 + \delta}\, |\pi_n(x)|\, z^n \leq d^{\,(\alpha \wedge 2 + \delta)/2 + 1} \sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} |x_1|^{\alpha \wedge 2 + \delta} \sum_{N=2}^{\infty} \pi_n^{(N)}(x)\, z_c^n, \tag{5.1.51}$$

cf. [34, Lemma 4.1]. We now apply (5.1.50) with $\zeta = \delta_1, \delta_2$, given by

$$\delta_1 \in \big(\delta\,,\, (\alpha \wedge 2) \wedge (2 - 2(\alpha \wedge 2))\big), \tag{5.1.52}$$

$$\delta_2 = \alpha \wedge 2 + \delta - \delta_1. \tag{5.1.53}$$

This yields

$$O(1) \int_0^\infty \frac{\mathrm{d}u}{u^{1+\delta_1}} \int_0^\infty \frac{\mathrm{d}v}{v^{1+\delta_2}} \sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} \sum_{N=2}^{\infty} [1 - \cos(u\, x_1)]\, [1 - \cos(v\, x_1)]\, \pi_n^{(N)}(x)\, z_c^n \tag{5.1.54}$$

as an upper bound of (5.1.51). We write the double integral appearing in (5.1.54) as the sum of four terms, $I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \sum_{N=2}^{\infty} \int_0^1 \frac{\mathrm{d}u}{u^{1+\delta_1}} \int_0^1 \frac{\mathrm{d}v}{v^{1+\delta_2}} \sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} [1 - \cos(\vec{u} \cdot x)]\, [1 - \cos(\vec{v} \cdot x)]\, \pi_n^{(N)}(x)\, z_c^n \tag{5.1.55}$$

with

$$\vec{u} = (u, 0, \ldots, 0) \in \mathbb{R}^d, \qquad \vec{v} = (v, 0, \ldots, 0) \in \mathbb{R}^d, \tag{5.1.56}$$

and I_2, I_3, I_4 are defined similarly:

$$I_2 = \int_0^1 \mathrm{d}u \int_1^\infty \mathrm{d}v \cdots, \quad I_3 = \int_0^1 \mathrm{d}u \int_1^\infty \mathrm{d}v \cdots, \quad I_4 = \int_1^\infty \mathrm{d}u \int_1^\infty \mathrm{d}v \cdots. \tag{5.1.57}$$

We now show that $I_1, \dots, I_4$ are all finite, which implies (5.1.41). The bound $I_4 < \infty$ simply follows from $1 - \cos t \leq 2$ and (5.1.44). In order to prove the bounds $I_1, I_2, I_3 < \infty$ we need the particular structure of the $\pi_n^{(N)}(x)$-terms.

We recall from (2.2.15) and (2.2.16) the definitions of

$$\tilde{G}_z(x) = z(D * G_z)(x), \qquad x \in \mathbb{Z}^d, \tag{5.1.58}$$

and

$$\tilde{B}(z) = \sup_{x \in \mathbb{Z}^d} (G_z * \tilde{G}_z)(x). \tag{5.1.59}$$

Fix $N \geq 1$. In Proposition 2.4 it is shown that for $z \geq 0$,

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)]\, \Pi_z^{(1)}(x) = 0 \tag{5.1.60}$$

and

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)]\, \Pi_z^{(N)}(x) \leq \frac{N}{2}(N+1) \left(\sup_x [1 - \cos(k \cdot x)]\, G_z(x) \right) \tilde{B}(z)^{N-1}, \quad N \geq 2. \tag{5.1.61}$$

These bounds are called *diagrammatic estimates*, because the lace expansion coefficients $\pi_z^{(N)}(x)$ are expressed in terms of diagrams, whose structure is heavily used in the derivation of the above bounds. The composition of the diagrams, and their decomposition into two-point functions as in (5.1.60)–(5.1.61), is sketched in Section 2.2 and described in detail in [102, Sections 3 and 4]. It is clear that a slight modification of this procedure proves the following bound:

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} [1 - \cos(\vec{v} \cdot x)]\, [1 - \cos(\vec{u} \cdot x)]\, \pi_n^{(N)}(x)\, z^n \\ &\quad \leq O(N^4)\, \tilde{B}(z)^{N-2} \left(\sup_x [1 - \cos(\vec{v} \cdot x)]\, G_z(x) \right) \\ &\quad\quad \times \left(\sup_y \sum_{x \in \mathbb{Z}^d} [1 - \cos(\vec{u} \cdot x)]\, G_z(x)\, G_z(y - x) \right). \end{aligned} \tag{5.1.62}$$

Given (5.1.62), it remains to show the following three bounds:

$$\tilde{B}(z_c) = \sup_{x \in \mathbb{Z}^d} (G_{z_c} * \tilde{G}_{z_c})(x) \quad \leq \quad O\big(L^{-d}\big); \tag{5.1.63}$$

$$\sup_x [1 - \cos(\vec{v} \cdot x)]\, G_{z_c}(x) \quad \leq \quad O\big(v^{\alpha \wedge 2}\big)\,; \tag{5.1.64}$$

$$\sup_y \sum_{x \in \mathbb{Z}^d} [1 - \cos(\vec{u} \cdot x)]\, G_{z_c}(x)\, G_{z_c}(y - x) \quad \leq \quad O\Big(u^{(d - 2(\alpha \wedge 2)) \wedge (\alpha \wedge 2)}\Big). \tag{5.1.65}$$

Suppose (5.1.63)–(5.1.65) were true, then

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} [1 - \cos(\vec{u} \cdot x)]\, [1 - \cos(\vec{v} \cdot x)]\, \pi_n^{(N)}(x)\, {z_c}^n \\ &\quad \leq O\big(N^4\big)\, O\big(L^{-d}\big)^{N-2}\, O\big(v^{\alpha \wedge 2}\big)\, O\Big(u^{(d - 2(\alpha \wedge 2)) \wedge (\alpha \wedge 2)}\Big). \end{aligned} \tag{5.1.66}$$

Since $\delta_1 < (\alpha \wedge 2) \wedge (d - 2(\alpha \wedge 2))$ and $\delta_2 < \alpha \wedge 2$, we obtain that I_1 is finite for L sufficiently large, as desired. Similarly, it follows that I_2 and I_3 are finite. It remains to prove (5.1.63)–(5.1.65), and we will use results from Chapter 3 to show it.

We start by referring to Proposition 3.9 for the fact that $f(z) \leq O(1)$ uniformly in $z \in [0, z_c)$ under the conditions of Theorem 5.2. Since the bound is uniform and $c_n(x) \geq 0$, we can conclude that even $f(z_c) < \infty$. Therefore, (5.1.63) indeed follows from (3.3.16) and the fact that $\beta = O(L^{-d})$, cf. Proposition 3.1. Furthermore,

$$\sup_x [1 - \cos(k \cdot x)]\, G_{z_c}(x) \leq O(1 - \hat{D}(k)), \tag{5.1.67}$$

by Lemma 3.4 and (3.3.12), hence also (5.1.64) follows. It remains to prove (5.1.65). Since

$$\begin{aligned} &\sup_y \sum_{x \in \mathbb{Z}^d} [1 - \cos(\vec{u} \cdot x)]\, G_{z_c}(x)\, G_{z_c}(y - x) \\ = \;& \sup_y \int_{[-\pi,\pi)^d} \mathrm{e}^{-il\cdot y} \Big(\hat{G}_{z_c}(l) - \frac{1}{2} \Big(\hat{G}_{z_c}(l - \vec{u}) + \hat{G}_{z_c}(l + \vec{u}) \Big) \Big)\, \hat{G}_{z_c}(l)\, \frac{\mathrm{d}l}{(2\pi)^d} \\ \leq \;& \int_{[-\pi,\pi)^d} \Big| \frac{1}{2}\, \Delta_{\vec{u}}\, \hat{G}_{z_c}(l) \Big|\, \hat{G}_{z_c}(l)\, \frac{\mathrm{d}l}{(2\pi)^d}\;, \end{aligned} \tag{5.1.68}$$

our bounds $f_2(z_c) \leq K$ and $f_3(z_c) \leq K$, together with $\lambda_{z_c} = 1$, imply that

$$\begin{aligned} &\sup_y \sum_{x \in \mathbb{Z}^d} [1 - \cos(\vec{u} \cdot x)]\, G_{z_c}(x)\, G_{z_c}(y - x) \\ &\leq 100 K^2\, \hat{C}_1(\vec{u})^{-1} \int_{[-\pi,\pi)^d} \Big(\hat{C}_1(l - \vec{u})\, \hat{C}_1(l + \vec{u}) + \hat{C}_1(l - \vec{u})\, \hat{C}_1(l) \\ &\qquad\qquad + \hat{C}_1(l)\, \hat{C}_1(l + \vec{u}) \Big)\, \hat{C}(l)\, \frac{\mathrm{d}l}{(2\pi)^d} \\ &= O(1)\, [1 - \hat{D}(\vec{u})] \int_{[-\pi,\pi)^d} \bigg(\frac{1}{[1 - \hat{D}(l - \vec{u})]\, [1 - \hat{D}(l + \vec{u})]\, [1 - \hat{D}(l)]} \\ &\qquad + \frac{1}{[1 - \hat{D}(l - \vec{u})]\, [1 - \hat{D}(l)]^2} + \frac{1}{[1 - \hat{D}(l + \vec{u})]\, [1 - \hat{D}(l)]^2} \bigg)\, \frac{\mathrm{d}l}{(2\pi)^d}. \end{aligned} \tag{5.1.69}$$

Chen and Sakai show that the integral term on the right hand side of (5.1.69) is bounded above by $O\big(u^{(d-3(\alpha\wedge 2))\wedge 0}\big)$, cf. [34, (4.30)–(4.33)]. Furthermore, $1 - \hat{D}(\vec{u}) \leq O\big(u^{\alpha\wedge 2}\big)$ by (5.1.1). The combination of the above inequalities implies (5.1.65), and hence the claim follows. □

5.1.4 Error bounds

The proof of Lemma 5.3 is the final piece in the proof of Theorem 5.2. Our proof of Lemma 5.3 makes use of the following lemma:

Lemma 5.7. *Consider a function g given by the power series $g(z) = \sum_{n=0}^{\infty} a_n z^n$, with z_c as radius of convergence.*

(i) If $|g(z)| \leq O(|z_c - z|^{-b})$ for some $b \geq 1$, then $|a_n| \leq O(z_c^{-n} \log(n))$ if $b = 1$, or $|a_n| \leq O(z_c^{-n}\, n^{b-1})$ if $b > 1$.

(ii) If $|g'(z)| \leq O(|z_c - z|^{-b})$ for some $b > 1$, then $|a_n| \leq O(z_c^{-n}\, n^{b-2})$.

The proof of assertion (i) is contained in [38, Lemma 3.2], and (ii) is a direct consequence of (i) since (i) implies that $|n\,a_n| \le O(z_c^{-n}\,n^{b-1})$. Lemma 5.7 is the key to the proof of Lemma 5.3.

Proof of Lemma 5.3. We recall

$$\Theta_z(k) = \sum_{n=0}^{\infty} \theta_n(k)\, z^n, \tag{5.1.70}$$

where

$$\Theta_z(k) = \frac{[1-z/z_c]\,E_z(k)}{\big([1-z/z_c]\,(A(k)+E_z(k))+B(k)\big)\,\big([1-z/z_c]\,A(k)+B(k)\big)}. \tag{5.1.71}$$

We fix $\varepsilon \in (0, (d(\alpha\wedge 2)^{-1}-2)\wedge 1)$ and aim to prove $|\theta_n(k)| \le O(z_c^{-n}\,n^{-\varepsilon})$, where the constant in the O-term is uniform for $k \in [-\pi,\pi)^d$. By Lemma 5.7 it is sufficient to show $|\partial_z \Theta_z(k)| \le O\big(|z_c - z|^{-(2-\varepsilon)}\big)$, and we prove this now.

Before bounding $\partial_z\Theta_z(k)$, we consider derivatives of $\hat\Pi_z(k)$ (the Fourier transform of $\Pi_z(x)$ introduced in (5.1.22)). The first derivative of $\partial_z\hat\Pi_z(k)$ is converging absolutely for $z \le z_c$, i.e.,

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n\,|\pi_n(x)|\,z_c^{n-1} < \infty, \tag{5.1.72}$$

cf. [88, Theorem 6.2.9] for a proof in the finite-range setting, and again Chapter 3 (in particular Proposition 3.1) for the extension to long-range systems. Moreover, we claim that

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\,|\pi_n(x)|\,z_c^{n-1} < \infty; \tag{5.1.73}$$

for $\varepsilon \in (0, (d(\alpha\wedge 2)^{-1}-2)\wedge 1)$. The bound (5.1.73) can be proved by considering *temporal* fractional derivatives, as introduced in [88, Section 6.3]. In particular, the proof of [88, Theorem 6.4.2] shows

$$\sup_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\,c_n(x)\,z_c^{n-1} \le O(1)\int_{[-\pi,\pi)^d}\sum_{n\ge 2} n(n-1)^{\varepsilon}\hat D(k)^{n-2}\frac{\mathrm{d}k}{(2\pi)^d}, \tag{5.1.74}$$

(see the first displayed identity in [88, p. 196]). On the one hand, $\hat D(k) = 1-(1-\hat D(k)) \le \mathrm{e}^{-(1-\hat D(k))} \le \mathrm{e}^{-c_1\,|k|^{\alpha\wedge 2}}$ for some constant $c_1>0$, by (5.1.1). On the other hand, $-\hat D(k) \le 1-c_2$ for a positive constant c_2, by (3.1.4)/(3.1.9). Together these bounds yield

$$\begin{aligned}\int_{[-\pi,\pi)^d} \hat D(k)^{n-2}\frac{\mathrm{d}k}{(2\pi)^d} \;&\le\; \int_{\substack{k\in[-\pi,\pi)^d:\\ \hat D(k)\ge 0}} \mathrm{e}^{-c_1\,(n-2)\,|k|^{\alpha\wedge 2}}\,\frac{\mathrm{d}k}{(2\pi)^d} \\ &\quad+ \int_{\substack{k\in[-\pi,\pi)^d:\\ \hat D(k)< 0}} (1-c_2)^{n-2}\frac{\mathrm{d}k}{(2\pi)^d} \\ &\le\; O(n^{-d/(\alpha\wedge 2)}) + (1-c_2)^{n-2} \le O(n^{-d/(\alpha\wedge 2)}).\end{aligned} \tag{5.1.75}$$

Hence the right hand side of (5.1.74) is less than or equal to

$$\sum_{n\geq 2} n(n-1)^{\varepsilon}\, O(n^{-d/(\alpha\wedge 2)}), \tag{5.1.76}$$

and this is finite if $1+\varepsilon-d/(\alpha\wedge 2)<-1$. Furthermore, the proof of [88, Corollary 6.4.3] shows that

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\,|\pi_n(x)|\, z_c^{n-1} \leq O(1)\left(\sup_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\, c_n(x)\, z_c^{n-1}\right) \tag{5.1.77}$$

under the conditions of Theorem 5.2. This proves (5.1.73).

We first prove

$$E_z(k) \leq O(|z_c-z|^{\varepsilon}) \tag{5.1.78}$$

by considering the power series representation of $\hat{\Pi}_z(k)$ in (5.1.29):

$$E_z(k) = \frac{1}{z_c-z}\sum_x\sum_{n\geq 2} \mathrm{e}^{ik\cdot x}\,\pi_n(x)\,(z_c^n-z^n) - \sum_x\sum_{n\geq 2}\mathrm{e}^{ik\cdot x}\,\pi_n(x)\, n\, z_c^{n-1}. \tag{5.1.79}$$

Since

$$\frac{z_c^n-z^n}{z_c-z} = \sum_{i=0}^{n-1} z^i\, z_c^{(n-1)-i}, \tag{5.1.80}$$

one has

$$E_z(k) = \sum_x\sum_{n\geq 2}\mathrm{e}^{ik\cdot x}\,\pi_n(x)\sum_{i=1}^{n-1}\left(z^i-z_c^i\right) z_c^{(n-1)-i}. \tag{5.1.81}$$

For every $\zeta,\varepsilon\in(0,1)$ and $n\geq 2$,

$$\begin{aligned}\left|1-\zeta^{n-1}\right| &= \left|(1-\zeta^{n-1})^{1-\varepsilon}\left(\frac{1-\zeta^{n-1}}{1-\zeta}\right)^{\varepsilon}(1-\zeta)^{\varepsilon}\right| \\ &\leq \left|\sum_{l=0}^{n-2}\zeta^l\right|^{\varepsilon}(1-\zeta)^{\varepsilon} \leq (n-1)^{\varepsilon}\,(1-\zeta)^{\varepsilon}\,.\end{aligned} \tag{5.1.82}$$

Applying this for $\zeta=z/z_c$, we obtain for $z<z_c$ and $0<i<n$,

$$\begin{aligned}\left|z^i-z_c^i\right| z_c^{(n-1)-i} &= \left|1-\left(\frac{z}{z_c}\right)^i\right| z_c^{n-1} \leq \left|1-\left(\frac{z}{z_c}\right)^{n-1}\right| z_c^{n-1} \\ &\leq \left|1-\frac{z}{z_c}\right|^{\varepsilon}(n-1)^{\varepsilon}\, z_c^{n-1}.\end{aligned} \tag{5.1.83}$$

Insertion into (5.1.81) yields

$$|E_z(k)| \leq (z_c-z)^{\varepsilon}\sum_x\sum_{n\geq 2} n(n-1)^{\varepsilon}\,|\pi_n(x)|\, z_c^{n-1} \leq O(|z_c-z|^{\varepsilon}), \tag{5.1.84}$$

where the last bound uses (5.1.73). We further differentiate (5.1.29) to get

$$\begin{aligned}\partial_z E_z(k) &= \frac{(z_c-z)\,\partial_z\big(\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)\big)+\big(\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)\big)}{(z_c-z)^2} \\ &= \frac{1}{z_c-z}\left(\frac{\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)}{z_c-z}-\partial_z\hat{\Pi}_z(k)\right).\end{aligned} \tag{5.1.85}$$

A calculation similar to (5.1.79)–(5.1.84) shows

$$|\partial_z E_z(k)| \le \left|\frac{E_z(k)}{z_c - z}\right| + \frac{1}{z_c - z}\left|\sum_x \sum_{n\ge 2} \mathrm{e}^{ik\cdot x}\,\pi_n(x)\,n\,\big(z_c^{n-1} - z^{n-1}\big)\right| \le O(|z_c - z|^{\varepsilon-1}). \tag{5.1.86}$$

We write D_1 and D_2 for the two factors in the denominator in (5.1.71). Then

$$\begin{aligned} z_c^2\,\partial_z\Theta_z(k) =& \frac{z_c}{D_1\,D_2}\,\big((z_c - z)\,\partial_z E_z(k) - E_z(k)\big) \\ & - \frac{z_c - z}{(D_1\,D_2)^2}\,E_z(k)\Big(\big(-A(k) - E_z(k) + (z_c - z)\,\partial_z E_z(k)\big)\,D_2 - D_1\,A(k)\Big). \end{aligned} \tag{5.1.87}$$

After further cancelation of D_1-, D_2-terms we are left with D_1 and D_2 in the denominator only, hence a lower bound on them suffices. Indeed, there is a constant $c > 0$ such that

$$|D_1| = \left|z_c\,\hat{G}_z(k)\right|^{-1} \ge z_c^{-1}\,\chi(z) \ge c\,(z_c - z)\,, \tag{5.1.88}$$

since $\gamma_{\mathrm{S}} = 1$ by Theorem 3.16 (cf. (1.4.9)). Furthermore,

$$|D_2| \ge c\,(z_c - z) \tag{5.1.89}$$

because D_2 is a linear function in $(z_c - z)$. The lower bounds on D_1 and D_2, together with the bounds on $E_z(k)$ and $\partial_z E_z(k)$ in (5.1.78) and (5.1.86), prove that (5.1.87) is uniformly bounded for all $z \le z_c$, and in particular

$$\partial_z\Theta_z(k) \le O(|z_c - z|^{-(2-\varepsilon)}). \tag{5.1.90}$$

Finally, assertion *(ii)* in Lemma 5.7 implies

$$\theta_n(k) \le O(z_c^{-n}\,n^{-\varepsilon}) \tag{5.1.91}$$

for all $\varepsilon \in (0, (d(\alpha\wedge 2)^{-1} - 2)\wedge 1)$, uniformly in k. □

5.2 Mean-r displacement

The *mean-r displacement* is defined as

$$\xi^{(r)}(n) := \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^r c_n(x)}{c_n}\right)^{1/r}, \tag{5.2.1}$$

where we recall that $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x) = \hat{c}_n(0)$. The mean-$r$ displacement for $r = 2$ coincides with the radius of gyration, and it is already well understood. For example, van der Hofstad and Slade [71] prove the following rather general version:

Theorem 5.8 (Radius of gyration [71])**.** *Consider self-avoiding walk with step distribution D in dimension $d > 4$ with spread-out parameter L sufficiently large, where D is given as in the finite-variance spread-out model satisfying (D1)–(D3). Then there is a constant $C > 0$ such that, as $n \to \infty$,*

$$\frac{1}{c_n}\sum_{x\in\mathbb{Z}^d} |x|^2 c_n(x) = C\,n\,(1 + o(1)). \tag{5.2.2}$$

In the sequel we prove a complementary result for $r < 2$.

Theorem 5.9 (Mean-r displacement). *Under the assumptions of Theorem 5.2, for any $r < \alpha \wedge 2$,*

$$\xi^{(r)}(n) \asymp \begin{cases} n^{1/(\alpha\wedge 2)} & \text{if } \alpha \neq 2, \\ (n \log n)^{1/2} & \text{if } \alpha = 2, \end{cases} \tag{5.2.3}$$

as $n \to \infty$.

In view of (5.2.2) we conjecture that (5.2.3) indeed holds for all $r > 0$, although our methods apply only for $r < \alpha \wedge 2$. When proving Theorem 5.9 we shall use methods similar to those developed in Section 5.1, and again a key ingredient is the equality in (5.1.50).

Proof of Theorem 5.9. In order to deal with the cases $\alpha = 2$ and $\alpha \neq 2$ simultaneously, we write

$$f_\alpha(n) = \begin{cases} (v_\alpha n)^{-1/(\alpha\wedge 2)} & \text{if } \alpha \neq 2, \\ (v_2 n \log \sqrt{n})^{-1/2} & \text{if } \alpha = 2, \end{cases} \tag{5.2.4}$$

and note that (5.2.3) can be rewritten as $\xi^{(r)}(n) \asymp f_\alpha(n)^{-1}$. Also, we write x_1 for the first component of the vector $x \in \mathbb{Z}^d$, and denote by $\vec{u}$ the vector $\vec{u} = (u, 0, \dots, 0) \in \mathbb{R}^d$, see also (5.1.56). We use reflection and rotation symmetry of c_n in the first line, and (5.1.50) in the second line to obtain

$$\begin{aligned} \frac{1}{c_n} \sum_{x\in\mathbb{Z}^d} |x|^r c_n(x) &\asymp \sum_{x\in\mathbb{Z}^d} |x_1|^r \frac{c_n(x)}{c_n} \\ &\asymp \sum_{x\in\mathbb{Z}^d} \int_0^\infty \frac{\mathrm{d}u}{u^{1+r}} \left[1 - \cos(\vec{u}\cdot x)\right] \frac{c_n(x)}{c_n} \\ &= \int_{f_\alpha(n)}^\infty \frac{\mathrm{d}u}{u^{1+r}} \sum_{x\in\mathbb{Z}^d} \left[1 - \cos(\vec{u}\cdot x)\right] \frac{c_n(x)}{c_n} \\ &\quad + \int_0^{f_\alpha(n)} \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u})}{\hat{c}_n(0)}\right). \end{aligned} \tag{5.2.5}$$

For the first integral on the right hand side of (5.2.5) we use $0 \le [1 - \cos(\vec{u}\cdot x)] \le 2$ yielding

$$0 \le \int_{f_\alpha(n)}^\infty \frac{\mathrm{d}u}{u^{1+r}} \sum_{x\in\mathbb{Z}^d} \left[1 - \cos(\vec{u}\cdot x)\right] \frac{c_n(x)}{c_n} \le \int_{f_\alpha(n)}^\infty \frac{\mathrm{d}u}{u^{1+r}} = O\big(f_\alpha(n)^{-r}\big). \tag{5.2.6}$$

For the second integral, we substitute u by $u_n := u\, f_\alpha(n)$ to obtain

$$\int_0^{f_\alpha(n)} \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u})}{\hat{c}_n(0)}\right) = f_\alpha(n)^{-r} \int_0^1 \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right), \tag{5.2.7}$$

where $\vec{u}_n = f_\alpha(n)\vec{u}$ (compare with k_n in (5.1.3)). Suppose we know

$$\int_0^1 \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) \asymp 1, \tag{5.2.8}$$

then it would follow that $c_n^{-1} \sum_x |x|^r c_n(x) \asymp f_\alpha(n)^{-r}$, as desired.

It remains to show (5.2.8) is indeed true. The idea is the following. If the ratio $\hat{c}_n(\vec{u}_n)/\hat{c}_n(0)$ is replaced by its limit $\exp\{-K_\alpha u^{\alpha\wedge 2}\}$ (cf. Theorem 5.2), then Taylor expansion shows

$$1 - \exp\{-K_\alpha u^{\alpha\wedge 2}\} = K_\alpha u^{\alpha\wedge 2} + O\big(u^{2(\alpha\wedge 2)}\big),$$

and since $\alpha \wedge 2 - (1+r) > -1$, the integral in (5.2.8) converges. However, a careful consideration of error terms makes the argument look slightly more complicated.

We write

$$h_n = -n(1-\hat{D}(\vec{u}_n))\, A(\vec{u}_n)^{-1}\, B(\vec{u}_n)[1-\hat{D}(\vec{u}_n)]^{-1}. \tag{5.2.9}$$

By (5.1.37),

$$\left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) = \big(1+O(n^{-\varepsilon})\big)\left[1 - \frac{A(0)}{A(\vec{u}_n)+B(\vec{u}_n)}\left(1+\frac{h_n}{n}\right)^n\right]. \tag{5.2.10}$$

Taylor expansion shows

$$n\log\left(1+\frac{h_n}{n}\right) = h_n + O\left(\frac{h_n^2}{n}\right)$$

and

$$\left(1+\frac{h_n}{n}\right)^n = \mathrm{e}^{n\log(1+h_n/n)} = \mathrm{e}^{h_n}\left(1+O\left(\frac{h_n^2}{n}\right)\right).$$

Insertion into (5.2.10) obtains

$$\left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) = \big(1+O(n^{-\varepsilon})\big)\,\frac{A(0)}{A(\vec{u}_n)+B(\vec{u}_n)}\left[\frac{A(\vec{u}_n)+B(\vec{u}_n)}{A(0)} - 1 + \left(1-\mathrm{e}^{h_n}\right) - O\left(\frac{h_n^2}{n}\right)\mathrm{e}^{h_n}\right]. \tag{5.2.11}$$

We remark that the limit in (5.1.4) is uniform in $u \in (0,1]$, and the bound (5.1.46) implies that $B(\vec{u}_n) \asymp [1-\hat{D}(\vec{u}_n)]$ uniformly in $u \in (0,1]$. We show below that the limit $A(\vec{u}_n) \to A(0)$ is also uniform. Consequently, also $\lim_{n\to\infty} h_n = -Ku^{\alpha\wedge 2}$ is a uniform limit, and this is important since we are integrating u over the interval $(0,1]$.

We finally show that

$$\frac{A(\vec{u}_n)+B(\vec{u}_n)}{A(0)} - 1 = \frac{A(\vec{u}_n)-A(0)+B(\vec{u}_n)}{A(0)} \le u^{\alpha\wedge 2}o(1) \tag{5.2.12}$$

as $n\to\infty$, uniformly in u. By (5.1.38), $B(\vec{u}_n) \asymp [1-\hat{D}(\vec{u}_n)] = O(1/n)\,u^{\alpha\wedge 2}$. We choose δ as in (5.1.40), so that in particular $0 \le (\alpha\wedge 2)+\delta \le 2$. Consequently,

$$\begin{aligned}\Big|A(\vec{u}_n)-A(0)\Big| &\le [1-\hat{D}(\vec{u}_n)] + \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}[1-\cos(\vec{u}_n\cdot x)]\,n\,|\pi_n(x)|\,z_c^{n-1}\\ &= u^{\alpha\wedge 2}\,O(1/n) + \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} O\big(|\vec{u}_n|^{(\alpha\wedge 2)+\delta}|x|^{(\alpha\wedge 2)+\delta}\big)\,n\,|\pi_n(x)|\,z_c^{n-1}\end{aligned}$$

Since $|\vec{u}_n|^{(\alpha\wedge 2)+\delta} \asymp u^{(\alpha\wedge 2)+\delta}/n^{1+\delta/(\alpha\wedge 2)}$ for $\alpha \neq 2$, and $|\vec{u}_n|^{(\alpha\wedge 2)+\delta} \asymp u^2/(n\log\sqrt{n})$ for $\alpha = 2$, we bound further

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} O\big(|\vec{u}_n|^{(\alpha\wedge 2)+\delta}|x|^{(\alpha\wedge 2)+\delta}\big)\, n\,|\pi_n(x)|\, z_c^{n-1} \\
&\le O\big(u^{(\alpha\wedge 2)+\delta}\big) \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} |x|^{(\alpha\wedge 2)+\delta}\,|\pi_n(x)|\, z_c^{n-1} \times \begin{cases} n^{-\delta/(\alpha\wedge 2)}, & \text{if } \alpha \neq 2,\\ (\log n)^{-1}, & \text{if } \alpha = 2,\end{cases}
\end{aligned} \tag{5.2.13}$$

and this is bounded above by $u^{\alpha\wedge 2}o(1)$ by appeal to Lemma 5.6. In particular, this implies that $A(\vec{u}_n) \to A(0)$ uniformly in u.

We have proven that the only non-vanishing contribution towards (5.2.11) comes from the term $1-\mathrm{e}^{h_n}$. Since the sequence h_n converges uniformly to the negative limit $-K_\alpha\, u^{\alpha\wedge 2}$, there is an n_0 such that for all $n \ge n_0$, $-2K_\alpha\, u^{\alpha\wedge 2} \le h_n \le -K_\alpha\; u^{\alpha\wedge 2}/2$. Consequently, $1-\mathrm{e}^{h_n}$ is positive for $n \ge n_0$, and $1-\mathrm{e}^{h_n} \le O\big(u^{\alpha\wedge 2}\big)$ by Taylor expansion. Therefore,

$$0 \le \left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) \le O\big(u^{\alpha\wedge 2}\big) \tag{5.2.14}$$

as $n \to \infty$, where the bounds on the error terms do not depend on n. Hence for $r < \alpha \wedge 2$, the integral

$$\int_0^1 \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) \tag{5.2.15}$$

converges, and is positive for sufficiently large n. The combination of (5.2.6), (5.2.7) and (5.2.14) implies the claim. □

5.3 Convergence to Brownian motion and α-stable processes

We recall the definition of $f_\alpha(n)$ from Section 5.2. Given an n-step self-avoiding walk w, define

$$X_n(t) = (2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\; f_\alpha(n)\, w(\lfloor nt \rfloor), \qquad t \in [0,1]. \tag{5.3.1}$$

We aim to identify the scaling limit of X_n, and the appropriate space to study the limit is the space of $\mathbb{R}^d$-valued càdlàg-functions $D([0,1],\mathbb{R}^d)$ equipped with the Skorokhod topology.

For $\alpha \in (0,2]$, $W^{(\alpha)}$ denotes the standard α-stable Lévy measure, normalized such that

$$\int \mathrm{e}^{ik\cdot B^{(\alpha)}(t)}\;\mathrm{d}W^{(\alpha)} = \mathrm{e}^{-|k|^\alpha t/(2d)}, \tag{5.3.2}$$

where $B^{(\alpha)}$ is a (càdlàg version of) standard symmetric α-stable Lévy motion (in the sense of [96, Definition 3.1.3]). Note that $W^{(2)}$ is the Wiener measure, and $B^{(2)}$ is Brownian motion. By $\langle\cdot\rangle_n$ we denote expectation with respect to the self-avoiding walk measure $\mathbb{Q}_n$ of Section 1.2.

Theorem 5.10 (Weak convergence to α-stable processes and Brownian motion)**.** *Under the assumptions in Theorem 5.2,*

$$\lim_{n\to\infty}\langle f(X_n)\rangle_n = \int f\,\mathrm{d}W^{(\alpha\wedge 2)}, \tag{5.3.3}$$

for every bounded continuous function $f\colon D([0,1],\mathbb{R}^d)\to\mathbb{R}$. *That is to say,* X_n *converges in distribution to an* α*-stable Lévy motion for* $\alpha<2$, *and to Brownian motion for* $\alpha\geq 2$. *Equivalently,* $\mathbb{Q}_n$ *converges weakly to* $W^{(\alpha\wedge 2)}$.

Slade [100, 101] proves convergence of the *nearest-neighbor* self-avoiding walk to Brownian motion in sufficiently high dimension, using a finite-memory cut-off. Hara and Slade [57] provide an alternative argument by using fractional derivative estimates. An account of the latter approach is contained in the monograph [88, Sect. 6.6], where also the uniform spread-out case is considered.

Our proof of Theorem 5.10 follows the line of arguments in [88, Section 6.6], adapted so as to deal with a different limit.

5.3.1 Convergence of finite dimensional distributions

In order to prove convergence in distribution, we need two properties: (i) the convergence of finite-dimensional distributions, and (ii) tightness of the family $\{X_n\}$. In this section we shall consider the former.

Convergence of finite-dimensional distributions means that for every $N=1,2,3,\dots$, any $0<t_1<\cdots<t_N\leq 1$, and any bounded continuous function $g\colon\mathbb{R}^{dN}\to\mathbb{R}$,

$$\lim_{n\to\infty}\big\langle g\big(X_n(t_1),\dots,X_n(t_N)\big)\big\rangle_n=\int g\big(B^{(\alpha\wedge 2)}(t_1),\dots,B^{(\alpha\wedge 2)}(t_N)\big)\,\mathrm{d}W^{(\alpha\wedge 2)}. \tag{5.3.4}$$

The distribution of a random variable is determined by its characteristic function, hence it suffices to consider functions g of the form

$$g(x_1,\dots,x_N)=\exp\{i\,\mathbf{k}\cdot(x_1,\dots,x_N)\}, \tag{5.3.5}$$

where $\mathbf{k}=\big(k^{(1)},\dots,k^{(N)}\big)\in(-\pi,\pi]^{dN}$ and $x_i\in\mathbb{R}^d$, $i=1,\dots,N$. We rather use the equivalent form

$$g(x_1,\dots,x_N)=\exp\{i\,\mathbf{k}\cdot(x_1,x_2-x_1,\dots,x_N-x_{N-1})\}, \tag{5.3.6}$$

which better fits in our setting.

For $\mathbf{n}=(n^{(1)},\dots,n^{(N)})\in\mathbb{N}^N$, with $n^{(1)}<\cdots<n^{(N)}$, we define

$$\begin{aligned}\hat{\mathbf{c}}_{\mathbf{n}}^{(N)}(\mathbf{k}):=\sum_{x_1,x_2,\dots,x_{n^{(N)}}}&\exp\left\{i\sum_{j=1}^{N}k^{(j)}\cdot\big(x_{n^{(j)}}-x_{n^{(j-1)}}\big)\right\}\\&\times\prod_{i=1}^{n^{(N)}}D(x_i-x_{i-1})\,\mathbb{1}_{\{(0,x_1,x_2,\dots,x_{n^{(N)}})\text{ is self-avoiding}\}}\end{aligned} \tag{5.3.7}$$

as the N-dimensional version of (1.2.2), with $n^{(0)}=0$. An alternative representation is

$$\hat{\mathbf{c}}_{\mathbf{n}}^{(N)}(\mathbf{k})=\sum_{w\in\mathcal{W}_{n^{(N)}}}\mathrm{e}^{i\mathbf{k}\cdot\Delta w(\mathbf{n})}\,W(w)\,\mathbb{1}_{\{w\text{ is self-avoiding}\}}, \tag{5.3.8}$$

where $\mathcal{W}_n=\bigcup_{x\in\mathbb{Z}^d}\mathcal{W}_n(x)=\{0\}\times\mathbb{Z}^{dn}$ is the set of n-step walks on $\mathbb{Z}^d$,

$$\mathbf{k}\cdot\Delta w(\mathbf{n})=\sum_{j=1}^{N}k^{(j)}\cdot\big(w_{n^{(j)}}-w_{n^{(j-1)}}\big)$$

and $W(w)=\prod_{i=1}^{|w|}D(w_i-w_{i-1})$ is the *weight* of the walk w ($|w|$ denotes the length).

From (5.1.3) we recall that $k_n = f_\alpha(n)\, k$, where $f_\alpha(n)$ was defined in (5.2.4). We fix a sequence b_n converging to infinity slowly enough such that

$$f_\alpha(n)^{\alpha\wedge 1}\, b_n = o(1), \tag{5.3.9}$$

for example $b_n = \log n$.

Theorem 5.11 (Finite-dimensional distributions)**.** *Let N be a positive integer, $k^{(1)},\dots,$ $k^{(N)} \in (-\pi,\pi]^d$, $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} \in \mathbb{R}$, and $g = (g_n)$ a sequence of real numbers satisfying $0 \le g_n \le b_n$. Denote*

$$\mathbf{k}_n = \big(k_n^{(1)},\dots,k_n^{(N)}\big) = f_\alpha(n)\,\big(k^{(1)},\dots,k^{(N)}\big),$$

$$n\mathbf{T} = \big(\lfloor nt^{(1)}\rfloor,\dots,\lfloor nt^{(N-1)}\rfloor,\lfloor nT\rfloor\big)$$

with $T = t^{(N)}(1-g_n)$. Under the conditions of Theorem 5.2,

$$\lim_{n\to\infty}\frac{\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat c_{nT}(0)} = \exp\left\{-K_\alpha\sum_{j=1}^N |k^{(j)}|^{\alpha\wedge 2}\,\big(t^{(j)}-t^{(j-1)}\big)\right\} \tag{5.3.10}$$

holds uniformly in g.

Before we start proving the theorem, let us emphasize that (5.3.10) has indeed the required form. Let $g_n \equiv 0$ in Theorem 5.11, so that $n\mathbf{T} = \big(\lfloor nt^{(1)}\rfloor,\dots,\lfloor nt^{(N)}\rfloor\big)$. Then

$$\begin{aligned}\Big\langle \exp\big\{i\,\mathbf{k}\cdot\Delta X_n(n\mathbf{T})\big\}\Big\rangle_n &= \Big\langle \exp\Big\{i\,(2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\,\mathbf{k}_n\cdot\Delta\bullet(n\mathbf{T})\Big\}\Big\rangle_n\\ &= \frac{\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}\Big((2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\,\mathbf{k}_n\Big)}{\hat c_{nT}(0)},\end{aligned}$$

and this converges to

$$\exp\left\{-\frac{1}{2d}\sum_{j=1}^N |k^{(j)}|^{\alpha\wedge 2}\,\big(t^{(j)}-t^{(j-1)}\big)\right\}$$

as $n\to\infty$, as we aim to show for (5.3.4). Thus the finite dimensional distributions of long-range self-avoiding walk converge to those of an α-stable Lévy motion, which proves that this is the only *possible* scaling limit.

Proof of Theorem 5.11. The proof is via induction over N, and is very much inspired by the proof of [88, Theorem 6.6.2], where finite-range models were considered. The flexibility in the last argument of $n\mathbf{T}$ is needed to perform the induction step. We shall further write $nt^{(j)}$ and nT instead of $\lfloor nt^{(j)}\rfloor$ and $\lfloor nT\rfloor$ for brevity.

To initialize the induction we consider the case $N=1$. Since $\hat{\mathbf{c}}^{(1)}_{n\mathbf{T}}(\mathbf{k}_n) = \hat c_{nT}(k_n^{(1)})$, the assertion for $N=1$ is a minor generalization of Theorem 5.2. In fact, if we replace n by nT, then instead of (5.1.4) we have

$$nT\,[1-\hat D(k_n)] = nt^{(1)}(1-g_n)\left[1-\hat D\big(f_\alpha(t^{(1)}n)\,k\,(t^{(1)})^{1/(\alpha\wedge 2)}\big)\right] \to |k|^{\alpha\wedge 2}\,t^{(1)} \quad \text{as } n\to\infty. \tag{5.3.11}$$

With an appropriate change in (5.1.37) we obtain (5.3.10) for $N=1$ from Theorem 5.2.

To advance the induction we prove (5.3.10) assuming that it holds when N is replaced by $N-1$. For a path $w\in\mathcal{W}_n$ and $0\le a\le b\le n$ it will be convenient to write

$$K_{[a,b]}(w) := \mathbb{1}_{\{(w_a,\dots,w_b)\text{ is self-avoiding}\}}. \tag{5.3.12}$$

We further consider the quantity $J_{[a,b]}(w)$ that arises in the algebraic derivation of the lace expansion as in [102, Sect. 3.2]. For our needs it suffices to know that

$$\sum_{w \in \mathcal{W}_n(x)} W(w) J_{[0,n]}(w) = \pi_n(x) \tag{5.3.13}$$

and, for any integers $0 \le m \le n$ and paths $w \in \mathcal{W}_n$,

$$K_{[0,n]}(w) = \sum_{I \ni m} K_{[0,I_1]}(w)\, J_{[I_1,I_2]}(w)\, K_{[I_2,n]}(w), \tag{5.3.14}$$

where the sum is over all intervals $I = [I_1, I_2]$ of integers with either $0 \le I_1 < m < I_2 \le n$ or $I_1 = m = I_2$. We refer to [102, (3.13)] for (5.3.13), and to [88, Lemma 5.2.5] for (5.3.14). By (5.3.8) and (5.3.14),

$$\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = \sum_{I \ni nt^{(N-1)}} \sum_{w \in \mathcal{W}_{nT}} \mathrm{e}^{i\mathbf{k}_n \cdot \Delta w(n\mathbf{T})}\, W(w)\, K_{[0,I_1]}(w)\, J_{[I_1,I_2]}(w)\, K_{[I_2,nT]}(w). \tag{5.3.15}$$

Let $\overset{\le}{\mathbf{c}}{}^{(N)}$ and $\overset{>}{\mathbf{c}}{}^{(N)}$ denote the contributions towards (5.3.15) corresponding to intervals I with length $|I| = I_2 - I_1 \le b_n$ and $|I| > b_n$, respectively. It will turn out that the latter contribution is negligible. We take n sufficiently large so that $(nt^{(N-1)} - nt^{(N-2)}) \vee (nt^{(N)} - nt^{(N-1)}) \ge b_n$ and

$$\begin{aligned}\overset{\le}{\mathbf{c}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = &\sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le b_n}} \hat{\mathbf{c}}^{(N-1)}_{(nt^{(1)},\dots,nt^{(N-2)},I_1)}\big(k_n^{(1)}, \dots, k_n^{(N-1)}\big) \times \hat{c}_{nT-I_2}(k_n^{(N)}) \\ &\times \sum_{w \in \mathcal{W}_{|I|}} \exp\big\{ik_n^{(N-1)} \cdot w_{nt^{(N-1)}-I_1} + ik_n^{(N)} \cdot (w_{I_2-I_1} - w_{nt^{(N-1)}-I_1})\big\}\, W(w)\, J_{[0,|I|]}(w).\end{aligned} \tag{5.3.16}$$

We use $\mathrm{e}^y = 1 + O(|y|^{\alpha\wedge 1})$ and (5.3.13) to see that the second line in (5.3.16) is equal to

$$\sum_x \big(1 + O(|f_\alpha(n)\, x|^{\alpha\wedge 1})\big)\, \pi_{|I|}(x). \tag{5.3.17}$$

By the induction hypothesis,

$$\begin{aligned}&\hat{\mathbf{c}}^{(N-1)}_{(nt^{(1)},\dots,nt^{(N-2)},I_1)}\big(k_n^{(1)}, \dots, k_n^{(N-1)}\big) \\ &\qquad = \hat{c}_{I_1}(0)\, \exp\left\{-K_\alpha \sum_{j=1}^{N-1} |k^{(j)}|^{\alpha\wedge 2}\, (t^{(j)} - t^{(j-1)})\right\} + o(1)\end{aligned} \tag{5.3.18}$$

and

$$\hat{c}_{nT-I_2}(k_n^{(N)}) = \hat{c}_{nT-I_2}(0)\, \exp\big\{-K_\alpha\, |k^{(N)}|^{\alpha\wedge 2}\, (t^{(N)} - t^{(N-1)})\big\} + o(1), \tag{5.3.19}$$

where the error terms are uniform in $|I| \le b_n$.

Substituting (5.3.17)–(5.3.19) into (5.3.16) yields

$$\overset{\le}{\mathbf{c}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = \exp\left\{-K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{\alpha\wedge 2}\, (t^{(j)} - t^{(j-1)})\right\} \overset{\le}{\mathbf{c}}{}^{(N)}_{n\mathbf{T}}(\mathbf{0}) + \Theta + o(1) \tag{5.3.20}$$

where

$$|\Theta| \leq \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \leq b_n}} \hat{c}_{I_1}(0)\, \hat{c}_{nT-I_2}(0) \sum_x O\big(|f_\alpha(n)\, x|^{\alpha\wedge 1}\big)\, \pi_{|I|}(x). \tag{5.3.21}$$

In (5.3.21) there are precisely $m-1$ ways to choose the interval $I \ni nt^{(N-1)}$ if $|I| = m$. We further bound

$$\begin{aligned} \frac{|\Theta|}{\hat{c}_{nT}(0)} &\leq \sum_{m=1}^{b_n} m \sum_x O\big(|f_\alpha(n)\, x|^{\alpha\wedge 1}\big)\, \pi_m(x)\, z_c^m \\ &\leq O(|f_\alpha(n)|^{\alpha\wedge 1}\, b_n) \sum_{m=1}^{\infty} \sum_x |x|^{\alpha\wedge 2}\, |\pi_m(x)|\, z_c^m = o(1), \end{aligned} \tag{5.3.22}$$

where Corollary 5.4 is used in the first inequality, $m \leq b_n$ in the second, and the last estimate uses (5.3.9) and Lemma 5.6. Recalling $\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}) = \overset{\leq}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}) + \overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k})$,

$$\frac{\overset{\leq}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)} = \exp\left\{-K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{\alpha\wedge 2}\, (t^{(j)} - t^{(j-1)})\right\} \left(1 - \frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{0})}{\hat{c}_{nT}(0)}\right) + \frac{|\Theta|}{\hat{c}_{nT}(0)} + \frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)}, \tag{5.3.23}$$

and it suffices to show $\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)/\hat{c}_{nT}(0) = o(1)$ as $n \to \infty$. By bounding $|\,\mathrm{e}^{i\mathbf{k}_n\cdot\Delta w(n\mathbf{T})}\,| \leq 1$ in (5.3.15), and using again (5.3.13) and Corollary 5.4,

$$\frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)} \leq O(1) \sum_{m=b_n+1}^{\infty} m \sum_x |\pi_m(x)|\, z_c^m, \tag{5.3.24}$$

which vanishes as $n \to \infty$ by (5.1.72) and the fact that $b_n \to \infty$ as $n \to \infty$. We have completed the advancement of the induction, and all error terms occurring are uniform in sequences $g = (g_n)$ that satisfy $0 \leq g_n \leq b_n$. This proves (5.3.10) for all $N \geq 1$. □

5.3.2 Tightness

In this subsection we prove tightness of the sequence X_n, the missing piece for the proof of Theorem 5.10. Indeed, tightness is implied by Theorem 5.9 and the following tightness criterion.

Proposition 5.12 (Tightness criterion [19])**.** *The sequence $\{X_n\}$ is tight in $D([0,1],\mathbb{R}^d)$ if the limiting process X has a.s. no discontinuity at $t = 1$ and there exist constants $C > 0$, $r > 0$ and $a > 1$ such that for $0 \leq t_1 < t_2 < t_3 \leq 1$ and for all n,*

$$\langle |X_n(t_2) - X_n(t_1)|^r\, |X_n(t_3) - X_n(t_2)|^r \rangle_n \leq C|t_3 - t_1|^a. \tag{5.3.25}$$

This proposition is a slight modification of Billingsley [19, Theorem 15.6], where (15.21) is replaced by the stronger moment condition on the bottom of page 128 (both references to Billingsley [19]).

Corollary 5.13 (Tightness)**.** *The sequence $\{X_n\}$ in* (5.3.1) *is tight in $D([0,1],\mathbb{R}^d)$.*

Proof. We first remark that α-stable Lèvy motion indeed has a version without jumps at fixed times, and hence no discontinuity at $t = 1$ occurs, see e.g. [78, Theorem 13.1]. Fix

$r = 3/4\,(\alpha \wedge 2)$ (in fact, any choice $r \in ((\alpha \wedge 2)/2, \alpha \wedge 2)$ is possible). Again we write nt for $\lfloor nt \rfloor$, for brevity. The left hand side of (5.3.25) can be written as

$$\frac{f_\alpha(n)^{2r}}{c_n\,(2dK_\alpha)^{2r/(\alpha\wedge 2)}} \sum_{w \in \mathcal{W}_n} |w(nt_2) - w(nt_1)|^r\, |w(nt_3) - w(nt_2)|^r\, W(w)\, K_{[0,n]}(w), \quad (5.3.26)$$

where $K_{[0,n]}(w)$ was defined in (5.3.12). Since

$$K_{[0,n]}(w) \le K_{[0,nt_1]}(w)\, K_{[nt_1,nt_2]}(w)\, K_{[nt_2,nt_3]}(w)\, K_{[nt_3,n]}(w) \quad (5.3.27)$$

and, by Corollary 5.4,

$$c_n^{-1} \le O(1)\, c_{nt_1}^{-1}\, c_{nt_2-nt_1}^{-1}\, c_{nt_3-nt_2}^{-1}\, c_{n-nt_3}^{-1}, \quad (5.3.28)$$

we can bound (5.3.26) from above by

$$\begin{aligned} &\langle |X_n(t_2) - X_n(t_1)|^r\, |X_n(t_3) - X_n(t_2)|^r \rangle_n \\ &\le O(1)\, f_\alpha(n)^{2r}\, \frac{1}{c_{nt_2-nt_1}} \sum_{w \in \mathcal{W}_{nt_2-nt_1}} |w(nt_2 - nt_1)|^r \\ &\qquad \times \frac{1}{c_{nt_3-nt_2}} \sum_{w \in \mathcal{W}_{nt_3-nt_2}} |w(nt_3 - nt_2)|^r \\ &= O(1)\, f_\alpha(n)^{2r} \left(\xi^{(r)}(nt_2 - nt_1)\right)^r \left(\xi^{(r)}(nt_3 - nt_2)\right)^r. \end{aligned} \quad (5.3.29)$$

By Theorem 5.9 and (5.2.4),

$$\left(\xi^{(r)}(nt^* - nt_*)\right)^r \le O(1)\, f_\alpha(n)^{-r}\, (t^* - t_*)^{r/(\alpha\wedge 2)} \quad (5.3.30)$$

for any $0 \le t_* < t^* \le 1$, so that

$$\langle |X_n(t_2) - X_n(t_1)|^r\, |X_n(t_3) - X_n(t_2)|^r \rangle_n \le O(1)\, (t_3 - t_1)^{2r/(\alpha\wedge 2)} = O(1)\, (t_3 - t_1)^{3/2}. \quad (5.3.31)$$

This proves tightness of the sequence $\{X_n\}$. □

Proof of Theorem 5.10. The convergence in distribution in Theorem 5.10 is implied by convergence of finite dimensional distributions and tightness of the sequence X_n, see e.g. [19, Theorem 15.1]. Hence, Theorem 5.11 and Corollary 5.13 imply Theorem 5.10. □

Appendix A
Diagrammatic bounds for the Ising model

This appendix is devoted to the proof of Proposition 3.6 for the Ising model. We proceed by considering the quantities $\pi_\Lambda^{(M)}$ ($M = 0, 1, 2, \dots$) defined in [95], which give rise to Π_M^Λ and R_{M+1}^Λ by [95, (1.12) and (1.13)]:

$$\delta_{0,x} + \Pi_M^\Lambda(x) = \sum_{N=0}^{M} (-1)^N \pi_\Lambda^{(N)}(x), \quad 0 \le \left| R_M^\Lambda(x) \right| \le \tau(z) \sum_{u,v} \pi_\Lambda^{(M)}(u)\, D(v-u)\, G(v,x). \tag{A.1}$$

We first discuss a bound on $\pi_\Lambda^{(N)}$, and use this to prove Proposition 3.6. Note that (A.2)–(A.3) is the Ising model analogue of Eq. (3.3.42)–(3.3.43) for percolation.

Proposition A.1 (Diagrammatic bounds for the Ising model)**.** *Suppose that, for the Ising model, $f(z) \le K$ for some $z \in (0, z_c)$, $K > 1$. Then there exists a constant $\bar{c}_K > 0$, such that*

$$\delta_{0,N} \le \sum_x \pi_\Lambda^{(N)}(x) \le \begin{cases} 1 + \bar{c}_K \beta^2 & (N = 0), \\ (\bar{c}_K \beta)^N & (N \ge 1), \end{cases} \tag{A.2}$$

and

$$\sum_x [1 - \cos(k \cdot x)] \pi_\Lambda^{(N)}(x) \le \hat{C}_{\lambda_z}(k)^{-1} (\bar{c}_K \beta)^{N \vee 1}, \tag{A.3}$$

uniformly in Λ.

This proposition is a variation of [95, Proposition 3.2]. However, it is important that the bounds of the type $\sum_x |x|^2 \pi_\Lambda^{(N)}(x)$ in [95] have been replaced by bounds involving the factor $1 - \cos(k \cdot x)$, as in (A.3). This replacement is a basic philosophy for this work. The following heuristic reasoning explains why the factor $|x|^2$ is not sufficient in the case of infinite variance spread-out models.

By (A.28) below, $\pi_z^{(0)}(x) \le G_z(x)^3$. Let us assume that $G_z(x) \approx C_{\lambda_z}(x)$, as suggested by Theorem 3.7. For $z = z_c$, and using that $C_1(x) \approx \text{const}/|x|^{d-(\alpha\wedge 2)}$, that would lead to

$$\sum_x |x|^2 \pi_{z_c}^{(0)}(x) \approx \sum_x |x|^2 \frac{1}{|x|^{3(d-(\alpha\wedge 2))}},$$

and this is finite if and only if $d < 3(d-(\alpha\wedge 2))-2$. In particular, this suggests that for $\alpha < 2$ and $2(\alpha \wedge 2) < d < 1 + 3/2(\alpha \wedge 2)$, $\sum_x |x|^2 \pi_{z_c}^{(0)}(x) = \infty$ but $\sum_x [1 - \cos(k \cdot x)] \pi_{z_c}^{(0)}(x) < \infty$. Thus, using $\sum_x |x|^2 \pi_{z_c}^{(0)}(x) < \infty$ as a criterion for $d > d_c$ suggests a wrong value for the critical dimension. Rather, it appears that we must assume $\sum_x |x|^{\alpha\wedge 2} \pi_{z_c}^{(0)}(x) < \infty$ instead.

We first show how Proposition A.1 implies Proposition 3.6, and afterwards discuss its proof.

Proof of Proposition 3.6 subject to Proposition A.1. We proceed as in the proof of Proposition 3.5. The bounds (3.3.52)–(3.3.53) follow immediately with $c_K = 2\bar{c}_K$, where the extra 1 in the $(N=0)$-case is compensated by the substraction of $\delta_{0,x}$, and the factor 2 comes from summing the geometric series (where we required β small enough to ensure $\bar{c}_K\beta \leq 1/2$). For the bounds on the remainder term R_M, we see by (A.1) that

$$\sum_x |R_M^\Lambda(x)| \leq K\hat{\pi}_\Lambda^{(M)}(0)\,\chi(z). \tag{A.4}$$

However, by (A.2), (3.3.54) follows if $z < z_c$ and $M = M(z)$ is so large that $(c_K\beta)^M\chi(z) \leq c_K\beta$. Finally, for (3.3.55), we use (2.1.50) below with $n = 3$ to see that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\,|R_M^\Lambda(x)| \leq & 7K[1-\hat{D}(k)]\hat{\pi}_\Lambda^{(M)}(0)\chi(z) + 7K\left(\hat{\pi}_\Lambda^{(M)}(0)-\hat{\pi}_\Lambda^{(M)}(k)\right)\chi(z)\\ &+ 7K\hat{\pi}_\Lambda^{(M)}(0)\left(\hat{G}_z(0)-\hat{G}_z(k)\right).\end{aligned} \tag{A.5}$$

For the first term, we use (3.3.12) and (A.2) to bound

$$7K[1-\hat{D}(k)]\hat{\pi}_\Lambda^{(M)}(0)\chi(z) \leq 14K(\bar{c}_K\beta)^M\chi(z)\hat{C}_{\lambda_z}(k)^{-1}.$$

For the second term, we use (A.3) to see that $\hat{\pi}_\Lambda^{(M)}(0) - \hat{\pi}_\Lambda^{(M)}(k) \leq \hat{C}_{\lambda_z}(k)^{-1}(\bar{c}_K\beta)^{M\vee 1}$. Finally, for the third term in (A.5), we use the upper bound on f_3 and the uniform bound $\hat{C}_{\lambda_z}(k) \leq (1-\lambda_z)^{-1} = \chi(z)$ to obtain

$$|\hat{G}_z(0)-\hat{G}_z(k)| = \frac{1}{2}|\Delta_k\hat{G}_z(0)| \leq 16K\hat{C}_{\lambda_z}(k)^{-1}\left(3\,(1-\lambda_z)^{-2}\right) = 48K\hat{C}_{\lambda_z}(k)^{-1}\chi(z)^2. \tag{A.6}$$

Together with (A.2), this yields the desired bound. □

We now prove Proposition A.1 subject to the diagrammatic bounds in [95], and this will occupy the remainder of the appendix. Our proof is an adaptation of the proof of [95, Prop. 3.2], with a modified bootstrap hypothesis. In particular, the factor $|x|^2$ at various places in that proof is replaced by the factor $1-\cos(k\cdot x)$ here. We fix $z \in (0, z_c)$ and throughout the remainder of the section omit it from the notation (e.g., we write τ for $\tau(z)$). Also we fix some subset Λ containing the origin. We keep in mind that we are interested in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$, and in fact our bounds hold uniformly in Λ. We elaborate on this after Prop. A.2 below. All sums below are taken over $\mathbb{Z}^d$, unless stated otherwise.

Writing again

$$\tilde{G}(x) = \tau(D*G)(x), \tag{A.7}$$

we note the basic estimate

$$G(x) \leq \delta_{0,x} + \tilde{G}(x) \tag{A.8}$$

resulting from the random-current representation and the source switching lemma, cf. [95, (4.2)].

As for self-avoiding walk, we write $B = (G*G)(0) = \sum_x G(x)^2$ for the *bubble diagram*, but in contrast to (2.2.16) we denote $\tilde{B} = (\tilde{G}*\tilde{G})(0)$ for the "non-vanishing bubble diagram". Under the hypothesis $f(z) \leq K$, we have that

$$B \leq 1 + O(\beta), \qquad \tilde{B} \leq O(\beta), \tag{A.9}$$

by (3.3.15) for the former, and (3.3.19) for the latter. Furthermore, it is easy to see that, by the Cauchy-Schwarz inequality, "open bubbles" are bounded by a "closed bubble", i.e., for all $x \in \mathbb{Z}^d$,

$$(G*G)(x) = \sum_v G(v)\,G(x-v) \leq B, \qquad (\tilde{G}*\tilde{G})(x) \leq \tilde{B}. \tag{A.10}$$

Here is an outline of the proof of Proposition A.1. We bound certain diagrams to be defined below in terms of B and $\tilde{B}$. In turn, these diagrams bound the lace expansion coefficients $\pi^{(j)}$, [95]. Hence, by exploiting (A.9), we prove a sufficient decay of the lace expansion coefficients subject to β being sufficiently small.

We now define various quantities needed to describe the bounding diagrams. All notation is chosen consistently with [95], which provides our basic estimates. In order to emphasize the diagrammatic structure, we write G and $\tilde{G}$ with two arguments, with the understanding that $G(y,x) = G(x-y)$, and for $\tilde{G}$ appropriately.

Let

$$\psi(y,x) := \sum_{j=0}^{\infty} (\tilde{G}^2)^{*j}(y,x) = \delta_{y,x} + \sum_{j=1}^{\infty} \sum_{\substack{u_0,u_1,\dots,u_j \in \\ \{x\}\times(\mathbb{Z}^d)^{j-1}\times\{y\}}} \prod_{l=1}^{j} \tilde{G}(u_{l-1},u_l)^2 \tag{A.11}$$

denote a "chain of bubbles", and

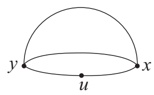
$$\tilde{\psi}(y,x) = \psi(y,x) - \delta_{y,x}. \tag{A.12}$$

If β is so small that $\tilde{B} < 1/2$ (which we shall assume from now on), then a basic calculation shows that

$$\tilde{\psi} := \sup_y \sum_x \tilde{\psi}(y,x) \le 2\tilde{B} = O(\beta). \tag{A.13}$$

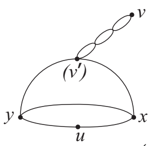
Let

$$P'^{(0)}_u(y,x) := G(y,x)^2 G(y,u)\, G(u,x) = \tag{A.14}$$



,

$$P''^{(0)}_{u,v}(y,x) := G(y,x)\, G(y,u)\, G(u,x) \sum_{v'} G(y,v')\, G(v',x)\, \psi(v',v) = \tag{A.15}$$



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In the last equalities of (A.14)–(A.15) we used the pictorial representation introduced in (2.1.40). A line between two points, say y and x, represents the two-point function $G(y,x)$, and vertices in brackets are summed over. The quantities $P'^{(0)}$ and $P''^{(0)}$ are the leading terms in the quantities P' and P'', defined in (A.22) below.

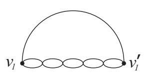
We further define

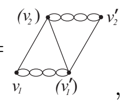
$$P^{(1)}(v_1,v_1') := 2\tilde{\psi}(v_1,v_1')\, G(v_1,v_1'), \tag{A.16}$$

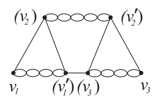
and, for $j = 2,3,\dots$,

$$P^{(j)}(v_1,v_j') := \sum_{\substack{v_2,\dots,v_j \\ v_1',\dots,v_{j-1}'}} G(v_1,v_2)\, G(v_2,v_1') \left(\prod_{i=1}^{j} \tilde{\psi}(v_1,v_1')\right) \times \left(\prod_{i=2}^{j-1} G(v_{i-1}',v_{i+1})\, G(v_{i+1},v_i')\right) G(v_j,v_{j-1}'). \tag{A.17}$$

The first three elements of the sequence look diagrammatically like

$P^{(1)}(v_1,v_1') =$ 

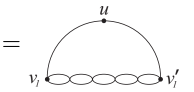
, $P^{(2)}(v_1,v_2') =$ 

, $P^{(3)}(v_1,v_3') =$ 

.

We now obtain quantities P' and P'' as variations on P. To this end, we define $P'^{(j)}_u(v_1, v'_j)$ by replacing one of the $2j-1$ two-point functions, say $G(z, z')$, on the right-hand side of (A.16)–(A.17) by the product of *two* two-point functions, $G(z,u)\,G(u,z')$, and then summing over all $2j-1$ choices of this replacement. For example,

$$P'^{(1)}_u(v_1, v'_1) = 2\tilde{\psi}(v_1, v'_1)\,G(v_1, u)\,G(u, v'_1) = \tag{A.18}$$



,

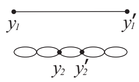
and

$$P'^{(2)}_u(v_1, v'_2) = \sum_{v_2, v'_1}\Big(\prod_{i=1}^{2}\tilde{\psi}(v_i, v'_i)\Big)\Big(G(v_1,u)\,G(u,v_2)\,G(v_2,v'_1)\,G(v'_1,v'_2) + G(v_1,v_2)\,G(v_2,u)\,G(u,v'_1)\,G(v'_1,v'_2) + G(v_1,v_2)\,G(v_2,v'_1)\,G(v'_1,u)\,G(u,v'_2)\Big). \tag{A.19}$$

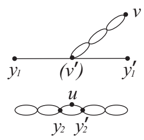
We define $P''^{(j)}_{u,v}(v_1, v'_j)$ similarly as follows. First we take *two* two-point functions in $P^{(j)}(v_1, v'_j)$, one of which (say, $G(y_1, y'_1)$ for some y_1, y'_1) is among the aforementioned $2j-1$ two-point functions, and the other (say, $\tilde{G}(y_2, y'_2)$ for some y_2, y'_2) is among those of which $\psi(v_i, v'_i) - \delta_{v_i, v'_i}$ for $i = 1, \dots, j$ are composed. The product $G(y_1, y'_1)\tilde{G}(y_2, y'_2)$ is then replaced by

$$\Big(\sum_{v'} G(y_1, v')\,G(v', y'_1)\,\psi(v', v)\Big)\Big(G(y_2, u)\tilde{G}(u, y'_2) + \tilde{G}(y_2, y'_2)\,\delta_{u, y'_2}\Big) + G(y_1, u)\,G(u, y'_1)\sum_{v'}\Big(G(y_2, v')\tilde{G}(v', y'_2) + \tilde{G}(y_2, y'_2)\,\delta_{v', y'_2}\Big)\,\psi(v', v). \tag{A.20}$$

In our pictorial representation,



is replaced by



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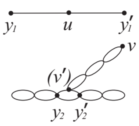
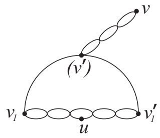


.

Finally, we define $P''^{(j)}_{u,v}(v_1, v'_j)$ by taking account of all possible combinations of $G(y_1, y'_1)$ and $\tilde{G}(y_2, y'_2)$. For example, we define $P''^{(1)}_{u,v}(v_1, v'_1)$ as

$$P''^{(1)}_{u,v}(v_1, v'_1) = \sum_{u', u'', v'}\Big(2\psi(v_1, u')\,\tilde{G}(u', u'')\Big(G(u', u)\,\tilde{G}(u, u'') + \tilde{G}(u', u'')\,\delta_{u, u''}\Big)\,\psi(u'', v'_1) \times G(v_1, v')\,G(v', v'_1)\psi(v', v) + (\text{permutation of } u \text{ and } v')\Big) \tag{A.21}$$

=



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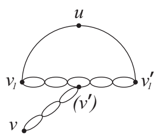
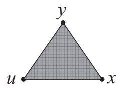
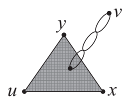


,

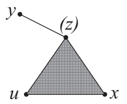
where the permutation term corresponds to the second diagram.

We let

$$P'_u(y,x) = \sum_{j\geq 0} P'^{(j)}_u(y,x) =$$  , $$P''_{u,v}(y,x) = \sum_{j\geq 0} P''^{(j)}_{u,v}(y,x) =$$  , (A.22)

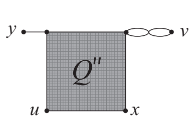
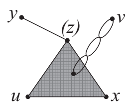
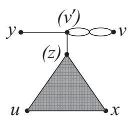
where $P'^{(0)}_u(y,x)$ and $P''^{(0)}_{u,v}(y,x)$ are the leading contributions to $P'_u(y,x)$ and $P''_{u,v}(y,x)$, respectively.

Finally, we define

$$Q'_u(y,x) = \sum_z \big(\delta_{y,z} + \tilde{G}(y,z)\big) P'_u(z,x) = \tag{A.23}$$  ,

$$\begin{aligned} Q''_{u,v}(y,x) &= \sum_z \big(\delta_{y,z} + \tilde{G}(y,z)\big) P''_{u,v}(z,x) \\ &\quad + \sum_{v',z} \big(\delta_{y,v'} + \tilde{G}(y,v')\big)\, \tilde{G}(v',z)\, P'_u(z,x)\, \psi(v',v), \end{aligned} \tag{A.24}$$

that is, pictorially,

$$Q''_{u,v}(y,x) =$$  $=$  $+$  . (A.25)

Based on the lace expansion, Sakai proved the following diagrammatic bound:

Proposition A.2 (Diagrammatic bounds [95, Prop. 4.1]). *For the ferromagnetic Ising model,*

$$\pi^{(0)}_\Lambda(x) \leq P'^{(0)}_0(0,x) \tag{A.26}$$

and, for $N \geq 1$,

$$\pi^{(N)}_\Lambda(x) \leq \sum_{\substack{b_1,\dots,b_j \\ v_1,\dots,v_j}} P'^{(0)}_{v_1}(0,\underline{b}_1) \left(\prod_{i=1}^{N-1} \tau D(b_i)\, Q''_{v_i,v_{i+1}}(\overline{b}_i, \underline{b}_{i+1}) \right) \tau D(b_j)\, Q'_{v_i,v_{i+1}}(\overline{b}_i, x), \tag{A.27}$$

where the sum is taken over vertices v_i and (directed) bonds $b_i = (\underline{b}_i, \overline{b}_i)$, $i = 1,\dots,j$. We denote $D(b_i) = D(\overline{b}_i - \underline{b}_i)$ and regard the empty product as 1 by convention. The bounds (A.26)–(A.27) holds uniformly in Λ.

Proposition A.2 is the Ising model analogue of the diagrammatic bounds in Proposition 2.2 for percolation and Proposition 2.4 for self-avoiding walk. It should be noted that Sakai [95] proved the bounds (A.26)–(A.27) on a finite graph Λ, where in particular all quantities on the right hand side are defined on Λ. By Griffith's second inequality [49], the two-point correlation function G_z is monotonically increasing in Λ, and thus so are P', Q' and Q''. Hence, the right hand side in (A.26)–(A.27) is monotonically increasing in Λ, and we consider the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$ as a uniform upper bound on $\pi^{(N)}_\Lambda(x)$. However, it is not obvious how to obtain the thermodynamic limit on the left hand side directly, since the quantities $\pi^{(N)}_\Lambda(x)$ are *not* monotone in Λ.

Proof of (A.2). We first show that $1 \le \sum_x \pi_\Lambda^{(0)}(x) \le 1+O(\beta^2)$. By the definition of $\pi_\Lambda^{(0)}(x)$ and (A.14), $\delta_{0,x} \le \pi_\Lambda^{(0)}(x) \le G(x)^3$. Whence

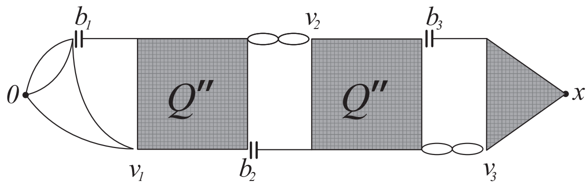
$$1 \le \sum_x \pi_\Lambda^{(0)}(x) \le 1 + \sum_{x\neq 0} G(x)^3 \le 1 + \Big(\sup_{x\neq 0} G(x)\Big) \sum_{x\neq 0} \tilde{G}^2(x). \tag{A.28}$$

The term $\sum_{x\neq 0} \tilde{G}^2(x)$ is bounded above by a non-vanishing bubble $\tilde{B}$, yielding a factor $O(\beta)$ by (A.9). The term $\sup_{x\neq 0} G(x)$ can be bounded as follows. We first apply (3.5.2), to obtain

$$\sup_{x\neq 0} G(x) \le \tau \|D\|_\infty + \|\tau D * \tilde{G}\|_\infty. \tag{A.29}$$

The first summand is bounded by $K\beta$, by our bound on f_1 and (3.2.4). Furthermore, $\|\tau D{*}\tilde{G}\|_\infty \le \|\tau D{*}\tilde{G}{*}\tilde{G}\|_\infty \le 8K^3\beta$ by (3.3.16). We thus obtain the bound on $\sum_x \pi_\Lambda^{(0)}(x)$.

We next consider the bound on $\sum_x \pi_\Lambda^{(N)}(x)$ for $N \ge 1$. Here is a diagrammatic representation of the bounds on $\sum_x \pi_\Lambda^{(N)}(x)$ for $N = 3$:



where all vertices v_1, v_2, v_3 and bonds b_1, b_2, b_3 are summed over. Since the diagrammatic bound (A.26)–(A.27) implies

$$\begin{aligned}\sum_x \pi_\Lambda^{(N)}(x) \le \Big(\sum_{v,x} P_v'^{(0)}(0,x)\Big)\Big(\sup_y \sum_{w,v,x} \tau D(w-y)\, Q''_{0,v}(w,x)\Big)^{N-1} \\ \Big(\sup_y \sum_{w,x} \tau D(w-y)\, Q'_0(w,x)\Big),\end{aligned} \tag{A.30}$$

it is sufficient to show that

(i) $\sum_{v,x} P_v'^{(0)}(0,x) \le O(1)$,

(ii) $\sup_y \sum_{w,x} \tau D(w-y)\, Q'_0(w,x) \le O(\beta)$,

(iii) $\sup_y \sum_{w,v,x} \tau D(w-y)\, Q''_{0,v}(w,x) \le O(\beta)$.

We will now prove these bounds one at a time.

(i) We first show that $\sum_{v,x} P_v'^{(0)}(0,x)$ is uniformly bounded. Indeed, by (A.10) and (A.14),

$$\sum_{v,x} P_v'^{(0)}(0,x) = \sum_{v,x} G(x)^2 G(v)\, G(v-x) \le \Big(\sup_y \sum_v G(v)\, G(v-y)\Big) \sum_x G(x)^2 \le B^2. \tag{A.31}$$

(ii) We bound

$$\sum_{w,x} \tau D(w-y)\, Q'_0(w,x) = \sum_{u,x} \Big(\sum_w \tau D(w-y)\big(\delta_{w,u} + \tilde{G}(u-w)\big)\Big) P'_0(u,x), \tag{A.32}$$

cf. (A.23). The factor β comes from the nonzero line segment $\sum_w \tau D(w-y)\big(\delta_{w,u}+\tilde{G}(u-w)\big)$, as we have seen in the discussion around (A.29).

It remains to show that $\sum_{u,x} P_0'(u,x) = \sum_{u,x}\sum_{j=0}^{\infty} P_0'^{(j)}(u,x)$ is uniformly bounded.

Claim A.3 (Bound on P')**.**

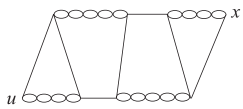
$$\sum_{u,x} P_0'(u,x) \le O(1). \tag{A.33}$$

Proof. To this end, it suffices to show

$$\sum_{u,x} P_0'^{(j)}(u,x) \le (2j-1)\, O(\beta)^j, \qquad (j \ge 1), \tag{A.34}$$

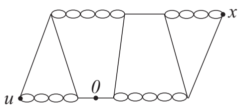
since the case $j=0$ has been treated in (A.31). The bound (A.34) will be achieved by decomposing the diagrams describing $P'^{(j)}$ into bubble diagrams, and we demonstrate this for the case $j=4$ explicitly.

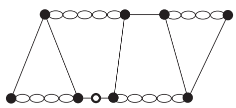
Recall from (A.17) that

$P^{(4)}(u,x) =$  , (A.35)

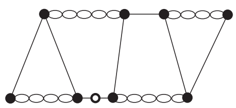
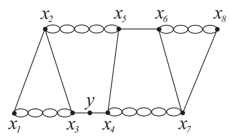
and we obtain $P'^{(4)}(u,x)$ from $P^{(4)}(u,x)$ by replacing one of the $7(=2j-1)$ factors of the form $G(u,v)$ by $\sum_w G(u,w)\,G(w,v)$. In terms of diagrams, there is an extra vertex added to either of the 7 straight lines in (A.35). This explains the factor $(2j-1)$ in (A.34).

In case this extra vertex falls to one of the horizontal lines, say the lower one, we bound as follows. We first extend our diagrammatical notation in the following way: we mark vertices that are summed over by a full dot, and fixed vertices (possibly with a supremum) are marked with an open dot, i.e.,

$\sum_{u,x}$ 

$=$  .

We decode the diagram as

 $= \sum_{\substack{x_1,x_2,x_3,x_4,\\ x_5,x_6,x_7,x_8}}$  ; (A.36)

and as a formula, this looks like

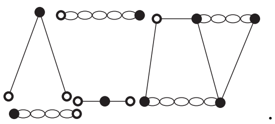
$$\begin{aligned}\sum_{\substack{x_1,x_2,x_3,x_4,\\ x_5,x_6,x_7,x_8}} & G(x_1,x_2)\,G(x_2,x_3)\,G(x_3,y)\,G(y,x_4)\,G(x_4,x_5)\,G(x_5,x_6)\\ &\times G(x_6,x_7)\,G(x_7,x_8)\,\tilde{\psi}(x_1,x_3)\,\tilde{\psi}(x_2,x_5)\,\tilde{\psi}(x_4,x_7)\,\tilde{\psi}(x_6,x_8).\end{aligned} \tag{A.37}$$

We use translation invariance to replace the sum in (A.37) by

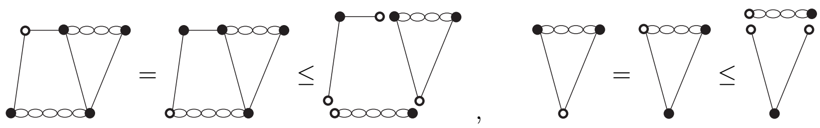
$$\sum_{\substack{x_1,x_2,y,x_4,\\ x_5,x_6,x_7,x_8}} \cdots \text{ (expression as in (A.37) with } x_3 \text{ fixed)} \tag{A.38}$$

The expression in (A.38) is bounded above by

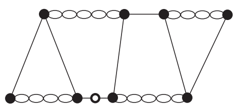
$$
\begin{aligned}
&\Bigg(\sum_{x_1} \tilde{\psi}(x_1, x_3)\Bigg) \Bigg(\sup_{\bar{x}_1} \sum_{x_2} G(\bar{x}_1, x_2)\, G(x_2, x_3)\Bigg) \\
&\times \Bigg(\sup_{\bar{x}_2} \sum_{x_5} \tilde{\psi}(\bar{x}_2, x_5)\Bigg) \Bigg(\sup_{\bar{x}_4} \sum_{y} G(x_3, y)\, G(y, \bar{x}_4)\Bigg) \\
&\times \Bigg(\sup_{x_5} \sum_{x_4, x_6, x_7, x_8} G(x_4, x_5)\, G(x_5, x_6)\, G(x_6, x_7)\, G(x_7, x_8)\, \tilde{\psi}(x_4, x_7)\, \tilde{\psi}(x_6, x_8)\Bigg)
\end{aligned}
$$

$=$  . (A.39)

For the remaining component on the right hand side, we again use translation invariance and bound further as

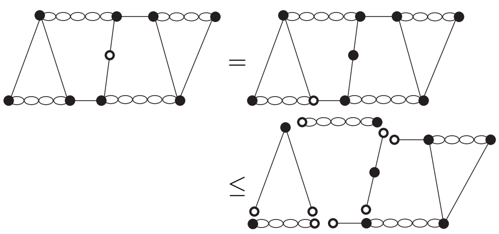
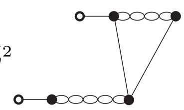
 $\le \tilde{\psi}\, B.$ (A.40)

Hence,

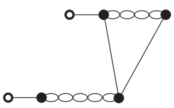
 $\le B^4 \tilde{\psi}^4,$ (A.41)

and this can be made smaller than $O(\beta)^4$, cf. (A.9) and (A.13).

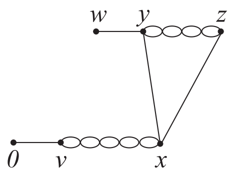
However, if the extra vertex falls to one of the vertical lines, then the details are slightly different:

 $\le B^2\, \tilde{\psi}^2$  . (A.42)

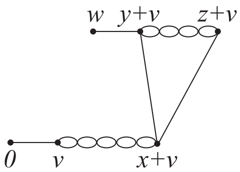
The remaining diagram in (A.42) is bounded by multiple use of translation invariance, as we will show now:



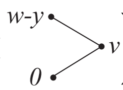
$$= \sup_{w} \sum_{v,x,y,z}$$



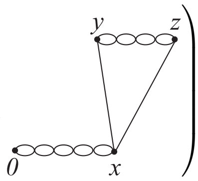
$$= \sup_{w} \sum_{x,y,z,v}$$



$$\leq \Bigg(\sup_{w,y} \sum_{v}$$



$$\Bigg) \Bigg(\sum_{x,y,z}$$



$$\Bigg) \leq B^2 \cdot \tilde{\psi}^2. \qquad \text{(A.43)}$$

This proves (A.34) for $j = 4$. The cases $j \notin \{0, 4\}$ are omitted, since the same methods will lead to the desired bounds. □

(iii) We now turn to the bounds involving Q'', i.e., we prove

$$\sup_{y} \sum_{w,v,x} \tau D(w-y) Q''_{0,v}(w,x) \leq O(\beta). \qquad \text{(A.44)}$$

Recalling the definition of Q'' in (A.24), (A.44) is established once we have shown

$$\sup_{y} \sum_{w,v,v',z,x} \tau D(w-y) \big(\delta_{w,v'} + \tilde{G}(w,v')\big)\, \tilde{G}(v',z)\, P'_0(z,x)\, \psi(v',v) \leq O(\beta) \qquad \text{(A.45)}$$

and

$$\sup_{y} \sum_{w,v,z,x} \tau D(w-y) \big(\delta_{w,z} + \tilde{G}(w,z)\big) P''_{0,v}(z,x) \leq O(\beta). \qquad \text{(A.46)}$$

A decomposition of the left hand side of (A.45) yields as an upper bound

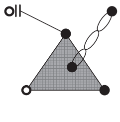
$$\Bigg(\sup_{z} \sum_{w,v'} \tau D(w-y) \big(\delta_{w,v'} + \tilde{G}(w,v')\big)\, \tilde{G}(v',z) \Bigg) \Bigg(\sup_{v'} \sum_{v} \psi(v',v) \Bigg) \Bigg(\sum_{z,x} P'_0(z,x) \Bigg), \qquad \text{(A.47)}$$

where the first term is bounded by $O(\beta)$, the second term is bounded by $1 + \tilde{\psi} = O(1)$ and the final term is bounded by $O(1)$, by Claim A.3.

It thus remains to show the following claim:

Claim A.4 (Bound on P'')**.** *The estimate (A.46) is true.*

Proof. In our pictorial representation, (A.46) can be expressed like



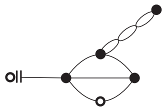
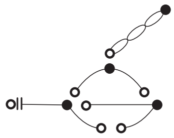
$$\leq O(\beta). \qquad \text{(A.48)}$$

Similarly to the proof of (A.33), it is sufficient to show

$$\sup_{y} \sum_{w,v,z,x} \tau D(w-y) \big(\delta_{w,z} + \tilde{G}(w,z)\big) P''^{(j)}_{0,v}(z,x) \leq O(\beta)^{j \vee 1} \qquad \text{(A.49)}$$

for $j = 0, 1, 2, \ldots$. We explicitly perform this bound for $j = 0, 1$, and omit the details for $j \geq 2$.

For $j = 0$, we bound

 $\leq$  , (A.50)

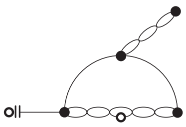
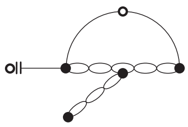
i.e.,

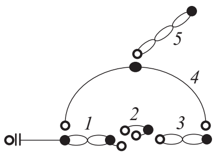
$$\sup_{y} \sum_{w,v,z,x} \tau D(w-y)\big(\delta_{w,z} + \tilde{G}(w,z)\big) P''^{(0)}_{0,v}(z,x) \leq O(\beta)\, B^2\, (1+\tilde{\psi}), \tag{A.51}$$

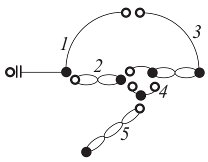
where the $O(\beta)$-factor arises from the open bubble involving the extra vertex, and the chain of bubbles hanging off from the top produces a factor $1 + \tilde{\psi}$.

For $j = 1$ we proceed similarly by recalling the definition of $P''^{(1)}$ in (A.21) and bound

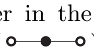
$$\sup_{y} \sum_{w,v,z,x} \tau D(w-y)\big(\delta_{w,z} + \tilde{G}(w,z)\big) P''^{(1)}_{0,v}(z,x)$$

$=$  $+$ 

$\leq$ 

$+$ 

,

where the numbers indicate the order in the decomposition. A calculation similar to (A.43) shows that  $\leq$ () () $= B(1+\tilde{\psi})$ (if the initial two-point function is dashed, then we obtain $\tilde{B}(1+\tilde{\psi})$ as an upper bound). Hence (A.49) for $j = 1$ follows. The terms for $j \geq 2$ are bounded in the same fashion. □

This completes the proof of (A.2).

Proof of (A.3). We now turn towards the proof of the bound (A.3) in Proposition A.1, which we restate here for convenience:

$$\sum_{x} [1 - \cos(k \cdot x)] \pi_{\Lambda}^{(N)}(x) \leq \hat{C}_{\lambda_z}(k)^{-1} (\bar{c}_K \beta)^{N \vee 1}.$$

We start by considering the case $N = 0$. By (A.26) and (A.14),

$$\sum_{x} [1 - \cos(k \cdot x)]\, \pi_{\Lambda}^{(0)}(x) \leq \sum_{x \neq 0} [1 - \cos(k \cdot x)]\, G^3(x). \tag{A.52}$$

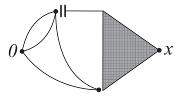
This is bounded above by

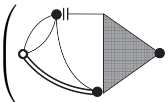
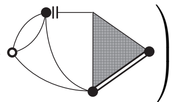
$$\Big(\sup_{x} [1 - \cos(k \cdot x)]\, G(x)\Big) \Bigg(\sum_{x \neq 0} G^2(x)\Bigg) \leq \Big(\sup_{x} [1 - \cos(k \cdot x)]\, G(x)\Big)\, \tilde{B}. \tag{A.53}$$

Then the desired bound follows from (A.9) and Lemma 3.4.

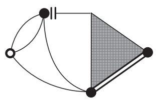
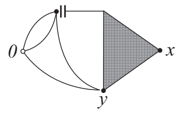
For $N > 0$, our strategy is to break the term $1 - \cos(k \cdot x)$ into parts using (2.1.50), which is reminiscent of the decomposition of squares in [95, (5.39)].

In the case $N = 1$ this allows for the following calculation. Recall from Prop. A.2 the upper bound on $\pi_\Lambda^{(1)}(x)$. An application of (2.1.50) for $n = 2$ yields

$$\sum_x [1 - \cos(k \cdot x)] \pi_\Lambda^{(1)}(x) \quad \leq \quad \sum_x [1 - \cos(k \cdot x)]$$ 

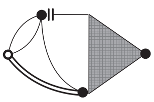
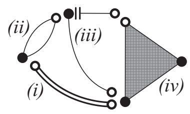
$$\leq \quad 5 \Big($$  $+$  $\Big).$ (A.54)

In (A.54) we extend our pictorial representation to incorporate factors of the form $[1 - \cos(k \cdot x)]$. Here a double line between two points, say y_1 and y_2, represents a factor $[1 - \cos(k \cdot (y_1 - y_2))]\, G(y_1 - y_2)$, while, as before, a normal line represents a factor $G(y_1 - y_2)$. For the second summand in (A.54) , there is not a single two-point function between the two endpoints of the double line. Here our understanding is that

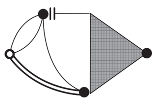
 $$= \sum_{x,y} [1 - \cos(k \cdot (x - y))]$$  . (A.55)

In other words, the double line between the two points y and x gives rise to the factor $[1 - \cos(k \cdot (x - y))]$.

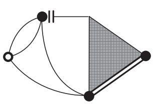
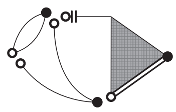
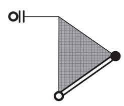
The first term in (A.54) is estimated like

 $\leq$  , (A.56)

which yields factors $B\hat{C}_{\lambda_z}(k)^{-1}$ arising from *(i)* by Lemma 3.4, B from *(ii)*, $\tilde{B}$ from *(iii)*, and $O(1)$ from *(iv)* by Claim A.3. Thus,

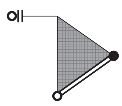
 $$\leq \hat{C}_{\lambda_z}(k)^{-1} O(\beta).$$ (A.57)

For the second term in (A.54) we bound

 $\leq$  $\leq \quad B^2 \cdot$  , (A.58)

The remaining factor $\sup_y \sum_{w,x} [1 - \cos(k \cdot x)]\, \tau D(w - y) Q_0'(w, x)$ is bounded by the following claim:

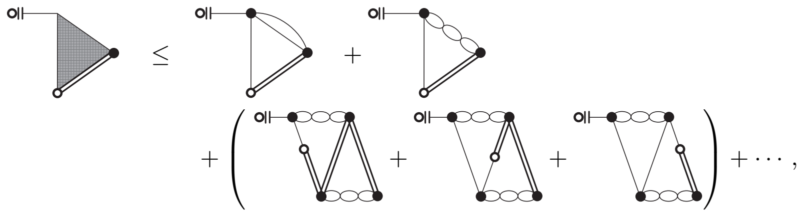
Claim A.5. *Under the assumptions of Proposition A.1,*

 $$= \sup_y \sum_{w,x} [1 - \cos(k \cdot x)]\, \tau D(w - y) Q_0'(w, x) \leq \hat{C}_{\lambda_z}(k)^{-1} O(\beta).$$ (A.59)

Proof. By (A.23),

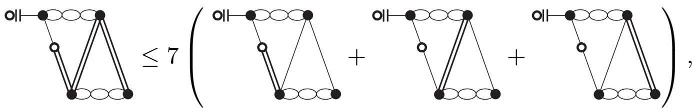
$$\begin{aligned}
&\sup_y \sum_{w,x} [1-\cos(k\cdot x)]\,\tau D(w-y) Q_0'(w,x) \\
&\quad = \sup_y \sum_{w,x,z} [1-\cos(k\cdot x)]\,\tau D(w-y) \sum_{j=0}^{\infty} \Big(\delta_{w,z} + \tilde{G}_z(w,z)\Big) P_0'^{(j)}(z,x).
\end{aligned} \qquad \text{(A.60)}$$

In diagrams, that is



(A.61)

where contributions according to $j = 0, 1, 2$ are shown explicitly and higher order contributions are indicated by dots. When we have a series of connected double lines (like in the first term in parenthesis), this indicates a factor $[1 - \cos(k \cdot (y_1 - y_2))]$, where y_1 is the starting point of the lines, and y_2 is the endpoint. We then use (2.1.50) to decompose the series of double lines. For example, for the first term in parenthesis we obtain



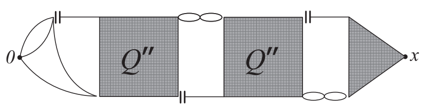
and a similar bound holds for the second term. With Lemma 3.4 it follows that the contribution from $j = 2$ in (A.61) (the term in parenthesis) is bounded by $O(\beta)^3 \hat{C}_{\lambda_z}(k)^{-1}$. The method can be generalized to $j \geq 3$ showing

$$\sup_y \sum_{w,x,z} [1-\cos(k\cdot x)]\,\tau D(w-y) \Big(\delta_{w,z} + \tilde{G}_z(w,z)\Big) P_0'^{(j)}(z,x) \leq O(j^2)\, O(\beta)^{j+1} \hat{C}_{\lambda_z}(k)^{-1}. \qquad \text{(A.62)}$$

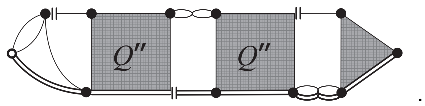
By (A.60), this is sufficient for (A.59). □

For $N > 1$, we proceed by distributing the spatial displacement $1 - \cos(k \cdot x)$ along the "bottom line" of the diagram. E.g., for $N = 3$, this yields

$$\sum_x [1-\cos(k\cdot x)] \pi_\Lambda^{(3)}(x) \quad \leq \quad \sum_x [1-\cos(k\cdot x)]$$

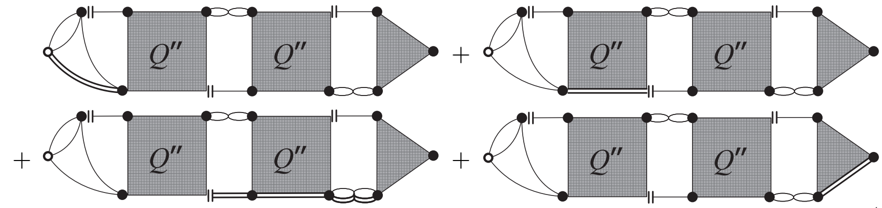


$=$



(A.63)

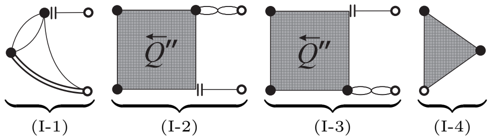
By (2.1.50), the right hand side of (A.63) is bounded above by 9 times



(A.64)

In the following we refer by (I), (II), (III) and (IV) to the four terms in (A.64), respectively. In fact, all 4 terms are bounded by $\hat{C}_{\lambda_z}(k)^{-1}\, O(\beta)^N$, as we will show now.

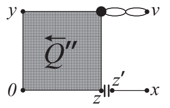
The bound on (IV) is an immediate consequence of (i) and (iii) below (A.30), and Claim A.5. For the bound on (I), we use translation invariance to obtain the factorization



(A.65)

The terms indicated by $\overleftarrow{Q}''$ in the diagram are obtaines from Q'' by shifting the two-point functions hanging off the left side of the Q''-box to the next factor on the left hand side, i.e. (compare with (A.25))

$$\overleftarrow{Q}''_{0,v}(y,x) = \sum_{z,z'} P''_{0,v}(y,z)\, \tau D(z'-z)\big(\delta_{z',x} + \tilde{G}(z',x)\big)$$
$$+ \sum_{z,z',w} \tilde{G}(y,w)\, P'_0(w,z)\, \psi(y,v)\, \tau D(z'-z)\big(\delta_{z',x} + \tilde{G}(z',x)\big) \qquad \text{(A.66)}$$

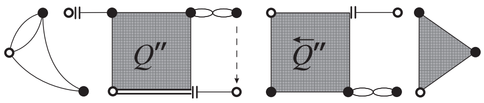
$=$ 

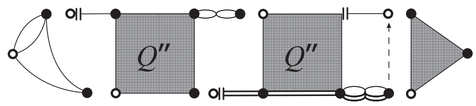
The first factor (I-1) is bounded by $\hat{C}_{\lambda_z}(k)^{-1}O(\beta)$ as in (A.56). The middle terms (I-2) and (I-3) are equal to $\sup_x \sum_{v,y} \overleftarrow{Q}''_{0,v}(y,x)$. Performing calculations as in (A.44)–(A.46), it can be shown that actually

$$\sup_x \sum_{v,y} \overleftarrow{Q}''_{0,v}(y,x) \le O(\beta), \qquad \text{(A.67)}$$

and this term occurs $N-1$ times in (I). The last term (I-4) is bounded by $O(1)$, cf. Claim A.3. The bounds on (I-1)–(I-4) show that (I) $\le \hat{C}_{\lambda_z}(k)^{-1}O(\beta)^N$.

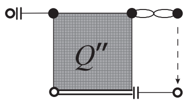
The terms (II) and (III) are bounded in a similar fashion by product structures:

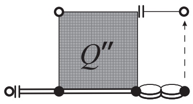
(II) $\le$  , (A.68)

(III) $\le$  . (A.69)

The term $\sum_{v,x} P_v'^{(0)}(0,x)$ on the left hand side is bounded by $O(1)$ by (A.31); the term $\sum_{u,x} P_0'(u,x)$ (the gray triangle on the right) is bounded by $O(1)$ by Claim A.3. The terms involving Q'' and $\overleftarrow{Q}''$ are bounded by $O(\beta)$ by (A.44) and (A.67), and together there are $N-2$ of these terms.

It remains to show that



$\leq \hat{C}_{\lambda_z}(k)^{-1} O(\beta)^2,$


$\leq \hat{C}_{\lambda_z}(k)^{-1} O(\beta)^2.$ (A.70)

Here the dashed arrow indicates that the supremum is taken over the *difference* between the two vertices at top and bottom of the arrow; see also [95, (5.46)]. In order to achieve the bounds in (A.70) we proceed as follows. First we use (2.1.50) to distribute the spatial displacement of $1-\cos(k\cdot x)$ to single two-point functions G or $\tilde{G}$. Secondly, from each of the emerging summands, we eliminate the term of the form $\sup_{x,y}[1-\cos(k\cdot(y-x))]G(x,y)$ (where x and y are chosen appropriately), and bound it by $\hat{C}_{\lambda_z}(k)^{-1}O(1)$, cf. Lemma 3.4. Finally, we bound the remaining quantity in the same fashion as in (A.44)–(A.46). Note that the removed bond is compensated by an extra bond hanging off the lower / upper right corner. The factor β^2 arises from the bubbles involving the two non-zero two-point functions hanging off the box. This finally leads to the required bound

$$(\mathrm{II}) + (\mathrm{III}) \leq \hat{C}_{\lambda_z}(k)^{-1} O(\beta)^N, \tag{A.71}$$

and thus proves (A.3). This completes the proof of Proposition A.1.

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Summary

The present thesis considers three models of statistical mechanics: percolation, self-avoiding walk and the Ising model. We consider these models on the hypercubic lattice $\mathbb{Z}^d$, with a coupling function D determining the local interaction: In percolation, a bond $\{x, y\}$ is occupied with probability proportional to $D(y-x)$; for self-avoiding walk, D describes the single step distribution of the underlying random walk; and for the Ising model, D is related to the spin-spin coupling. The most prominent example for D is the nearest-neighbor case, where D denotes the uniform distribution on the nearest neighbors of 0 on the hypercubic lattice. However, also various other forms of D are considered, and special attention is given to cases where D has infinite variance, so-called *long-range* models.

In Chapter 1 we introduce the above models and describe their basic features. In particular, we discuss the fact that all three models obey a phase transition, and we introduce the notion of critical exponents to describe the critical behavior. A priori these critical exponents depend on the dimension d. However, it is believed that there is an upper critical dimension d_c, such that the critical exponents have the same value for all $d > d_c$ (the *mean-field* value), but they do change when $d < d_c$. This thesis provides rigorous results for the high-dimensional case $d > d_c$.

Our tool to analyze the high-dimensional behavior of these spatial systems is the *lace expansion*. In Chapter 2 we sketch the lace expansion for percolation and self-avoiding walk: We show that the lace expansion obtains a combinatoric identity involving certain lace expansion coefficients, and demonstrate how the so-called diagrammatic bounds on these lace expansion coefficients are obtained. These diagrammatic bounds are the main ingredient in the analysis of the lace expansion of Chapter 3. The main result of this chapter is the infrared bound of Theorem 3.7. As a consequence, various critical exponents are shown to exist, and to take on their mean-field behavior, see Theorem 3.16.

Chapter 4 focusses on a somewhat different problem, namely the comparison of percolation on two different graphs: the infinite lattice $\mathbb{Z}^d$ and the (finite) d-dimensional torus, both in sufficiently high dimension. The main result of this chapter is Theorem 4.2, whose statement may be rephrased as follows: Let $|\mathcal{C}_{\max}|$ be the size of the largest connected component obtained for critical percolation on the d-dimensional torus with V vertices, where d is sufficiently large. Then $|\mathcal{C}_{\max}| \approx V^{2/3}$. The $V^{2/3}$-asymptotic is reminiscent of the Erdős-Rényi random graph model, i.e., percolation on the complete graph. In that the spatial model (percolation on the torus) shows the same asymptotics as the non-spatial model (percolation on the complete graph). The key to the proof of Theorem 4.2 is a coupling argument developed in Section 4.2.

Chapter 5 studies a long-range version of self-avoiding walk, where the single step distribution has the form $D(x) \propto |x|^{-(d+\alpha)}$. The main result of the chapter is Theorem 5.10, stating that a rescaled version of long-range self-avoiding walk in dimension $d > 2(\alpha \wedge 2)$ converges to α-stable Lévy motion if $\alpha < 2$, and to Brownian motion if $\alpha \geq 2$.

About the author

Markus Heydenreich was born on August 5, 1976 in Berlin, Germany. He completed secondary school at *Heinrich-Hertz-Oberschule Berlin* in summer 1996. In compliance with his service duty, Markus did social work with blind and deaf-blind people in Hanover, Germany, during 1997. In April 1998 he enrolled in the program 'Mathematics with Business Studies and Computer Science' at the department of Mathematics of *Technische Universität Berlin*, from which he graduated 'with distinction' in September 2004. During that period, Markus had part-time employments at DaimlerChrysler Research and Technology, at the German Children and Youth Foundation, and as a teaching assistant at the department of Mathematics of TU Berlin. Besides, Markus was an elected member of the council of the department of Mathematics, and he chaired the educational committee of the department. The academic year 2000/2001 he spent at *City University London*, UK. His *Diplomarbeit* (Master's thesis) on the parabolic Anderson model was supervised by Prof. Jürgen Gärtner, and led to publication [48].

In October 2004, Markus moved to the Netherlands and started as a Ph.D. student at *Technische Universiteit Eindhoven* and *EURANDOM*. Under the guidance of Prof. Remco van der Hofstad, he worked on mean-field behavior of statistical mechanical models. The results of this study are presented in the papers [64, 65, 67], as well as in the present thesis. Besides he was engaged in a joint project with electrical engineering about probabilistic aspects of digital-to-analog conversion, which resulted in the publications [66] and [94]. He presented his research in Berlin, Hilversum, Vancouver, Mainz, Delft, Münster, München, Leiden, Tübingen, Oberwolfach, Utrecht, Bath, Aachen, and Eindhoven. Starting January 2009, Markus will work as a postdoctoral researcher at *Vrije Universiteit Amsterdam*.

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